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Aspects of Analytic Number Theory: The Universality of the Riemann Zeta-Function

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Abstract. These notes deal with Voronin's universality theorem which states, roughly speaking, that any non-vanishing analytic function can be uniformly approximated by certain shifts of the Riemann zeta-function. We start with a brief introduction to the classical theory of the zeta-function. Then we give a self-contained proof of the universality theorem. We conclude with several interesting applications of this remarkable property and discuss some related problems and extensions.

Keywords. Riemann zeta-function, universality, value-distribution.

2000 MSC. 11M06, 11M26, 11M99, 30E10.

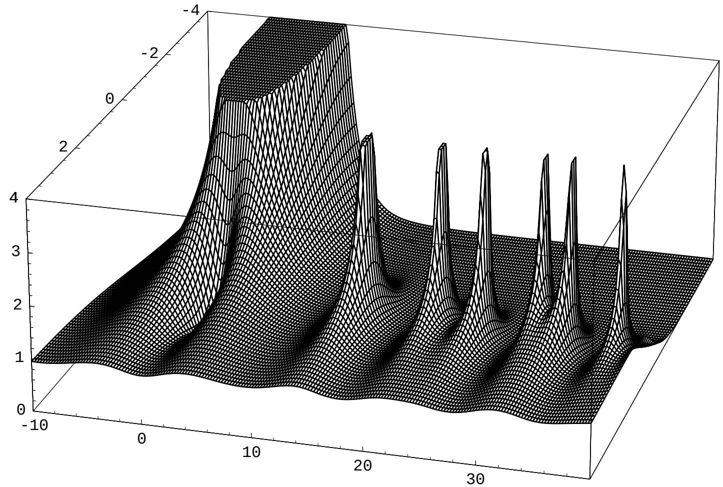

FIGURE 1. The reciprocal of the absolute value of $\zeta(\sigma + it)$ for $\sigma \in [-4,4], t \in [-10,40]$. The zeros of $\zeta(s)$ appear as poles.

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The theme of this course is an astonishing approximation property of the famous Riemann zeta-function, so the topic is settled in the intersection of complex analysis and analytic number theory. Arithmetical problems may often sound simple in their formulation; however, their treatment often needs sophisticated machinery and challenging ideas. Since the path-breaking works of Dirichlet and Riemann from the middle of the nineteenth century, analytic methods have become an important tool in number theory. The proof of the celebrated prime number theorem by investigating the distribution of zeros of the zeta-function is just one example. One of the most spectacular properties of the zeta-function is Voronin's universality theorem which states that any non-vanishing analytic function can be uniformly approximated by certain shifts of the zeta-function. Here we give a (more or less) complete proof of this remarkable result and discuss some of its applications, e.g., hypertranscendence and a criterion for the truth of the famous yet unproved Riemann hypothesis. Finally, we discuss some extensions and related open problems.

These self-contained lecture notes are mainly based on the original paper of Voronin [67], resp. the presentation in the monograph [33] of Karatsuba & Voronin with slight modifications. Thanks to Bagchi [1], Reich [56], and Laurinčikas [36], there is another, more sophisticated probabilistic approach to universality which allows slightly more general results. For the sake of simplicity we have chosen the down to earth approach of Voronin. Many of the additional results can be found in [61]. For the background in zeta-function theory (and for help with respect to the exercises) we refer to the classical monograph [63] of Titchmarsh and the online notes [62].

Jörn Steuding, Würzburg, February 2009.

CONTENTS

1. Introduction

Here we introduce the main actor, the Riemann zeta-function, and present first classical results on its amazing value-distribution due to Bohr as well as the remarkable universality theorem of Voronin. For historical details we refer to [61].

1.1. The Riemann zeta-function is universal. The Riemann zeta-function is a function of a complex variable $s = \sigma + it$,[*] for $\sigma > 1$ given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}; \tag{1.1}$$

here and in the sequel the letter p always denotes a prime number and the product is taken over all primes. The series and the product are prototypes of so-called Dirichlet series, resp. Euler products. They both converge absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset. The identity linking both, the series and the product was discovered by Euler in 1737 and can be regarded as an analytic version of the unique prime factorization of integers. The Euler product gives a first glance on the intimate connection between the zeta-function and the distribution of prime numbers. An immediate consequence is Euler's proof of the infinitude of the primes. Assuming that there were only finitely many primes, the product in (1.1) is finite, and therefore convergent for $s = 1$, contradicting the fact that the Dirichlet series defining $\zeta(s)$ reduces to the divergent harmonic series as $s \to 1+$. Hence, there exist infinitely many

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[*] This mixture of latin and greek letters is tradition in analytic number theory.

prime numbers. This fact is well known since Euclid's elementary proof, but the analytic access gives deeper knowledge on the distribution of the prime numbers as we shall see in the second chapter.

However, the main theme of these notes is a remarkable approximation property of Riemann's zeta-function, called *universality*.

By Weierstrass' celebrated approximation theorem we know that any continuous function, defined on a closed interval, can be uniformly approximated by polynomials. The set of continuous functions is rather big whereas the set of polynomials is comparably small. This makes the Weierstrass theorem remarkable. One may not believe that it is possible to approximate any continuous function on a bounded interval by a *single* function! Actually, the Riemann zeta-function has this astonishing approximation property! More precisely, shifts of its logarithm $s \mapsto \log \zeta(s + i\tau)$ can approximate any continuous function defined on a bounded interval. Of course, this approximation cannot be realized in the half-plane of absolute convergence of the zeta defining series. For our purpose we note that our protagonist, $\zeta(s)$, can be analytically continued to the whole complex plane except for a simple pole at $s = 1$, e.g.,

$$\zeta(s) = (1 - 2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}; \tag{1.2}$$

here the series on the right converges for $\sigma > 0$ (see also Exercise 1 below). Voronin's famous universality theorem states that *any non-vanishing analytic function g can be approximated uniformly by certain purely imaginary shifts of the zeta-function in the vertical strip $\frac{1}{2} < \sigma < 1$.* (A precise formulation will be given below.) For instance, for any positive ϵ, there exists a real number τ, which might be extremely large, such that the inequality

$$|\zeta(s + \tfrac{3}{4} + i\tau) - g(s)| < \epsilon$$

holds on any disk $|s| \leq r$, where $0 < r < \frac{1}{4}$ is fixed. For an illustrative example take $g = \exp$ and see the figure in Section 4.3. The same statement holds when the closed disk is replaced by a closed line segment on the imaginary axis and in this case we only need that the function g is continuous and has no zeros. This follows from a slightly more advanced version of Voronin's theorem (see Theorem 3.11). And if we want to get rid of the non-vanishing assumption, we can approximate by the logarithm of the zeta-function and this leads to the aforementioned extension of Weierstrass' approximation theorem. Another application of universality is related to the famous Riemann hypothesis, one of the seven millennium problems, about the distribution of zeros of the zeta-function (Theorem 4.3).

The first *universal* object in the mathematical literature was discovered by Fekete in 1914/15; he proved the existence of a real power series with the property that for any continuous function g on the unit interval, there exists a sequence of partial sums which approximates g uniformly. In 1926, G.D. Birkhoff [5] proved the existence of an entire function f with the property that to any given entire function g, there exists a sequence of complex numbers a_n such that $f(z + a_n) \to g(z)$ uniformly on compacta in $\mathbb{C}$, as $n \to \infty$. Universality is a frequent phenomenon in analysis, often appearing when analytical processes diverge or behave irregularly in some sense. The Riemann zeta-function and its relatives are so far the only known explicit examples of universal objects. In the following section we shall give a precise statement; however, we start with a brief look how this surprising and deep result has been developed.

1.2. Survey on value-distribution theory. The zeros of the zeta-function are of special interest (for reasons we will explain in the following chapter). It seems rather difficult to localize zeros or any other concrete values taken by the zeta-function, whereas it is much easier to study how often the values lie in a given set. Having this idea in mind, Harald Bohr refined former studies on the value-distribution of the Riemann zeta-function by applying diophantine, geometric, and probabilistic methods.

In the half-plane of absolute convergence $\sigma > 1$ we have

$$0 < |\zeta(s)| \leq \zeta(\sigma). \tag{1.3}$$

Thus the values of $\zeta(s)$ in half-planes $\sigma \geq \sigma_0 > 1$ are lying in the disk of radius $\zeta(\sigma_0)$ centered in the origin. It can be shown that $\zeta(s)$ assumes quite many of the complex values inside this disk when t varies in $\mathbb{R}$. On the other side $\zeta(\sigma)$ tends to infinity as $\sigma \to 1+$, and indeed Bohr [6] succeeded in proving that in any strip $1 < \sigma < 1 + \epsilon$, $\zeta(s)$ takes any non-zero value infinitely often. We sketch his argument. Define $\log \zeta(s)$ for any $s \in \mathbb{C}$ by choosing the principal branch of the logarithm on the intersection of the real axis with the half-plane of absolute convergence, and for other points $s = \sigma + it$ let $\log \zeta(\sigma + it)$ be the value obtained from $\log \zeta(2)$ by continuous variation along the line segments $[2, 2 + it]$ and $[2 + it, \sigma + it]$, provided that the path does not cross a zero or pole of $\zeta(s)$; if it does, then take $\log \zeta(\sigma + it) = \lim_{\epsilon \to 0+} \log \zeta(\sigma + i(t + \epsilon))$. For $\sigma > 1$,

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) = -\sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{sk}}.$$

For any fixed prime p and $\sigma > 1$, the set of values taken by the inner sum in the series representation on the right-hand side is a convex curve while t runs through $\mathbb{R}$. Adding up all these curves and using some facts from the theory of diophantine approximation, it follows that $\log \zeta(s)$ takes any complex value in $1 < \sigma < 1 + \epsilon$ which leads to Bohr's result.

The situation to the left of the abscissa of convergence is much more complicated. Here, Bohr studied finite Euler products

$$\zeta_M(s) := \prod_{p\leq M}\left(1-\frac{1}{p^s}\right)^{-1}.$$

As M tends to infinity, these products do not converge any longer but they approximate $\zeta(s)$ in the mean (we will meet this ingenious idea later again). The value-distribution of finite Euler products is treatable by the theory of diophantine approximation, and by their approximation property this leads to information on the values taken by the zeta-function. In a series of papers Bohr and his collaborators discovered that the asymptotic behaviour of $\zeta(s)$ is ruled by probability laws on every vertical line to the right of $\sigma = \frac{1}{2}$. In particular, Bohr & Courant [8] proved that for any fixed $\sigma \in (\frac{1}{2}, 1]$ the set of values $\zeta(\sigma + it)$ with $t \in \mathbb{R}$ lies dense in the complex plane. Later, Bohr refined these results significantly by applying probabilistic methods. Let $\mathcal{R}$ be an arbitrary fixed rectangle in the complex plane whose sides are parallel to the real and the imaginary axes, and let $\mathcal{G}$ be the half-plane $\sigma > \frac{1}{2}$ where all points are removed which have the same imaginary part as and smaller real part than one of the possible zeros of $\zeta(s)$ in this region. Then a remarkable limit theorem due to Bohr & Jessen [9, 10] states that for any fixed $\sigma > \frac{1}{2}$ the limit

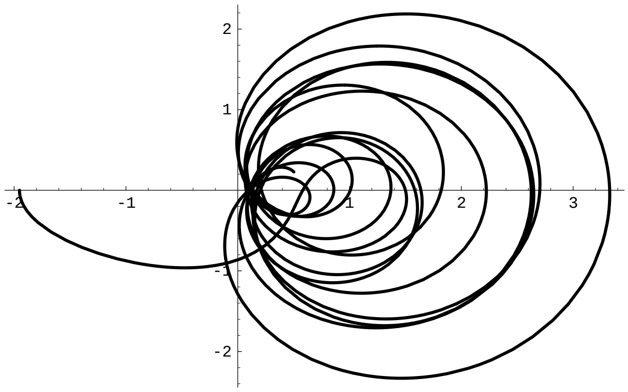

FIGURE 2. $\zeta(\frac{3}{5} + it)$ for $t \in [0, 60]$. The curve visits any neighbourhood of any point in the complex plane as t runs through the set of real numbers, and so the picture would be completely black in the end.

$$\lim_{T\to\infty} \frac{1}{T}\,\mathrm{meas}\,\{\tau \in [0, T] \,:\, \sigma + i\tau \in \mathcal{G},\, \log\zeta(\sigma + i\tau) \in \mathcal{R}\}$$

exists. Here and in the sequel $\mathrm{meas}\, A$ stands for the Lebesgue measure of a measurable set A. This limit value may be regarded as the *probability* how many values of $\log\zeta(\sigma + it)$ belong to the rectangle $\mathcal{R}$. Next, for any complex number c, denote by $N_c(\sigma_1, \sigma_2, T)$ the number of c-values of $\zeta(s)$, i.e., the roots of the equation $\zeta(s) = c$, inside the region $\sigma_1 < \sigma < \sigma_2$, $0 < t \leq T$ (counting multiplicities). From the limit theorem mentioned above Bohr & Jessen deduced

Theorem 1.1. *Let* c *be a complex number* $\neq 0$*. Then, for any* σ_1 *and* σ_2 *satisfying* $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$*, the limit* $\lim_{T\to\infty} \frac{1}{T} N_c(\sigma_1, \sigma_2, T)$ *exists and is positive.*

In 1935, Jessen & Wintner proved limit theorems similar to the one above by using more advanced methods from probability theory (infinite convolutions of probability measures). We do not mention further developments of Bohr's ideas by his successors Borchsenius, Jessen, and Wintner but refer for more details on Bohr's contribution and results of his collaborators to the monograph of Laurinčikas [36] and the survey of Matsumoto [48]. Bohr's line of investigations appears to have been almost abandoned for some time. Only in 1972, Voronin [66] obtained some significant generalizations of Bohr's denseness result.

Theorem 1.2. *For any fixed numbers* $s_1, \ldots, s_n$ *with* $\frac{1}{2} < \operatorname{Re} s_k < 1$ *for* $1 \leq k \leq n$ *and* $s_k \neq s_\ell$ *for* $k \neq \ell$*, the set*

$$\{(\zeta(s_1 + it), \ldots, \zeta(s_n + it)) : t \in \mathbb{R}\}$$

is dense in $\mathbb{C}^n$*. Moreover, for any fixed number* s *with* $\frac{1}{2} < \sigma < 1$*,*

$$\{(\zeta(s + i\tau), \zeta'(s + i\tau), \ldots, \zeta^{(n-1)}(s + i\tau)) : \tau \in \mathbb{R}\}$$

is dense in $\mathbb{C}^n$*.*

What about the value-distribution of the zeta-function on the line $\sigma = \frac{1}{2}$? It is conjectured but yet unproved that also *the set of values of* $\zeta(s)$ *taken on this vertical line is dense in* $\mathbb{C}$. However, Garunkštis & Steuding [18] have shown that the second statement of Theorem 1.2 is false for $\sigma = \frac{1}{2}$ whenever $n \geq 2$. Selberg (unpublished) proved that the values taken by an appropriate normalization of the Riemann zeta-function on this line are normally distributed: let $\mathcal{R}$ be an arbitrary fixed rectangle in the complex plane whose sides are parallel to the real and the imaginary axes, then

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ t \in (0, T] \, : \, \frac{\log \zeta\left(\frac{1}{2} + it\right)}{\sqrt{\frac{1}{2} \log\log T}} \in \mathcal{R} \right\}$$
$$= \frac{1}{2\pi} \iint_{\mathcal{R}} \exp\left(-\tfrac{1}{2}(x^2 + y^2)\right) \mathrm{d}x \, \mathrm{d}y.$$

The value-distribution on the line $\sigma = \frac{1}{2}$ is somehow special for several reasons (more about that in the following chapter).

In 1975, Voronin [67] proved his remarkable universality theorem:

Theorem 1.3. *Let* $0 < r < \frac{1}{4}$ *and suppose that* $g(s)$ *is a non-vanishing continuous function on the disk* $|s| \leq r$*, which is analytic in the interior. Then, for any*

$\epsilon > 0$, there exists a positive real number τ such that

$$\max_{|s|\leq r} \left| \zeta\left(s + \tfrac{3}{4} + i\tau\right) - g(s) \right| < \epsilon.$$

Moreover, the set of such τ has positive lower density:

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{|s|\leq r} \left| \zeta\left(s + \tfrac{3}{4} + i\tau\right) - g(s) \right| < \epsilon \right\} > 0.$$

Thus, any suitable target function can be approximated as good as we please an infinity of times. We say that $\zeta(s)$ is *universal* since appropriate shifts approximate uniformly any element of a huge class of functions. We may interpret the absolute value of an analytic function as an analytic landscape over the complex plane. Then the universality theorem states that any finite analytic landscape can be found (up to an arbitrarily small error) in the analytic landscape of the Riemann zeta-function. This is indeed a remarkable property of the zeta-function!

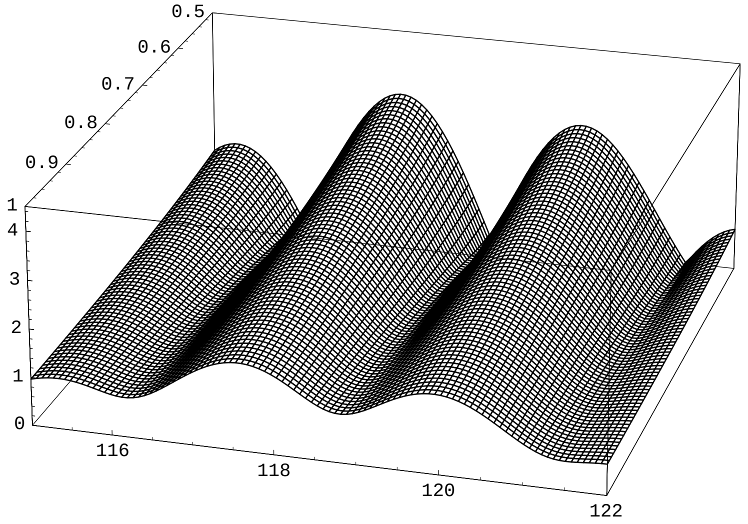

FIGURE 3. Some summits of the Himalaya or the analytic landscape of $\zeta(s)$ for $\sigma \in [\frac{1}{2}, 1], t \in [115, 122]$.

We shall give a more or less self-contained proof of Voronin's universality theorem in Chapter 3. A reader who is familiar with the basic theory of the Riemann zeta-function may directly jump to Chapter 3; for anybody else we provide the essentials of this theory in the following chapter. However, for the remaining part of this chapter we shall investigate a weaker, nevertheless still interesting approximation property than universality.

1.3. A weak approximation theorem. What might have been Voronin's intention for his studies which had led him to the discovery of this astonishing universality property?[†] One reason for Voronin's investigations might have been Bohr's concept of almost periodicity and its applications to the Riemann hypothesis (see Section 4.2). Another starting point for Voronin could have been the wish to extend Theorem 1.2; the universality theorem can be seen as an infinite dimensional analogue of the second part of this theorem. To illustrate this, we sketch how Theorem 1.2 can be used to obtain some *weak* form of the universality theorem.

Assume we are given an analytic target function $g(s)$ defined on $|s|\leq r$, where r is a positive real number. Our main tool is the Taylor series expansion

$$g(s)=\sum_{k=0}^{\infty}\frac{g^{(k)}(0)}{k!}s^k,$$

valid for all s with $|s|\leq r$. By Cauchy's formula,

$$g^{(k)}(0)=\frac{k!}{2\pi i}\oint_{|s|=r}\frac{g(s)}{s^{k+1}}\,\mathrm{d}s,$$

where the integral is taken over the circle $|s|=r$ in counterclockwise direction. Hence,

$$|g^{(k)}(0)|\leq k!Mr^{-k},$$

where $M:=\max_{|s|=r}|g(s)|$. Let $\delta\in(0,1)$. Then

$$\left|\frac{g^{(k)}(0)}{k!}s^k\right|\leq M\delta^k \qquad \text{for} \quad |s|\leq\delta r.$$

For any positive ϵ, there exists a positive integer n such that

$$\Sigma_1:=\left|g(s)-\sum_{0\leq k<n}\frac{g^{(k)}(0)}{k!}s^k\right|<\frac{\epsilon}{3} \qquad \text{for} \quad |s|\leq\delta r. \tag{1.4}$$

By Theorem 1.2 there exists a positive real number τ such that

$$|\zeta^{(k)}(\tfrac{3}{4}+i\tau)-g^{(k)}(0)|<\frac{\epsilon}{3} \qquad \text{for} \quad 0\leq k<n. \tag{1.5}$$

Thus,

$$\begin{aligned}\Sigma_2 \;&:=\; \left|\sum_{0\leq k<n}\frac{\zeta^{(k)}(\frac{3}{4}+i\tau)}{k!}s^k-\sum_{0\leq k<n}\frac{g^{(k)}(0)}{k!}s^k\right|\\ &<\; \frac{\epsilon}{3}\sum_{0\leq k<n}\frac{(\delta r)^k}{k!}<\frac{\epsilon}{3}\exp(\delta r) \qquad \text{for} \quad |s|\leq\delta r. \end{aligned}\tag{1.6}$$

[†] Voronin died in 1996.

Of course, we also have the Taylor series expansion

$$\zeta(s+\tfrac{3}{4}+i\tau)=\sum_{k=0}^{\infty}\frac{\zeta^{(k)}(\frac{3}{4}+i\tau)}{k!}s^k$$

for $|s|\leq r$. Put

$$M(\tau)=\max_{|s|=r}|\zeta(s+\tfrac{3}{4}+i\tau)|.$$

Then, again by Cauchy's formula,

$$\left|\frac{\zeta^{(k)}(\frac{3}{4}+i\tau)}{k!}s^k\right|\leq M(\tau)\delta^k\qquad\text{for}\quad|s|\leq\delta r.$$

Hence,

$$\begin{aligned}\Sigma_3\quad&:=\quad\left|\zeta(s+\tfrac{3}{4}+i\tau)-\sum_{0\leq k<n}\frac{\zeta^{(k)}(\frac{3}{4}+i\tau)}{k!}s^k\right|\\
&=\quad\left|\sum_{k=n}^{\infty}\frac{\zeta^{(k)}(\frac{3}{4}+i\tau)}{k!}s^k\right|\leq M(\tau)\frac{\delta^n}{1-\delta}\qquad\text{for}\quad|s|\leq\delta r.\end{aligned}\tag{1.7}$$

Putting (1.4)-(1.7) together, we find

$$|\zeta(s+\tfrac{3}{4}+i\tau)-g(s)|\leq\Sigma_1+\Sigma_2+\Sigma_3<\frac{\epsilon}{3}+\frac{\epsilon}{3}\exp(\delta r)+M(\tau)\frac{\delta^n}{1-\delta}.$$

Now choose $\delta>0$ such that

$$M(\tau)\frac{\delta^n}{1-\delta}=\frac{\epsilon}{3}(2-\exp(\delta r));\tag{1.8}$$

this is possible since the left-hand side tends to zero as $\delta\to0$, while the right-hand side tends to $\frac{\epsilon}{3}>0$, resp. when $\delta\to1$ the left-hand side tends to infinity, but the right-hand side remains finite. We thus have proved

Theorem 1.4. *Let g be analytic for $|s|\leq r$. Then, for any $\epsilon>0$ there exist real numbers $\tau>0$ and $\delta=\delta(\epsilon,g,\tau)\in(0,1)$ such that*

$$\max_{|s|\leq\delta r}|\zeta(s+\tfrac{3}{4}+i\tau)-g(s)|<\epsilon.$$

It is remarkable that there is no restriction on g to be non-vanishing on the disk $|s|\leq r$ as in the universality theorem. Indeed, the statement contradicts the Riemann hypothesis if g is not identically vanishing but has a zero inside the disk $|s|\leq\delta r$ (since any such zero would generate via Rouché's theorem many zeros of $\zeta(s+\frac{3}{4}+i\tau)$, as we shall show in a later section). However, it seems that there is an *inner mechanism* which prevents to obtain such an extraordinarily good approximation of the target function. We observe that a small ϵ leads to a big τ and the smaller we have to choose δ via (1.8). This follows from the fact that the zeta-function is unbounded on any vertical line with real part $\sigma<1$ and so the quantity $M(\tau)$ is increasing to infinity as $\tau\to\infty$.

A quantitative version of Theorem 1.4 can be found in the recent article by Garunkštis et al. [17]; the main tool to obtain explicit bounds for the values τ and δ is another result of Voronin, a so-called *multidimensional* Ω*-theorem*, which combines analytic and diophantine approximation properties (see also [33]). It is believed that Voronin himself was aware about statements like Theorem 1.4. For more details on the history of Voronin's theorem we refer to the nice survey articles of Laurinčikas [39] and Matsumoto [49].

Mathematics is not a spectator sport! The following exercises may help to dive deeper into zeta-function theory. Although $\zeta(s)$ is defined as an absolutely convergent series in the half-plane $\sigma > 1$, the distribution of values taken near the vertical line $\sigma = 1$ is interesting:

Exercise 1. For $\sigma > 0$ prove the representation (1.2) and deduce that $\zeta(s) < 0$ for $s \in (0,1)$.

Exercise 2. Prove inequality (1.3). Can you make use of formula (1.2) to estimate the growth of $\zeta(\sigma + it)$ for fixed $\sigma > 0$ as $t \to \infty$?

2. Zeta-function theory

In this chapter we give some hints for the importance of the Riemann zeta-function for analytic number theory. We start with a survey on the remarkable link between prime numbers and zeros of $\zeta(s)$. Later we prove the prime number theorem as well as density estimates for the number of hypothetical zeros off the critical line $\sigma = \frac{1}{2}$. Besides we develop parts of the machinery which is needed to prove Voronin's universality theorem. For historical details and more references we refer to [53].

2.1. Primes and zeros. It was the young Gauss who conjectured in 1791 for the number $\pi(x)$ of primes $p \leq x$ the asymptotic formula[‡]

$$\pi(x) \sim \mathrm{li}(x), \tag{2.1}$$

where the logarithmic integral is given by

$$\mathrm{li}(x) = \lim_{\epsilon \to 0+} \left\{ \int_0^{1-\epsilon} + \int_{1+\epsilon}^x \right\} \frac{\mathrm{d}u}{\log u} = \int_2^x \frac{\mathrm{d}u}{\log u} - 1.04\ldots.$$

Gauss' conjecture states that, in first approximation, the number of primes $\leq x$ is asymptotically $\frac{x}{\log x}$. By elementary means, Chebyshev proved around 1850 that $0.921\ldots \leq \pi(x)\frac{\log x}{x} \leq 1.055\ldots$ for sufficiently large x. Furthermore, he showed that if the limit

$$\lim_{x \to \infty} \pi(x) \frac{\log x}{x}$$

[‡] We write $f(x) \sim g(x)$, if $\lim_{x\to\infty} f(x)/g(x) = 1$.

exists, the limit is equal to one, which supports conjecture (2.1). Riemann was the first to investigate the Riemann zeta-function as a function of a complex variable. In his only one but outstanding paper [58] on number theory from 1859 he outlined how Gauss' conjecture could be proved by using the function $\zeta(s)$. However, at Riemann's time the theory of functions was not developed sufficiently far, but the open questions concerning the zeta-function pushed the research in this field quickly forward. We shall briefly discuss Riemann's memoir; some of the sketched results will later be proved in detail.

First of all, by partial summation,

$$\zeta(s) = \sum_{n\leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + s\int_N^\infty \frac{[u]-u}{u^{s+1}}\,\mathrm{d}u; \tag{2.2}$$

here and in the sequel $[u]$ denotes the maximal integer less than or equal to u. This gives an analytic continuation for $\zeta(s)$ to the half-plane $\sigma > 0$ except for a simple pole at $s = 1$ with residue 1. This process can be continued to the left half-plane and shows that $\zeta(s)$ is analytic throughout the whole complex plane except for $s = 1$. Riemann discovered the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{2.3}$$

where $\Gamma(s)$ denotes Euler's Gamma-function. In view of the Euler product (1.1) it is easily seen that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$. It follows from the functional equation and from basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\sigma < 0$ exactly at the so-called trivial zeros $s = -2n$ with $n \in \mathbb{N}$. All other zeros of $\zeta(s)$ are said to be nontrivial, and we denote them by $\rho = \beta + i\gamma$. Obviously, they have to lie inside the so-called critical strip $0 \leq \sigma \leq 1$, and it is easily seen that they are non-real. The functional equation (2.3) and the identity $\zeta(\overline{s}) = \overline{\zeta(s)}$ show some symmetries of $\zeta(s)$. In particular, the nontrivial zeros of

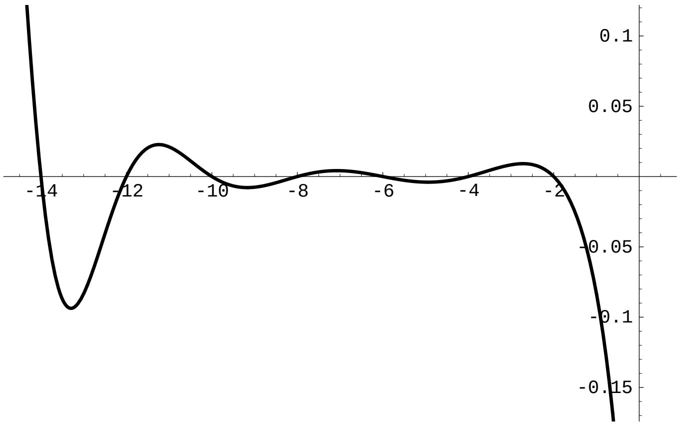

FIGURE 4. $\zeta(s)$ in the range $s \in [-14.5, 0.5]$.

$\zeta(s)$ are distributed symmetrically with respect to the real axis and to the vertical line $\sigma = \frac{1}{2}$. It was Riemann's ingenious contribution to number theory to point out how the distribution of these nontrivial zeros is linked to the distribution of prime numbers. Riemann conjectured the asymptotics for the number $N(T)$ of nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma \leq T$ (counted according multiplicities). This conjecture was proved in 1895 by von Mangoldt [46, 47] who found more precisely[§]

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{2.4}$$

Riemann worked with the function $t \mapsto \zeta(\frac{1}{2} + it)$ and wrote that very likely all roots t are real, i.e., all nontrivial zeros lie on the so-called critical line $\sigma = \frac{1}{2}$. This is the famous, yet unproved Riemann hypothesis which we rewrite equivalently as

Riemann's hypothesis. $\zeta(s) \neq 0$ *for* $\sigma > \frac{1}{2}$.

