SECOND ORDER OPTIMALITY CONDITIONS FOR BANG-BANG BILINEAR CONTROL PROBLEMS*

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Abstract. In this article, a bilinear optimal control problem subject to a semilinear elliptic equation is investigated. Sufficient optimality conditions of second-order for the bang-bang control problem are established. These allow to prove local optimality of stationary points together with stability of a Tikhonov regularization of the bang-bang problem.

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1. Introduction.

In this paper we consider the following control problem

$$(\mathrm{P}_\gamma) \qquad \min_{\alpha\le u\le\beta} J_\gamma(u) = \frac{1}{2}\|y_u - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)}, \tag{1.1}$$

where $0 < \alpha < \beta < +\infty$, $\gamma \ge 0$, $y_d \in L^2(\Omega)$, and y_u is the solution of the semilinear state equation

$$\begin{cases} Ay + a(x,y) = 0 & \text{in } \Omega, \\ \partial_{\nu_A} y + uy = g & \text{on } \Gamma. \end{cases} \tag{1.2}$$

In the above equation, A stands for the partial differential operator

$$Ay = \sum_{i,j=1}^n \partial_{x_j}[a_{ij}(x)\partial_{x_i} y]$$

and

$$\partial_{\nu_A} y = \sum_{i,j=1}^n a_{ij}(x)\partial_{x_i} y \nu_j,$$

where $\nu(x) = (\nu_j(x))_{j=1}^n$ denotes the outward unit normal vector to Γ at the point x.

In the subsequent analysis, we focus on the case that the parameter γ is set to zero. Then the functional J is not longer strictly convex with respect to the control u, which makes the analysis much more challenging. In the case $\gamma = 0$ the optimal control problem often exhibits solution of bang-bang type, where at locally optimal controls $\bar u$ the control constraints are active almost everywhere, i.e. $\bar u(x) \in \{\alpha, \beta\}$ a.e. on Γ. If one views this case $\gamma = 0$ as the reference case, then problem (P_γ) with $\gamma > 0$ constitutes a Tikhonov regularization of (P_0).

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The second-order sufficient optimality conditions for (P_γ) with $\gamma > 0$ can be proved by using standard arguments; see, for instance, [1], [3]. The analysis corresponding to the case $\gamma = 0$ is more complicated by the fact that the Legendre-Clebsch condition is not satisfied. A first result was proved in [2] for the case where the control appears linearly in the right hand side of the equation; see also [4]. In this paper we will investigate second-order sufficient optimality conditions for problem (P_0). The result of [2] cannot be applied to (P_0) because of the bilinear character of the coupling of control and state in (1.2). We present a new second-order condition, which contains a term accounting for the bilinear term of the state equation.

The second-order condition allows to prove local growth of the cost functional J_0. This enables to study the stability of solutions of (P_γ) for $\gamma \to 0$. In previous work, control problems with linear state equations were analyzed. Under a strengthened strict complementarity condition, convergence rates with respect to $\gamma \to 0$ for the error in the controls were proven in [9, 10]. This condition was also used in [5] to prove discretization error estimates for finite element discretizations. As mentioned in Remark 6.4 below, it is an open question to apply the strengthened complementarity condition for the problem (P_0). Here, the bilinear structure prohibits an analysis as in [9, 10].

The plan of the paper is as follows. In section 2, the assumptions are established and the differentiability properties of the relation control-to-state and the cost functional are proved. Section 3 is devoted to the proof of the existence of a solution for the control problem and to the derivation of the first-order optimality conditions. The second-order analysis is carried out in sections 4 and 5 for the cases $\gamma > 0$ and $\gamma = 0$, respectively. In section 6, the stability analysis with respect to γ is studied. Finally, in section 7, we extend the results to the case of distributed controls.