In support of his conjecture, Riemann calculated some zeros; the first one with positive imaginary part is $\rho = \frac{1}{2} + i14.134\ldots$.[¶] Furthermore, Riemann conjectured that there exist constants A and B such that

$$\tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \exp(A + Bs)\prod_{\rho}\left(1 - \frac{s}{\rho}\right)\exp\left(\frac{s}{\rho}\right),$$

where the product on the right is taken over all nontrivial zeros (the trivial zeta zeros are cancelled by the poles of the Gamma-factor). This was shown by Hadamard [22] in 1893 (on behalf of his theory of product representations of entire functions). Finally, Riemann conjectured the so-called explicit formula which states that

$$\begin{aligned} \pi(x) + \sum_{n=2}^{\infty} \frac{\pi(x^{\frac{1}{n}})}{n} &= \mathrm{li}(x) - \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \left(\mathrm{li}(x^{\rho}) + \mathrm{li}(x^{1-\rho})\right) \\ &\quad + \int_x^{\infty} \frac{\mathrm{d}u}{u(u^2-1)\log u} - \log 2 \end{aligned} \tag{2.5}$$

for any $x \geq 2$ not being a prime power (otherwise a term $\frac{1}{2k}$ has to be added on the left-hand side, where $x = p^k$); the appearing integral logarithm is defined by

$$\mathrm{li}(x^{\beta+i\gamma}) = \int_{(-\infty+i\gamma)\log x}^{(\beta+i\gamma)\log x} \frac{\exp(z)}{z + \delta i\gamma}\,\mathrm{d}z,$$

[§] We write $f(x) = O(g(x))$, if $\limsup_{x\to\infty} |f(x)|/g(x) < \infty$; equivalently, we also write $f \ll g$.

[¶] In 1932, Siegel published an account of Riemann's work on the zeta-function found in Riemann's private papers in the archive of the university library in Göttingen. It became evident that behind Riemann's speculation there was extensive analysis and computation.

where $\delta=+1$ if $\gamma>0$ and $\delta=-1$ otherwise. The explicit formula was proved by von Mangoldt [46] in 1895 as a consequence of both product representations for $\zeta(s)$, the Euler product (1.1) and the Hadamard product.

Building on these ideas, Hadamard [23] and de la Vallée-Poussin [64] found (independently) in 1896 the first proof of Gauss' conjecture (2.1), the celebrated prime number theorem. For technical reasons it is of advantage to work with the logarithmic derivative of $\zeta(s)$ which is for $\sigma>1$ given by

$$\frac{\zeta'}{\zeta}(s)=-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s},$$

where the von Mangoldt Λ-function is defined by

$$\Lambda(n)=\begin{cases}\log p & \text{if } n=p^k \text{ with } k\in\mathbb{N},\\ 0 & \text{otherwise.}\end{cases}\tag{2.6}$$

A lot of information concerning the prime counting function $\pi(x)$ can be recovered from information about

$$\psi(x):=\sum_{n\leq x}\Lambda(n)=\sum_{p\leq x}\log p+O\left(x^{\frac{1}{2}}\log x\right).\tag{2.7}$$

Partial summation gives $\pi(x)\sim\frac{\psi(x)}{\log x}$. First of all, we shall express $\psi(x)$ in terms of the zeta-function. If c is a positive constant, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}\,\mathrm{d}s=\begin{cases}1 & \text{if } x>1,\\ 0 & \text{if } 0<x<1.\end{cases}\tag{2.8}$$

This yields the so-called Perron formula: for $x\notin\mathbb{Z}$ and $c>1$,

$$\psi(x)=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,\mathrm{d}s.\tag{2.9}$$

Moving the path of integration to the left, we find that the latter expression is equal to the corresponding sum of residues, that are the residues of the integrand at the pole of $\zeta(s)$ at $s=1$, at the zeros of $\zeta(s)$, and at the additional pole of the integrand at $s=0$. The main term turns out to be

$$\operatorname{Res}_{s=1}\left\{-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\right\}=\lim_{s\to1}(s-1)\bigg(\frac{1}{s-1}+O(1)\bigg)\frac{x^s}{s}=x,$$

whereas each nontrivial zero ρ gives the contribution

$$\operatorname{Res}_{s=\rho}\left\{-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\right\}=-\frac{x^\rho}{\rho}.$$

By the same reasoning, the trivial zeros altogether contribute

$$\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n}=\tfrac{1}{2}\log\left(1-\frac{1}{x^2}\right).$$

Incorporating the residue at $s = 0$, this leads to the *exact* explicit formula

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \tfrac{1}{2}\log\left(1 - \frac{1}{x^2}\right) - \log(2\pi),$$

which is equivalent to Riemann's formula (2.5). This formula is valid for any $x \notin \mathbb{Z}$. Notice that the right-hand side of this formula is not absolutely convergent. If $\zeta(s)$ would have only finitely many nontrivial zeros, the right-hand side would be a continuous function of x, contradicting the jumps of $\psi(x)$ for prime powers x. Going on it is more convenient to cut the integral in (2.9) at $t = \pm T$ which leads to the truncated version

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho} + O\left(\frac{x}{T}(\log(xT))^2\right), \tag{2.10}$$

valid for all values of x. Next we need information on the distribution of the nontrivial zeros. Already the non-vanishing of $\zeta(s)$ on the line $\sigma = 1$ yields the asymptotic relations $\psi(x) \sim x$, resp. $\pi(x) \sim \mathrm{li}\,(x)$, which is Gauss' conjecture (2.1) and sufficient for many applications. However, more precise asymptotics with a remainder term can be obtained by a zero-free region inside the critical strip. The largest known zero-free region for $\zeta(s)$ was found by Vinogradov [65] and Korobov [35] (independently) in 1958 who proved

$$\zeta(s) \neq 0 \qquad \text{in} \quad \sigma \geq 1 - \frac{c}{(\log(|t| + 3))^{\frac{1}{3}}(\log\log(|t| + 3))^{\frac{2}{3}}},$$

where c is some positive absolute constant. In combination with the Riemann-von Mangoldt formula (2.4) one can estimate the sum over the nontrivial zeros in (2.10). Balancing out T and x, we obtain the prime number theorem with the sharpest known remainder term:

Theorem 2.1. *There exists an absolute positive constant C such that for sufficiently large x*

$$\pi(x) = \mathrm{li}\,(x) + O\left(x \exp\left(-C\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right)\right).$$

We shall give a complete proof of the prime number theorem with a slightly weaker remainder term in Section 2.6.

By the explicit formula (2.10) the impact of the Riemann hypothesis on the prime number distribution becomes visible. In 1900, von Koch [34] showed that for fixed $\theta \in [\frac{1}{2}, 1)$

$$\pi(x) - \mathrm{li}\,(x) \ll x^{\theta+\epsilon} \qquad \Longleftrightarrow \qquad \zeta(s) \neq 0 \quad \text{for} \quad \sigma > \theta\,; \tag{2.11}$$

equivalently, one can replace the left-hand side by $\psi(x) - x$. Here and in the sequel ϵ stands for an arbitrary small positive constant, not necessarily the same at each appearance. With regard to known zeros of $\zeta(s)$ on the critical line it turns out that an error term with $\theta < \frac{1}{2}$ is impossible. Thus, the Riemann hypothesis states that *the prime numbers are as uniformly distributed as possible*!

Many computations were done to find a counterexample to the Riemann hypothesis. Van de Lune, te Riele & Winter [45] localized the first 1 500 000 001 zeros, all lying without exception on the critical line; moreover they all are simple! By observations like this it is conjectured, that *all or at least almost all zeros of the zeta-function are simple*. This is the so-called essential simplicity hypothesis.

A classical density theorem due to Bohr & Landau [11] states that *most* of the zeros lie *arbitrarily close to* the critical line. Denote by $N(\sigma, T)$ the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ for which $\beta > \sigma$ and $0 < \gamma \le T$ (counting multiplicities). Bohr & Landau proved

$$N(\sigma, T) \ll T = o(N(T)) \tag{2.12}$$

for any fixed $\sigma > \frac{1}{2}$.[‖] Hence, almost all zeros are clustered around the critical line. The strongest unconditional estimate that holds throughout the right half of the critical strip is due to Gritsenko [21]:

Theorem 2.2. *For any fixed σ with $\frac{1}{2} < \sigma < 1$,*

$$N(\sigma, T) \ll T^{\frac{12}{5}(1-\sigma)} (\log T)^{\frac{91}{5}}.$$

Comparing with Theorem 1.1, we see that zero is an exceptional value of the zeta-function. The location of zeros appears to be completely different from any other value.

On the other hand, Hardy [24] showed that infinitely many zeros lie on the critical line. Refining a mollifying technique of Selberg, Levinson [43] localized more than one third of the nontrivial zeros of the zeta-function on the critical line, and as Heath-Brown [26] and Selberg (unpublished) discovered, they are all simple. Introducing Kloosterman sums, Conrey [13] was able to choose longer mollifiers in order to show that more than two fifths of the zeros are simple and on the critical line.

In the remainder of this chapter we shall prove the prime number theorem as well as a density theorem, both not as sharp as those mentioned above. Besides, we introduce much of the analytic machinery needed for the proof of Voronin's universality theorem in the following chapter.

2.2. The approximate functional equation. By the Riemann integral convergence criterion the series defining zeta converges absolutely for $\sigma > 1$. Since,

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[‖] Here we write $f(x) = o(g(x))$, if $\lim_{x\to\infty} f(x)/g(x) = 0$.

for $\sigma \geq \sigma_0 > 1$,

$$\left|\sum_{n=1}^{\infty}\frac{1}{n^s}\right| \leq \sum_{n=1}^{\infty}\frac{1}{n^{\sigma_0}} \leq 1+\sum_{n=2}^{\infty}\int_{n-1}^{n}\frac{\mathrm{d}u}{u^{\sigma_0}} = 1+\int_1^{\infty} u^{-\sigma_0}\,\mathrm{d}u = 1+\frac{1}{\sigma_0-1}, \tag{2.13}$$

the series in question converges uniformly in any half-plane $\sigma \geq \sigma_0$ with $\sigma_0 > 1$. Thus, by a well-known theorem of Weierstrass, $\zeta(s)$, being the limit of a uniformly convergent sequence of analytic functions, is analytic in its half-plane of absolute convergence.

Lemma 2.3. *$\zeta(s)$ is analytic for $\sigma > 1$ and satisfies identity (1.1), i.e.,*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}.$$

Proof. It remains to show the identity between the series and the product. By the geometric series expansion and the unique prime factorization of the integers,

$$\prod_{p\leq x}\left(1-\frac{1}{p^s}\right)^{-1}=\prod_{p\leq x}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\ldots\right)=\sum_{\substack{n\\ p|n\Rightarrow p\leq x}}\frac{1}{n^s};$$

as usual, we write $d|n$ if the integer d divides the integer n, and $d\nmid n$ otherwise. Since

$$\left|\sum_{n=1}^{\infty}\frac{1}{n^s}-\sum_{\substack{n\\ p|n\Rightarrow p\leq x}}\frac{1}{n^s}\right|\leq\sum_{n>x}\frac{1}{n^\sigma}\leq\int_x^{\infty}u^{-\sigma}\,\mathrm{d}u=\frac{x^{1-\sigma}}{\sigma-1}$$

tends to zero as $x\to\infty$, we get the desired identity by sending $x\to\infty$. •

Next we shall derive not only an analytic continuation for $\zeta(s)$ to the half-plane $\sigma > 0$ but also a rather good approximation which will be very useful later on. At $s = 1$ the zeta-function defining series reduces to the harmonic series. To obtain an analytic continuation for $\zeta(s)$ we have to seperate this singularity. For that purpose we apply Abel's partial summation:

Lemma 2.4. *Let $\lambda_1 < \lambda_2 < \ldots$ be a divergent sequence of real numbers, define for $\alpha_n \in \mathbb{C}$ the function $A(u) = \sum_{\lambda_n\leq u}\alpha_n$, and let $F : [\lambda_1,\infty) \to \mathbb{C}$ be a continuous differentiable function. Then*

$$\sum_{\lambda_n\leq x}\alpha_n F(\lambda_n)=A(x)F(x)-\int_{\lambda_1}^{x}A(u)F'(u)\,\mathrm{d}u.$$

Proof. We have

$$A(x)F(x)-\sum_{\lambda_n\leq x}\alpha_nF(\lambda_n)=\sum_{\lambda_n\leq x}\alpha_n(F(x)-F(\lambda_n))=\sum_{\lambda_n\leq x}\int_{\lambda_n}^{x}\alpha_nF'(u)\,\mathrm{d}u.$$

Since $\lambda_1 \leq \lambda_n \leq u \leq x$, interchanging integration and summation yields the assertion. •

Next we apply partial summation to the partial sums of the Dirichlet series defining zeta. Let $N < M$ be positive integers and $\sigma > 1$. Then, application of Lemma 2.4 with $F(u) = u^{-s}, \alpha_n = 1$ and $\lambda_n = n$ yields

$$\begin{aligned}\sum_{N<n\leq M} \frac{1}{n^s} &= M^{1-s} - N^{1-s} + s\int_N^M \frac{[u]}{u^{s+1}}\,\mathrm{d}u \\ &= \frac{1}{s-1}(N^{1-s} - M^{1-s}) + s\int_N^M \frac{[u]-u}{u^{s+1}}\,\mathrm{d}u.\end{aligned}$$

The integral exists for $\sigma > 0$. Sending $M \to \infty$ we obtain

Theorem 2.5. *For $\sigma > 0$,*

$$\zeta(s) = \sum_{n\leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + s\int_N^\infty \frac{[u]-u}{u^{s+1}}\,\mathrm{d}u.$$

In particular, $\zeta(s)$ has an analytic continuation to the half-plane $\sigma > 0$ except for a simple pole at $s = 1$ with residue 1.

Putting $N = 1$ in the formula of Theorem 2.5, we obtain the analytic continuation (2.2) for $\zeta(s)$. Our next aim is to derive from the representation of the theorem a very useful approximation of $\zeta(s)$ inside the critical strip.

Let $f(u)$ be any function with continuous derivative on the interval $[a, b]$. Using the lemma on partial summation with $\alpha_n = 1$ if $n \in (a, b]$, and $\alpha_n = 0$ otherwise, we get

$$\begin{aligned}\sum_{a<n\leq b} f(n) &= ([b]-[a])f(b) - \int_a^b ([u]-[a])f'(u)\,\mathrm{d}u \\ &= [b]f(b) - [a]f(a) - \int_a^b [u]f'(u)\,\mathrm{d}u.\end{aligned}$$

Obviously,

$$-\int_a^b [u]f'(u)\,\mathrm{d}u = \int_a^b \left(u - [u] - \tfrac{1}{2}\right) f'(u)\,\mathrm{d}u - \int_a^b \left(u - \tfrac{1}{2}\right) f'(u)\,\mathrm{d}u.$$

Applying partial integration to the last integral on the right-hand side, we deduce Euler's summation formula:

Lemma 2.6. *Assume that $f : [a, b] \to \mathbb{R}$ has a continuous derivative. Then*

$$\begin{aligned}\sum_{a<n\leq b} f(n) &= \int_a^b f(u)\,\mathrm{d}u + \int_a^b \left(u - [u] - \tfrac{1}{2}\right) f'(u)\,\mathrm{d}u \\ &\quad + \left(a - [a] - \tfrac{1}{2}\right) f(a) - \left(b - [b] - \tfrac{1}{2}\right) f(b).\end{aligned}$$

Next, we replace in Euler's summation formula the function $u-[u]-\frac{1}{2}$ by its Fourier series expansion.

Lemma 2.7. *For $u \in \mathbb{R} \setminus \mathbb{Z}$,*

$$\left| u-[u]-\tfrac{1}{2}-\sum_{\substack{|m|\leq M\\ m\neq 0}} \frac{\exp(-2\pi imu)}{2\pi im}\right| \leq \frac{1}{2\pi M(u-[u])},$$

and, for $u \in \mathbb{R}$,

$$\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \frac{\exp(-2\pi imu)}{2\pi im} = \begin{cases} u-[u]-\frac{1}{2} & \text{if} \quad u \notin \mathbb{Z}, \\ 0 & \text{if} \quad u \in \mathbb{Z}, \end{cases}$$

where the terms with ±m have to be added together; the partial sums are uniformly bounded in u and M.

Proof. By symmetry and periodicity it suffices to consider the case $0<u\leq \frac{1}{2}$. Since

$$\int_u^{\frac{1}{2}} \exp(-2\pi imx)\,\mathrm{d}x = \frac{(-1)^{m+1}+\exp(-2\pi imu)}{2\pi im} \qquad \text{for} \quad 0\neq m\in\mathbb{Z},$$

we obtain

$$\begin{aligned} \sum_{\substack{|m|\leq M\\ m\neq 0}} \frac{\exp(-2\pi imu)}{2\pi im}-u+\tfrac{1}{2} &= \int_u^{\frac{1}{2}} \sum_{|m|\leq M} \exp(2\pi imx)\,\mathrm{d}x \\ &= \int_u^{\frac{1}{2}} \frac{\sin((2M+1)\pi x)}{\sin(\pi x)}\,\mathrm{d}x. \qquad (2.14) \end{aligned}$$

By the mean-value theorem there exists $\xi \in (u,\frac{1}{2})$ such that the latter integral equals

$$\int_u^{\xi} \frac{\sin((2M+1)\pi x)}{\sin(\pi u)}\,\mathrm{d}x.$$

This implies both formulas of the lemma. It remains to show that the partial sums of the Fourier series are uniformly bounded in u and M. Substituting $y=(2M+1)\pi x$ in (2.14), we get

$$\begin{aligned} \int_u^{\frac{1}{2}} \frac{\sin((2M+1)\pi x)}{\sin(\pi x)}\,\mathrm{d}x &= \int_u^{\frac{1}{2}} \frac{\sin((2M+1)\pi x)}{\pi x}\,\mathrm{d}x \\ &\quad + \int_u^{\frac{1}{2}} \sin((2M+1)\pi x)\left(\frac{1}{\sin(\pi x)}-\frac{1}{\pi x}\right)\mathrm{d}x \\ &\ll \int_0^{\infty} \frac{\sin(y)}{y}\,\mathrm{d}y + \int_0^{\frac{1}{2}} \left|\frac{1}{\sin(\pi x)}-\frac{1}{\pi x}\right|\,\mathrm{d}x \end{aligned}$$

with an implicit constant not depending on u and M. Both integrals on the right exist, which gives the uniform boundedness. •

Further, we need the following estimate of exponential integrals.

Lemma 2.8. *Assume that* $F : [a,b] \to \mathbb{R}$ *has a continuous non-vanishing derivative and that* $G : [a,b] \to \mathbb{R}$ *is continuous. If* G/F' *is monotonic on* $[a,b]$, *then*

$$\left| \int_a^b G(u) \exp(iF(u))\,\mathrm{d}u \right| \le 4 \left| \frac{G}{F'}(a) \right| + 4 \left| \frac{G}{F'}(b) \right| .$$

Proof. First, we assume that $F'(u) > 0$ for $a \le u \le b$. Since $(F^{-1}(v))' = F'(F^{-1}(v))^{-1}$, substituting $u = F^{-1}(v)$ leads to

$$\int_a^b G(u) \exp(iF(u))\,\mathrm{d}u = \int_{F(a)}^{F(b)} \frac{G(F^{-1}(v))}{F'(F^{-1}(v))} \exp(iv)\,\mathrm{d}v.$$

By the monotonicity of G/F', application of the mean-value theorem gives

$$\begin{aligned} & \mathrm{Re} \left\{ \int_{F(a)}^{F(b)} \frac{G(F^{-1}(v))}{F'(F^{-1}(v))} \exp(iv)\,\mathrm{d}v \right\} \\ = \; & \frac{G}{F'}(F(a)) \int_{F(a)}^{\xi} \cos v \,\mathrm{d}v + \frac{G}{F'}(F(b)) \int_{\xi}^{F(b)} \cos v \,\mathrm{d}v \end{aligned}$$

with some $\xi \in [F(a), F(b)]$. This gives the desired estimate. The same idea applies to the imaginary part. The case $F'(u) < 0$ can be treated analogously. •

Now we are in the position to prove the van der Corput summation formula:

Theorem 2.9. *For any given* $\eta > 0$ *there exists a positive constant* $C = C(\eta)$, *depending only on* η, *with the following property: if* $f : [a,b] \to \mathbb{R}$ *is a function with continuous derivative,* $g : [a,b] \to [0,\infty)$ *is differentiable, and* f', g *and* $|g'|$ *are all monotically decreasing, then*

$$\sum_{a<n\le b} g(n) \exp(2\pi i f(n)) = \sum_{f'(b)-\eta<m<f'(a)+\eta} \int_a^b g(u) \exp(2\pi i(f(u) - mu))\,\mathrm{d}u + \mathcal{E},$$

where

$$|\mathcal{E}| \le C(\eta) \left(|g'(a)| + g(a) \log(|f'(a)| + |f'(b)| + 2) \right).$$

Van der Corput's summation formula looks rather technical but the idea is simple as we shall shortly explain. The integral

$$\int_a^b g(u) \exp(2\pi i(f(u) - mu))\,\mathrm{d}u$$

is (up to a constant factor) the Fourier transform of $g(u)\exp(2\pi i f(u))$ at $u = m$; therefore, one may interpret Theorem 2.9 as an approximate version of Poisson's summation formula (see (2.16) below).

Proof of Theorem 2.9. We apply Euler's summation formula with the function $F(u) = g(u)\exp(2\pi i f(u))$. Using the Fourier series expansion of Lemma 2.7, we get

$$\begin{aligned}\sum_{a<n\leq b} g(n)\exp(2\pi i f(n)) &= \int_a^b g(u)\exp(2\pi i f(u))\,\mathrm{d}u + O(g(a))\\ &\quad + \int_a^b \sum_{m\neq 0} \frac{\exp(-2\pi i m u)}{2\pi i m}\,(g(u)\exp(2\pi i f(u)))'\,\mathrm{d}u.\end{aligned}$$

The series on the right-hand side converges uniformly on each compact subset, which is free of integers. Moreover, the partial sums are uniformly bounded. Hence, we may interchange summation and integration. This yields

$$\begin{aligned}\sum_{a<n\leq b} g(n)\exp(2\pi i f(n)) &= \int_a^b g(u)\exp(2\pi i f(u))\,\mathrm{d}u\\ &\quad + \sum_{m\neq 0}\frac{1}{m}\left(\mathcal{I}_1(m) + \frac{1}{2\pi i}\mathcal{I}_2(m)\right) + O(g(a)),\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}\mathcal{I}_1(m) &:= \int_a^b f'(u)g(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u,\\ \mathcal{I}_2(m) &:= \int_a^b g'(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u.\end{aligned}$$

Partial integration gives

$$\begin{aligned}\mathcal{I}_1(m) &= \left[\frac{\exp(2\pi i(f(u)-mu))g(u)}{2\pi i}\right]_{u=a}^{b}\\ &\quad - \int_a^b \frac{\exp(2\pi i f(u))}{2\pi i}(g(u)\exp(-2\pi i m u))'\,\mathrm{d}u,\\ &= O(g(a)) - \frac{1}{2\pi i}\mathcal{I}_2(m) + m\int_a^b g(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{\substack{f'(b)-\eta<m<f'(a)+\eta\\ m\neq 0}} \frac{1}{m}\left(\mathcal{I}_1(m)+\frac{1}{2\pi i}\mathcal{I}_2(m)\right)\\ = \sum_{\substack{f'(b)-\eta<m<f'(a)+\eta\\ m\neq 0}} \int_a^b g(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u\\ +O\left(\sum_{\substack{f'(b)-\eta<m<f'(a)+\eta\\ m\neq 0}} \frac{g(a)}{|m|}\right).\end{aligned}$$

Now assume that $m > f'(a)+\eta$ and $f'(b) > 0$. Then $f'(u) > 0$ for $a \leq u \leq b$. Using Lemma 2.8 with $F(u) = 2\pi(f(u) - mu)$ and $G = gf'$, we find

$$\mathcal{I}_1(m) \ll \left|\frac{g(a)f'(a)}{f'(a)-m}\right|.$$

Hence,

$$\sum_{\substack{m>f'(a)+\eta\\ m\neq 0}} \left|\frac{\mathcal{I}_1(m)}{m}\right| \ll g(a)\sum_{0<m\leq 2|f'(a)|}\frac{1}{m} + g(a)\sum_{m>|f'(a)|}\frac{|f'(a)|}{m^2}.$$

The contribution arising from $m < f'(b)-\eta$ can be treated similarly. This gives

$$\sum_{\substack{m\notin[f'(b)-\eta,f'(a)+\eta]\\ m\neq 0}} \left|\frac{\mathcal{I}_1(m)}{m}\right| \ll g(a)\log(|f'(a)|+|f'(b)|+2).$$

Now assume $m > f'(a)+\eta$ and $m \neq 0$. Then, by the mean-value theorem, we get for the-real part

$$\operatorname{Re}\mathcal{I}_2(m) = -\int_a^b |g'(u)|\cos 2\pi(f(u)-mu)\,\mathrm{d}u = g'(a)\int_a^\xi \cos 2\pi(f(u)-mu)\,\mathrm{d}u$$

with some $\xi \in (a,b)$. Partial integration yields

$$\begin{aligned}\int_a^\xi \cos 2\pi(f(u)-mu)\,\mathrm{d}u &= \left[-\mathrm{Re}\,\frac{\exp(2\pi i(f(u)-mu))}{2\pi i m}\right]_{u=a}^{\xi}\\ &\quad +\mathrm{Re}\,\frac{1}{m}\int_a^\xi f'(u)\exp(2\pi i(f(u)-mu))\,\mathrm{d}u\\ &\ll \frac{1}{|m|}\left(1+\frac{|f'(a)|}{|f'(a)-m|}\right).\end{aligned}$$

Therefore,

$$\sum_{m>f'(a)+\eta, m\neq 0}\left|\frac{\operatorname{Re}\mathcal{I}_2(m)}{m}\right| \ll g'(a).$$

With slight modifiactions this method applies also to the imaginary part of $\mathcal{I}_2(m)$ and the case $m \leq f'(b) - \eta$. Further, if $0 \notin [f'(b) - \eta, f'(a) + \eta]$, then Lemma 2.8 gives

$$\int_a^b g(u) \exp(2\pi i f(u)) \, \mathrm{d}u \ll g(a).$$

In view of (2.15) the theorem follows from the above estimates under the condition $f'(b) > 0$. If this condition is not fulfilled, then we may argue with $f(u) - (1 - [f'(b)])u$ in place of $f(u)$. •

Now we apply van der Corput's summation formula to the zeta-function. Let $\sigma > 0$. By Theorem 2.5 we have

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \sum_{x < n \leq N} \frac{\exp(-it \log n)}{n^\sigma} + \frac{N^{1-s}}{s-1} + s \int_N^\infty \frac{[u] - u}{u^{s+1}} \, \mathrm{d}u.$$

Setting $g(u) = u^{-\sigma}$ and $f(u) = -\frac{t}{2\pi} \log u$, we get $f'(u) = -\frac{t}{2\pi u}$. Assume that $|t| \leq 4x$, then $|f'(u)| \leq \frac{7}{8}$. With the choice $\eta = \frac{1}{10}$ the interval $(f'(b) - \eta, f'(a) + \eta)$ contains only the integer $m = 0$. Thus Theorem 2.9 yields

$$\sum_{x < n \leq N} \frac{\exp(-it \log n)}{n^\sigma} = \int_x^N u^{-s} \, \mathrm{d}u + O(x^{-\sigma}) = \frac{N^{1-s} - x^{1-s}}{1-s} + O(x^{-\sigma}).$$

In addition with

$$s \int_N^\infty \frac{[u] - u}{u^{s+1}} \, \mathrm{d}u \ll |s| N^{-\sigma}$$

and letting $N \to \infty$, we deduce

Theorem 2.10. *We have, uniformly for $\sigma \geq \sigma_0 > 0, |t| \leq 4x$,*

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + O\left(x^{-\sigma}\right).$$

This so-called approximate functional equation was found by Hardy & Littlewood; the name comes from the appearing quantities s and $1 - s$ as in the functional equation (2.3). There are better approximate functional equations known, where the approximation is realized by shorter sums with a smaller error term.

2.3. The functional equation. Now we shall prove the functional equation (2.3) for Riemann's zeta-function:

Theorem 2.11. *For any $s \in \mathbb{C}$,*

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Riemann [58] himself gave two proofs of the functional equation. In the meantime several rather different proofs were found. Here we follow Riemann's original approach which relies on the functional equation of the theta-function which is given by the infinite series

$$\theta(x)=\sum_{n\in\mathbb{Z}}\exp(-\pi xn^2).$$

Recall the Poisson summation formula: if $f:\mathbb{R}\to\mathbb{R}$ is twice differentiable with $f(z)\ll z^{-2}$ as $z\to\pm\infty$, and f'' is integrable over $\mathbb{R}$, then

$$\sum_{n\in\mathbb{Z}}f(n+\alpha)=\sum_{m\in\mathbb{Z}}\hat{f}(m)\exp(2\pi im\alpha)\tag{2.16}$$

for all $\alpha\in\mathbb{R}$, where $\hat{f}$ denotes the Fourier transform of f. We shall apply the Poisson summation formula with the function $f(z):=\exp(-\pi\frac{z^2}{x})$, where $x>0$. First, we compute the Fourier transform by quadratic substitution:

$$\begin{aligned}\hat{f}(y)&=\int_{-\infty}^{+\infty}\exp(-\pi(\tfrac{z^2}{x}+2iyz))\,\mathrm{d}z\\&=x\exp(-\pi xy^2)\int_{-\infty}^{+\infty}\exp(-\pi x(w+iy)^2)\,\mathrm{d}w.\qquad(2.17)\end{aligned}$$

To gon on we evaluate the integral

$$I(\lambda):=\int_{-\infty}^{+\infty}\exp(-\pi x(w+\lambda)^2)\,\mathrm{d}w,$$

where λ is any complex number. For this aim we consider the integral

$$\int_{\mathcal{R}}\exp(-x\omega^2)\,\mathrm{d}\omega,$$

where $\mathcal{R}$ is the rectangular contour with vertices $\pm r,\pm r+i\mathrm{Im}\,\lambda$, and r is a positive real number. By Cauchy's theorem, the integral is equal to zero. On the line $\mathrm{Re}\,\omega=r$, the integrand tends uniformly to zero as $r\to\infty$. Hence, $I(\lambda)=I(0)$, and thus the integral $I(\lambda)$ does not depend on λ. This gives in (2.17)

$$\hat{f}(y)=x\exp(-\pi xy^2)\int_{-\infty}^{+\infty}\exp(-\pi xw^2)\,\mathrm{d}w=C\sqrt{x}\exp(-\pi xy^2),$$

where

$$C:=\int_{-\infty}^{+\infty}\exp(-\pi z^2)\,\mathrm{d}z.$$

Now applying Poisson's summation formula leads to

$$\sum_{n\in\mathbb{Z}}\exp(-\pi\tfrac{(n+\alpha)^2}{x})=C\sqrt{x}\sum_{m\in\mathbb{Z}}\exp(-\pi xm^2+2\pi im\alpha).$$

Choosing $\alpha = 0$ and $x = 1$, both sums are equal; thus, $C = 1$ and we have just proved the functional equation for the theta-function:

Theorem 2.12. *For any $x > 0$,*

$$\theta(x) = \frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right).$$

Now we are ready to give the

Proof of Theorem 2.11. For $\mathrm{Re}\, z > 0$, the Gamma-function may be defined by Euler's integral

$$\Gamma(z) = \int_0^\infty u^{z-1}\exp(-u)\,\mathrm{d}u.$$

Substituting $u = \pi n^2 x$ leads to

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\frac{1}{n^s} = \int_0^\infty x^{\frac{s}{2}-1}\exp(-\pi n^2 x)\,\mathrm{d}x. \tag{2.18}$$

Summing up over all $n \in \mathbb{N}$ yields

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^\infty \frac{1}{n^s} = \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1}\exp(-\pi n^2 x)\,\mathrm{d}x.$$

On the left-hand side we find the Dirichlet series defining $\zeta(s)$; in view of its range of convergence, the latter formula is valid only for $\sigma > 1$. On the right-hand side we may interchange summation and integration, justified by absolute convergence. Thus we obtain

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^\infty \exp(-\pi n^2 x)\,\mathrm{d}x.$$

We split the integral at $x = 1$ and get

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \left\{\int_0^1 + \int_1^\infty\right\} x^{\frac{s}{2}-1}\omega(x)\,\mathrm{d}x, \tag{2.19}$$

where the series $\omega(x)$ is given in terms of the theta-function:

$$\omega(x) := \sum_{n=1}^\infty \exp(-\pi n^2 x) = \tfrac{1}{2}\left(\theta(x) - 1\right).$$

In view of the functional equation for the theta-function,

$$\omega\left(\frac{1}{x}\right) = \tfrac{1}{2}\left(\theta\left(\frac{1}{x}\right) - 1\right) = \sqrt{x}\omega(x) + \tfrac{1}{2}(\sqrt{x} - 1),$$

we find by the substitution $x \mapsto \frac{1}{x}$ that the first integral in (2.19) is equal to

$$\int_1^\infty x^{-\frac{s}{2}-1}\omega\left(\frac{1}{x}\right)\,\mathrm{d}x = \int_1^\infty x^{-\frac{s+1}{2}}\omega(x)\,\mathrm{d}x + \frac{1}{s-1} - \frac{1}{s}.$$

Substituting this in (2.19) yields

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty\left(x^{-\frac{s+1}{2}}+x^{\frac{s}{2}-1}\right)\omega(x)\,\mathrm{d}x. \tag{2.20}$$

Since $\omega(x)\ll\exp(-\pi x)$, the last integral converges for all values of s, and thus (2.20) holds by analytic continuation throughout the complex plane. The right-hand side remains unchanged by $s\mapsto 1-s$. This proves the functional equation for zeta. •

To indicate the power of the functional equation we consider the growth of the zeta-function on vertical lines. A standard application of the Phragmén–Lindelöf principle (see [61, 63]) to the entire function

$$\tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{2.21}$$

in combination with Stirling's formula shows that for any vertical strip $\sigma_1\leq\sigma\leq\sigma_2$ of bounded width there exists a positive constant c such that

$$\zeta(\sigma+it)\ll t^c\qquad\text{as}\quad t\to\infty. \tag{2.22}$$

In the particular case of the critical line this easily yields the bound

$$\zeta\left(\tfrac{1}{2}+it\right)\ll t^{\frac{1}{4}+\epsilon}\qquad\text{as}\quad t\to\infty.$$

Better estimates are known. Using rather advanced methods (lattice points, estimates for exponential series, etc.), Huxley [30] obtained the exponent $\frac{32}{205}+\epsilon$. The yet unproved Lindelöf hypothesis states

$$\zeta\left(\tfrac{1}{2}+it\right)\ll t^{\epsilon}\qquad\text{as}\quad t\to\infty; \tag{2.23}$$

note that the truth of the Riemann hypothesis would imply the latter estimate.

Another application of the functional equation yields a proof of the Riemann–von Mangoldt formula (2.4). For this aim one applies the argument principle to the function given by (2.21). However, in the sequel we are mainly concerned with zero-counting functions in rectangles to the right of the critical line.

2.4. The mean-square and applications. Using the approximate functional equation, we shall derive a mean-square formula for $\zeta(s)$ in the half-plane $\sigma>\frac{1}{2}$. Such mean-square formulae are important tools in the theory of the Riemann zeta-function. For example, they provide information on the number of hypothetical zeros off the critical line as we shall see below.

Theorem 2.13. *For* $\sigma>\frac{1}{2}$,

$$\int_1^T|\zeta(\sigma+it)|^2\,\mathrm{d}t=\zeta(2\sigma)T+O(T^{2-2\sigma}\log T).$$

Proof. By the approximate functional equation,

$$\zeta(\sigma+it)=\sum_{n<t}\frac{1}{n^{\sigma+it}}+O(t^{-\sigma}).$$

Using $\zeta(\overline{s})=\overline{\zeta(s)}$, we get

$$\int_1^T\left|\sum_{n<t}\frac{1}{n^{\sigma+it}}\right|^2\mathrm{d}t=\int_1^T\sum_{m,n<t}\frac{1}{n^{\sigma+it}m^{\sigma-it}}\,\mathrm{d}t=\sum_{m,n<T}\frac{1}{(mn)^\sigma}\int_\tau^T\left(\frac{m}{n}\right)^{it}\mathrm{d}t$$

with $\tau:=\max\{m,n\}$. The diagonal terms $m=n$ give the contribution

$$\sum_{n<T}\frac{T-n}{n^{2\sigma}}=T\left(\zeta(2\sigma)-\sum_{n\geq T}\frac{1}{n^{2\sigma}}\right)-\sum_{n<T}\frac{1}{n^{2\sigma-1}}=\zeta(2\sigma)T+O(T^{2-2\sigma}).$$

The non-diagonal terms $m\neq n$ contribute

$$\sum_{\substack{m,n<T\\ m\neq n}}\frac{1}{(mn)^\sigma}\frac{\left(\frac{m}{n}\right)^{iT}-\left(\frac{m}{n}\right)^{i\tau}}{i\log\frac{n}{m}}\ll\sum_{0<m<n<T}\frac{1}{(mn)^\sigma\log\frac{n}{m}}.$$

If $m<\frac{n}{2}$ then $\log\frac{n}{m}>\log 2>0$, and hence

$$\sum_{n<T}\sum_{m<\frac{n}{2}}\frac{1}{(mn)^\sigma\log\frac{n}{m}}\ll\left(\sum_{n<T}\frac{1}{n^\sigma}\right)^2\ll T^{2-2\sigma}.$$

If $m\geq\frac{n}{2}$ we write $n=m+r$ with $1\leq r\leq\frac{n}{2}$. By the Taylor series expansion of the logarithm,

$$\log\frac{n}{m}=-\log\left(1-\frac{r}{n}\right)>\frac{r}{n}.$$

This gives

$$\sum_{n<T}\sum_{r\leq\frac{n}{2}}\frac{1}{(mn)^\sigma\log\frac{n}{m}}\ll\sum_{n<T}n^{1-2\sigma}\sum_{r\leq\frac{n}{2}}\frac{1}{r}\ll T^{2-2\sigma}\log T.$$

Collecting together, the assertion of the theorem follows. •

In view of the simple pole of the zeta-function, the mean-square formula above cannot hold on the critical line because $\zeta(2\sigma)$ is unbounded as $\sigma\to\frac{1}{2}+$. Hardy & Littlewood [25] have shown

$$\int_0^T|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t=T\log T+O(T).$$

The asymptotics of the fourth power moment were found by Ingham; the asymptotics of the sixth moment and all higher moments are unsettled.

From the above theorem we can deduce some remarkable information on the distribution of zeros of $\zeta(s)$. This observation dates back to Littlewood [44]. For this purpose we need the following integrated version of the argument principle, also known as Littlewood's lemma:

Lemma 2.14. *Let $A < B$ and let $f(s)$ be analytic on $\mathcal{R} := \{s \in \mathbb{C} : A \leq \sigma \leq B, |t| \leq T\}$. Suppose that $f(s)$ does not vanish on the right edge $\sigma = B$ of $\mathcal{R}$. Let $\mathcal{R}'$ be $\mathcal{R}$ minus the union of the horizontal cuts from the zeros of f in $\mathcal{R}$ to the left edge of $\mathcal{R}$, and choose a single-valued branch of $\log f(s)$ in the interior of $\mathcal{R}'$. Denote by $\nu(\sigma, T)$ the number of zeros $\rho = \beta + i\gamma$ of $f(s)$ inside the rectangle with $\beta > \sigma$ including zeros with $\gamma = T$ but not those with $\gamma = -T$. Then*

$$\int_{\partial\mathcal{R}} \log f(s)\,\mathrm{d}s = -2\pi i \int_A^B \nu(\sigma, T)\,\mathrm{d}\sigma.$$

We give a sketch of the simple proof. Cauchy's theorem implies $\int_{\partial\mathcal{R}'} \log f(s)\,\mathrm{d}s = 0$, and so the left-hand side of the formula of the lemma, $\int_{\partial\mathcal{R}}$, is minus the sum of the integrals around the paths hugging the cuts. Since the function $\log f(s)$ jumps by $2\pi i$ across each cut (assuming for simplicity that the zeros of f in $\mathcal{R}$ are simple and have different height; the general case is no harder), $\int_{\partial\mathcal{R}}$ is $-2\pi i$ times the total length of the cuts, which is the right-hand side of the formula in the lemma.

Littlewood's lemma can be used in various ways to obtain estimates for the number of zeros of the zeta-function in certain regions of the complex plane. We start with a weak version of the Riemann-von Mangoldt formula (2.4) for the number $N(T)$ of nontrivial zeros $\rho = \beta + i\gamma$ with imaginary part $\gamma \in (0, T]$.

Theorem 2.15. *For sufficiently large T,*

$$N(T+1) - N(T) \ll \log T.$$

Proof. Jensen's formula states that if $f(s)$ is an analytic function for $|s| \leq R$ with zeros $s_1, \ldots, s_m$ (according their multiplicities) and $f(0) \neq 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(r \exp(i\theta))|\,\mathrm{d}\theta = \log \frac{r^m |f(0)|}{|s_1 \cdot \ldots \cdot s_m|} \tag{2.24}$$

for $r < R$ (this is a variant of the Poisson integral formula). This applied with $f(s) = \zeta(2 + iT + s)$ leads to the bound

$$\log \frac{r^m |f(0)|}{|s_1 \cdot \ldots \cdot s_m|} \ll \log T,$$

where $r \in [3, 4]$ is chosen such that $\zeta(2 + iT + s)$ is non-zero. Since any zero ρ of $\zeta(s)$ with $|\gamma - T| \leq 1$ has distance at most $r > \sqrt{5}$ to $2 + iT$, it follows from

(2.22) that

$$N(T+1)-N(T)\le\sum_{|\gamma-T|\le 1}1 = \sum_{|\gamma-T|\le 1}\log\frac{r}{|\rho-2-iT|}\frac{1}{\log\frac{r}{\sqrt{5}}}$$
$$\ll \sum_{|\gamma-T|\le 1}\log\frac{r}{|\rho-2-iT|}\ll\log T.$$

The theorem is proved. •

Next we are interested in an estimate for the number $N(\sigma,T)$ of zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ with $\beta>\sigma, 0<\gamma\le T$. Application of Littlewood's lemma with fixed $\sigma_0>\frac{1}{2}$ yields

$$\begin{aligned}2\pi\int_{\sigma_0}^{1}N(\sigma,T)\,\mathrm{d}\sigma &= \int_0^T\log|\zeta(\sigma_0+it)|\,\mathrm{d}t-\int_0^T\log|\zeta(2+it)|\,\mathrm{d}t\\ &\quad+\int_2^{\sigma_0}\arg\zeta(\sigma+iT)\,\mathrm{d}\sigma-\int_2^{\sigma_0}\arg\zeta(\sigma)\,\mathrm{d}\sigma.\end{aligned}\tag{2.25}$$

The main contribution comes from the first integral on the right-hand side. The last integral does not depend on T and so it is bounded. Since $\zeta(s)$ has an Euler product representation, the logarithm has a Dirichlet series representation:

$$\log\zeta(s)=-\sum_p\log\left(1-\frac{1}{p^s}\right)=\sum_{p,k}\frac{1}{kp^{ks}}\qquad\text{for}\quad\sigma>1,\tag{2.26}$$

where k runs through the positive integers; here we choose that branch of the logarithm which is real on the positive real axis. We obtain

$$\int_0^T\log|\zeta(2+it)|\,\mathrm{d}t=\mathrm{Re}\left\{\sum_{p,k}\frac{1}{kp^{2k}}\int_0^T\exp(-itk\log p)\,\mathrm{d}t\right\}\ll\sum_{n=2}^{\infty}\frac{1}{n^2}\ll 1.$$

It remains to estimate $\arg\zeta(\sigma+iT)$. We may assume that T is not the ordinate of any zero. Since $\arg\zeta(2)=0$ and

$$\arg\zeta(s)=\arctan\left(\frac{\mathrm{Im}\,\zeta(s)}{\mathrm{Re}\,\zeta(s)}\right),$$

where

$$\mathrm{Re}\,\zeta(2+it)=\sum_{n=1}^{\infty}\frac{\cos(it\log n)}{n^2}\ge 1-\sum_{n=2}^{\infty}\frac{1}{n^2}>1-\int_1^{\infty}\frac{\mathrm{d}u}{u^2}=0,$$

we have by the argument principle

$$|\arg\zeta(2+iT)|\le\tfrac{\pi}{2}.$$

Now assume that $\mathrm{Re}\,\zeta(\sigma+iT)$ vanishes q times in the range $\frac{1}{2}\le\sigma\le 2$. Devide the interval $[\frac{1}{2}+iT,2+iT]$ into $q+1$ parts, throughout each of which $\mathrm{Re}\,\zeta(s)$

is of constant sign. Hence, again by the argument principle, in each part the variation of $\arg\zeta(s)$ does not exceed π. This gives

$$|\arg\zeta(s)| \leq \left(q+\tfrac{3}{2}\right)\pi \qquad \text{for} \quad \sigma \geq \tfrac{1}{2}.$$

Further, q is the number of zeros of the function

$$g(z) = \tfrac{1}{2}\left(\zeta(z+iT)+\zeta(z-iT)\right)$$

for $\operatorname{Im} z = 0$ and $\frac{1}{2} \leq \operatorname{Re} z \leq 2$. Thus, $q \leq n(\frac{3}{2})$, where $n(r)$ denotes the number of zeros of $g(z)$ for $|z-2| \leq r$. Obviously,

$$\int_0^2 \frac{n(r)}{r}\,\mathrm{d}r \geq \int_{\frac{3}{2}}^2 \frac{n(r)}{r}\,\mathrm{d}r \geq n\left(\tfrac{3}{2}\right)\int_{\frac{3}{2}}^2 \frac{\mathrm{d}r}{r} = n\left(\tfrac{3}{2}\right)\log\tfrac{4}{3}.$$

By Jensen's formula (2.24),

$$\int_0^2 \frac{n(r)}{r}\,\mathrm{d}r = \frac{1}{2\pi}\int_0^{2\pi} \log|\zeta(2+r\exp(i\theta))|\,\mathrm{d}\theta - \log|\zeta(2)|.$$

In view of (2.22) we obtain

$$q \leq n\left(\tfrac{3}{2}\right) \leq \frac{1}{\log\frac{4}{3}}\int_0^2 \frac{n(r)}{r}\,\mathrm{d}r \ll \log T.$$

This yields

$$\arg\zeta(\sigma+iT) \ll \log T \qquad \text{uniformly for} \quad \sigma \geq \tfrac{1}{2},$$

and, consequently, the same bound holds by integration with respect to $\frac{1}{2} \leq \sigma \leq 2$. The restriction on T not to be an imaginary-part of a zero of $\zeta(s)$ can be removed by considerations of continuity. Therefore, we may replace (2.25) by

$$\int_{\sigma_0}^1 N(\sigma,T)\,\mathrm{d}\sigma = \frac{1}{2\pi}\int_0^T \log|\zeta(\sigma_0+it)|\,\mathrm{d}t + O(\log T). \tag{2.27}$$

Now we need another fact due to Jensen, namely the Jensen's inequality, which states that for any continuous function $f(u)$ on $[a,b]$,

$$\frac{1}{b-a}\int_a^b \log f(u)\,\mathrm{d}u \leq \log\left(\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right).$$

Hence, we obtain

$$\int_0^T \log|\zeta(\sigma+it)|\,\mathrm{d}t \leq \frac{T}{2}\log\left(\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^2\,\mathrm{d}t\right) \ll T$$

by applying Theorem 2.13. Thus, for any fixed $\sigma_0 > \frac{1}{2}$,

$$\int_{\sigma_0}^1 N(\sigma,T)\,\mathrm{d}\sigma \ll T.$$

Let $\sigma_1 = \frac{1}{2} + \frac{1}{2}(\sigma_0 - \frac{1}{2})$, then we get

$$N(\sigma_0, T) \leq \frac{1}{\sigma_0 - \sigma_1} \int_{\sigma_1}^{\sigma_0} N(\sigma, T)\,\mathrm{d}\sigma \leq \frac{2}{\sigma_0 - \frac{1}{2}} \int_{\sigma_1}^{1} N(\sigma, T) \ll T.$$

Because of (2.27) we have proved estimate (2.12) from Section 2.1.

2.5. A density theorem. In the last section we have proved a first estimate for the number of hypothetical zeros to the right of the critical line. Now we give the proof of a stronger density theorem due to Hoheisel [29]:

Theorem 2.16. *For any fixed* $\sigma \in (\frac{1}{2}, 1)$,

$$N(\sigma, T) \ll T^{4\sigma(1-\sigma)}(\log T)^{10}.$$

For the proof we need the following simple but powerful lemma, also called Gallagher's lemma:

Lemma 2.17. *Let* $f(t)$ *be a continuously differentiable complex-valued function on the interval* $[a, b]$. *Let* $t_0 = a < t_1 < \ldots < t_{k-1} < t_k = b$ *and denote by* δ *the minimum of all differences* $t_{j+1} - t_j$. *Then*

$$\sum_{j=1}^{k} |f(t_j)|^2 \leq \frac{1}{\delta} \int_a^b |f(t)|^2\,\mathrm{d}t + 2\left(\int_a^b |f(t)|^2\,\mathrm{d}t \int_a^b |f'(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Proof. Denote by $\chi_j(t)$ the characteristic function on the interval $[t_j, t_{j+1}]$, i.e., $\chi_j(t) = 1$ for $t \in [t_j, t_{j+1}]$ and $\chi_j(t) = 0$ otherwise. Further let

$$\lambda_j(t) = \frac{1}{t_{j+1} - t_j} \int_a^t \chi_j(\tau)\,\mathrm{d}\tau.$$

Then, by partial integration,

$$\begin{aligned} \int_{t_j}^{t_{j+1}} \lambda_j(t)\left(|f(t)|^2\right)'\,\mathrm{d}t &= \lambda_j(t)|f(t)|^2\Big|_{t=t_j}^{t_{j+1}} - \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} |f(t)|^2 \chi_j(t)\,\mathrm{d}t \\ &= |f(t_{j+1})|^2 - \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} |f(t)|^2\,\mathrm{d}t. \end{aligned}$$

It follows that

$$|f(t_{j+1})|^2 \leq \frac{1}{\delta} \int_{t_j}^{t_{j+1}} |f(t)|^2\,\mathrm{d}t + 2\int_{t_j}^{t_{j+1}} |f(t)|\,|f'(t)|\,\mathrm{d}t.$$

Now the assertion of the lemma follows from summation over j and application of the Cauchy–Schwarz inequality. •

Now we are in the position to give the

Proof of Theorem 2.16. For $2 \leq V \leq T$ let $N_1(\sigma, V)$ count the zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\beta \geq \sigma$ and $\frac{1}{2}V < \gamma \leq V$. Taking $x = V$ in Theorem 2.10, we have

$$\zeta(s) = \sum_{k \leq V} \frac{1}{k^s} + \frac{V^{1-s}}{s-1} + O\left(V^{-\sigma}\right)$$

for $\frac{1}{2}V < t \leq V$ and $\frac{1}{2} \leq \sigma \leq 1$. Now define the Dirichlet polynomial

$$M_X(s) := \sum_{m \leq X} \frac{\mu(m)}{m^s},$$

where $X = V^{2\sigma-1}$ and $\mu(m)$ is the Möbius μ-function, defined by the representation

$$\zeta(s)^{-1} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}, \tag{2.28}$$

valid for $\sigma > 1$. In particular, it follows that $\mu(m)$ is equal to $(-1)^\ell$ if m is the product of ℓ different primes, and equal to zero otherwise. Now let

$$\zeta(s) M_X(s) = P(s) + R(s),$$

where

$$P(s) := \sum_{m \leq X} \frac{\mu(m)}{m^s} \sum_{k \leq V} \frac{1}{k^s} = \sum_{n \leq XV} \frac{a(n)}{n^s}$$

with

$$a(n) := \sum_{\substack{m \mid n \\ m \leq X, n \leq mV}} \mu(m) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad 1 < n \leq X, \end{cases} \tag{2.29}$$

and

$$R(s) \ll |M_X(s)| V^{-\sigma}.$$

Note that $M_X(s)$ *mollifies* $\zeta(s)^{-1}$. We shall use $P(s)$ as a *zero-detector*. Let $s = \rho = \beta + i\gamma$ be a zero of the zeta-function with $\frac{1}{2}V < \gamma \leq V$. Then,

$$\begin{aligned} 1 &\leq \left| \sum_{X < n \leq XV} \frac{a(n)}{n^\rho} \right| + O(|M_X(\rho)| V^{-\beta}), \\ 1 &\ll \left| \sum_{X < n \leq XV} \frac{a(n)}{n^\rho} \right|^2 + O(|M_X(\rho)|^2 V^{-2\beta}). \end{aligned}$$

Then, summing up both sides of the latter inequality over all zeros leads to

$$N_1(V) \ll \sum_{\substack{\sigma \leq \beta \leq 1 \\ \frac{1}{2}V < \gamma \leq V}} \left(\left| \sum_{X < n \leq XV} \frac{a(n)}{n^\rho} \right|^2 + |M_X(\rho)|^2 V^{-2\sigma} \right). \tag{2.30}$$

Now we divide the interval $[\frac{1}{2}V, V]$ into subintervals of length 1 of the form $[2m+n-1, 2m+n]$, where $n = 1, 2$ and $\frac{1}{4}V - 1 \le m \le \frac{1}{2}V$. Then, we may continue as follows

$$\begin{aligned}\sum_{\substack{\sigma\le\beta\le 1\\ \frac{1}{2}V<\gamma\le V}} &\le \sum_{\frac{1}{4}V-1\le m\le\frac{1}{2}V}\sum_{n=1}^{2}\sum_{2m+n-1<\gamma\le 2m+n}\\ &\le 2\max_{1\le n\le 2}\sum_{\frac{1}{4}V-1\le m\le\frac{1}{2}V}\sum_{2m+n-1<\gamma\le 2m+n};\end{aligned}$$

here we have omitted the terms to be summed for better readability. By Theorem 2.15 there are only $\ll \log V$ many zeros with $2m+n-1 < \gamma \le 2m+n$. Now denote by $\sum_{\rho}'$ the largest of the related sums according to $2m+n-1 < \gamma \le 2m+n$. Then

$$\sum_{\substack{\sigma\le\beta\le 1\\ \frac{1}{2}V<\gamma\le V}} \ll \log V \sum_{\rho}{}' ,$$

resp. in (2.30)

$$N_1(V) \ll \log V \sum_{\rho}{}' \left(\left| \sum_{X<n\le XV} \frac{a(n)}{n^{\rho}} \right|^2 + \left| \sum_{m\le X} \frac{\mu(m)}{m^{\rho}} \right|^2 V^{-2\sigma} \right). \tag{2.31}$$

First of all we shall give a bound for

$$S(Y) := \sum_{\rho}{}' \left| \sum_{Y<n\le U} \frac{b(n)}{n^{\rho}} \right|^2 ,$$

where $U \le 2Y$, $V \ge Y \ge 1$ and

$$b(n) \ll \sum_{d|n} 1 =: d(n); \tag{2.32}$$

the arithmetic function $d(n)$ is called the divisor function since it counts the number of positive divisors of n. By partial summation, for fixed $\rho = \beta + i\gamma$,

$$\sum_{Y<n\le U} \frac{b(n)}{n^{\rho}} = \int_Y^U C(u)\,\mathrm{d}u^{-\beta} \qquad \text{with} \quad C(u) := \sum_{Y<n\le u} \frac{b(n)}{n^{i\gamma}}.$$

Applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\left| \sum_{Y<n\le U} \frac{b(n)}{n^{\rho}} \right| &\ll Y^{-\beta-1} \int_Y^U |C(u)|\,\mathrm{d}u + Y^{-\beta}|C(U)|,\\ \left| \sum_{Y<n\le U} \frac{b(n)}{n^{\rho}} \right|^2 &\ll Y^{-2\beta-1} \int_Y^U |C(u)|^2\,\mathrm{d}u + Y^{-2\beta}|C(U)|^2.\end{aligned}$$

This leads to

$$S(Y) \ll Y^{-2\sigma} {\sum_{\rho}}' \left| \sum_{Y<n\leq W} \frac{b(n)}{n^{i\gamma}} \right|^2 ,$$

where $W \leq U$ is such that the latter expression is maximal. Since all differences $\gamma_{r+1} - \gamma_r$ of imaginary parts of counted zeros $\rho_r = \beta_r + i\gamma_r$ are ≥ 1, we deduce from Lemma 2.17 the estimate

$$S(Y) \ll Y^{-2\sigma}(I_1 + \sqrt{I_1 I_2}),$$

where

$$I_1 := \int_{\frac{1}{2}V}^{V} \left| \sum_{Y<n\leq W} b(n) n^{it} \right|^2 \,\mathrm{d}t \ , \quad I_2 := \int_{\frac{1}{2}V}^{V} \left| \sum_{Y<n\leq W} b(n) \log n \cdot n^{it} \right|^2 \,\mathrm{d}t.$$

Taking (2.29) into account, $|a(n)|$ satisfies condition (2.32) on $b(n)$. By elementary estimates one can show that

$$\sum_{n\leq x} d(n)^k \ll_k x(\log x)^{2^k-1}, \tag{2.33}$$

where the implicit constant depends only on k. This yields

$$\begin{aligned} I_1 &\ll (V+Y)\log V \sum_{Y<n\leq 2Y} d(n)^2 \ll (VY+Y^2)(\log V)^5, \\ I_2 &\ll (VY+Y^2)(\log V)^7. \end{aligned}$$

Now dividing the first sum on the right-hand side of (2.31) into $\ll \log V$ sums (as above), application of the latter estimates yields

$$\log V {\sum_{\rho}}' \left| \sum_{X<n\leq VX} \frac{a(n)}{n^{\rho}} \right|^2 \ll (VX^{1-2\sigma} + (VX)^{2-2\sigma})(\log V)^9.$$

Similarly, we get for the second term

$$V^{-2\sigma}(\log T)^2 {\sum_{\rho}}' \left| \sum_{m\leq X} \frac{\mu(m)}{m^{\rho}} \right|^2 \ll V^{-2\sigma}(V + X^{2-2\sigma})(\log V)^9.$$

Substituting this in (2.31) with $X = V^{2\sigma-1}$, we obtain

$$N_1(V) \ll V^{4\sigma(1-\sigma)}(\log V)^9.$$

Using this with $V = T^{1-n}$ and summing up over all $n \in \mathbb{N}$, finishes the proof of the theorem. •

The density hypothesis states

$$N(\sigma, T) \ll T^{(2+\epsilon)(1-\sigma)}$$

for all $\epsilon > 0$ *and sufficiently large* T. Gritsenko's theorem 2.2 falls not too far behind this open conjecture. One can show that the Lindelöf hypothesis (2.23) implies the density hypothesis. However, already Theorem 2.16 can serve in quite many applications as substitute for the Riemann hypothesis.

2.6. The prime number theorem. Now we shall prove the prime number theorem with a slightly weaker remainder term than in Theorem 2.1. For this aim we need to establish a zero-free region for $\zeta(s)$ inside the critical strip. We may argue only for $s = \sigma + it$ from the upper half-plane, since the zeros are symmetrically distributed with respect to the real axis.