2. Assumptions and Preliminary Results. In the sequel Ω will denote a bounded domain in $\mathbb{R}^n$, $1 \le n \le 3$, with a Lipschitz boundary Γ. Concerning the operator A defined in Ω, we assume that $a_{ij} \in L^\infty(\Omega)$ for $1 \le i, j \le n$, and there exists some $\Lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \ge \Lambda\,|\xi|^2 \text{ for a.a. } x \in \Omega \quad\text{and}\quad \forall \xi \in \mathbb{R}^n.$$

Moreover, the function $a : \Omega \times \mathbb{R} \to \mathbb{R}$ introduced in (1.1) is a Carathéodory function of class C^2 with respect to the second variable, and satisfying

$$\begin{cases} a(\cdot, 0) \in L^{\bar p}(\Omega) \text{ for some } \bar p > \dfrac{n}{2} \\ \dfrac{\partial a}{\partial y}(x, y) \ge 0 \quad \text{for a.a. } x \in \Omega \quad\text{and}\quad \forall y \in \mathbb{R} \\ \forall M > 0\ \exists C_M > 0 \text{ s.t. } \displaystyle\sum_{j=1}^{2} \left| \frac{\partial^j a}{\partial y^j}(x, y) \right| \le C_M \quad \text{for a.a. } x \in \Omega \quad\text{and}\quad |y| \le M. \end{cases} \tag{2.1}$$

Moreover, we assume some uniform continuity of $\dfrac{\partial^2 a}{\partial y^2}$: For every $M > 0$ and $\varepsilon > 0$, there exists $\rho_{\varepsilon,M} > 0$ such that

$$\left| \frac{\partial^2 a}{\partial y^2}(x, y_2) - \frac{\partial^2 a}{\partial y^2}(x, y_1) \right| < \varepsilon \text{ if } \ |y_i| \le M \text{ and } |y_2 - y_1| < \rho_{\varepsilon,M} \ \text{ for a.a. } x \in \Omega. \tag{2.2}$$

Additionally, we assume that the boundary datum of (1.2) g belongs to $L^{\bar q}(\Gamma)$ for some $\bar q > n-1$.

Hereafter, the set of admissible controls will be denoted by

$$\mathbb{K} = \{u \in L^\infty(\Gamma) : \alpha \le u(x) \le \beta \text{ for a.a. } x \in \Gamma\}.$$

We also define the open set of $L^\infty(\Gamma)$

$$\mathcal{A} = \{u \in L^\infty(\Gamma) : \exists c_u > 0 \text{ such that } u(x) \ge c_u \text{ for a.a. } x \in \Gamma\}.$$

Under the above assumptions we have the following result.

THEOREM 2.1. *There exists a number $\mu \in (0,1]$ such that for every control $u \in \mathcal{A}$ the equation* (1.2) *has a unique solution in $H^1(\Omega) \cap C^{0,\mu}(\bar\Omega)$. Moreover, the mapping $G : \mathcal{A} \to H^1(\Omega) \cap C^{0,\mu}(\bar\Omega)$, $u \mapsto y_u$, is of class C^2. Its first derivative $z_v = G'(u)v$ is given by the solution z of*

$$\begin{cases} Az + \dfrac{\partial a}{\partial y}(x, y_u) z = 0 & \text{in } \Omega, \\ \partial_{\nu_A} z + uz + v y_u = 0 & \text{on } \Gamma, \end{cases} \tag{2.3}$$

while the second derivative $w_{v_1,v_2} = G''(u)(v_1, v_2)$ associated with directions $v_i \in L^\infty(\Omega)$, $i = 1, 2$, is the solution w of

$$\begin{cases} Aw + \dfrac{\partial a}{\partial y}(x, y_u) w + \dfrac{\partial^2 a}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} = 0 & \text{in } \Omega, \\ \partial_{\nu_A} w + uw + v_1 z_{v_2} + v_2 z_{v_1} = 0 & \text{on } \Gamma, \end{cases} \tag{2.4}$$

where $z_{v_i} = G'(u)v_i$.

Proof. For the proof of the existence and uniqueness of $y_u \in H^1(\Omega) \cap C^{0,\mu}(\bar\Omega)$ the reader is referred to [7, Theorem 6.6].