Lemma 2.18. *For* $t \geq 8, 1 - \frac{1}{2}(\log t)^{-1} \leq \sigma \leq 2$,

$$\zeta(s) \ll \log t \qquad \textit{and} \qquad \zeta'(s) \ll (\log t)^2.$$

Proof. Let $1 - (\log t)^{-1} \leq \sigma \leq 3$. If $n \leq t$, then

$$|n^s| = n^\sigma \geq n^{1-(\log t)^{-1}} = \exp\left(\left(1 - \frac{1}{\log t}\right)\log n\right) \gg n.$$

Thus, the approximate functional equation, Theorem 2.10, implies

$$\zeta(s) \ll \sum_{n \leq t} \frac{1}{n} + t^{-1} \ll \log t$$

(the bound for the sum is an easy exercise in analysis; in Exercise 5 below one shall prove an asymptotic formula (2.39)). The estimate for $\zeta'(s)$ follows immediately from Cauchy's formula,

$$\zeta'(s) = \frac{1}{2\pi i} \oint_{|z-s|=r} \frac{\zeta(z)}{(z-s)^2}\, \mathrm{d}z$$

with $r > 0$ sufficiently small, or alternatively, by (carefull) differentiation of the formula of Theorem 2.5. •

In view of the Euler product representation of zeta we find for $\sigma > 1$

$$|\zeta(\sigma + it)| = \exp(\mathrm{Re}\, \log \zeta(s)) = \exp\left(\sum_{p,k} \frac{\cos(kt \log p)}{kp^{k\sigma}}\right).$$

Since

$$17 + 24\cos\alpha + 8\cos(2\alpha) = (3 + 4\cos\alpha)^2 \geq 0,$$

it follows that

$$\zeta(\sigma)^{17} |\zeta(\sigma + it)|^{24} |\zeta(\sigma + 2it)|^8 \geq 1. \tag{2.34}$$

This inequality is the main idea for our following observations. By the approximate functional equation, Theorem 2.10, we have

$$\zeta(\sigma) \ll \frac{1}{\sigma - 1}$$

for sufficiently small $\sigma > 1$. Assuming that $\zeta(1+it)$ has a zero for $t = t_0 \neq 0$, we have $|\zeta(\sigma + it_0)| \ll \sigma - 1$ as $\sigma \to 1+$, which leads to

$$\lim_{\sigma \to 1+} \zeta(\sigma)^{17} |\zeta(\sigma + it_0)|^{24} = 0,$$

contradicting (2.34). Thus $\zeta(1+it) \neq 0$. A simple refinement of this argument allows a lower estimate for the modulus of $\zeta(1+it)$: for $t \geq 1$ and $1 < \sigma < 2$, we deduce from (2.34) and Lemma 2.18

$$\frac{1}{|\zeta(\sigma+it)|} \leq \zeta(\sigma)^{\frac{17}{24}} |\zeta(\sigma + 2it)|^{\frac{1}{3}} \ll (\sigma-1)^{-\frac{17}{24}} (\log t)^{\frac{1}{3}}.$$

Furthermore, with Lemma 2.18,

$$\zeta(1+it) - \zeta(\sigma+it) = -\int_1^\sigma \zeta'(u+it)\,\mathrm{d}u \ll |\sigma - 1| (\log t)^2. \tag{2.35}$$

Hence

$$\begin{aligned} |\zeta(1+it)| &\geq |\zeta(\sigma+it)| - c_1(\sigma-1)(\log t)^2 \\ &\geq c_2(\sigma-1)^{\frac{17}{24}}(\log t)^{-\frac{1}{3}} - c_1(\sigma-1)(\log t)^2, \end{aligned}$$

where c_1, c_2 are certain positive constants. Chosing a constant $B > 0$ such that $A := c_2 B^{\frac{17}{24}} - c_1 B > 0$ and putting $\sigma = 1 + B(\log t)^{-8}$, we obtain

$$|\zeta(1+it)| \geq \frac{A}{(\log t)^6}. \tag{2.36}$$

This gives an estimate on the left of the line $\sigma = 1$. It also allows an estimate inside the critical strip:

Lemma 2.19. *For $t \geq 8$, there exists a positive constant δ such that*

$$\zeta(s) \neq 0 \qquad \textit{for} \qquad \sigma \geq 1 - \delta \min\{1, (\log t)^{-8}\}.$$

Proof. In view of Lemma 2.18 estimate (2.35) holds for $1 - \delta(\log t)^{-8} \leq \sigma \leq 1$. Using (2.36), it follows that

$$|\zeta(\sigma + it)| \geq \frac{A - c_1\delta}{(\log t)^6},$$

where the term on the right is positive for sufficiently small δ. •

Now we are in the position to prove the prime number theorem. We shall work with the logarithmic derivative of $\zeta(s)$. Since $\zeta(s)$ does not vanish in the half-plane $\sigma > 1$, the logarithmic derivative $\frac{\zeta'}{\zeta}(s)$ is analytic for $\sigma > 1$. Partial summation gives

$$-\frac{\zeta'}{\zeta}(s) = s \int_1^\infty \psi(x) \frac{\mathrm{d}x}{x^{s+1}}.$$

For the definition of ψ see (2.7). We would like to isolate $\psi(x)$ from this formula. For this purpose we shall prove Formula (2.8) by some kind of Fourier transformation.

Lemma 2.20. *Let c and y be positive and real. Then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s}\,\mathrm{d}s = \begin{cases} 0 & \textit{if} \quad 0<y<1, \\ \frac{1}{2} & \textit{if} \quad y=1, \\ 1 & \textit{if} \quad y>1. \end{cases}$$

Proof. If $y=1$, then the integral in question equals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{c+it} = \frac{1}{\pi}\lim_{T\to\infty}\int_0^T \frac{c}{c^2+t^2}\,\mathrm{d}t = \frac{1}{\pi}\lim_{T\to\infty}\arctan(T/c) = \tfrac{1}{2},$$

by well-known properties of the arctan-function. Now assume that $0<y<1$ and $r>c$. Since the integrand is analytic in $\sigma>0$, Cauchy's theorem implies, for $T>0$,

$$\int_{c-iT}^{c+iT}\frac{y^s}{s}\,\mathrm{d}s = \left\{\int_{c-iT}^{r-iT}+\int_{r-iT}^{r+iT}+\int_{r+iT}^{c+iT}\right\}\frac{y^s}{s}\,\mathrm{d}s.$$

It is easily seen that

$$\begin{aligned} \int_{r\pm iT}^{c\pm iT}\frac{y^s}{s}\,\mathrm{d}s &\ll \frac{1}{T}\int_r^c y^\sigma\,\mathrm{d}\sigma \ll \frac{y^c}{T|\log y|}, \\ \int_{r-iT}^{r+iT}\frac{y^s}{s}\,\mathrm{d}s &\ll \frac{y^r}{r}+y^r\int_1^T\frac{\mathrm{d}t}{t} \ll y^r\left(\frac{1}{r}+\log T\right). \end{aligned}$$

Now sending first r and then T to infinity, the first case follows. Finally, if $y>1$, then we bound the corresponding integrals over the rectangular contour with corners $c\pm iT, -r\pm iT$, analogously. Now the pole of the integrand at $s=0$ with residue

$$\mathrm{Res}_{\,s=0}\frac{y^s}{s} = \lim_{s\to 0}\frac{y^s}{s}\cdot s = 1$$

gives the values $2\pi i$ for the integral in this case. •

Now we continue our study of the logarithmic derivative of the zeta-function. For $x\notin\mathbb{Z}$ and $c>1$ we have

$$\int_{c-i\infty}^{c+i\infty}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\frac{x^s}{s}\,\mathrm{d}s = \sum_{n=1}^{\infty}\Lambda(n)\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{n}\right)^s\frac{\mathrm{d}s}{s};$$

here interchanging integration and summation is allowed by the absolute convergence of the series. In view of Lemma 2.20 it follows that

$$\sum_{n\le x}\Lambda(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\frac{x^s}{s}\,\mathrm{d}s,$$

resp.

$$\psi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,\mathrm{d}s;$$

this is known as Perron's formula (2.9). Since

$$\int_{c\pm iT}^{c\pm i\infty} \frac{y^s}{s}\,\mathrm{d}s = \frac{y^s}{s\log y}\bigg|_{s=c\pm iT}^{c\pm i\infty} + \frac{1}{\log y}\int_{c\pm iT}^{c\pm i\infty} \frac{y^s}{s^2}\,\mathrm{d}s \ll \frac{y^c}{T|\log y|}$$

for $0 < y \neq 1$ and $T > 0$, it follows that

$$\int_{c\pm iT}^{c\pm i\infty} \left(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}\right) \frac{x^s}{s}\,\mathrm{d}s \ll \frac{x^c}{T}\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^c\left|\log\frac{x}{n}\right|} \ll \frac{x^c}{T}\left|\frac{\zeta'}{\zeta}(c)\right| + \frac{x(\log x)^2}{T} + \log x.$$

This yields

$$\begin{aligned}\psi(x) &= -\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,\mathrm{d}s \\ &\quad + O\left(\frac{x^c}{T}\left|\frac{\zeta'}{\zeta}(c)\right| + \frac{x(\log x)^2}{T} + \log x\right),\end{aligned} \tag{2.37}$$

which holds for arbitrary x. To find an asymptotic formula for the integral above we move the path of integration to the left. Here we may get contributions from the poles of the integrand, i.e., the residues at the nontrivial zeros of $\zeta(s)$, and at the pole of $\zeta(s)$ at $s = 1$. For our purpose it is sufficient to exclude the zeros of the zeta-function. In view of the zero-free region of Lemma 2.19 we put $c = 1+\lambda$ with $\lambda = \delta(\log T)^{-8}$, where δ is given by Lemma 2.19, and integrate over the boundary of the rectangle $\mathcal{R}$ given by the corners $1 \pm \lambda \pm iT$. By this choice $\zeta(s)$ does not vanish in and on the boundary of $\mathcal{R}$. Hence,

$$\begin{aligned}&\int_{c-iT}^{c+iT} \left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,\mathrm{d}s \\ &= \left\{\int_{1+\lambda-iT}^{1-\lambda-iT} + \int_{1-\lambda-iT}^{1-\lambda+iT} + \int_{1-\lambda+iT}^{1+\lambda-iT}\right\}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,\mathrm{d}s \\ &\quad + 2\pi i\,\mathrm{Res}_{\,s=1}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}.\end{aligned}$$

For the logarithmic derivative of $\zeta(s)$ we have

$$-\frac{\zeta'}{\zeta}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = \frac{1}{s-1} + O(1)$$

as $s \to 1$. Thus, we obtain for the residue at $s = 1$

$$\mathrm{Res}_{\,s=1}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s} = x.$$

It remains to bound the integrals. For the horizontal integrals we deduce from Lemma 2.19 that

$$\int_{1-\lambda\pm iT}^{1+\lambda\pm iT} \left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,\mathrm{d}s \ll \frac{x^{1+\lambda}}{T}.$$

Further, for the vertical integral,

$$\int_{1-\lambda-iT}^{1+\lambda+iT}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}\,\mathrm{d}s \ll x^{1-\lambda}(\log T)^9.$$

Collecting together, we deduce from (2.37)

$$\psi(x)=x+O\left(\frac{x^{1+\lambda}}{T\lambda}+x^{1-\lambda}(\log T)^9+\frac{x(\log x)^2}{T}+\log x\right).$$

Choosing $T=\exp(\delta^{\frac{1}{10}}(\log x)^{\frac{1}{9}})$, we arrive at

$$\psi(x)=x+O\left(x\exp(-c(\log x)^{\frac{1}{9}})\right).$$

Setting

$$\theta(x):=\sum_{p\le x}\log p,$$

since

$$\psi(x)-\theta(x)=\sum_{\substack{p^k\le x\\ k\le 2}}\log p \ll x^{\frac{1}{2}}(\log x)^2,$$

it follows that

$$\theta(x)=x+O\left(x\exp(-c(\log x)^{\frac{1}{9}})\right).$$

Applying now partial summation, Lemma 2.4, we find

$$\begin{aligned}\pi(x)=\sum_{p\le x}\log p\cdot\frac{1}{\log p} &= \frac{\theta(x)}{\log x}-\int_2^x \theta(u)\left(\frac{1}{\log u}\right)'\mathrm{d}u\\ &= \frac{x}{\log x}-\int_2^x u\left(\frac{1}{\log u}\right)'\mathrm{d}u\\ &\quad +O\left(x\exp\left(-c(\log x)^{\frac{1}{9}}\right)\right).\end{aligned}$$

Now partial integration leads to the prime number theorem with remainder term:

Theorem 2.21. *There exists a positive constant c such that for $x\ge 2$*

$$\pi(x)=\mathrm{li}\,(x)+O\left(x\exp\left(-c(\log x)^{\frac{1}{9}}\right)\right).$$

Thus, the simple pole of the zeta-function is not only the key in Euler's proof of the infinitude of primes but also gives the main term of the asymptotic formula in the prime number theorem. We see that the primes are not too irregularly distributed. For example, the prime number theorem implies that, if p_n denotes the n-th prime number (in ascending order), then $p_n\sim n\log n$.

We conclude this section by giving a sketch of von Koch's equivalent (2.11) for the Riemann hypothesis. By partial summation we obtain for $\sigma>1$

$$-\frac{\zeta'}{\zeta}(s)=\frac{s}{s-1}+s\int_1^\infty\frac{\psi(u)-u}{u^{s+1}}\,\mathrm{d}u.$$

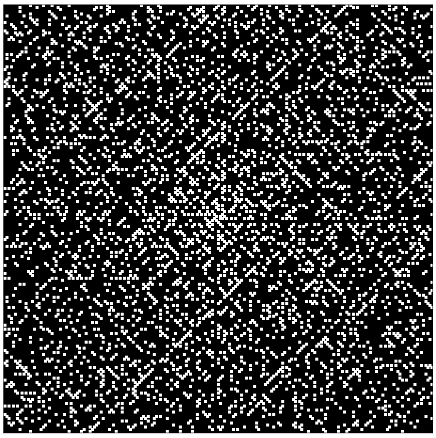

FIGURE 5. This is Ulam's spiral: the first 65 000 positive integers are listed in a spiral in ascending order, the primes are coloured white, the composite numbers black.

If $\psi(x)-x \ll x^{\theta+\epsilon}$, then the integral above converges for $\sigma > \theta$, giving an analytic continuation for

$$\frac{\zeta'}{\zeta}(s)-\frac{1}{s-1}$$

to the half-plane $\sigma > \theta$, and, in particular, $\zeta(s)$ does not vanish there. For the converse implication we assume that all nontrivial zeros $\rho = \beta + i\gamma$ satisfy $\beta \leq \theta$. Then it follows from (2.10) that

$$\psi(x)-x \ll x^{\theta}\sum_{|\gamma|\leq T}\frac{1}{|\gamma|}+\frac{x}{T}(\log(xT))^2. \tag{2.38}$$

By Theorem 2.15 we have $N(T+1)-N(T) \ll \log T$, and therefore

$$\sum_{|\gamma|\leq T}\frac{1}{|\gamma|} \ll \sum_{m=1}^{[T]+1}\frac{\log m}{m} \ll (\log T)^2.$$

Substituting this in (2.38) leads to

$$\psi(x)-x \ll x^{\theta}(\log T)^2+\frac{x}{T}(\log(xT))^2.$$

Now the choice $T = x^{1-\theta}$ finishes the proof of this implication. So the Riemann hypothesis is true if and only if the error term in the prime number theorem is $O(x^{\frac{1}{2}+\epsilon})$. In this case there cannot be too long intervals free of primes. Note that it is an open question whether there is always a prime in between two consecutive squares. Some examples may convince the reader that this is a reasonable conjecture, however, this statement does not even follow from Riemann's hypothesis.

Practice makes perfect! We continue with some exercises. Already Euler found an explicit formula for the values of the zeta-function at all positive even integers in terms of Bernoulli numbers (see [63]).

Exercise 3. Recall the product representation

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z}{k^2}\right)$$

and deduce some zeta-values: $\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \ldots$ by expanding the infinite product.

On the contrary, not too much is known about zeta-values at positive odd integers. In 1978, Apéry proved that $\zeta(3)$ is irrational, however, it is not known whether this value is transcendental or whether $\zeta(5)$ is irrational.

It is conjectured that all zeros of the zeta-function are simple. A classical theorem of Speiser states that the Riemann hypothesis is true if and only if $\zeta'(s)$ is non-vanishing in $0 < \sigma < \frac{1}{2}$.

Exercise 4. Prove that any zero of $\zeta'(s)$ on the critical line is a multiple zero of $\zeta(s)$.

The following three exercises can be solved with Abel's partial summation (Lemma 2.4).

Exercise 5. Prove the following asymptotic formulas

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right), \tag{2.39}$$

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1); \tag{2.40}$$

here γ is the Euler–Mascheroni constant $\gamma := \lim_{N\to\infty} \frac{1}{N}(\sum_{n=1}^{N} \frac{1}{n} - \log N) = 0.557\ldots$.

Exercise 6. Prove formula (2.33). For this one may count the lattice points $(a, b) \in \mathbb{Z}^2$ under a hyperbola: $\sum_{n\leq x} d(n) = \sum_{ab\leq x} 1$. For the second moment observe that

$$\sum_{n\leq x} d(n)^2 = \sum_{ab\leq x} d(ab) \leq \sum_{a\leq x} d(a) \sum_{b\leq x/a} d(b).$$

Another approach uses contour integration of the function $\zeta(s)^k \frac{x^s}{s}$, following the lines of proof of the prime number theorem.

The next exercise is about twin primes, that are pairs of primes of the form $p, p + 2$. It is unknown whether there are infinitely many twin primes. Brun showed for the number $\pi_2(x)$ of twin primes $p, p + 2$ with $p \leq x$ the estimate $\pi_2(x) \ll x(\log x)^{-2}$.

Exercise 7. Deduce from Brun's estimate that the sum over the reciprocals of all twin primes converges although the sum of the reciprocals over all primes diverges (see (2.40)).

Relevant information about the zeta-function is contained in its order of growth along vertical lines as well as in the distribution of its zeros. For the next exercises one may consult [62, 63]:

Exercise 8. Apply the Phragmén-Lindelöf principle in order to prove estimate (2.22) with an explicit constant c and give the details for the proof of Theorem 2.15.

Exercise 9. Prove the Riemann-von Mangoldt formula (2.4).

3. Universality theorems

In this chapter we shall prove the famous universality theorem of Voronin; besides we indicate how to derive other remarkable universality theorems by similar means (e.g., Reich's universality theorem 3.11 below). The method of proof is a mixture of techniques from function theory, analytic number theory, and basic functional analysis.

3.1. Voronin's universality theorem.

Now we are going to prove Voronin's universality theorem, that is Theorem 1.3 from the introduction: *Let* $0 < r < \frac{1}{4}$ *be fixed and suppose that* $g(s)$ *is a non-vanishing continuous function on the disk* $|s| \le r$ *which is analytic in the interior. Then, for any* $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{|s|\le r} \left| \zeta\left(s + \tfrac{3}{4} + i\tau\right) - g(s) \right| < \epsilon \right\} > 0.$$

The Euler product for the zeta-function is the key to prove the universality theorem in spite of the fact that it does not converge in the region of universality. However, as already Bohr observed, an appropriate truncated Euler product approximates $\zeta(s)$ in a certain mean-value sense inside the critical strip; this is related to the use of modified truncated Euler products in Voronin's proof (see (3.23) below). Another important tool in the proof are approximation theorems, one for numbers and one for functions. This is not too surprising since universality is an approximation property. Last but not least we shall make use of the prime number theorem and classical function theory.

It is more convenient to work with series than with products. Therefore, we consider the logarithms of the functions in question. Since $g(s)$ has no zeros in $|s| \le r$, its logarithm exists and we may define an analytic function $f(s)$ by $g(s) = \exp f(s)$ for $|s| < r$. Conversely, if $f(s)$ is analytic, then $g(s) = \exp f(s)$ is analytic and non-vanishing. Now we formulate

Theorem 3.1. *Let* $0 < r < \frac{1}{4}$ *and suppose that* $f(s)$ *is a continuous function on the disk* $|s| \le r$*, which is analytic in the interior. Then, for any* $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{|s|\le r} \left| \log \zeta\left(s + \tfrac{3}{4} + i\tau\right) - f(s) \right| < \epsilon \right\} > 0.$$

Note that the zeros of the zeta-function are negligible since they form a set of density zero by Theorem 2.2 whereas the set of approximating τ has positive lower density. Once Theorem 3.1 is proved, Voronin's universality theorem 1.3 follows. To see this we observe

$$\begin{aligned} & g(s)-\zeta\left(s+\tfrac{3}{4}+i\tau\right) \\ & \quad = \quad g(s)\left(1-\exp\left(\log\zeta\left(s+\tfrac{3}{4}+i\tau\right)-f(s)\right)\right). \end{aligned}$$

Assume that f is also analytic on the boundary. Then, taking the maximum over all values of s with $|s|\leq r$, the desired estimate follows from the expansion $\exp z-1=z+O(|z|^2)$. If f is not analytic on the boundary, we may conclude with a simple continuity argument. Hence, it suffices to prove Theorem 3.1. Its lengthy proof is organized as follows: a rearrangement theorem in a certain Hilbert space (Theorem 3.2 in §3.2) allows to approximate the target function by the logarithms of certain truncated Euler products (Theorem 3.6 in §3.3). The transition to the logarithm of the zeta-function is realized by diophantine approximation theory (§3.4) and integration in order to obtain a set of the desired translates τ having positive lower density (§3.5).

3.2. Rearrangement of conditionally convergent series. A series $\sum_n a_n$ of real numbers a_n is said to be conditionally convergent, if $\sum_n |a_n|$ is divergent but $\sum_n a_n$ is convergent for an appropriate rearrangement of the terms a_n. Riemann proved that any conditionally convergent series can be rearranged such that its sum converges to an arbitrary preassigned real number or infinity. For instance, to any given $c\in\mathbb{R}$ there exists a permutation σ of $\mathbb{N}$ such that

$$\sum_{n\in\mathbb{N}}\frac{(-1)}{\sigma(n)}=c.$$

In some sense, conditionally convergent series are *universal* with respect to $\mathbb{R}$.

It is the aim of this section to extend Riemann's rearrangement theorem to Hilbert spaces. In what follows let $\mathcal{H}$ be a Hilbert space and denote, as usual, its inner product by $\langle x,y\rangle$ and its norm by $\|x\|=\sqrt{\langle x,x\rangle}$.

Theorem 3.2. *Assume that a series $\sum_n u_n$ of vectors in a real Hilbert space $\mathcal{H}$ satisfies*

$$\sum_{n=1}^{\infty}\|u_n\|^2<\infty,$$

and for any $e\in\mathcal{H}$ with $\|e\|=1$ the series $\sum_n\langle u_n,e\rangle$ converges conditionally (with some rearrangement). Then for any $v\in\mathcal{H}$ there is a permutation σ of $\mathbb{N}$ such that

$$\sum_{n=1}^{\infty}u_{\sigma(n)}=v$$

in the norm of $\mathcal{H}$.

This theorem is due to Pechersky who proved it on demand of Voronin. The proof is slightly more complicated than the one for Riemann's rearrangement theorem. We start with

Lemma 3.3. *Under the assumptions of Theorem 3.2, for any $v \in \mathcal{H}$ and any $\epsilon > 0$ there exist a positive integer N and numbers $\epsilon_1, \ldots, \epsilon_N$, equal to 0 or 1, such that*

$$\Big\| s - \sum_{n=1}^{N} \epsilon_n u_n \Big\| < \epsilon.$$

Proof. We choose an integer m such that

$$\sum_{n=m}^{\infty} \|u_n\|^2 < \tfrac{1}{9}\epsilon^2.$$

Denote by P_m the set of all linear combinations

$$\sum_{n=m}^{N} \lambda_n u_n \qquad \text{with} \quad \lambda_n \in [0,1] \quad \text{and} \quad N = m, m+1, m+2, \ldots.$$

Obviously, P_m is convex. Let $\overline{P_m}$ be the closure of P_m with respect to the norm of $\mathcal{H}$; consequently $\overline{P_m}$ is a closed convex set. First of all we show that $\overline{P_m}$ coincides with $\mathcal{H}$.

The seperation theorem for linear operators states that if X is a normed linear space and D is a convex subset of X which is closed in the norm of X, then for any $s \in X \setminus D$ there exist $\epsilon > 0$ and a linear functional F on X such that

$$F(x) \leq F(s) - \epsilon \qquad \text{for all} \quad x \in D.$$

The proof follows from the well-known theorem of Hahn-Banach, which relates linear functionals to convex sets. A simple consequence is that for any proper convex subset D of real Hilbert space $\mathcal{H}$, which is closed in the norm of $\mathcal{H}$, there exists a vector $e \in \mathcal{H}$ with $\|e\| = 1$ such that

$$\sup_{x \in D} \langle x, e \rangle < \infty.$$

We return to our problem: suppose that $\overline{P_m} \neq \mathcal{H}$, then, by the above reasoning, there exists $e \in \mathcal{H}$ with $\|e\| = 1$ such that $\sup_{x \in \overline{P_m}} \langle x, e \rangle < \infty$. Since, by assumption, the series $\sum_{n \geq m}^{\infty} \langle u_n, e \rangle$ converges conditionally with some arrangement of the terms, the subseries consisting of the positive terms is divergent. Thus, for any C there exist an N and a sequence $\epsilon_m, \ldots, \epsilon_N$, each ϵ_n being equal to 0 or 1, such that

$$\sum_{n=m}^{N} \epsilon_n \langle u_n, e \rangle \ > C.$$

Since $\sum_{m}^{N} \epsilon_n u_n \in \overline{P_m}$, it follows that $\sup_{x \in \overline{P_m}} \langle x, e \rangle = \infty$, giving the contradiction.

So we have shown $\overline{P_m} = \mathcal{H}$. Consequently, there exist $N \geq m$ and $\lambda_m, \ldots, \lambda_N \in [0, 1]$ such that

$$\Big\| v - \sum_{n=m}^{N} \lambda_n u_n \Big\| < \tfrac{1}{3}\epsilon.$$

By induction we can construct $\epsilon_m, \ldots, \epsilon_N$, equal to 0 or 1, such that for any M with $m \leq M \leq N$ the inequality

$$\Big\| \sum_{n=m}^{M} \lambda_n u_n - \sum_{n=m}^{M} \epsilon_n u_n \Big\| \leq \sum_{n=m}^{M} \|u_n\|^2$$

holds. We may set $\epsilon_m = 1$ and suppose that $\epsilon_m, \ldots, \epsilon_M$ have been chosen so that the last inequality is fulfilled. With ϵ_{M+1}, equal to 0 or 1, satisfying

$$(\lambda_{M+1} - \epsilon_{M+1}) \Big\langle \sum_{n=m}^{M} (\lambda_n - \epsilon_n) u_n, u_{M+1} \Big\rangle \leq 0,$$

we get

$$\Big\| \sum_{n=m}^{M+1} \lambda_n u_n - \sum_{n=m}^{M+1} \epsilon_n u_n \Big\|^2 \leq \Big\| \sum_{n=m}^{M} (\lambda_n - \epsilon_n) u_n \Big\|^2 + \|u_{M+1}\|^2 \leq \sum_{n=m}^{M+1} \|u_n\|^2.$$

Hence, we can find a sequence of numbers $\epsilon_m, \ldots, \epsilon_N$, each being 0 or 1, such that

$$\Big\| \sum_{n=m}^{N} \lambda_n u_n - \sum_{n=m}^{N} \epsilon_n u_n \Big\|^2 \leq \sum_{n=m}^{N} \|u_n\|^2 < \tfrac{1}{9}\epsilon^2.$$

Thus,

$$\Big\| v - \sum_{n=m}^{N} \epsilon_n u_n \Big\| \leq \Big\| v - \sum_{n=m}^{N} \lambda_n u_n \Big\| + \Big\| \sum_{n=m}^{N} \lambda_n u_n - \sum_{n=m}^{N} \epsilon_n u_n \Big\| < \tfrac{2}{3}\epsilon,$$

which proves the lemma. •

The next step is

Lemma 3.4. *Under the assumptions of Theorem 3.2, there exists a permutation* $\{n_k\}$ *of* $\mathbb{N}$ *such that some subsequence of the partial sums of the series* $\sum_k u_{n_k}$ *converges to* v *in the norm of* $\mathcal{H}$.

Proof. We construct the sequence $n_1, n_2, \ldots$ as follows. First let $n_1 = 1$. Applying Lemma 3.3 to the series $\sum_{n \geq 2} u_n$, yields the existence of a finite set $T_1 \subset \{2, 3, \ldots\}$ such that

$$\Big\| v - u_1 - \sum_{n \in T_1} u_n \Big\| < \tfrac{1}{2}.$$

Now write the indices in T_1 in an arbitrary order. If $2 \notin T_1$, then write also 2. Denote by T_2 the set of all indices we have so far, and define $N_1 = \max\{n \in T_2\}$.

Applying Lemma 3.3 to the series $\sum_{n=N_1+1}^{\infty} u_n$, shows that there exists a finite set $T_3 \subset \{N_1+1, N_1+2, \ldots\}$ such that

$$\Big\| v - \sum_{n\in T_2} u_n - \sum_{n\in T_3} u_n \Big\| < \tfrac{1}{4}.$$

Now write out the indices of first T_2 and then T_3, each in arbitrary order, write 3 if $3 \notin T_2 \cup T_3$. Continuing this process, the assertion of the lemma follows. •.

Further, we have to prove

Lemma 3.5. *Let $v_1, \ldots, v_N$ be arbitrary elements in a real Hilbert space $\mathcal{H}$. Then there exists a permutation σ of the set $\{1, \ldots, N\}$ such that*

$$\max_{1\leq m\leq N} \Big\| \sum_{k=1}^{m} v_{\sigma(k)} \Big\| \leq \left(\sum_{n=1}^{N} \|v_n\|^2 \right)^{\frac{1}{2}} + 2 \Big\| \sum_{n=1}^{N} v_n \Big\|.$$

Proof. First, suppose that

$$\sum_{n=1}^{N} v_n = 0.$$

Then we shall construct by induction a permutation $\{n_1, \ldots, n_N\}$ of $\{1, \ldots, N\}$ such that

$$\max_{1\leq m\leq N} \Big\| \sum_{k=1}^{m} v_{n_k} \Big\| \leq \left(\sum_{n=1}^{N} \|v_n\|^2 \right)^{\frac{1}{2}}. \tag{3.1}$$

For this aim put $n_1 = 1$ and suppose that $n_1, \ldots, n_j$ with $1 \leq j \leq N-1$ have been chosen, satisfying

$$\max_{1\leq m\leq j} \Big\| \sum_{k=1}^{m} v_{n_k} \Big\|^2 \leq \sum_{n=1}^{j} \|v_n\|^2.$$

Then we may choose n_{j+1} from the remaining numbers such that

$$\left\langle \sum_{k=1}^{j} v_{n_k}, v_{n_{j+1}} \right\rangle \leq 0.$$

Such an n_{j+1} exists since otherwise

$$\sum_{i\neq n_k} \Big\langle \sum_{k=1}^{i} v_{n_k}, v_i \Big\rangle = \Big\langle \sum_{k=1}^{j} v_{n_k}, -\sum_{k=1}^{j} v_{n_k} \Big\rangle > 0.$$

Hence,

$$\begin{aligned}\Big\|\sum_{k=1}^{j+1} v_{n_k}\Big\|^2 &= \sum_{k=1}^{j} \|v_{n_k}\|^2 + \|v_{n_{j+1}}\|^2 + 2\left\langle \sum_{k=1}^{j} v_{n_k}, v_{n_{j+1}}\right\rangle \\ &\leq \sum_{k=1}^{j+1} \|v_{n_k}\|^2.\end{aligned}$$

This yields a permutation $\{n_1, n_2, \ldots, n_N\}$ of $\{1, 2, \ldots, N\}$ which satisfies (3.1) under the assumption $\sum_{n=1}^{N} v_n = 0$.

For arbitrary $v_1, \ldots, v_N$ define

$$v_{N+1} = -\sum_{n=1}^{N} v_n,$$

and apply the already proved case for $v_1, \ldots, v_N, v_{N+1}$. This leads to a permutation $\{n_1, n_2, \ldots, n_{N+1}\}$ of $\{1, 2, \ldots, N+1\}$ with

$$\max_{1\leq m\leq N+1} \Big\|\sum_{n=1}^{m} v_{\sigma(n)}\Big\| \leq \left(\sum_{n=1}^{N} \|v_n\|^2\right)^{\frac{1}{2}} + \Big\|\sum_{n=1}^{N} v_n\Big\|.$$

Removing ν_{N+1} from the set $\{v_{n_1}, \ldots, v_{n_N}, v_{n_{N+1}}\}$ we get an N-tuple of vectors which satisfies the inequality of the lemma. •

Now we are in the position for the

Proof of Theorem 3.2. By Lemma 3.4 we may assume that some subsequence of the partial sums of the series $\sum_k u_k$ converges to v in the norm of $\mathcal{H}$. We define

$$U_n = \sum_{k=1}^{n} u_k,$$

and suppose that a sequence of partial sums U_{n_j} converges to v. For each $j \in \mathbb{N}$ there is a permutation σ of the set of vectors $\{U_{n_j+1}, \ldots, U_{n_{j+1}}\}$ in such a way that the value of

$$m_j := \max_{1\leq m\leq n_{j+1}-n_j} \Big\|\sum_{n=n_j+1}^{n_j+m} u_{\sigma(n)}\Big\|$$

is minimal. By Lemma 3.5 it follows that

$$m_j \leq \left(\sum_{n=n_j+1}^{\infty} \|u_n\|^2\right)^{\frac{1}{2}} + 2\|U_{n_{j+1}} - U_{n_j}\|,$$

which tends to zero as $j \to \infty$. Hence, the corresponding series converges to v in the norm of $\mathcal{H}$. Theorem 3.2 is proved. •

In the sequel we shall apply Pechersky's rearrangement theorem 3.2 to the following Hilbert space. Let R be a positive real number, then the so-called Hardy space $\mathcal{H}_2^R$ is the set of functions $f(s)$ which are analytic for $|s| < R$ and for which

$$\|f\| := \lim_{r\to R-} \iint_{|s|<r} |f(s)| \,\mathrm{d}\sigma\,\mathrm{d}t < \infty.$$

We define on $\mathcal{H}_2^R$ an inner product by

$$\langle f, g\rangle \ = \mathrm{Re} \iint_{|s|\leq R} f(s)\overline{g(s)}\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{3.2}$$

Hence $\mathcal{H}_2^R$ is a real Hilbert space.

3.3. Finite Euler products. Let Ω denote the set of all sequences of real numbers indexed by the primes, that are all infinite vectors of the form $\omega := (\omega_2, \omega_3, \ldots)$ with $\omega_p \in \mathbb{R}$. Then we define for any finite subset M of the set of all primes, any $\omega \in \Omega$ and complex s

$$\zeta_M(s,\omega) = \prod_{p\in M} \left(1 - \frac{\exp(-2\pi i\omega_p)}{p^s}\right)^{-1}.$$

Obviously, $\zeta_M(s,\omega)$ is an analytic function in s without zeros in the half-plane $\sigma > 0$. Consequently, its logarithm exists and equals

$$\log \zeta_M(s,\omega) = -\sum_{p\in M} \log\left(1 - \frac{\exp(-2\pi i\omega_p)}{p^s}\right);$$

here as for $\log\zeta(s)$ we may take the principal branch of the logarithm on the positive real axis.

The first step in the proof of Theorem 3.1 is to show

Theorem 3.6. *Let $0 < r < \frac{1}{4}$ and suppose that $f(s)$ is continuous on $|s| \leq r$ and analytic in the interior. Further, let $\omega_0 = \left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots\right)$. Then for any $\epsilon > 0$ and any $y > 0$ there exists a finite set M of prime numbers, containing at least all primes $p \leq y$, such that*

$$\max_{|s|\leq r} \left|\log \zeta_M\left(s + \tfrac{3}{4}, \omega_0\right) - f(s)\right| < \epsilon.$$

Proof. Since $f(s)$ is continuous for $|s| \leq r$, there exists $\kappa > 1$ such that $\kappa^2 r < \frac{1}{4}$ and

$$\max_{|s|\leq r} \left| f\left(\frac{s}{\kappa^2}\right) - f(s)\right| < \frac{\epsilon}{2}. \tag{3.3}$$

The function $f\left(\frac{s}{\kappa^2}\right)$ is bounded on the disc $|s| \leq \kappa r =: R$, and thus belongs to the Hardy space $\mathcal{H}_2^R$.

Denote by p_k the kth prime number. We consider the series

$$\sum_{k=1}^{\infty} u_k(s) \qquad \text{with} \quad u_k(s) := \log\left(1-\exp(-2\pi i\omega_{p_k})p_k^{-s-\frac{3}{4}}\right)^{-1}.$$

First, we shall prove that for every $v \in \mathcal{H}_2^R$ there exists a rearrangement of the series $\sum u_k(s)$ for which

$$\sum_{k=1}^{\infty} u_{j_k}(s) = v(s).$$

In view of the Taylor expansion of the logarithm the series $\sum_k u_k(s)$ differs from

$$\sum_{k=1}^{\infty} \eta_k(s) \qquad \text{with} \quad \eta_k(s) := \exp\left(-\tfrac{2\pi i k}{4}\right) p_k^{-s-\frac{3}{4}}$$

by an absolutely convergent series. Hence, it suffices to verify the conditions of the rearrangement theorem 3.2 for the series $\sum_k \eta_k(s)$. Since $R < \frac{1}{4}$,

$$\sum_{k=1}^{\infty} \|\eta_k(s)\|^2 \ll \sum_p \frac{1}{p^{\frac{3}{2}-2R}} < \infty.$$

Further, we have to check that for any $\phi \in \mathcal{H}_2^R$ with $\|\phi\|^2 = 1$ the series

$$\sum_{k=1}^{\infty} \langle \eta_k, \phi \rangle \tag{3.4}$$

is conditionally convergent for some rearrangement of its terms. By the Cauchy-Schwarz inequality,

$$\sum_{k=1}^{\infty} \langle \eta_k, \phi \rangle \leq \left\| \sum_{k=1}^{\infty} \eta_k \right\|^{\frac{1}{2}} \cdot \|\phi\|^{\frac{1}{2}} = \left\| \sum_{k=1}^{\infty} \eta_k \right\|^{\frac{1}{2}} < \infty,$$

and so it is sufficient to show that there exist two subseries of (3.4), where one is diverging to $+\infty$ and the other one to $-\infty$.

By (3.2),

$$\langle \eta_k, \phi \rangle = \operatorname{Re}\left\{ \exp\left(-\tfrac{2\pi i k}{4}\right) \iint_{|s|\leq R} p_k^{-s-\frac{3}{4}} \overline{\phi(s)}\, \mathrm{d}\sigma\, \mathrm{d}t \right\}. \tag{3.5}$$

Now define

$$\Delta(x) = \iint_{|s|\leq R} \exp\left(-x\left(s+\tfrac{3}{4}\right)\right) \overline{\phi(s)}\, \mathrm{d}\sigma\, \mathrm{d}t,$$

then the integral appearing on the right of (3.5) equals $\Delta(\log p_k)$. Further, let $\phi(s) = \sum_{m=0}^{\infty} \alpha_m s^m$. Then we may express $\Delta(x)$ in terms of the Taylor coefficients α_m as follows:

$$\begin{aligned}\Delta(x) &= \exp\left(-\tfrac{3x}{4}\right) \iint_{|s|\leq R} \exp(-sx)\overline{\phi(s)}\,\mathrm{d}\sigma\,\mathrm{d}t \\ &= \exp\left(-\tfrac{3x}{4}\right) \iint_{|s|\leq R} \sum_{n=0}^{\infty} \frac{(-sx)^n}{n!} \sum_{m=0}^{\infty} \overline{\alpha_m}\,\overline{s}^m\,\mathrm{d}\sigma\,\mathrm{d}t \\ &= \exp\left(-\tfrac{3x}{4}\right) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\overline{\alpha_m} \iint_{|s|\leq R} \overline{s}^m s^n\,\mathrm{d}\sigma\,\mathrm{d}t.\end{aligned}$$

We compute

$$\begin{aligned}\iint_{|s|\leq R} \overline{s}^m s^n\,\mathrm{d}\sigma\,\mathrm{d}t &= \int_0^R \int_0^{2\pi} \rho^{m+n}\exp(i\theta(n-m))\,\mathrm{d}\theta\,\mathrm{d}\rho \\ &= \begin{cases} 2\pi\frac{R^{2m+2}}{2m+2} & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}\end{aligned}$$

This yields

$$\Delta(x) = \pi R^2 \exp\left(-\frac{3x}{4}\right)\sum_{m=0}^{\infty} \frac{\beta_m}{m!}(xR)^m, \tag{3.6}$$

where

$$\beta_m = (-1)^m \frac{\overline{\alpha_m} R^m}{m+1}.$$

Since $\|\phi\| = 1$, we get

$$\begin{aligned}1 = \iint_{|s|\leq R} |\phi(s)|^2\,\mathrm{d}\sigma\,\mathrm{d}t &= \sum_{m=0}^{\infty} |\alpha_m|^2 \iint_{|s|\leq R} |s|^{2m}\,\mathrm{d}\sigma\,\mathrm{d}t \\ &= \pi R^2 \sum_{m=0}^{\infty} \frac{|\alpha_m|^2}{m+1} R^{2m}.\end{aligned}$$

Hence,

$$0 < \sum_{m=0}^{\infty} |\beta_m|^2 \ll 1, \tag{3.7}$$

which implies that β_m is bounded. Consequently, the function $F(z)$, given by

$$F(z) = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} z^m,$$

defines an entire function in z.

Next we shall show that for any $\delta > 0$ there exists a sequence of positive real numbers z_j, tending to $+\infty$, for which

$$|F(z_j)| > \exp(-(1+2\delta)z_j). \tag{3.8}$$

Suppose the contrary. Then there is some $\delta \in (0,1)$ and a constant B such that $|F(z)| < B\exp(-(1+2\delta)z)$ for any $z \geq 0$. It follows that

$$|\exp((1+\delta)z)F(z)| < B\exp(-\delta|z|) \qquad \text{for} \quad z \geq 0; \tag{3.9}$$

since $|\beta_m| \ll 1$, this estimate even holds for $z < 0$ by a suitable change of the constant B.

Here we shall apply two theorems from Fourier analysis. First, recall the theorem of Paley-Wiener: given an entire function $G(z)$, then the relation

$$G(z) = \int_{-\alpha}^{\alpha} g(\xi)\exp(i\xi z)\,\mathrm{d}\xi \tag{3.10}$$

holds for some square integrable function $g(\xi)$ if and only if

$$\int_{-\infty}^{\infty} |G(z)|^2\,\mathrm{d}z < \infty,$$

and $G(z)$ has an analytic continuation throughout the complex plane satisfying $G(z) \ll \exp((\alpha+\epsilon)|z|)$ for any $\epsilon > 0$, where the implicit constant may depend on ϵ (this characterizes all transcendent functions of fixed exponential type $\leq \alpha$). Plancherel's theorem states that for any such $G(z)$ with (3.10) also

$$g(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(z)\exp(-i\xi z)\,\mathrm{d}z$$

holds almost everywhere in $\mathbb{R}$.

Application of the theorem of Paley-Wiener with $G(z) = \exp((1+\delta)z)F(z)$ yields with regard to (3.9) the representation

$$\exp((1+\delta)z)F(z) = \int_{-3}^{3} f(\xi)\exp(i\xi z)\,\mathrm{d}\xi,$$

where $f(\xi)$ is a square integrable function with support on the interval $[-3,3]$ (not to be confused with our target function). Further, Plancherel's theorem implies

$$f(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(z)\exp((1+\delta)z - i\xi z)\,\mathrm{d}z$$

almost everywhere. Hence, $f(\xi)$ is analytic in a strip covering the real axis. Since the support of $f(\xi)$ lies inside a compact interval, the integral above has to be zero outside this interval. Hence, $F(z)$ has to vanish identically, contradicting the existence of a sequence of positive real numbers z_j diverging to $+\infty$ with (3.8).

Let $x_j = \frac{z_j}{R}$. Then it follows from (3.6) and (3.8) that

$$|\Delta(x_j)| > \pi R^2 \exp\left(-\tfrac{3}{4}x_j\right) F(x_j R) \geq \pi R^2 \exp\left(-x_j\left(\tfrac{3}{4} + R(1+2\delta)\right)\right).$$

Thus, for sufficiently small $\delta' > 0$ we obtain the existence of a sequence of positive real numbers x_j, tending to $+\infty$, such that

$$|\Delta(x_j)| > \exp(-(1-\delta')x_j). \tag{3.11}$$

Now we shall approximate F and Δ by polynomials. Let $N_j = [x_j] + 1$ and assume that $x_j - 1 \leq x \leq x_j + 1$. Since $|\beta_m| \ll 1$,

$$\sum_{m=N_j^2+1}^{\infty} \frac{\beta_m}{m!}(xR)^m \ll \frac{(xR)^{N_j^2}}{(N_j^2)!} \sum_{m=0}^{\infty} \frac{(xR)^m}{m!} \ll \frac{N_j^{N_j^2} \exp(N_j)}{(N_j^2)!} \ll \exp(-2x_j),$$

by Stirling's formula. Trivially,

$$\sum_{m=N_j^2+1}^{\infty} \frac{1}{m!}\left(-\tfrac{3x}{4}\right)^m \ll \exp(-2x_j)$$

for the same x. Hence,

$$F(xR) = \left\{\sum_{m=0}^{N_j^2} + \sum_{m=N_j^2+1}^{\infty}\right\} \frac{\beta_m}{m!}(xR)^m = P_j(x) + O(\exp(-2x_j))$$

and analogously

$$\exp\left(-\tfrac{3x}{4}\right) = \tilde{P}_j(x) + O(\exp(-2x_j)),$$

where P_j and $\tilde{P}_j$ are polynomials of degree $\leq N_j^2$. This yields in view of (3.6)

$$\Delta(x) = Q_j(x) + o(\exp(-x_j)) \qquad \text{for} \quad x_j - 1 \leq x \leq x_j + 1,$$

where $Q_j = P_j\tilde{P}_j$ is a polynomial of degree $\leq N_j^4$.

In order to find lower bounds for $\Delta(x)$ we have to apply a classical theorem of A. A. Markov which states that if Q is a polynomial of degree N with real coefficients which satisfies the inequality

$$\max_{-1\leq x\leq 1} |Q(x)| \leq 1,$$

then

$$\max_{-1\leq x\leq 1} |Q'(x)| \leq N^2.$$

For the first, lets assume that Q_j is a real polynomial. Choose $\xi \in [x_j - 1, x_j + 1]$ such that

$$|Q_j(\xi)| = \max_{x_j-1\leq x\leq x_j+1} |Q_j(x)|.$$

Then Markov's inequality implies

$$\max_{x_j-1\leq x\leq x_j+1} |Q_j'(x)| \leq N_j^8 |Q_j(\xi)|.$$

For $|x-\xi| \le \frac{\delta}{\xi^2}$ with sufficiently small δ satisfying $0<\delta<N_j^{-8}$, it follows that

$$\begin{aligned} |Q_j(x)| &\ge |Q_j(\xi)| - |x-\xi| \max_{x_j-1\le x\le x_j+1} |Q_j'(x)| \\ &\ge |Q_j(\xi)| - O(|x-\xi|\xi^2|Q_j(\xi)|) \\ &\ge \tfrac{1}{2}|Q_j(\xi)| \ge \tfrac{1}{2}|Q_j(x_j)|. \end{aligned}$$

Hence, for $x \in [\xi-\frac{\delta}{\xi^2}, \xi+\frac{\delta}{\xi^2}] \cap [x_j-1, x_j+1]$

$$\begin{aligned} |\Delta(x)| &\ge |Q_j(x)| + o(\exp(-x_j)) \\ &\ge \tfrac{1}{2}|Q_j(x_j)| + o(\exp(-x_j)) \ge \tfrac{1}{2}|\Delta(x_j)| + o(\exp(-x_j)). \end{aligned}$$

We have assumed that Q_j has real coefficients. If this is not true, then the above reasoning may be applied to both, the real part and the imaginary part of Q_j. Hence, for sufficiently large x_j, the intervals $[x_j-1, x_j+1]$ contain intervals $[\alpha, \alpha+\beta]$ of length $\ge \frac{1}{200}N_j^{-8}$ all of whose points satisfy at least one of the inequalities

$$|\mathrm{Re}\,\Delta(x)| > \frac{1}{200}\exp(-(1-\delta')x)\ , \quad |\mathrm{Im}\,\Delta(x)| > \frac{1}{200}\exp(-(1-\delta')x). \tag{3.12}$$

In order to prove the divergence of a subseries of (3.4) we note that one of the inequalities in (3.12) is satisfied infinitely often as $x\to\infty$; we may assume that it is the one with the real part. By the prime number theorem 2.21, the interval $[\exp(\alpha), \exp(\alpha+\beta)]$ contains

$$\begin{aligned} &\int_{\exp(\alpha)}^{\exp(\alpha+\beta)} \frac{\mathrm{d}u}{\log u} + O\left(\exp\left(\alpha+\beta-c\alpha^{\frac{1}{9}}\right)\right) \\ &\gg \frac{\exp(\alpha)}{\alpha}\left(\exp(\beta)-1+O\left(\exp\left(\beta-c\alpha^{\frac{1}{9}}\right)\right)\right) \end{aligned}$$

many primes, where $c>0$ is some absolute constant. Under these prime numbers $p_k \in [\exp(\alpha), \exp(\alpha+\beta)]$ we choose those with $k\equiv 0 \bmod 4$. Since $\omega_{p_k} = \frac{k}{4}$, we deduce from (3.5) and (3.11)

$$\sum_{\substack{k\equiv 0 \bmod 4 \\ \alpha\le\log p_k\le\alpha+\beta}} \langle \eta_k, \phi\rangle = \sum_{\substack{k\equiv 0 \bmod 4 \\ \alpha\le\log p_k\le\alpha+\beta}} \mathrm{Re}\,\Delta(\log p_k) \gg \exp(\tfrac{1}{2}\delta' x_j),$$

which diverges with $x_j\to\infty$.

Thus, we have shown that the series (3.4) satsifies the conditions of Theorem 3.2. Hence, there exists a rearrangement of the series $\sum_k u_k(s)$ such that

$$\sum_{k=1}^{\infty} u_{j_k}(s) = f\left(\frac{s}{\kappa^2}\right), \tag{3.13}$$

where f is our target function. Before we can finish the proof of Theorem 3.6 we have to prove the following easy

Lemma 3.7. *Suppose that* $G(s)$ *is analytic on* $|s-s_0|\leq R$ *and*

$$\iint_{|s-s_0|\leq r}|G(s)|^2\,\mathrm{d}\sigma\,\mathrm{d}t=M.$$

Then, for any fixed r *satisfying* $r<R$ *and any* s *with* $|s-s_0|\leq r$,

$$|G(s)|\leq\frac{1}{R-r}\left(\frac{M}{\pi}\right)^{\frac{1}{2}}.$$

Proof. By Cauchy's formula,

$$G(s)^2=\frac{1}{2\pi i}\oint_{|z-s|=\rho}\frac{G(z)^2}{z-s}\,\mathrm{d}z=\frac{1}{2\pi}\int_0^{2\pi}G^2(s+\rho\exp(i\theta))\,\mathrm{d}\theta$$

for any $\rho<R$. Taking the absolute modulus and integrating with respect to ρ, we obtain

$$|G(s)|^2\int_0^{R-r}\rho\,\mathrm{d}\rho\leq\frac{1}{2\pi}\int_0^{2\pi}\int_0^{R-r}|G(s+\rho\exp(i\theta))|^2\rho\,\mathrm{d}\rho\,\mathrm{d}\theta=\frac{M}{2\pi}.$$

This yields the assertion. •

We return to the proof of Theorem 3.6. According to (3.13),

$$\lim_{n\to\infty}\sum_{k=1}^n u_{j_k}(s)=f\left(\frac{s}{\kappa^2}\right)$$

in the norm of $\mathcal{H}_2^R$. This implies

$$\lim_{n\to\infty}\iint_{|s|\leq R}\left|f\left(\frac{s}{\kappa^2}\right)-\sum_{k=1}^n u_{j_k}(s)\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t=0$$

uniformly on $|s|\leq R$. Thus, application of Lemma 3.7 shows that for sufficiently large m

$$\max_{|s|\leq R}\left|f\left(\frac{s}{\kappa^2}\right)-\sum_{k=1}^m u_{j_k}(s)\right|<\tfrac{1}{2}\epsilon.$$

Hence, there exists a finite set M, containing without loss of generality all primes $p\leq y$, such that

$$\log\zeta_M\left(s+\tfrac{3}{4},\omega_0\right)=\sum_{k=1}^m u_{j_k}(s).$$

approximates $g(s)$. More precisely, in view of (3.3) it follows that

$$\max_{|s|\leq r}\left|\log\zeta_M\left(s+\tfrac{3}{4},\omega_0\right)-f(s)\right|$$
$$\leq\max\nolimits_{|s|\leq r}\left|\log\zeta_M\left(s+\tfrac{3}{4},\omega_0\right)-f\left(\tfrac{s}{\kappa^2}\right)\right|+\max\nolimits_{|s|\leq r}\left|f\left(\tfrac{s}{\kappa^2}\right)-f(s)\right|<\epsilon.$$

This finishes the proof of Theorem 3.6. •

Before we continue with the proof of Voronin's universality theorem, we need some arithmetical tools from the theory of diophantine approximation.

3.4. Diophantine approximation. In the theory of diophantine approximations one investigates how *good* an irrational number can be approximated by rational numbers. This has plenty of applications in various fields of mathematics and natural sciences.

For abbreviation we denote vectors of $\mathbb{R}^N$ by $\mathbf{x} = (x_1, \ldots, x_N)$, we define $\tau\mathbf{x} = (\tau x_1, \ldots, \tau x_N)$ for $\tau \in \mathbb{R}$ and $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \ldots + x_N y_N$. Further, for $\mathbf{x} \in \mathbb{R}^N$ and $\gamma \subset \mathbb{R}^N$ we write $\mathbf{x} \in \gamma \bmod 1$ if there exists $\mathbf{y} \in \mathbb{Z}^N$ such that $\mathbf{x} - \mathbf{y} \in \gamma$. Moreover, we shall introduce the notion of Jordan volume of a region $\gamma \subset \mathbb{R}^N$. Therefore, we consider the sets of parallelepipeds γ_1 and γ_2 with sides parallel to the axes and of volume Γ_1 and Γ_2 with $\gamma_1 \subset \gamma \subset \gamma_2$; if there are γ_1 and γ_2 such that $\limsup_{\gamma_1} \Gamma_1$ coincides with $\liminf_{\gamma_2} \Gamma_2$, then γ has the Jordan volume

$$\Gamma = \limsup_{\gamma_1} \Gamma_1 = \liminf_{\gamma_2} \Gamma_2.$$

The Jordan sense of volume is more restrictive than the one of Lebesgue, but if the Jordan volume exists it is also defined in the sense of Lebesgue and equal to it.

Weyl [69] proved

Theorem 3.8. *Let $a_1, \ldots, a_N \in \mathbb{R}$ be linearly independent over the field of rational numbers, write $\mathbf{a} = (a_1, \ldots, a_N)$, and let γ be a subregion of the N-dimensional unit cube with Jordan volume Γ. Then*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\{\tau \in (0, T) \,:\, \tau\mathbf{a} \in \gamma \bmod 1\} = \Gamma.$$

Proof. From the definition of the Jordan volume it follows that for any $\epsilon > 0$ there exist two finite sets of open parallelepipeds $\{\prod_j^-\}$ and $\{\prod_j^+\}$ inside the unit cube such that

$$\overline{\bigcup \prod\nolimits_j^-} \subset \operatorname{int}(\gamma) \subset \overline{\gamma} \subset \bigcup \prod\nolimits_j^+ \tag{3.14}$$

and

$$\operatorname{meas}\left(\bigcup \prod\nolimits_j^+ \setminus \bigcup \prod\nolimits_j^-\right) < \epsilon;$$

here, as usual, $\overline{M}$ denotes the closure of the set M, and $\operatorname{int}(M)$ its interior. Denote by $\mathbf{1}^\pm$ the characteristic function of $\bigcup \prod_j^\pm$, i.e.

$$\mathbf{1}^\pm(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \mathbf{x} \in \bigcup \prod_j^\pm, \\ 0 & \text{if} \quad \mathbf{x} \notin \bigcup \prod_j^\pm. \end{cases}$$

Further, let $\mathbf{1}$ be the characteristic function of $\gamma \bmod 1$. Consequently,

$$0 \leq \mathbf{1}^-(\mathbf{x}) \leq \mathbf{1}(\mathbf{x}) \leq \mathbf{1}^+(\mathbf{x}) \leq 1,$$

and

$$\int_{[0,1]^N} (\mathbf{1}^+(\mathbf{x}) - \mathbf{1}^-(\mathbf{x}))\,\mathrm{d}\mathbf{x} < \epsilon,$$

where the integral is N-dimensional with $\mathrm{d}\mathbf{x} = \mathrm{d}x_1 \cdots \mathrm{d}x_N$. Define

$$\Phi(x) = \left\{ \begin{array}{cl} 0 & \text{if} \quad |x| \geq \frac{1}{2}, \\ c\exp\left(-\left(\frac{1}{x+\frac{1}{2}} + \frac{1}{x-\frac{1}{2}}\right)\right) & \text{if} \quad |x| < \frac{1}{2}, \end{array}\right.$$

where c is defined via

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi(x)\,\mathrm{d}x = 1.$$

Consequently, $\Phi(x)$ is an infintely differentiable function, and hence the functions, given by

$$\tilde{\mathbf{1}}^{\pm}(\mathbf{x}) = \delta^{-N} \int_{[0,1]^N} \mathbf{1}^{\pm}(\mathbf{y})\Phi\left(\frac{x_1 - y_1}{\delta}\right) \cdots \Phi\left(\frac{x_N - y_N}{\delta}\right)\,\mathrm{d}\mathbf{y}$$

for $0 < \delta < 1$, are infinitely differentiable functions too. From (3.14) it follows that for sufficiently small δ we have

$$0 \leq \tilde{\mathbf{1}}^-(\mathbf{x}) \leq \mathbf{1}(\mathbf{x}) \leq \tilde{\mathbf{1}}^+(\mathbf{x}) \leq 1,$$

and

$$0 \leq \int_{[0,1]^N} (\tilde{\mathbf{1}}^+(\mathbf{x}) - \tilde{\mathbf{1}}^-(\mathbf{x}))\,\mathrm{d}\mathbf{x} < 2\epsilon. \tag{3.15}$$

We conclude

$$\int_0^T \tilde{\mathbf{1}}^-(\tau\mathbf{a})\,\mathrm{d}\tau \leq \text{meas}\,\{\tau \in (0,T) \,:\, \tau\mathbf{a} \in \gamma \bmod 1\} \leq \int_0^T \tilde{\mathbf{1}}^+(\tau\mathbf{a})\,\mathrm{d}\tau \tag{3.16}$$

and

$$0 \leq \int_0^T \tilde{\mathbf{1}}^+(\tau\mathbf{a})\,\mathrm{d}\tau - \int_0^T \tilde{\mathbf{1}}^-(\tau\mathbf{a})\,\mathrm{d}\tau \leq 2\epsilon T.$$

Both integrands above are infinitely differentiable functions which are 1-periodic in each variable. Thus, we have the Fourier expansion

$$\tilde{\mathbf{1}}^{\pm}(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^N} c_{\mathbf{n}}^{\pm} \exp(2\pi i\,\mathbf{n}\cdot\mathbf{x}),$$

where

$$c_{\mathbf{n}}^{\pm} = \int_{[0,1]^N} \tilde{\mathbf{1}}^{\pm}(\mathbf{x}) \exp(-2\pi i\,\mathbf{n}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Note that $c_{\mathbf{0}}^{\pm}$ is the volume of $\bigcup \prod_j^{\pm}$. Integration by parts gives

$$c_{\mathbf{n}}^{\pm} \ll \prod_{j=1}^{N} (|n_j| + 1)^{-k} \qquad \text{for} \quad k = 1, 2, \ldots,$$

where the implicit constant depends only on k. This shows that the Fourier series converges absolutely, and hence, for every $\epsilon > 0$, there exists a finite set $\mathcal{M} \subset \mathbb{Z}^N$ such that

$$\tilde{\mathbf{1}}^{\pm}(\mathbf{x}) = \sum_{\mathbf{n}\in\mathcal{M}} c_{\mathbf{n}}^{\pm} \exp(2\pi i\, \mathbf{n}\cdot\mathbf{x}) + R(\mathbf{x}) \qquad \text{with} \quad |R(\mathbf{x})| < \epsilon.$$

This yields

$$\frac{1}{T}\int_0^T \tilde{\mathbf{1}}^{\pm}(\tau\mathbf{a})\,\mathrm{d}\tau = \frac{1}{T}\int_0^T \sum_{\mathbf{n}\in\mathcal{M}} c_{\mathbf{n}}^{\pm} \exp(2\pi i\tau\, \mathbf{n}\cdot\mathbf{a})\,\mathrm{d}\tau + \theta\epsilon$$

with some θ satisfying $|\theta| < 1$. Consequently,

$$\frac{1}{T}\int_0^T \tilde{\mathbf{1}}^{\pm}(\tau\mathbf{a})\,\mathrm{d}\tau = c_{\mathbf{0}}^{\pm} + \sum_{\mathbf{0}\neq\mathbf{n}\in\mathcal{M}} c_{\mathbf{n}}^{\pm} \frac{1}{T}\int_0^T \exp(2\pi i\tau\, \mathbf{n}\cdot\mathbf{a})\,\mathrm{d}\tau + \theta\epsilon.$$

Since the a_n are linearly independent over $\mathbb{Q}$, we have $\mathbf{n}\cdot\mathbf{a} \neq 0$ for $\mathbf{n} \neq 0$. It follows for such $\mathbf{n}$ that

$$\int_0^T \exp(2\pi i\tau\, \mathbf{n}\cdot\mathbf{a}))\,\mathrm{d}\tau \ll 1.$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \tilde{\mathbf{1}}^{\pm}(\tau\mathbf{a})\,\mathrm{d}\tau = c_{\mathbf{0}}^{\pm}.$$

Thus, we get in (3.16)

$$\begin{aligned} c_{\mathbf{0}}^{-} - \epsilon &\leq \liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\{\tau \in (0,T) \,:\, \tau\mathbf{a} \in \gamma \bmod 1\} \\ &\leq \limsup_{T\to\infty} \frac{1}{T} \operatorname{meas}\{\tau \in (0,T) \,:\, \tau\mathbf{a} \in \gamma \bmod 1\} \leq c_{\mathbf{0}}^{+} + \epsilon \end{aligned}$$

for any positive ϵ. From (3.15) it follows that $0 \leq c_{\mathbf{0}}^{+} - c_{\mathbf{0}}^{-} \leq 2\epsilon$. Now sending $\epsilon \to 0$, the theorem is proved. •

As an immediate consequence of Theorem 3.8 we get the classical inhomogeneous Kronecker approximation theorem:

Corollary 3.9. *Let $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$, let $\beta_1, \ldots, \beta_N$ be arbitrary real numbers, and let q be a positive number. Then there exists a number $\tau > 0$ and integers $x_1, \ldots, x_N$ such that*

$$|\tau\alpha_n - \beta_n - x_n| < \frac{1}{q} \qquad \textit{for} \quad 1 \leq n \leq N.$$

We conclude with the notion of uniform distribution modulo 1. Let $\gamma(\tau)$ be a continuous function with domain of definition $[0,\infty)$ and range $\mathbb{R}^N$. Then the curve $\gamma(\tau)$ is said to be uniformly distributed mod 1 in $\mathbb{R}^N$ if

$$\lim_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\,\{\tau\in(0,T)\,:\,\gamma(\tau)\in\prod \bmod 1\}=\prod_{j=1}^N(\beta_j-\alpha_1)$$

for every parallelepiped $\prod=[\alpha_1,\beta_1]\times\ldots\times[\alpha_N,\beta_N]$ with $0\leq\alpha_j<\beta_j\leq 1$ for $1\leq j\leq N$. In a sense, a curve is uniformly distributed mod 1 if the number of values which lie in any given measurable subset of the unit cube is proportional to the measure of the subset.