Let us sketch the proof of the differentiability properties of G. Let us define $Y := H^1(\Omega) \cap C^{0,\mu}(\bar\Omega)$ and

$$V := \{y \in Y : \ Ay \in L^{\bar p}(\Omega),\ \partial_{\nu_A} y \in L^{\bar q}(\Gamma)\}.$$

Y and V are Banach spaces when they are endowed with the norms

$$\|y\|_Y = \|y\|_{H^1_0(\Omega)} + \|y\|_{C^{0,\mu}(\bar\Omega)}$$

and

$$\|y\|_V := \|y\|_Y + \|Ay\|_{L^{\bar p}(\Omega)} + \|\partial_{\nu_A} y\|_{L^{\bar q}(\Gamma)}.$$

Let us define the function $\mathcal{F} :\ \mathcal{A} \times V \to L^{\bar p}(\Omega) \times L^{\bar q}(\Gamma)$ by

$$\mathcal{F}(u, y) = (Ay + a(\cdot, y),\ \partial_{\nu_A} y + uy - g).$$

By the assumptions in Section 1, $\mathcal{F}$ is of class C^2. The partial derivative with respect to y is given by

$$\frac{\partial \mathcal{F}}{\partial y}(u, y) z = \left(Ay + \frac{\partial a}{\partial y}(\cdot, y) z,\ \partial_{\nu_A} z + uz \right).$$

By the already mentioned result [7, Theorem 6.6], the operator $\frac{\partial \mathcal{F}}{\partial y}(u,y)$ is an isomorphism from V to $L^{\bar p}(\Omega) \times L^{\bar q}(\Gamma)$. Then the statement of the theorem follows by the implicit function theorem. ∎

Using this result it is straightforward to prove the differentiability of the cost functional.

THEOREM 2.2. *For every $u \in \mathcal{A}$ the cost functional $J_\gamma : \mathcal{A} \subset L^\infty(\Gamma) \longrightarrow \mathbb{R}$ is of class C^2, and the derivatives are given by*

$$J'_\gamma(u)v = \int_\Omega (y_u - y_d) z_v dx + \gamma \int_\Gamma uv dx$$

and

$$J''_\gamma(u)(v_1,v_2) = \int_\Omega \left(z_{v_1} z_{v_2} + (y_u - y_d) w_{v_1,v_2}\right) dx + \gamma \int_\Gamma v_1 v_2 dx.$$

For a given $u \in \mathcal{A}$ let us define the adjoint state φ_u as the solution of the adjoint equation

$$\begin{cases} A^*\varphi + \dfrac{\partial a}{\partial y}(x,y_u)\varphi = y_u - y_d & \text{in } \Omega, \\ \partial_{\nu_{A^*}}\varphi + u\varphi = 0 & \text{on } \Gamma, \end{cases} \tag{2.5}$$

With the help of the adjoint state we can rewrite the expressions for J'_γ and J''_γ as follows.

COROLLARY 2.3. *Let $u \in \mathcal{A}$ and be $v_1, v_2 \in L^\infty(\Gamma)$ given. Then it holds*

$$J'_\gamma(u)v_1 = \int_\Gamma (\gamma u - y_u\varphi_u) v_1 dx$$

and

$$J''_\gamma(u)(v_1,v_2) = \int_\Omega \left(1 - \frac{\partial^2 a}{\partial y^2}(x,y_u)\varphi_u\right) z_{v_1} z_{v_2} dx + \int_\Gamma \left(\gamma v_1 v_2 - (v_1 z_{v_2} + v_2 z_{v_1})\varphi_u\right) dx.$$

Let us remark that if $v, v_1, v_2 \in L^2(\Gamma)$ only, then z_v and w_{v_1,v_2} are still well-defined in $H^1(\Omega)$. But they are not in $L^\infty(\Omega)$ unless $n = 2$.

LEMMA 2.4. *Let $u \in \mathcal{A}$ and $v \in L^2(\Gamma)$ be given. Then we have the a-priori bound*

$$\|z_v\|_{H^1(\Omega)} \le c\, \|v\|_{L^2(\Gamma)}$$

with constant $c > 0$ depending monotonically on $\|y_u\|_{C(\bar\Omega)}$ but independent of v.