In questions about uniform distribution mod 1 one is interested in the fractional part only. Hence, we define for a curve $\gamma(\tau)=(\gamma_1(\tau),\ldots,\gamma_N(\tau))$ in $\mathbb{R}^N$

$$\{\gamma(\tau)\}=(\gamma_1(\tau)-[\gamma_1(\tau)],\ldots,\gamma_N(\tau)-[\gamma_N(\tau)]);$$

recall that $[x]$ denotes the integral part of $x\in\mathbb{R}$.

Theorem 3.10. *Suppose that the curve $\gamma(\tau)$ is uniformly distributed* mod 1 *in $\mathbb{R}^N$. Let D be a closed and Jordan measurable subregion of the unit cube in $\mathbb{R}^N$ and let Ω be a family of complex-valued continuous functions defined on D. If Ω is uniformly bounded and equicontinuous, then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\{\gamma(\tau)\})\mathbf{1}_D^\gamma(\tau)\,\mathrm{d}\tau=\int_D f(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

uniformly with respect to $f\in\Omega$, where $\mathbf{1}_D^\gamma(\tau)$ is equal to 1 *if $\gamma(\tau)\in D$* mod 1, *and equal to zero otherwise.*

Proof. By the definition of the Riemann integral as a limit of Riemann sums, we have for any Riemann integrable function F on the unit cube in $\mathbb{R}^N$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F(\{\gamma(\tau)\})\,\mathrm{d}\tau=\int_{[0,1]^N}F(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{3.17}$$

By the assumptions on Ω, for any $\epsilon>0$ there exist $f_1,\ldots,f_n\in\Omega$ such that for every $f\in\Omega$ there is an f_j among them satisfying

$$\sup_{\mathbf{x}\in D}|f(\mathbf{x})-f_j(\mathbf{x})|<\epsilon.$$

By (3.17) there exists T_0 such that for any $T>T_0$ and for each function $f_1,\ldots,f_n$ one has

$$\left|\int_D f_j(\mathbf{x})\,\mathrm{d}\mathbf{x}-\frac{1}{T}\int_0^T f_j(\{\gamma(\tau)\})\mathbf{1}_D^\gamma(\tau)\,\mathrm{d}\tau\right|<\epsilon.$$

Now, for any $f \in \Omega$,

$$\begin{aligned}
&\left| \int_D f(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{1}{T}\int_0^T f(\{\gamma(\tau)\})\mathbf{1}_D^\gamma(\tau)\,\mathrm{d}\tau \right| \\
&\leq \left| \int_D f_j(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{1}{T}\int_0^T f_j(\{\gamma(\tau)\})\mathbf{1}_D^\gamma(\tau)\,\mathrm{d}\tau \right| + \left| \int_D (f(\mathbf{x}) - f_j(\mathbf{x}))\,\mathrm{d}\mathbf{x} \right| \\
&\quad + \frac{1}{T}\left| \int_0^T (f_j(\{\gamma(\tau)\}) - f(\{\gamma(\tau)\}))\mathbf{1}_D^\gamma(\tau)\,\mathrm{d}\tau \right|
\end{aligned}$$

By the appropriate choice of f_j it follows from the estimates above that this is bounded by 3ϵ. Since $\epsilon > 0$ is arbitrary, the assertion of the theorem follows. •

Besides, there is also the notion of uniformly distributed sequences, defined in a similar way. It was proved by Hlawka [28] that the imaginary parts of the zeros of the zeta-function are uniformly distributed modulo 1.

3.5. Approximation in the mean — end of proof. We choose $\kappa > 1$ and $\epsilon_1 \in (0,1)$ such that $\kappa r < \frac{1}{4}$ and

$$\max_{|s|\leq r} \left| f\left(\frac{s}{\kappa}\right) - f(s) \right| < \epsilon_1.$$

Let $Q = \{p \leq z\}$ and $\mathcal{E} = \{s : -\kappa r < \sigma \leq 2, -1 \leq t \leq T\}$. We shall estimate

$$\mathcal{I} := \int_T^{2T} \iint_{\mathcal{E}} \left| \zeta_Q^{-1}\left(s + \tfrac{3}{4} + i\tau, \mathbf{0}\right) \zeta\left(s + \tfrac{3}{4} + i\tau\right) - 1 \right|^2 \mathrm{d}\sigma\,\mathrm{d}t\,\mathrm{d}\tau, \tag{3.18}$$

where $\mathbf{0} = (0, 0, \ldots)$. By Theorem 2.10,

$$\zeta(s + i\tau) = \sum_{n \leq T} \frac{1}{n^{s+i\tau}} + O(T^{-\sigma}).$$

This gives

$$\begin{aligned}
\mathcal{I} &= \iint_{\mathcal{E}+\frac{3}{4}} \int_T^{2T} |\zeta_Q^{-1}(s + i\tau, \mathbf{0})\zeta(s + i\tau) - 1|^2\,\mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\ll \iint_{\mathcal{E}+\frac{3}{4}} \int_T^{2T} \left| \zeta_Q^{-1}(s + i\tau, \mathbf{0}) \sum_{n \leq T} \frac{1}{n^{s+i\tau}} - 1 \right|^2 \mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\quad + \iint_{\mathcal{E}+\frac{3}{4}} \int_T^{2T} T^{-\sigma} |\zeta_Q^{-1}(s + i\tau, \mathbf{0})|^2\,\mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{d}t,
\end{aligned} \tag{3.19}$$

where $\mathcal{E} + \frac{3}{4}$ is the set of all s with $s - \frac{3}{4} \in \mathcal{E}$. By definition,

$$\zeta_Q^{-1}(s, \mathbf{0}) = \prod_{p \in Q} \left(1 - \frac{1}{p^s}\right) = \sum_{\substack{m=1 \\ p|m \Rightarrow p \in Q}}^{\infty} \frac{\mu(m)}{m^s};$$

recall that $\mu(m)$ is Möbius μ-function defined by (2.28). We may bound the second term appearing on the right-hand side of (3.19) by

$$T^{-2\left(\frac{3}{4}-\kappa r\right)}\max_{s\in\mathcal{E}+\frac{3}{4}}\int_T^{2T}|\zeta_Q^{-1}(s+i\tau,\mathbf{0})|^2\,\mathrm{d}\tau\ll T^{2\kappa r-\frac{1}{2}}\left|\zeta_Q^{-1}\left(\tfrac{3}{4}-\kappa r,\mathbf{0}\right)\right|^2.$$

Furthermore, for $T>z$ a simple computation gives

$$\zeta_Q^{-1}(s,\mathbf{0})\sum_{n\le T}\frac{1}{n^s}=1+\sum_{z<k\le z^zT}\frac{b_k}{k^s}\qquad\text{with}\quad b_k=\sum_{\substack{m|k\\ p|m\Rightarrow p\in Q;k\le mT}}1.$$

By estimate (2.29) for the divisor function, we have

$$|b_k|\le d(k)\ll k^{\epsilon}\qquad\text{for any}\quad\epsilon>0. \tag{3.20}$$

Hence, for $T>z$

$$\begin{aligned}\int_T^{2T}\left|\zeta_Q^{-1}(s+i\tau,\mathbf{0})\sum_{n\le T}\frac{1}{n^{s+i\tau}}-1\right|^2\mathrm{d}\tau&=\int_T^{2T}\left|\sum_{z<k\le z^zT}\frac{b_k}{k^{s+i\tau}}\right|^2\mathrm{d}\tau\\ =\ T\sum_{z<k\le z^zT}\frac{|b_k|^2}{k^{2\sigma}}&+O\left(\sum_{0<\ell<k\le z^zT}\frac{|b_kb_\ell|}{(k\ell)^\sigma}\left|\int_T^{2T}\left(\frac{k}{\ell}\right)^{i\tau}\mathrm{d}\tau\right|\right).\end{aligned}$$

Using estimate (3.20) with $\epsilon=\frac{\epsilon_1}{2}$, the above is bounded by

$$\begin{aligned}&T\sum_{k>z}\frac{d^2(k)}{k^{2\sigma}}+\sum_{0<\ell<k\le z^zT}\frac{d(k)d(\ell)}{(k\ell)^\sigma\log\frac{k}{\ell}}\\ &\ll\ Tz^{1-2\sigma+\epsilon_1}+(z^zT)^{\epsilon_1}\sum_{0<\ell<k\le z^zT}\frac{1}{(k\ell)^\sigma\log\frac{k}{\ell}}.\end{aligned}$$

The sum on the right can be estimated by $((z^zT)^{2-2\sigma}+1)\log^2(z^zT)$ similarly as we did in the proof of Theorem 2.13. Thus, we finally arrive at

$$\begin{aligned}&\iint_{\mathcal{E}+\frac{3}{4}}\int_T^{2T}\left|\zeta_Q^{-1}(s+i\tau,\mathbf{0})\sum_{n\le T}\frac{1}{n^{s+i\tau}}-1\right|^2\mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{d}t\\ \ll&\ \iint_{\mathcal{E}+\frac{3}{4}}\left(Tz^{1-2\sigma+\epsilon_1}+(z^zT)^{\epsilon_1}((z^zT)^{2-2\sigma}+1)\log^2(z^zT)\right)\mathrm{d}\sigma\,\mathrm{d}t\\ \ll&\ z^{2\kappa r+\epsilon_1-\frac{1}{2}}T.\end{aligned}$$

In view of (3.19) we conclude that for any $\epsilon_2>0$

$$\mathcal{I}\ll\epsilon_2^4T, \tag{3.21}$$

provided that z and T are sufficiently large, say $z > z_0$ and $T > T_0$, depending only on ϵ_2. Define

$$\mathcal{A}_T = \left\{ \tau \in [T, 2T] \, : \, \iint_{\mathcal{E}+\frac{3}{4}} |\zeta_Q^{-1}(s + i\tau, \mathbf{0})\zeta(s + i\tau) - 1|^2 \, \mathrm{d}\sigma \, \mathrm{d}t < \epsilon_2^2 \right\}.$$

Then it follows from (3.19) and (3.21) that for sufficiently large z and T

$$\operatorname{meas} \mathcal{A}_T > (1 - \epsilon_2)T, \tag{3.22}$$

which is surprisingly large. Application of Lemma 3.7 gives for $\tau \in \mathcal{A}_T$

$$\max_{|s| \le r} |\zeta_Q^{-1}(s + i\tau, \mathbf{0})\zeta(s + i\tau) - 1| < C\epsilon_2,$$

where C is a positive constant, depending only on κ. For sufficiently small ϵ_2 we deduce

$$\max_{|s| \le r} \left|\log \zeta\left(s + \tfrac{3}{4} + i\tau\right) - \log \zeta_Q\left(s + \tfrac{3}{4} + i\tau, \mathbf{0}\right)\right| < 2C\epsilon_2, \tag{3.23}$$

provided $\tau \in \mathcal{A}_T$.

By Theorem 3.6 there exists a sequence of finite sets of prime numbers $M_1 \subset M_2 \subset \ldots$ such that $\cup_{k=1}^{\infty} M_k$ contains all primes and

$$\lim_{k \to \infty} \max_{|s| \le \kappa r} \left|\log \zeta_{M_k}\left(s + \tfrac{3}{4}, \omega_0\right) - f\left(\frac{s}{\kappa}\right)\right| = 0. \tag{3.24}$$

Let $\omega' = (\omega_2', \omega_3', \ldots)$ and $\omega = (\omega_2, \omega_3, \ldots)$. By the continuity of $\log \zeta_M\left(s + \frac{3}{4}, \omega\right)$ with respect to ω, for any $\epsilon_1 > 0$ there exists a positive δ with the property that whenever

$$\|\omega_p - \omega_p'\| < \delta \qquad \text{for all} \quad p \in M_k, \tag{3.25}$$

then

$$\max_{|s| \le \kappa r} \left|\log \zeta_{M_k}\left(s + \tfrac{3}{4}, \omega_0\right) - \log \zeta_{M_k}\left(s + \tfrac{3}{4}, \omega'\right)\right| < \epsilon. \tag{3.26}$$

Setting

$$\mathcal{B}_T = \left\{ \tau \in [T, 2T] \, : \, \left\|\tau \frac{\log p}{2\pi} - \omega_p\right\| < \delta \right\},$$

we get

$$\begin{aligned} &\frac{1}{T} \int_{\mathcal{B}} \iint_{|s| \le \kappa r} \left|\log \zeta_Q\left(s + \tfrac{3}{4} + i\tau, \mathbf{0}\right) - \log \zeta_{M_k}\left(s + \tfrac{3}{4} + i\tau, \mathbf{0}\right)\right|^2 \, \mathrm{d}\sigma \, \mathrm{d}t \, \mathrm{d}\tau \\ &\quad = \iint_{|s| \le \kappa r} \frac{1}{T} \int_{\mathcal{B}_T} \left|\log \zeta_Q\left(s + \tfrac{3}{4} + i\tau, \mathbf{0}\right) - \log \zeta_{M_k}\left(s + \tfrac{3}{4} + i\tau, \mathbf{0}\right)\right|^2 \, \mathrm{d}\tau \, \mathrm{d}\sigma \, \mathrm{d}t. \end{aligned}$$

Putting $\omega(\tau) = \left(\tau \frac{\log 2}{2\pi}, \tau \frac{\log 3}{2\pi}, \ldots\right)$, we may rewrite the inner integral as

$$\int_{\mathcal{B}_T} \left|\log \zeta_Q\left(s + \tfrac{3}{4}, \omega(\tau)\right) - \log \zeta_{M_k}\left(s + \tfrac{3}{4}, \omega(\tau)\right)\right|^2 \, \mathrm{d}\tau.$$

The logarithms of the prime numbers are linearly independent over $\mathbb{Q}$ (this follows easily from the unique prime factorization of the integers). Thus, by Weyl's theorem 3.8, the curve $\gamma(\tau) = \big(\tau\frac{\log 2}{2\pi}, \tau\frac{\log 3}{2\pi}, \ldots, \tau\frac{\log p_N}{2\pi}\big)$ is uniformly distributed mod 1. Application of Theorem 3.10 yields

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_{\mathcal{B}_T}&\left|\log\zeta_Q\left(s+\tfrac{3}{4},\omega(\tau)\right)-\log\zeta_{M_k}\left(s+\tfrac{3}{4},\omega(\tau)\right)\right|^2\,\mathrm{d}\tau\\ =&\int_{\mathcal{D}}\left|\log\zeta_Q\left(s+\tfrac{3}{4},\omega\right)-\log\zeta_{M_k}\left(s+\tfrac{3}{4},\omega\right)\right|^2\,\mathrm{d}\mu\end{aligned}$$

uniformly in s for $|s|\le\kappa r$, where $\mathcal{D}$ is the subregion of the unit cube in $\mathbb{R}^N$ given by the inequalities (3.25) and $\mathrm{d}\mu$ is the Lebesgue measure. By the definition of $\zeta_M(s,\omega)$ it follows that for $M_k\subset Q$

$$\zeta_Q(s,\omega)=\zeta_{M_k}(s,\omega)\zeta_{Q\setminus M_k}(s,\omega),$$

and thus

$$\begin{aligned}&\int_{\mathcal{D}}\left|\log\zeta_Q\left(s+\tfrac{3}{4},\omega\right)-\log\zeta_{M_k}\left(s+\tfrac{3}{4},\omega\right)\right|^2\,\mathrm{d}\mu\\ &=\int_{\mathcal{D}}\left|\log\zeta_{Q\setminus M_k}\left(s+\tfrac{3}{4},\omega\right)\right|^2\,\mathrm{d}\mu=\operatorname{meas}\mathcal{D}\cdot\int_{[0,1]^N}\left|\log\zeta_{Q\setminus M_k}\left(s+\tfrac{3}{4},\omega\right)\right|^2\,\mathrm{d}\mu.\end{aligned}$$

Since

$$\log\zeta_{Q\setminus M_k}\left(s+\tfrac{3}{4},\omega\right)=\sum_{p\in Q\setminus M_k}\sum_{n=1}^{\infty}\frac{\exp(-2\pi i n\omega_p)}{np^{n\left(s+\frac{3}{4}\right)}},$$

we obtain

$$\int_{[0,1]^N}\left|\log\zeta_{Q\setminus M_k}\left(s+\tfrac{3}{4},\omega\right)\right|^2\,\mathrm{d}\mu=\sum_{p\in Q\setminus M_k}\sum_{n=1}^{\infty}\frac{1}{n^2p^{2n\sigma+\frac{3n}{2}}}.$$

If M_k contains all primes $\le y_k$, then

$$\sum_{p\in Q\setminus M_k}\sum_{n=1}^{\infty}\frac{1}{n^2p^{2n\sigma+\frac{3n}{2}}}\ll y_k^{2\kappa r-\frac{1}{2}}.$$

Hence, we finally get

$$\begin{aligned}\frac{1}{T}\int_{\mathcal{B}_T}\iint_{|s|\le\kappa r}&\left|\log\zeta_Q\left(s+\tfrac{3}{4}+i\tau,\mathbf{0}\right)-\log\zeta_{M_k}\left(s+\tfrac{3}{4}+i\tau,\mathbf{0}\right)\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t\,\mathrm{d}\tau\\ &\ll y_k^{2\kappa r-\frac{1}{2}}\operatorname{meas}\mathcal{D}.\end{aligned}$$

As already noticed above, the curve $\gamma(\tau)$ is uniformly distributed mod 1. Hence, application of Theorem 3.8 shows that

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}\mathcal{B}_T=\operatorname{meas}\mathcal{D},$$

which implies for sufficiently large y_k

$$\operatorname{meas}\left\{\tau\in\mathcal{B}_T\,:\,\iint_{|s|\leq\kappa r}\left|\log\zeta_Q\left(s+\tfrac{3}{4}+i\tau,\mathbf{0}\right)-\log\zeta_{M_k}\left(s+\tfrac{3}{4}+i\tau,\mathbf{0}\right)\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t\right.$$
$$\left.<y_k^{\kappa r-\frac{1}{4}}\right\}>\tfrac{1}{2}\operatorname{meas}\mathcal{D}\cdot T.$$

Now application of Theorem 3.10 yields

$$\operatorname{meas}\left\{\tau\in\mathcal{B}_T\,:\,\max_{|s|\leq\kappa r}\left|\log\zeta_Q\left(s+\tfrac{3}{4}+i\tau,\mathbf{0}\right)-\log\zeta_{M_k}\left(s+\tfrac{3}{4}+i\tau,\mathbf{0}\right)\right|\right.$$
$$\left.<y_k^{\frac{1}{5}\left(\kappa r-\frac{1}{4}\right)}\right\}>\tfrac{1}{2}\operatorname{meas}\mathcal{D}\cdot T. \tag{3.27}$$

If we now take $0<\epsilon_2<\frac{1}{2}\operatorname{meas}\mathcal{D}$, then (3.22) implies

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\ \mathcal{A}_T\cap\mathcal{B}_T>0.$$

Thus, in view of (3.24) we may approximate $f\left(\frac{s}{\kappa}\right)$ by $\log\zeta_{M_k}\left(s+\frac{3}{4},\mathbf{0}\right)$ (independent on τ), with (3.26) and (3.27) the latter function by $\log\zeta_Q\left(s+\frac{3}{4},\mathbf{0}\right)$, and finally with regard to (3.23) by $\log\zeta\left(s+\frac{3}{4}+i\tau\right)$ on a set $\mathcal{A}_T\cap\mathcal{B}_T$ of τ with positive measure. This finishes the proof of Theorem 3.1 (as well as Voronin's universality theorem 1.3). •

3.6. Reich's discrete universality theorem and other related results. Reich [56] and Bagchi [1] improved Voronin's result significantly in replacing the disk by an arbitrary compact set in the right half of the critical strip with connected complement, and by giving a lucid proof in the language of probability theory. The strongest version of Voronin's theorem has the form:

Theorem 3.11. *Suppose that $\mathcal{K}$ is a compact subset of the strip $\frac{1}{2}<\sigma<1$ with connected complement, and let $g(s)$ be a non-vanishing continuous function on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then, for any $\epsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]\,:\,\max_{s\in\mathcal{K}}|\zeta(s+i\tau)-g(s)|<\epsilon\right\}>0.$$

The topological restriction on $\mathcal{K}$ is necessary. This follows from basic facts in approximation theory. Notice that the interior of a compact line segment $\mathcal{K}$ is empty and therefore the target function g only needs to be continuous and zero-free for such sets $\mathcal{K}$. The restriction on g to be non-vanishing cannot be removed as we shall show in Section 4.2. The domain in which the uniform approximation of admissible target functions takes place is called the strip of universality. In the case of the zeta-function this strip of universality is the open right half of the critical strip. It is impossible to extend the universality property of the zeta-function to any region covering the critical line, since there are *too many* zeros of the zeta-function on the critical line (see also Garunkštis & Steuding [18]).

An interesting variation of Voronin's theorem is due to Reich. In [56] he introduced the concept of *discrete* universality by restricting the approximating shifts to arithmetic progressions. Surprisingly, this still leads to a positive lower density for the number of solutions to the corresponding approximation problem. Here is Reich's theorem in its strongest form:

Theorem 3.12. *Suppose that $\mathcal{K}$ is a compact subset of the strip $\frac{1}{2} < \sigma < 1$ with connected complement, and let $g(s)$ be a non-vanishing continuous function on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then, for any $\Delta > 0$ and any $\epsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N}\sharp\left\{1 \leq n \leq N \,:\, \max_{s\in\mathcal{K}} |\zeta(s+in\Delta) - g(s)| < \epsilon\right\} > 0.$$

Neither does Voronin's theorem imply Reich's theorem nor the other way around (by our current knowledge). Nevertheless, his argument follows in the main parts along the lines of Voronin's proof. The integral $\mathcal{I}$ given by (3.18) has to be replaced by the sum

$$\frac{1}{N}\sum_{n=1}^{N} |\zeta(s+in\Delta) - \zeta_Q(s+in\Delta,\mathbf{0})|^2.$$

Using Gallagher's lemma 2.17 (see also Montgomery [50]), the latter expression can be bounded by the corresponding integrals:

$$\begin{aligned}\limsup_{N\to\infty} &\frac{1}{N}\sum_{n=1}^{N} |\zeta(s+in\Delta) - \zeta_Q(s+in\Delta,\mathbf{0})|^2 \\ \ll\ & \limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta(s+iu) - \zeta_Q(s+iu,\mathbf{0})|^2 \,\mathrm{d}u \\ & + \left(\Delta \limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta(s+iu) - \zeta_Q(s+iu,\mathbf{0})|^2 \,\mathrm{d}u\right)^{\frac{1}{2}} \times \\ & \qquad \times \left(\limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta'(s+iu) - \zeta_Q'(s+iu,\mathbf{0})|^2 \,\mathrm{d}u\right)^{\frac{1}{2}},\end{aligned}$$

and the right-hand side can be treated as before. The remaining parts of the proof are very similar.

We shall briefly mention another line of investigation. Recently, Kaczorowski, Laurinčikas & Steuding [32] studied shifts of universal Dirichlet series with respect to universality and their value-distribution. Assume that $\mathcal{K}_1,\ldots,\mathcal{K}_n$ are disjoint compact subsets of $\frac{1}{2} < \sigma < 1$ with connected complements. Let $g(s)$ be any non-vanishing continuous function, defined on $\bigcup_{j=1}^n \mathcal{K}_j$, which is analytic in the interior. If now for any $\epsilon > 0$, there exists a real number τ such that

$$\max_{s\in\cup_{j=1}^n \mathcal{K}_j} |\zeta(s+i\tau) - g(s)| < \epsilon,$$

then also

$$\max_{1\le j\le n}\max_{s\in\mathcal{K}_j}|\zeta(s+i\tau)-g_j(s)|<\epsilon,$$

where the $g_j(s)$ are defined as restriction of $g(s)$ on $\mathcal{K}_j$. Equivalently, one can consider all $g_j(s)$ being defined on some compact subset $\mathcal{K}$ of $\frac{1}{2}<\sigma<1$ with connected complement and study shifts of $\zeta(s)$. Let $\lambda_1,\ldots,\lambda_n$ be complex numbers, $\mathcal{K}$ be any compact set, and define $\mathcal{K}_j:=\{s+\lambda_j\,:\,s\in\mathcal{K}\}$. Then the shifts

$$\zeta_{\lambda_j}(s):=\zeta(s+\lambda_j)$$

are said to be *jointly* universal with respect to $\lambda_1,\ldots,\lambda_n$ if, for every compact $\mathcal{K}$ with connected complement and for which the sets $\mathcal{K}_j$ are disjoint subsets of $\frac{1}{2}<\sigma<1$, every family of (non-vanishing) continuous functions $g_j(s)$ defined on $\mathcal{K}$ which are analytic in the interior, and for any $\epsilon>0$, we have

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]\,:\,\max_{1\le j\le n}\max_{s\in\mathcal{K}}\,:\,\zeta_{\lambda_j}(s+i\tau)-g_j(s)|<\epsilon\right\}>0.$$

Clearly, the assumption on the $\mathcal{K}_j$ to be disjoint is necessary. According to the foregoing remarks we see that $\zeta(s)$ and $\zeta(s+i\lambda)$ can approximate uniformly any pair of suitable target functions on sufficiently small disks simultaneously provided that $\lambda\neq 0$. This simultaneous approximation property may be regarded as a first example of a phenomenon which is called joint universality.

A rolling stone gathers no moss! Here are the exercises for this chapter. As an immediate consequence of Voronin's universality theorem one can obtain universality for certain relatives of $\zeta(s)$:

Exercise 10. Show that $\zeta(s)^{-1}$ is universal.

It might be a good exercise to prove all technical details of the lengthy proof of Voronin's theorem which we left for the reader:

Exercise 11. Work out all estimates from Section 3.5.

Exercise 12. Show that the logarithm of the prime numbers are linearly independent over $\mathbb{Q}$.

The latter assertion was essential for the application of Weyl's approximation theorem 3.8. Here is another application. Although the zeta-function has no zeros in its half-plane of absolute convergence it assumes arbitrarily small values:

Exercise 13. Use Kronecker's approximation theorem, Corollary 3.9, to show that

$$\inf_{\sigma>1}|\zeta(\sigma+it)|=0.$$

The following two exercises might be not too easy; help can be found in [56, 61].

Exercise 14. Prove Theorem 3.11. For this aim use Mergelyan's celebrated approximation theorem.

Exercise 15. Prove Reich's discrete universality theorem 3.12.

4. Applications, extensions, and open problems

The universality property of the Riemann zeta-function allows several interesting applications, maybe the most important one is functional independence. Besides we shall also present Bagchi's theorem which connects universality with the Riemann hypothesis. Moreover, we discuss several generalizations and open questions.

4.1. Functional independence. We state some consequences of universality. To begin with we extend Bohr's classical result about the denseness of the set of values taken by $\zeta(s)$ on a vertical line $\sigma \in (\frac{1}{2}, 1)$. The following theorem is essentially Voronin's theorem 1.2 from the introduction:

Theorem 4.1. *Let* $\frac{1}{2} < \sigma < 1$ *be fixed, then the sets*

$$\{(\log\zeta(\sigma+it), (\log\zeta(\sigma+it))', \ldots, (\log\zeta(\sigma+it))^{(n-1)}) \,:\, t \in \mathbb{R}\}$$

and

$$\{(\zeta(\sigma+it), \zeta'(\sigma+it), \ldots, \zeta^{(n-1)}(\sigma+it)) \,:\, t \in \mathbb{R}\}$$

lie everywhere dense in $\mathbb{C}^n$.

Proof. Suppose that we are given a vector $(b_0, b_1, \ldots, b_{n-1}) \in \mathbb{C}^n$. Let

$$r = \tfrac{1}{4} - \tfrac{1}{2}\min\left\{\sigma - \tfrac{1}{2}, 1-\sigma\right\}$$

and define

$$f(s) = \sum_{k=0}^{n-1} \frac{b_k}{k!} s^k.$$

Obviously, $f^{(k)}(0) = b_k$ for $k = 0, 1, \ldots, n-1$. By Cauchy's formula, we have for any analytic function $g(s)$ on $|s| \le \rho$

$$g^{(k)}(0) = \frac{k!}{2\pi i}\oint_{|s|=\rho} \frac{g(s)}{s^{k+1}}\,\mathrm{d}s. \tag{4.1}$$

By Voronin's universality theorem 1.3 the function $f(s)$ can be approximated to arbitrary precision on the disk $|s| \le r$ by $\log\zeta\left(s + \frac{3}{4} + i\tau\right)$ for some τ. Hence, taking

$$g(s) = f(s) - \log\zeta\left(s + \tfrac{3}{4} + i\tau\right)$$

and $\rho < r$ in (4.1), shows that $(\log\zeta(\sigma+it), \log\zeta'(\sigma+it), \ldots, \log\zeta(\sigma+it)^{(n-1)})$ with fixed $\sigma \in (\frac{1}{2}, 1)$ lies for some values of t as close to $(f(0), f'(0), \ldots, f^{(n-1)}(0)) = (b_0, b_1, \ldots, b_{n-1})$ as we want. This implies the statement for the first set.

We use induction on m to prove that for any $(m{+}1)$-tuple $(a_0, a_1, \ldots, a_m) \in \mathbb{C}^{m+1}$ with $a_0 \neq 0$, there exists $(b_0, b_1, \ldots, b_m) \in \mathbb{C}^{m+1}$ for which

$$\exp\left(\sum_{k=0}^{m} b_k s^k\right) \equiv \sum_{k=0}^{m} \frac{a_k}{k!} s^k \mod s^{m+1};$$

here the notation $\equiv \mod s^{m+1}$ means that the power series expansion of the difference of both sides of $\equiv$ consists of no terms with s^k for $k < m+1$. For $m = 0$ one only has to choose $b_0 = \log a_0$. By the induction assumption we may assume that with some α

$$\exp\left(\sum_{k=0}^{m} b_k s^k\right) \equiv \sum_{k=0}^{m} \frac{a_k}{k!} s^k + \alpha s^{m+1} \mod s^{m+2}.$$

Thus,

$$\exp\left(\sum_{k=0}^{m} b_k s^k + \beta s^{m+1}\right) \equiv (1 + \beta s^{m+1}) \left(\sum_{k=0}^{m} \frac{a_k}{k!} s^k + \alpha s^{m+1}\right) \mod s^{m+2}.$$

Now, let $b_{m+1} = \beta$ be the solution of the equation

$$\beta a_0 + \alpha = \frac{a_{m+1}}{(m+1)!},$$

which exists by the restriction on a_0 to be non-vanishing. This leads to

$$\exp\left(\sum_{k=0}^{m+1} b_k s^k\right) \equiv \sum_{k=0}^{m+1} \frac{a_k}{k!} s^k \mod s^{m+2},$$

and hence the claim.

Finally, let

$$g(s) := \exp\left(\sum_{k=0}^{n-1} b_k s^k\right) \equiv \sum_{k=0}^{n-1} \frac{a_k}{k!} s^k \mod s^n.$$

By Voronin's theorem 1.3 there exists a sequence τ_j, tending with j to infinity, such that

$$\lim_{j\to\infty} \max_{|s|\le r} \left|\zeta\left(s + \tfrac{3}{4} + i\tau_j\right) - g(s)\right| = 0$$

for some $r \in (0, \frac{1}{4})$. In view of (4.1) we obtain

$$\lim_{j\to\infty} \max_{|s|\le r-\epsilon} \left|\zeta^{(k)}\left(s + \tfrac{3}{4} + i\tau_j\right) - g^{(k)}(s)\right| = 0$$

for $k = 1, \ldots, n-1$ and any $\epsilon \in (0, r)$. By the same reasoning as above, this proves the theorem. •

Further, universality implies functional independence:

Theorem 4.2. *Let* $\mathbf{z} = (z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n$. *If* $F_0(\mathbf{z}), F_1(\mathbf{z}), \ldots, F_N(\mathbf{z})$ *are continuous functions, not all identically zero, then there exists some* $s \in \mathbb{C}$ *such that*

$$\sum_{k=0}^{N} s^k F_k(\zeta(s), \zeta'(s), \ldots, \zeta^{(n-1)}(s)) \neq 0.$$

In particular it follows that the zeta-function is hypertranscendental, i.e., $\zeta(s)$ does not satisfy any algebraic differential equation. This solves one of Hilbert's famous problems which he posed at the International Congress of Mathematicians in Paris 1900. The first proof of the hypertranscendence of the zeta-function was given by Stadigh and in a more general setting by Ostrowski, both, of course, by a different reasoning (see [54]).

Proof. First, we shall show that if $F(\mathbf{z})$ is a continuous function and

$$F(\zeta(s), \zeta'(s), \ldots, \zeta^{(n-1)}(s)) = 0$$

identically in $s \in \mathbb{C}$, then F vanishes identically.

Suppose the contrary, i.e. $F(\mathbf{z}) \not\equiv 0$. Then there exists $\mathbf{a} \in \mathbb{C}^n$ for which $F(\mathbf{a}) \neq 0$. Since F is continuous, there exist a neighbourhood U of $\mathbf{a}$ and a positive ϵ such that

$$|F(\mathbf{z})| > \epsilon \qquad \text{for} \quad \mathbf{z} \in U.$$

Choosing an arbitrary $\sigma \in (\frac{1}{2}, 1)$, application of Theorem 4.1 yields the existence of some t for which

$$(\zeta(\sigma + it), \zeta'(\sigma + it), \ldots, \zeta^{(n-1)}(\sigma + it)) \in U,$$

giving the desired contradiction. This proves our claim, resp. the assertion of the theorem with $N = 0$.

Without loss of generality we may assume that $F_0(\mathbf{z})$ is not identically zero. As above there exist an open bounded set U and a positive ϵ such that

$$|F_0(\mathbf{z})| > \epsilon \qquad \text{for} \quad \mathbf{z} \in U.$$

Denote by M the maximum of all indices m for which

$$\sup_{\mathbf{z} \in U} |F_m(\mathbf{z})| \neq 0.$$

For $M = 0$ the assertion of the theorem follows from the special case from above. Otherwise, we may choose an open subset $V \subset U$ such that

$$\inf_{\mathbf{z} \in V} |F_M(\mathbf{z})| > \epsilon$$

for some positive ϵ. By Theorem 4.1, there exists a sequence t_j, tending with j to infinity, such that

$$(\zeta(\sigma + it_j), \zeta'(\sigma + it_j), \ldots, \zeta^{(n-1)}(\sigma + it_j)) \in V.$$

This implies

$$\lim_{j\to\infty}\left|\sum_{k=0}^{M}(\sigma+it_j)^k F_k(\zeta(\sigma+it_j),\zeta'(\sigma+it_j),\ldots,\zeta^{(n-1)}(\sigma+it_j))\right|=\infty.$$

This proves the theorem. •

4.2. Self-recurrence and the Riemann hypothesis. It is a natural question to ask whether the condition on $g(s)$ to be non-vanishing is necessary or *is it possible to approximate uniformly functions having a zero by the zeta-function?* The answer is negative.

To see this assume that $g(s)$ is an analytic function on the disk $|s|\le r$ with a zero ξ in the interior of the disk but no zero on the boundary. An application of Rouché's theorem shows that whenever the inequality

$$\max_{|s|=r}\left|\zeta\left(s+\tfrac{3}{4}+i\tau\right)-g(s)\right|<\min_{|s|=r}|g(s)| \tag{4.2}$$

holds, $\zeta\left(s+\frac{3}{4}+i\tau\right)$ has a zero inside $|s|<r$ too. The zeros of an analytic function lie either discretely distributed or the function vanishes identically, and thus inequality (4.2) holds if the left-hand side is sufficiently small. If now for any $\epsilon>0$

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\left\{\tau\in[0,T]\,:\,\max_{|s|\le r}\left|\zeta\left(s+\tfrac{3}{4}+i\tau\right)-g(s)\right|<\epsilon\right\}>0,$$

then we expect $\gg T$ many τ in the interval $[0,T]$ each of which corresponds via (4.2) to a complex zero of $\zeta(s)$ in the strip $\frac{3}{4}-r<\sigma<\frac{3}{4}+r$ up to level T (for a rigorous proof one has to consider the densities of values τ satisfying (4.2) which can be done along the lines of the proof of Theorem 4.3 below). This contradicts density theorem 2.16 which gives

$$N\left(\tfrac{3}{4}-r,T\right)=o(T).$$

Thus, a given function with a zero cannot be approximated uniformly by the zeta-function (in the sense of Voronin's theorem)!

The above reasoning shows that the location of the complex zeros of Riemann's zeta-function is intimately related to the universality property. This observation is essential for the following observation which links universality with the Riemann hypothesis.

Bohr introduced the fruitful notion of almost periodicity to analysis. An analytic function $f(s)$, defined on some vertical strip $a<\sigma<b$, is called almost periodic if, for any positive ε and any α,β with $a<\alpha<\beta<b$, there exists a length $\ell=\ell(f,\alpha,\beta,\varepsilon)>0$ such that in every interval (t_1,t_2) of length ℓ there is a number $\tau\in(t_1,t_2)$ such that

$$|f(\sigma+it+i\tau)-f(\sigma+it)|<\varepsilon\qquad\text{for any}\quad\alpha\le\sigma\le\beta,t\in\mathbb{R}.$$

Bohr [7] proved that any Dirichlet series is almost-periodic in its half-plane of absolute convergence. Bohr discovered an interesting relation between the Riemann hypothesis and almost periodicity; indeed, his aim in introducing the concept of almost periodicity might have been Riemann's hypothesis. His approach failed for the Riemann zeta-function but he succeeded for some relatives, namely Dirichlet L-functions associated to some arithmetical functions called characters (defined in §4.4). Bohr showed that if χ is a non-principal character, then the analogue of Riemann's hypothesis for the Dirichlet L-function $L(s,\chi)$ is equivalent to the almost periodicity of $L(s,\chi)$ in the half-plane $\sigma > \frac{1}{2}$. The condition on the character looks artificial but is necessary for Bohr's reasoning. His argument relies in the main part on diophantine approximation applied to the coefficients of the Dirichlet series representation. The Dirichlet series for $L(s,\chi)$ with a non-principal character χ converges throughout the critical strip, however the Dirichlet series for the zeta-function does not.

More than half a century later Bagchi [1] proved that the Riemann hypothesis is true if and only if for any compact subset $\mathcal{K}$ of the strip $\frac{1}{2} < \sigma < 1$ with connected complement and for any $\epsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{s\in\mathcal{K}} |\zeta(s+i\tau) - \zeta(s)| < \epsilon \right\} > 0.$$

In [2], Bagchi generalized this result in various directions; in particular for Dirichlet L-functions to arbitrary characters. One implication of his proof relies essentially on Voronin's universality theorem which, of course, was unknown to Bohr. Later, Bagchi [3] gave another proof in the language of topological dynamics, independent of universality, and therefore this property, equivalent to Riemann's hypothesis, is called strong recurrence. Following [61] we extend Bagchi's result slightly to

Theorem 4.3. *Let $\theta \geq \frac{1}{2}$. Then $\zeta(s)$ is non-vanishing in the half-plane $\sigma > \theta$ if and only if, for any $\epsilon > 0$, any z with $\theta < \mathrm{Re}\ z < 1$, and for any $0 < r < \min\{\mathrm{Re}\ z - \theta, 1 - \mathrm{Re}\ z\}$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{|s-z|\leq r} |\zeta(s+i\tau) - \zeta(s)| < \epsilon \right\} > 0.$$

Proof. If Riemann's hypothesis is true, we can apply Voronin's universality theorem 1.3 with $g(s) = \zeta(s)$, which implies the strong recurrence. More generally, the non-vanishing of $\zeta(s)$ for $\sigma > \theta$ would allow to approximate $\zeta(s)$ by shifts $\zeta(s+i\tau)$ uniformly on appropriate subsets of the strip $\theta < \sigma < 1$. The idea for the proof of the other implication is that if there is at least one zero to the right of the line $\sigma = \theta$, then the strong recurrence property implies the existence of *many* zeros, in fact *too many* with regard to density theorem 2.16.

Suppose that there exists a zero ξ of $\zeta(s)$ with $\mathrm{Re}\,\xi > \theta$. Without loss of generality we may assume that $\mathrm{Im}\,\xi > 0$. We shall show that there exists a disk with center ξ and radius r, satisfying the conditions of the theorem, and a positive ϵ such that

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{|s-z|\le r} |\zeta(s+i\tau) - \zeta(s)| < \epsilon \right\} = 0. \tag{4.3}$$

Locally, the zeta-function has the expansion

$$\zeta(s) = c(s-\xi)^m + O\left(|s-\xi|^{m+1}|\right) \tag{4.4}$$

with some non-zero $c \in \mathbb{C}$ and $m \in \mathbb{N}$. Now assume that for a neighbourhood $\mathcal{K}_\delta := \{s \in \mathbb{C} \,:\, |s-\xi| \le \delta\}$ of ξ the relation

$$\max_{s\in\mathcal{K}_\delta} |\zeta(s+i\tau) - \zeta(s)| < \epsilon \le \min_{|s|=\delta} |\zeta(s)| \tag{4.5}$$

holds; the second inequality is fulfilled for sufficiently small ϵ (by an argument already discussed above). Then Rouché's theorem implies the existence of a zero ρ of $\zeta(s)$ in

$$\mathcal{K}_\delta + i\tau := \{s \in \mathbb{C} \,:\, |s - i\tau - \xi| \le \delta\}.$$

We say that the zero ρ of $\zeta(s)$ is *generated* by the zero ξ. With regard to (4.4) and (4.5) the zeros ξ and $\rho = \beta + i\gamma$ are intimately related; more precisely,

$$\epsilon > |\zeta(\rho) - \zeta(\rho - i\tau)| = |\zeta(\rho - i\tau)| \ge |c| \cdot |\rho - i\tau - \xi|^m - O(\delta^{m+1}).$$

Hence,

$$|\rho - i\tau - \xi| \le \left(\frac{\epsilon}{|c|}\right)^{\frac{1}{m}} + O\left(\delta^{1+\frac{1}{m}}\right).$$

In particular,

$$\tfrac{1}{2} < \operatorname{Re}\xi - 2\left(\frac{\epsilon}{|c|}\right)^{\frac{1}{m}} < \beta < 1,$$

and

$$|\gamma - (\tau + \operatorname{Im}\xi)| < 2\left(\frac{\epsilon}{|c|}\right)^{\frac{1}{m}},$$

for sufficiently small ϵ and $\delta = o(\epsilon^{m+1})$. Next we have to count the generated zeros in terms of τ. Two different shifts τ_1 and τ_2 can lead to the same zero ρ, but their distance is bounded by

$$|\tau_1 - \tau_2| < 4\left(\frac{\epsilon}{|c|}\right)^{\frac{1}{m}}.$$

If we now write

$$\mathcal{I}(T) := \bigcup_j \mathcal{I}_j(T) := \left\{ \tau \in [0,T] \,:\, \max_{s\in\mathcal{K}_\delta} |\zeta(s+i\tau) - \zeta(s)| < \epsilon \right\},$$

where the $\mathcal{I}_j(T)$ are disjoint intervals, it follows that there are

$$\geq \left[\tfrac{1}{4} \left(\frac{|c|}{\epsilon} \right)^{\frac{1}{m}} \operatorname{meas} \mathcal{I}_j(T) \right] + 1 > \tfrac{1}{4} \left(\frac{|c|}{\epsilon} \right)^{\frac{1}{m}} \operatorname{meas} \mathcal{I}_j(T)$$

many distinct zeros according to $\tau \in \mathcal{I}_j(T)$, generated by ξ. The number of generated zeros is a lower bound for the number of all zeros. For the number of all zeros having real part $> \operatorname{Re}\xi - 2(\frac{\epsilon}{|c|})^{\frac{1}{m}}$ up to level T this yields

$$\sharp \left\{ \rho = \beta + i\gamma \,:\, \beta > \operatorname{Re}\xi - 2\left(\frac{\epsilon}{|c|} \right)^{\frac{1}{m}}, 0 < \gamma < T + \operatorname{Im}\xi + 2\left(\frac{\epsilon}{|c|} \right)^{\frac{1}{m}} \right\}$$

$$\geq \tfrac{1}{4} \left(\frac{|c|}{\epsilon} \right)^{\frac{1}{m}} \operatorname{meas} \mathcal{I}(T).$$

This in combination with density theorem 2.16 yields

$$\operatorname{meas} \mathcal{I}(T) = o(T),$$

which implies (4.3). The theorem is proved. •

4.3. The effectivity problem. The known proofs of universality theorems are ineffective, giving neither an estimate for the first approximating shift τ nor bounds for the positive lower density. There are attempts by Good, Laurinčikas, and Garunkštis which we shall now shortly discuss.

If the Riemann hypothesis is true, then

$$\log \left| \zeta\left(\tfrac{1}{2} + it \right) \right| = O\left(\frac{\log t}{\log\log t} \right) \tag{4.6}$$

as $t \to \infty$. This is a significant improvement of the bound for $\zeta(s)$ on the critical line predicted by the Lindelöf hypothesis, but we may ask whether it is the correct order? On the contrary, Montgomery [51] proved, for fixed $\frac{1}{2} < \sigma < 1$, there exists an absolute positive constant C such that

$$\max_{T^{\frac{1}{3}(\sigma - \frac{1}{2})} < t \leq T} |\zeta(\sigma + it)| \geq \exp\left(C \frac{(\log t)^{1-\sigma}}{(\log\log t)^{\sigma}} \right); \tag{4.7}$$

the same estimate is valid for $\sigma = \frac{1}{2}$ under assumption of the truth of the Riemann hypothesis. By a different method, Balasubramanian & Ramachandra [4] obtained the same estimate for $\sigma = \frac{1}{2}$ unconditionally. These results were only slight improvements of earlier results, however, some probabilistic heuristics suggest these estimates to be best possible, i.e., the quantity in (4.7) describes the exact order of growth of $\zeta(s)$.[**]

[**] The recent random matrix model predicts significantly larger values: in analogy to large deviations for characteristic polynomials one may expect that the estimate in (4.6) gives the true order.

The proofs of the universality theorem, neither Voronin's original one nor Bagchi's probabilistic proof and its variations, do not give any information about the question how soon a given target function is approximated by $\zeta(s+i\tau)$ within a given range of accuracy, and Montgomery's approach does not give us any idea of the shape of the set of values of $\zeta(s)$ on vertical lines. Good [20] combined Voronin's universality theorem with the work of Montgomery on extreme values of the zeta-function. This enabled him to complement Voronin's qualitative picture with Montgomery's quantitative estimates. Recently, Garunkštis [16] proved another, more satisfying effective universality theorem along the lines of Voronin's proof and building on Good's ideas. In particular, his remarkable result shows that if $f(s)$ is analytic in $|s| \leq 0.06$ with $\max_{|s|\leq 0.06} |f(s)| \leq 1$, then for any $0 < \epsilon < \frac{1}{2}$ there exists a

$$0 \leq \tau \leq \exp\left(\exp\left(10\epsilon^{-13}\right)\right) \tag{4.8}$$

such that

$$\max_{|s|\leq 0.0001} \left|\log\zeta\left(s+\tfrac{3}{4}+i\tau\right) - f(s)\right| < \epsilon,$$

and further

$$\begin{aligned} \liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{\tau \in [0,T] \,:\, \max_{|s|\leq 0.0001} \left|\log\zeta\left(s+\tfrac{3}{4}+i\tau\right) - f(s)\right| < \epsilon\right\} \\ \geq \exp\left(-\epsilon^{-13}\right). \end{aligned} \tag{4.9}$$

The original theorem is too complicated to be given here. Laurinčikas found another approach which gives conditional effective results subject to certain assumptions on the speed of convergence of a related limit distribution. However, the rate of convergence of weakly convergent probability measures in the space of analytic functions is not understood very well.

Following [59, 61], we shall investigate the converse problem, namely effective upper bounds for the *upper* density of universality:

Theorem 4.4. *Suppose that $g(s)$ is a non-constant, non-vanishing analytic function defined on $|s| \leq r$, where $r \in (0, \frac{1}{4})$. Then, for any sufficiently small $\epsilon > 0$,*

$$\begin{aligned} \overline{\mathrm{d}}(\epsilon, r, g) &:= \limsup_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{\tau \in [0,T] \,:\, \max_{|s|\leq r} \left|\zeta\left(s+\tfrac{3}{4}+i\tau\right) - g(s)\right| < \epsilon\right\}. \\ &= o(\epsilon). \end{aligned}$$

Thus, the decay of $\overline{\mathrm{d}}(\epsilon, r, g)$ with $\epsilon \to 0$ is more than linear in ϵ for any suitable function g.

Proof. Assume that $g(s)$ is a non-constant, non-vanishing analytic function defined on $B_r := \{s \,:\, |s| \leq r\}$. Then there exists a complex number c in the interior of $g(B_r)$ (which is not empty since $g(s)$ is not constant) such that

$$g(s) = c + \gamma(s - \lambda_c) + O\left(|s-\lambda_c|^2\right) \tag{4.10}$$

for some λ_c of modulus less than r and some $\gamma \neq 0$; this means that λ_c is a c-value of $g(s)$ of multiplicity one. To see this suppose that for all c in the interior of $g(B_r)$ the local expansion is different from (4.10), i.e., $g'(s)$ vanishes identically in the interior. Then g is a constant function, contradicting the assumption of the theorem.

Now suppose that

$$\max_{|s|=r} \left|\left\{\zeta\left(s+\tfrac{3}{4}+i\tau\right)-c\right\}-\{g(s)-c\}\right| < \min_{|s|=r}|g(s)-c|.$$

Then, by Rouché's theorem, $\zeta(z)$ has at least one c-value ρ_c in $\{z = s+\frac{3}{4}+i\tau : |s|<r\}$. We rewrite the latter inequality as

$$\max_{|s|\leq r} \left|\zeta\left(s+\tfrac{3}{4}+i\tau\right)-g(s)\right| < \epsilon \leq \min_{|s|=r}|g(s)-c|. \tag{4.11}$$

By Voronin's universality theorem 1.3 the first inequality holds for a set of τ with positive lower density. The second one follows for sufficiently small ϵ from the fact that $c = g(\lambda_c)$ has positive distance to the boundary of $g(B_r)$. Thus, a c-value of $g(s)$ *generates* many c-values of $\zeta(z)$.

Assume that $\rho_c = s_j+\frac{3}{4}+i\tau_j$ with $|s_j|<r$ for $j=1,2$. It follows from (4.11) that

$$|g(s_j)-c| = |g(s_j)-g(\lambda_c)| < \epsilon. \tag{4.12}$$

Since $g'(\lambda_c)=\gamma\neq 0$, there exists a neighborhood of c where the inverse function g^{-1} exists and is a one-valued continuous function. By continuity, (4.12) implies

$$|s_j-\lambda_c| < \varepsilon = \varepsilon(\epsilon), \tag{4.13}$$

where $\varepsilon(\epsilon)$ tends with ϵ to zero; since $g(s)$ behaves locally as a linear function by (4.10), we have $\varepsilon(\epsilon)\asymp\epsilon$. Now (4.13) implies

$$|\tau_2-\tau_1| = |s_1-s_2| \leq |s_1-\lambda_c|+|\lambda_c-s_2| < 2\varepsilon. \tag{4.14}$$

Denote by $\mathcal{I}_j(T)$ the disjoint intervals in $[0,T]$ such that (4.11) is valid exactly for

$$\tau \in \bigcup_j \mathcal{I}_j(T) =: \mathcal{I}(T).$$

Inequality (4.14) implies that in every interval $\mathcal{I}_j(T)$ lie at least

$$1+\left[\frac{1}{2\varepsilon}\operatorname{meas}\mathcal{I}_j(T)\right] \geq \frac{1}{2\varepsilon}\operatorname{meas}\mathcal{I}_j(T)$$

c-values ρ_c of $\zeta(s)$ in the strip $\frac{1}{2}<\sigma<1$. Thus, the number $\mathcal{N}_c(T)$ of these c-values ρ_c (counting multiplicities) satisfies the estimate

$$2\varepsilon\mathcal{N}_c(T) \geq \operatorname{meas}\mathcal{I}(T). \tag{4.15}$$

Next we locate the real parts of these c-values more precisely. Obviously, by (4.13),

$$\operatorname{Re}\lambda_c+\tfrac{3}{4}-\varepsilon < \operatorname{Re}\rho_c = \operatorname{Re}s_j+\tfrac{3}{4} < \operatorname{Re}\lambda_c+\tfrac{3}{4}+\varepsilon.$$

Clearly, for sufficiently small ε this range for the c-values lies in the interior of the strip of universality. Hence, if we let $N_c(\sigma_1,\sigma_2,T)$ count all c-values of $\zeta(s)$ in the region $\sigma_1<\sigma<\sigma_2$, $0<t\leq T$ (counting multiplicities), then we can rewrite (4.15) as

$$\operatorname{meas}\mathcal{I}(T)\leq 2\varepsilon N_c\left(\operatorname{Re}\lambda_c+\tfrac{3}{4}-\varepsilon,\operatorname{Re}\lambda_c+\tfrac{3}{4}+\varepsilon,T\right). \tag{4.16}$$

In view of the universality theorem 1.3 there exists an increasing sequence (T_k) with $\lim_{k\to\infty}T_k=\infty$ such that for any $\delta>0$

$$\operatorname{meas}\mathcal{I}(T_k)\geq(\overline{\mathrm{d}}(\epsilon,r,g)-\delta)T_k.$$

Consequently, this together with (4.16) leads to

$$(\overline{\mathrm{d}}(\epsilon,r,g)-\delta)T_k\leq 2\varepsilon N_c\left(\operatorname{Re}\lambda_c+\tfrac{3}{4}-\varepsilon,\operatorname{Re}\lambda_c+\tfrac{3}{4}+\varepsilon,T\right).$$

Sending $\delta\to 0$, yields

$$\overline{\mathrm{d}}(\epsilon,r,g)\leq\limsup_{T\to\infty}\frac{2\varepsilon}{T}N_c\left(\operatorname{Re}\lambda_c+\tfrac{3}{4}-\varepsilon,\operatorname{Re}\lambda_c+\tfrac{3}{4}+\varepsilon,T\right). \tag{4.17}$$

Here we shall use a classical theorem of Bohr & Jessen [10]: for any complex $c\neq 0$,

$$\lim_{T\to\infty}\frac{1}{T}N_c\left(\operatorname{Re}\lambda_c+\tfrac{3}{4}-\varepsilon,\operatorname{Re}\lambda_c+\tfrac{3}{4}+\varepsilon,T\right)=o(1) \tag{4.18}$$

as $\epsilon\to 0$. Substituting this in (4.17) implies (4.4) and the assertion of the theorem follows. •

As substitute of the deep result of Bohr & Jessen we can give an alternative argument at the expense of a slightly weaker estimate as follows. For this purpose define

$$\ell(s)=\begin{cases}\frac{1}{1-c}(\zeta(s)-c) & \text{if}\quad c\neq 1,\\ 2^{s-1}(\zeta(s)-1) & \text{if}\quad c=1.\end{cases}$$

Then the c-values of $\zeta(s)$ correspond one-to-one to the zeros of $\ell(s)$ (having the same multiplicity) and

$$\ell(\sigma+it)=1+\lambda^{-\sigma-it}+O(\Lambda^{-\sigma}) \tag{4.19}$$

with some constants λ,Λ satisfying $1<\lambda<\Lambda$, as $\sigma\to\infty$. Hence, there exists a real number $\sigma_2>1$ such that there are no zeros of $\ell(s)$ to the right of σ_2-1. Now let $N_c(\sigma,T)$ count the number of c-values of $\zeta(s)$ with real-part greater than σ and imaginary part in $(0,T]$, resp. the zeros of $\ell(s)$ (counting multiplicities). Then Littlewood's lemma 2.14 yields

$$\int_{\sigma_1}^{\sigma_2}N_c(\sigma,T)\,\mathrm{d}\sigma=\frac{1}{2\pi i}\int_{\mathcal{R}}\log\ell(s)\,\mathrm{d}s+O(1), \tag{4.20}$$

where $\mathcal{R}$ is the rectangular contour with vertices $\sigma_1,\sigma_2,\sigma_1+iT,\sigma_2+iT$ with $\frac{1}{2}<\sigma_1<1<\sigma_2$. Here the error term arises from the pole of $\zeta(s)$ (to define here $\log\ell(s)$ we choose the principal branch of the logarithm on the real axis whereas for other points s the value of the logarithm is obtained by continuous variation).

A standard application of Jensen's inequality (as in Section 2.4) shows that the right-hand side of (4.20) can be replaced by

$$\frac{1}{2\pi}\int_0^T \log|\zeta(\sigma_1+it)|\,\mathrm{d}t + O(T) \le \frac{T}{4\pi}\log\left(\frac{1}{T}\int_0^T |\zeta(\sigma_1+it)|^2\,\mathrm{d}t\right) + O(T).$$

The right-hand side can be estimated by the mean-square theorem 2.13. This gives in (4.20)

$$\sum_{\substack{\mathrm{Re}\,\rho_c>\sigma_1\\ 0<\mathrm{Im}\,\rho_c\le T}} (\mathrm{Re}\,\rho_c - \sigma_1) \ll T,$$

as $T \to \infty$; here the sum on the left-hand side is taken over all c-values ρ_c of $\zeta(s)$ (not necessarily generated by λ_c). Since, for $\frac{1}{2} < \sigma_1 < \sigma_3$,

$$N_c(\sigma_3, T) \le \frac{1}{\sigma_3 - \sigma_1}\sum_{\substack{\mathrm{Re}\,\rho_c>\sigma_1\\ 0<\mathrm{Im}\,\rho_c\le T}} (\mathrm{Re}\,\rho_c - \sigma_1),$$

we may estimate

$$N_c\left(\mathrm{Re}\,\lambda_c + \tfrac{3}{4} - \varepsilon, \mathrm{Re}\,\lambda_c + \tfrac{3}{4} + \varepsilon, T\right) \le N_c(\tfrac{1}{2}(\tfrac{1}{2} + \mathrm{Re}\,\lambda_c + \tfrac{3}{4} - \varepsilon), T) \ll T.$$

Thus, we deduce from (4.17) the bound $\overline{\mathrm{d}}(\epsilon, r, g) = O(\epsilon)$, which is slightly weaker than the bound from the theorem.