Proof. The inequality follows by testing the weak formulation of (2.3) by z_v. ∎

Remark 2.5. *Let us observe that the functional $J'(u)$ as well as the bilinear form $J''(u)$ can be extended to continuous linear and bilinear forms on $L^2(\Gamma)$ and $L^2(\Gamma) \times L^2(\Gamma)$, respectively. We will use this fact later without explicit mentioning.*

We will frequently need to work with weakly converging sequences of controls. For later reference, we will prove here that the associated states will converge strongly.

LEMMA 2.6. *Let (u_k) denote a sequence in $\mathbb{K}$ converging weakly in $L^1(\Gamma)$ to u. Then, the associated sequence of states (y_{u_k}) converges strongly in $H^1(\Omega) \cap C(\bar\Omega)$ to y_u.*

Proof. First, let us observe that the boundedness of (u_k) in $L^\infty(\Gamma)$ implies that (u_k) converges weakly to u in every space $L^q(\Gamma)$ for any $1 \le q < \infty$. In particular, we have that $u_k \rightharpoonup u$ in $L^{\bar q}(\Gamma)$. Now, let us denote $y_k := y_{u_k}$. The difference $y_k - y_u$ satisfies the equation

$$\begin{cases} A(y_k - y_u) + \displaystyle\int_0^1 \frac{\partial a}{\partial y}(x, y_u + \theta(y_k - y_u))d\theta(y_k - y_u) = 0 & \text{in } \Omega, \\ \partial_{\nu_A}(y_k - y_u) + u_k(y_k - y_u) + y_u(u_k - u) = 0 & \text{on } \Gamma, \end{cases}.$$

Since $\frac{\partial a}{\partial y} \ge 0$ and $u_k \ge \alpha > 0$ holds, we obtain that (y_k) is uniformly bounded in $H^1(\Omega) \cap L^\infty(\Omega)$ by well-known regularity results. Using the results of [7] we find the uniform boundedness of (y_k) in $C^{0,\mu}(\bar\Omega)$. Then after extracting subsequences if necessary we obtain $y_k \rightharpoonup \tilde y$ in $H^1(\Omega)$ and $y_k \to \tilde y$ in $C(\bar\Omega)$. This allows to pass to the limit in the weak formulation of (1.2), which proves that $\tilde y$ satisfies (1.2), and hence $\tilde y = y_u$. As the limit y_u is independent of the chosen subsequences, the convergence of the whole sequence (y_k) to y_u in the sense already obtained follows. Moreover, we have $y_k u_k \rightharpoonup y_u u$ in $L^{\bar q}(\Gamma)$. By compactness of the embedding of $L^{\bar q}(\Gamma)$ in $H^{-1/2}(\Gamma)$, we obtain the strong convergence of $y_k u_k \to y_u u$ in $H^{-1/2}(\Gamma)$, and consequently it holds $y_k \to y_u$ in $H^1(\Omega)$ strongly. □

LEMMA 2.7. *Let (u_k) denote a sequence in $\mathbb{K}$ converging weakly in $L^1(\Gamma)$ to $u \in \mathbb{K}$. Then the associated sequence (φ_{u_k}) of adjoint states converges strongly in $H^1(\Omega) \cap C(\bar\Omega)$ to φ_u.*

Proof. By Lemma 2.6, we have $y_{u_k} \to y_u$ strongly in $H^1(\Omega) \cap C(\bar\Omega)$. The convergence of the adjoints states can now be proven by analogous arguments as in the proof of Lemma 2.6. □

LEMMA 2.8. *Let (u_k) denote a sequence in $\mathbb{K}$ converging strongly in $L^1(\Gamma)$ to u. Let (v_k) be a sequence converging weakly in $L^2(\Gamma)$. Then, the sequence $(G'(u_k)v_k)$ converges strongly in $H^1(\Omega)$ to $G'(u)v$. If additionally (v_k) converges weakly in $L^{\bar q}(\Gamma)$ to v, then the sequence $(G'(u_k)v_k)$ converges strongly in $C(\bar\Omega)$.*

Proof. By the results of Lemma 2.6, we have that $y_{u_k}v_k$ converges weakly in $L^2(\Gamma)$ to $y_u v$. If $v_k \rightharpoonup v$ in $L^{\bar q}(\Gamma)$ then $y_{u_k}v_k \rightharpoonup y_u v$ in $L^{\bar q}(\Gamma)$ as well. The claim can now be proven using arguments similar to those used in the proof of Lemma 2.6 above. □

3. Existence of solutions and necessary optimality conditions. Let us briefly comment on the existence of solutions of (P_γ).