We return to the problem of effectivity in the universality theorem for $\zeta(s)$. Comparing the lower bound (4.9) of Garunkštis from the beginning with the upper bounds of Theorem 4.4, we may ask which estimate is more close to the truth. If a given function $g(s)$ is sufficiently *nice*, i.e., if its logarithm $f(s)$ satisfies the condition of Garunkštis' theorem, then

$$\exp\left(-\epsilon^{-13}\right) \ll \underline{\mathrm{d}}(\epsilon, g, r) \le \overline{\mathrm{d}}(\epsilon, g, r) = o(\epsilon).$$

Given a positive ϵ and a sufficiently small disk $\mathcal{K}$ located in the right half of the critical strip, in principle, estimate (4.8) allows to find algorithmically an approximating τ such that

$$\max_{s\in\mathcal{K}} |\zeta(s+i\tau) - g(s)| < \epsilon;$$

unfortunately, we cannot expect a reasonable running time for such an algorithm when ϵ is small. Anyway, this idea was considered in a project by Garunkštis, Šleževičienė–Steuding & Steuding. For certain *smooth* functions $g(s)$ and rather *large* values for ϵ approximating shifts τ were computed. Quite many of these τ were found but it is impossible to deduce any information about the density of universality as long as the running time of the underlying algorithm cannot be significantly improved. Nevertheless, we shall illustrate this attempt toward effective universality by some data.

Consider the exponential function on a small disk centered at the origin. For example, we have

$$\max_{|s|\leq 0.006} \left|\zeta\left(s+\tfrac{3}{4}+12\,963\,i\right)-\exp(s)\right| < 0.05.$$

The shift τ is a positive integer since the discrete variant of universality, Reich's theorem 3.12, was used in order to simplify the algorithm.

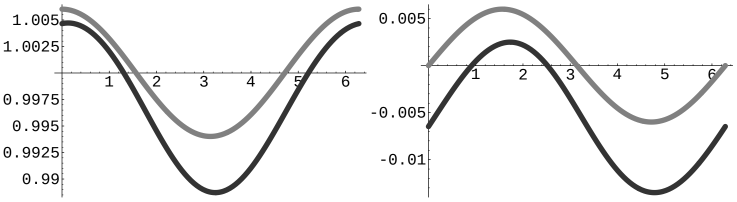

FIGURE 6. $\zeta(s+\frac{3}{4}+12\,963\,i) \approx \exp(s)$ for $s = 0.006\exp(i\phi)$ with $0 \leq \phi \leq 2\pi$. On the left the real parts, on the right the imaginary parts are plotted; the zeta-function is given in black, exp in grey.

We conclude with another application of discrete universality. The argument in the proof of Theorem 4.4 which gave us a factor ϵ for the upper bound does not apply if we consider discrete shifts and so, in general, we do not get an upper bound which tends with ϵ to zero. However, via Reich's discrete universality theorem 3.12 and (4.18) one can prove

$$\limsup_{N\to\infty} \frac{1}{N}\sharp\left\{1 \leq n \leq N \,:\, \max_{s\in\mathcal{K}} |\zeta(s+in\Delta)-g(s)| < \epsilon\right\} = o(1) \tag{4.21}$$

as $\epsilon \to 0$ for any real $\Delta > 0$ and any suitable function g on $\mathcal{K}$. This is of interest with respect to an estimate of Reich concerning small values of Dirichlet series on arithmetic progressions. Let $F(s)$ be a Dirichlet series, not identically zero, which has a half-plane of absolute convergence $\sigma > \sigma_a$, an analytic continuation to $\sigma > \sigma_m$ ($\sigma_m < \sigma_a$) except for at most a finite number of poles on the line $\sigma = \sigma_a$, such that its mean square exists and $F(s)$ is of finite order of growth in any closed strip in $\sigma_m < \sigma < \sigma_a$. Reich [57] proved under these assumptions for any $\sigma > \sigma_m, \sigma \neq \sigma_a$, any sufficiently small $\epsilon > 0$, and any real Δ, neither being equal to zero nor of the form $2\pi\ell\log(\frac{q}{r})^{-1}$ with positive integers ℓ, q, r and $q \neq r$, that the relation

$$\limsup_{N\to\infty} \frac{1}{N}\sharp\left\{1 \leq n \leq N \,:\, |F(\sigma+in\Delta)| < \epsilon\right\} < 1$$

holds. In particular, it follows that $F(\sigma+i\Delta n)$ cannot converge to zero as $n \to \infty$, and hence $s_n = \sigma + i\Delta n$ cannot be a sequence of zeros of $F(s)$. It should be noticed that Reich's theorem also includes estimates for c-values on arithmetic progressions (since with $F(s)$ also $F(s)-c$ satisfies the conditions).

In the special case of the Riemann zeta-function we note the following improvement of Reich's theorem:

Corollary 4.5. *Let c be any constant and $\sigma \in (\frac{1}{2}, 1)$, and $\Delta > 0$ be real. Then*

$$\lim_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \sharp \left\{1 \leq n \leq N \, : \, |\zeta(\sigma + in\Delta) - c| < \epsilon\right\} = 0.$$

In particular, there does not exist an arithmetic progression $s_n = \sigma + i\Delta n$ (with σ and Δ as in the theorem) on which $\zeta(s)$ converges to any complex number c.

We sketch the easy proof. Let $g(s)$ be a non-constant, non-vanishing, analytic function defined on a small disk centered at $\sigma \in (\frac{1}{2}, 1)$ such that its closure lies inside the strip of universality for the zeta-function. Further assume that

$$|g(s) - c| < \epsilon;$$

this choice for $g(s)$ is certainly possible for any given complex number c. By the triangle inequality,

$$|\zeta(\sigma + in\Delta) - c| \leq |\zeta(\sigma + in\Delta) - g(s)| + |g(s) - c|$$

for any s. Hence, applying (4.21) yields

$$\limsup_{N \to \infty} \frac{1}{N} \sharp \left\{1 \leq n \leq N \, : \, |\zeta(\sigma + in\Delta) - c| < \epsilon\right\} = o(1)$$

as $\epsilon \longrightarrow 0$. This is the assertion of the corollary. •

There are remarkable results for a related problem. Extending a classical result of Putnam on the impossibility of an infinite vertical arithmetic progressions of zeros (or even approximate zeros), van Frankenhuijsen [15] recently proved that

$$\zeta(\sigma + in\Delta) = 0 \qquad \text{for} \quad 0 < |n| < N$$

with fixed $\sigma, \Delta > 0$ and $N \geq 2$ cannot hold for

$$N \geq 60 \left(\frac{\Delta}{2\pi}\right)^{\frac{1}{\sigma} - 1} \log \Delta.$$

It is conjectured that there are no arithmetic progressions at all. Moreover, there are even no zeros known of the form $\frac{1}{2} + i\gamma$ and $\frac{1}{2} + i2\gamma$. It is conjectured that the ordinates of the nontrivial zeros of $\zeta(s)$ are linearly independent over $\mathbb{Q}$.

4.4. *L*-functions and joint universality. A special role in number theory is played by multiplicative arithmetical functions and their associated generating functions. Multiplicative functions respect the multiplicative structure of $\mathbb{N}$: an arithmetic function f is called multiplicative if $f(1) = 1$ and

$$f(m \cdot n) = f(m) \cdot f(n)$$

for all coprime integers m, n; if the latter identity holds for all integers, f is said to be completely multiplicative.

Let q be a positive integer. A Dirichlet character $\chi \bmod q$ is a non-vanishing group homomorphism from the group $(\mathbb{Z}/q\mathbb{Z})^*$ of prime residue classes modulo q to $\mathbb{C}$. The character, which is identically one, is called principal, and is denoted by χ_0. By setting $\chi(a) = 0$ on the non-prime residue classes any Dirichlet character extends via $\chi(n) = \chi(a)$ for $n \equiv a \bmod q$ to a completely multiplicative arithmetical function. For $\sigma > 1$, the Dirichlet L-function attached to a character $\chi \bmod q$ is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

The zeta-function $\zeta(s)$ may be regarded as the Dirichlet L-function to the principal character $\chi_0 \bmod 1$. It is possible that for values of n coprime with q the character $\chi(n)$ may have a period less than q. If so, we say that χ is imprimitive, and otherwise primitive; the principal character is not regarded as a primitive character. Every non-principal imprimitive character is induced by a primitive character. Two characters are non-equivalent if they are not induced by the same character. The characters to a common modulus are pairwise non-equivalent. If $\chi \bmod q$ is induced by a primitive character $\chi^* \bmod q^*$, then

$$L(s,\chi) = L(s,\chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right). \tag{4.22}$$

Being twists of the Riemann zeta-function with multiplicative characters, Dirichlet L-functions share many properties with the zeta-function. For instance, there is an analytic continuation to the complex plane, only with the difference that $L(s,\chi)$ is regular at $s = 1$ if and only if χ is non-principal. Furthermore, L-functions to primitive characters satisfy a functional equation of the Riemann-type; namely,

$$\left(\frac{q}{\pi}\right)^{\frac{s+\delta}{2}} \Gamma\left(\frac{s+\delta}{2}\right) L(s,\chi) = \frac{\tau(\chi)}{i^\delta \sqrt{q}} \left(\frac{q}{\pi}\right)^{\frac{1+\delta-s}{2}} \Gamma\left(\frac{1+\delta-s}{2}\right) L(1-s,\overline{\chi}),$$

where $\delta := \frac{1}{2}(1-\chi(-1))$ and

$$\tau(\chi) := \sum_{a \bmod q} \chi(a) \exp\left(\frac{2\pi i a}{q}\right)$$

is the Gauss sum attached to χ. One finds similar zero-free regions (with the exception of hypothetical Siegel zeros on the real line), density theorems, and also for Dirichlet L-functions it is expected that the analogue of the Riemann hypothesis holds; the so-called Generalized Riemann hypothesis states that *neither* $\zeta(s)$ *nor any* $L(s,\chi)$ *has a zero in the half-plane* $\sigma > \frac{1}{2}$.

Dirichlet L-functions were constructed by Dirichlet to investigate the distribution of primes in arithmetic progressions. The main ingredient in his approach are the orthogonality relations for characters linking prime residue classes with character sums:

$$\frac{1}{\varphi(q)}\sum_{a \bmod q}\chi(a)=\begin{cases}1 & \text{if } \chi=\chi_0,\\ 0 & \text{otherwise,}\end{cases} \tag{4.23}$$

and its dual variant

$$\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\chi(a)=\begin{cases}1 & \text{if } a\equiv 1 \bmod q,\\ 0 & \text{otherwise,}\end{cases}$$

valid for a coprime with q, where $\varphi(q)$ is Euler's φ-function which counts the number of prime residue classes $\bmod\, q$. By the latter relation a suitable linear combination of characters can be used as indicator function of prime residue classes modulo q. Using similar techniques as for $\zeta(s)$, one can prove a prime number theorem for arithmetic progressions: if $\pi(x;a \bmod q)$ denotes the number of primes $p\leq x$ in the residue class $a \bmod q$, then, for a coprime with q,

$$\pi(x;a \bmod q)\sim\frac{1}{\varphi(q)}\pi(x). \tag{4.24}$$

This shows that the primes are uniformly distributed in the prime residue classes. One can prove also an asymptotic formula with error term and the theorem of Page-Siegel-Walfisz gives an asymptotic formula which is uniform in a small region of values q. Under assumption of the Generalized Riemann hypothesis one has

$$\pi(x;a \bmod q)=\frac{1}{\varphi(q)}\,\mathrm{li}\,(x)+O\left(x^{\frac{1}{2}}\log(qx)\right)$$

for $x\geq 2, q\geq 1$, and a coprime with q, the implicit constant being absolute. There are plenty of results which hold if Riemann's hypothesis is true. Often one can replace this assumption by the celebrated theorem of Bombieri-Vinogradov which states that, for any $A\geq 1$,

$$\sum_{q\leq Q}\max_{\substack{a \bmod q\\ (a,q)=1}}\max_{y\leq x}\left|\pi(y;a \bmod q)-\frac{1}{\varphi(q)}\mathrm{li}(y)\right|\ll\frac{x}{(\log x)^A}+Qx^{\frac{1}{2}}(\log Qx)^6.$$

This shows that the error term in the prime number theorem (4.24) is, on average over $q\leq x^{\frac{1}{2}}(\log x)^{-A-7}$, of comparable size as predicted by the Riemann hypothesis. All these results can be found in [33, 62].

We return to universality. Voronin [68] proved that a collection of Dirichlet L-functions to non-equivalent characters can uniformly approximate simultaneously non-vanishing analytic functions. This is called *joint* universality and its strongest version is given in:

Theorem 4.6. *Let* $\chi_1 \bmod q_1,\ldots,\chi_\ell \bmod q_\ell$ *be pairwise non-equivalent Dirichlet characters,* $\mathcal{K}_1,\ldots,\mathcal{K}_\ell$ *be compact subsets of the strip* $\frac{1}{2}<\sigma<1$ *with connected complements. Further, for each* $1\leq j\leq\ell$, *let* $g_j(s)$ *be a continuous* *non-vanishing function on $\mathcal{K}_j$ which is analytic in the interior of $\mathcal{K}_j$. Then, for any $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] \, : \, \max_{1\le j\le \ell} \max_{s\in\mathcal{K}_j} |L(s+i\tau, \chi_j) - g_j(s)| < \varepsilon \right\} > 0.$$

The proof of this joint universality theorem can be found in [61]. The proof uses the orthogonality relation (4.23). Although this relation is a rather simple fact, the resulting independence is essential for joint universality. Consider a character χ mod q induced by another character χ^* mod q^*. Because of (4.22) it follows that both $L(s, \chi^*)$ and $L(s, \chi)$ cannot approximate uniformly a given function jointly.

Another type of universality was discovered by Bagchi. In [1], he proved universality for Dirichlet L-functions with respect to the characters; more precisely, if $\mathcal{K}$ is a compact subset of $\frac{1}{2} < \sigma < 1$ with connected complement and $g(s)$ is a non-vanishing continuous function on $\mathcal{K}$, which is analytic in the interior, then, for any sufficiently large prime number p and any $\varepsilon > 0$, there exist a Dirichlet character χ mod p such that

$$\max_{s\in\mathcal{K}} \left| L\left(s + \tfrac{3}{4}, \chi\right) - g(s) \right| < \varepsilon;$$

moreover, the latter inequality holds for more than cp characters χ mod p, where c is a positive constant (recall that there are $\varphi(p) = p - 1$ characters χ mod p).

Another interesting class of universal L-functions are those built from modular forms. First we recall some basic facts from the theory of automorphic forms which can be found in the book [31] of Iwaniec. Denote by $\mathbb{H}$ the upper half-plane $\{z := x + iy \in \mathbb{C} \, : \, y > 0\}$, and let k and N be positive integers, k being even. The subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) \, : \, c \equiv 0 \bmod N \right\}$$

of the full modular group $\mathsf{SL}_2(\mathbb{Z})$ is called Hecke subgroup of level N or congruence subgroup mod N. A holomorphic function $f(z)$ on $\mathbb{H}$ is said to be a cusp form of weight k and level N, if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

and if f vanishes at all cusps. The vanishing of f at the cusps is equivalent with the boundedness of the mapping

$$z := x + iy \ \mapsto \ y^k |f(z)|^2$$

on $\mathbb{H}$. In this case, f possesses for $z \in \mathbb{H}$ a Fourier expansion:

$$f(z) = \sum_{n=1}^{\infty} c(n) \exp(2\pi i n z). \tag{4.25}$$

The cusp forms on $\Gamma_0(N)$ of weight k form a finite dimensional complex vector space, denoted by $S_k(\Gamma_0(N))$, with the Petersson inner product, defined by

$$\langle f, g\rangle = \int_{\mathbb{H}/\Gamma_0(N)} f(z)\overline{g(z)} y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

for $f, g \in S_k(\Gamma_0(N))$. Suppose that $M|N$. If $f \in S_k(\Gamma_0(M))$ and $dM|N$, then $z \mapsto f(dz)$ is a cusp form on $\Gamma_0(N)$ of weight k too. The forms which may be obtained in this way from divisors M of the level N with $M \neq N$ span a subspace $S_k^{\mathrm{old}}(\Gamma_0(N))$, called the space of oldforms. Its orthogonal complement with respect to the Petersson inner product is denoted $S_k^{\mathrm{new}}(\Gamma_0(N))$. For $n \in \mathbb{N}$ we define the Hecke operator $T(n)$ by

$$T(n)f = \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right)$$

for $f \in S_k(\Gamma_0(N))$. The operators $T(n)$ are multiplicative, i.e., $T(mn) = T(m)T(n)$ for coprime m, n, and they encode plenty of arithmetic information about modular forms. The theory of Hecke operators implies the existence of an orthogonal basis of $S_k^{\mathrm{new}}(\Gamma_0(N))$ made of eigenfunctions of the operators $T(n)$ for n coprime with N. By the multiplicity-one principle of Atkin & Lehner, the elements f of this basis are in fact eigenfunctions of all $T(n)$, i.e., there exist complex numbers $\lambda_f(n)$ for which

$$T(n)f = \lambda_f(n) f \qquad \text{and} \qquad c(n) = \lambda_f(n) c(1) \qquad \text{for all} \quad n \in \mathbb{N}.$$

Furthermore, it follows that the first Fourier coefficient $c(1)$ of such an f is non-zero. Such a simultaneous eigenfunction is said to be an eigenform. A newform is defined to be an eigenform that does not come from a space of lower level and is normalized to have $c(1) = 1$. The newforms form a finite set which is an orthogonal basis of the space $S_k^{\mathrm{new}}(\Gamma_0(N))$. For instance, Ramanujan's cusp form

$$\sum_{n=1}^{\infty} \tau(n) \exp(2\pi i n z) := \exp(2\pi i z) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n z))^{24} \tag{4.26}$$

is a normalized eigenform of weight 12 to the full modular group, and hence a newform of level 1. Ramanujan [55] conjectured that the coefficients $\tau(n)$ are multiplicative and satisfy the estimate $|\tau(p)| \leq 2p^{\frac{11}{2}}$ for every prime number p. The multiplicativity was proved by Mordell [52], in particular by the beautiful formula

$$\tau(m)\tau(n) = \sum_{d|(m,n)} d^{11} \tau\left(\frac{mn}{d^2}\right).$$

The estimate was shown by Deligne. More precisely, Deligne [14] proved for the coefficients of any newform f of weight k the estimate

$$|c(n)| \leq n^{\frac{k-1}{2}} d(n). \tag{4.27}$$

In the 1930s, Hecke [27] started investigations on modular forms and Dirichlet series with a Riemann-type functional equation; his studies were completed by Atkin & Lehner (for newforms). Here we shall focus on newforms. Given a newform f with Fourier expansion (4.25), we define the associated L-function by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}. \tag{4.28}$$

In view of the classical bound $d(n) \ll n^{\epsilon}$ it follows from (4.27) that the series (4.28) converges absolutely for $\sigma > \frac{k+1}{2}$. By the theory of Hecke operators, it turns out that the Fourier coefficients of newforms are multiplicative. Hence, in the half-plane of absolute convergence, there is an Euler product representation:

$$L(s,f) = \prod_{p|N} \left(1 - \frac{c(p)}{p^s}\right)^{-1} \prod_{p\nmid N} \left(1 - \frac{c(p)}{p^s} + \frac{1}{p^{2s+1-k}}\right)^{-1}. \tag{4.29}$$

Hecke, resp. Atkin & Lehner, proved that $L(s,f)$ has an analytic continuation to an entire function and satisfies the functional equation

$$N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(s,f) = \omega(-1)^{\frac{k}{2}} N^{\frac{k-s}{2}}(2\pi)^{s-k}\Gamma(k-s)L(k-s,f),$$

where $\omega = \pm 1$ is the eigenvalue of the Atkin-Lehner involution $\left(\begin{smallmatrix} 0 & -N \\ 1 & 0 \end{smallmatrix}\right)$ on $S_k(\Gamma_0(N))$. Hecke proved a converse theorem which gives a characterization of these L-functions by their functional equation; this beautiful result generalizes Hamburger's theorem for the Riemann zeta-function (see Titchmarsh [63], §2.13).

Laurinčikas & Matsumoto obtained a universality theorem for L-functions attached to normalized eigenforms of the full modular group. Laurinčikas, Matsumoto & Steuding [41] extended this result to newforms:

Theorem 4.7. *Suppose that f is a newform of weight k and level N. Let $\mathcal{K}$ be a compact subset of the strip $\frac{k}{2} < \sigma < \frac{k+1}{2}$ with connected complement, and let $g(s)$ be a continuous non-vanishing function on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then, for any $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{s\in\mathcal{K}} |L(s+i\tau, f) - g(s)| < \epsilon \right\} > 0.$$

Laurinčikas & Matsumoto [40] obtained also a joint universality theorem for L-functions associated with newforms twisted by characters. Let $f \in S_k(\Gamma_0(N))$

be a newform with Fourier expansion (4.25) and let χ be a Dirichlet character $\bmod\, q$ where q is coprime with N. The twisted L-function is defined by

$$L_{\chi}(s,f)=\sum_{n=1}^{\infty}\frac{c(n)}{n^s}\chi(n).$$

As in the non-twisted case (4.28), this Dirichlet series has an Euler product and extends to an entire function.

Theorem 4.8. *Let $q_1,\dots,q_n$ be positive integers coprime with N and let $\chi_1 \bmod q_1,\dots,\chi_n \bmod q_n$ be pairwise non-equivalent character. Further, for $1\le j\le n$, let g_j be a continuous function on $\mathcal{K}_j$ which is non-vanishing in the interior, where $\mathcal{K}_j$ is a compact subset of the strip $\{s\in\mathbb{C}:\frac{k}{2}<\sigma<\frac{k+1}{2}\}$ with connected complement. Then, for any $\epsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]\,:\,\max_{1\le j\le n}\max_{s\in\mathcal{K}_j}|L_{\chi_j}(s+i\tau,f)-g_j(s)|<\varepsilon\right\}>0.$$

The proof relies on a joint limit theorem due to Laurinčikas and some kind of prime number theorem for the coefficients of cusp forms with respect to arithmetic progressions, namely

$$\sum_{\substack{p\le x\\ p\equiv a \bmod q}} c(p)^2p^{1-k}\sim\frac{1}{\varphi(q)}\frac{x}{\log x}, \tag{4.30}$$

where a is coprime with q. The proof of the latter formula uses ideas of Rankin.

By Wiles' celebrated proof of the Shimura-Taniyama conjecture for semistable modular forms [70] (which led to the proof of Fermat's last theorem), and the proof by Breuil et al. [12] of the general case, every L-function attached to an elliptic curve over the rationals is the L-function to some newform of weight 2 for some congruence subgroup. Consequently, Theorem 4.7 yields the universality of L-functions associated with elliptic curves. Laurinčikas & Steuding [42] used Theorem 4.8 to give an example of jointly universal L-functions associated with elliptic curves. Here one may choose any finite family of elliptic curves of the form

$$E_m\quad:\qquad Y^2=X^3-m^2X\qquad\text{with squarefree}\quad m\in\mathbb{N};$$

these curves were first studied in Tunnell's work on the congruent number problem. For this family one can avoid Wiles' proof of the Shimura-Taniyama-Weil conjecture and show more or less directly that the L-function associated with E_1 corresponds to a newform $f\in S_2(\Gamma_0(32))$ and that the L-function to E_m is a twist of E_1 with the Kronecker symbol $\left(\frac{m}{\cdot}\right)$.

4.5. The Linnik-Ibragimov conjecture. Meanwhile universality has been proved for quite many Dirichlet series. We list some significant examples.

A number field $\mathbb{K}$ is a finite algebraic extension of $\mathbb{Q}$. The Dedekind zeta-function of a number field $\mathbb{K}$ is given by

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathsf{a}} \frac{1}{\mathrm{N}(\mathsf{a})^s} = \prod_{\mathsf{p}} \left(1 - \frac{1}{\mathrm{N}(\mathsf{p})^s}\right)^{-1},$$

where the sum is taken over all non-zero integral ideals, the product is taken over all prime ideals of the ring of integers of $\mathbb{K}$, and $\mathrm{N}(\mathsf{a})$ is the norm of the ideal a. The Riemann zeta-function may be regarded as the Dedekind zeta-function for $\mathbb{Q}$. Universality for the Dedekind zeta-function was first obtained by Voronin [68] and Gonek [19] for some special cases, and in full generality by Reich [56]. Here the strip of universality is restricted to $1 - \min\{\frac{1}{2}, \frac{1}{d}\} < \sigma < 1$, where d is the degree of $\mathbb{K}$ over $\mathbb{Q}$. This restriction depends on the mean-square half-plane for $\zeta_{\mathbb{K}}(s)$; it is conjectured that for any Dedekind zeta-function the strip of universality can be extended to the open right half of the critical strip. In some cases more is known, namely, if $\mathbb{K}$ is abelian (e.g., a subfield of a cyclotomic field), then $\zeta_{\mathbb{K}}(s)$ splits into a product of Dirichlet L-functions to pairwise non-equivalent characters. Using the joint universality for these L-functions, it is easy to deduce the unrestricted universality of $\zeta_{\mathbb{K}}(s)$ in $\frac{1}{2} < \sigma < 1$.

There are other interesting examples which are *strongly* universal: they can approximate functions *with* zeros on a set of positive lower density. The first example is the logarithm of the Riemann zeta-function as we have seen by Theorem 3.1 (and of course the same argument applies to all universal Euler products as for example $\zeta_{\mathbb{K}}(s)$). Next we present a completely different example.

For $0 < \alpha \leq 1, \lambda \in \mathbb{R}$, the Lerch zeta-function is given by

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{\exp(2\pi i \lambda n)}{(n+\alpha)^s}.$$

This series converges absolutely for $\sigma > 1$. The analytic properties of $L(\lambda, \alpha, s)$ are quite different, if $\lambda \in \mathbb{Z}$ or not. If $\lambda \notin \mathbb{Z}$, the series converges for $\sigma > 0$ and $L(\lambda, \alpha, s)$ can be continued analytically to the whole complex plane. For $\lambda \in \mathbb{Z}$ the Lerch zeta-function becomes the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s};$$

this function has an analytic continuation to $\mathbb{C}$ except for a simple pole at $s = 1$ with residue 1. Denote by $\{\lambda\}$ the fractional part of a real number λ. Setting

$$\lambda^+ = 1 - \{\lambda\} \qquad \text{and} \qquad \lambda^- = \left\{ \begin{array}{cl} 1 & \text{if} \quad \lambda \in \mathbb{Z}, \\ \{\lambda\} & \text{otherwise}, \end{array} \right.$$

one can prove the functional equation

$$\begin{aligned}L(\lambda,\alpha,1-s) &= \frac{\Gamma(s)}{(2\pi)^s}\Big(\exp\left(2\pi i\left(\tfrac{s}{4}-\alpha\lambda^-\right)\right)L(-\alpha,\lambda^-,s)\\ &\qquad +\exp\left(2\pi i\left(-\tfrac{s}{4}+\alpha\lambda^+\right)\right)L(\alpha,\lambda^+,s)\Big).\end{aligned}$$

Twists with additive characters destroy the point symmetry of Riemann-type functional equations. Gonek [19] and Bagchi [1] (independently) obtained strong universality for the Hurwitz zeta-function $\zeta(s,\alpha)$ if α is transcendental or rational $\neq \frac{1}{2}, 1$. Laurinčikas [37] extended this result by proving that the Lerch zeta-function $L(\lambda,\alpha,s)$ is strongly universal if λ is not an integer and α is transcendental. All examples of strongly universal Dirichlet series do *not* have an Euler product and have *many* zeros in their region of universality; indeed, the property of approximating analytic functions with zeros is intimately related to the distribution of zeros of the Dirichlet series in question. Euler products for which the analogue of Riemann's hypothesis is expected should not be capable of approximating functions with zeros.

Roughly speaking, there are two methods to prove universality. Firstly, one can try to mimic Voronin's proof or Bagchi's probabilistic approach. This sounds more simple than it actually is, because one has to assure many analytic and arithmetic properties of the function in question. The second way is to find a representation as a linear combination or a product of jointly universal functions. All known proofs of universality of the first type depend on a certain kind of *independence*. For instance, the logarithms of the prime numbers are linearly independent over $\mathbb{Q}$ (we used this property in the proof of Voronin's universality theorem when we applied Weyl's refinement of Kronecker's approximation theorem). Another example are the numbers $\log(n+\alpha)$ with non-negative integral n which are linearly independent over $\mathbb{Q}$ if α is transcendental. In order to prove universality for the Hurwitz zeta-function, the first type of proof yields the result aimed at for transcendental α. If α is rational $\neq \frac{1}{2}, 1$, one can find a representation of $\zeta(s,\alpha)$ as a linear combination of non-equivalent Dirichlet L-functions for which we have the joint universality theorem. In the cases $\alpha = \frac{1}{2}$ and $\alpha = 1$ the Hurwitz zeta-function has an Euler product representation and is equal to the Riemann zeta-function for $\alpha = 1$, resp., for $\alpha = \frac{1}{2}$,

$$\zeta\left(s,\tfrac{1}{2}\right) = 2^s L(s,\chi),$$

where χ is the unique character $\bmod 2$. In both rational cases the Hurwitz zeta-function is universal but does not have the strong universality property. It is an interesting open problem whether $\zeta(s,\alpha)$ *is universal or even strongly universal if* α *is algebraic irrational.*

It was conjectured by Linnik and Ibragimov that *all reasonable functions given by Dirichlet series and analytically continuable to the left of the half-plane of absolute convergence are universal.* Here we need to explain what is meant by '*reasonable*'. For example, put $a(n) = 1$ if $n = 2^k$ with $k \in \mathbb{N}_0$, and $a(n) = 0$ otherwise. Then

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{k=0}^{\infty} \frac{1}{2^{ks}} = (1 - 2^{-s})^{-1},$$

and obviously, this function is far away from being universal. So one has to ask for *natural* conditions needed to prove universality. In [60, 61] a rather general universality theorem for an axiomatically defined set of L-functions was proved. If one is willing to accept some widely believed conjectures from number theory (e.g., the Ramanujan-Petersson conjecture, Langlands conjectures), then this class contains all arithmetically relevant L-functions. A satisfying joint universality result for this class is not yet found.

No pains - no gains! The following exercises may be used to repeat the whole content of these notes; for some help we refer to [33, 62]:

Exercise 16. Prove the prime number theorem for arithmetic progressions.

Exercise 17. Prove universality for a Dirichlet L-function $L(s, \chi)$.

Exercise 18. Use Reich's discrete universality theorem 3.12 in order to prove that the Riemann hypothesis is true if and only if for any $\epsilon > 0$, any real number $\Delta > 0$, any z with $\frac{1}{2} < \mathrm{Re}\ z < 1$, and any $0 < r < \min\{\mathrm{Re}\ z - \frac{1}{2}, 1 - \mathrm{Re}\ z\}$,

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq n \leq N \,:\, \max_{|s-z|\leq r} |\zeta(s + i\Delta n) - \zeta(s)| < \epsilon \right\} > 0.$$

Extend the assertion to Dirichlet L-functions.

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