THEOREM 3.1. *For every $\gamma \ge 0$ the problem (P_γ) is solvable.*

Proof. Since the lower bound α on the controls is positive, the state equation 1.2 is uniquely solvable for every $u \in \mathbb{K}$ by Theorem 2.1. Let (u_k) be a minimizing sequence for (P_γ). Due to the control constraints it is uniformly bounded in $L^\infty(\Gamma)$. By extracting a subsequence, if necessary, we obtain $u_k \rightharpoonup \bar u$ in $L^{\bar q}(\Gamma)$. The set $\mathbb{K}$ is weakly closed, hence $\bar u \in \mathbb{K}$ follows. By Lemma 2.6 the associated sequence (y_{u_k}) of states converges to $y_{\bar u}$ in $H^1(\Omega) \cap C(\bar\Omega)$. Using weakly lower semicontinuity of norms, we find that $\bar u$ is a solution of (P_γ). □

Let us remark that problem (P_γ) is not uniquely solvable in general. Global and local solutions of (P_γ) can be characterized by the following first-order necessary optimality condition.

THEOREM 3.2. *Let $\bar u$ be a local solution of (P_γ). Let $\bar y := y_{\bar u}$ denote the associated state and $\bar\varphi := \varphi_{\bar u}$ denote the associated adjoint state, cf.* (2.5). *Then the following* *system is satisfied:*

$$\begin{cases} A\bar{y} + a(x,\bar{y}) = 0 & \text{in } \Omega, \\ \partial_{\nu_A}\bar{y} + \bar{u}\bar{y} = g & \text{on } \Gamma, \end{cases} \tag{3.1}$$

$$\begin{cases} A^*\bar{\varphi} + \dfrac{\partial a}{\partial y}(x,\bar{y})\bar{\varphi} = \bar{y} - y_d & \text{in } \Omega, \\ \partial_{\nu_{A^*}}\bar{\varphi} + \bar{u}\bar{\varphi} = 0 & \text{on } \Gamma, \end{cases} \tag{3.2}$$

$$\int_\Gamma (\gamma\bar{u} - \bar{y}\bar{\varphi})(u - \bar{u})dx \geq 0 \quad \forall u \in \mathbb{K}. \tag{3.3}$$

Proof. By standard results on first-order necessary optimality conditions we have $J'_\gamma(\bar{u})(u - \bar{u}) \geq 0$ for all $u \in \mathbb{K}$. Using Corollary 2.3 we obtain the optimality system stated above. ∎

Under the assumptions of the theorem, we have the following pointwise representation of locally optimal controls.

COROLLARY 3.3. *Let the assumptions of Theorem 3.2 be satisfied. If $\gamma > 0$ then it holds for a.a. $x \in \Gamma$*

$$\bar{u}(x) = \text{proj}_{[\alpha,\beta]}\left(\frac{1}{\gamma}\bar{y}(x)\bar{\varphi}(x)\right). \tag{3.4}$$

If $\gamma = 0$ then it holds for a.a. $x \in \Gamma$

$$\bar{u}(x) \begin{cases} = \alpha & \text{if } \bar{y}(x)\bar{\varphi}(x) < 0, \\ \in [\alpha,\beta] & \text{if } \bar{y}(x)\bar{\varphi}(x) = 0, \\ = \beta & \text{if } \bar{y}(x)\bar{\varphi}(x) > 0. \end{cases} \tag{3.5}$$

Relation (3.5) gives rise to the notion of bang-bang controls: if it holds $\bar{y}\bar{\varphi} \neq 0$ a.e. on Γ, then the optimal control $\bar{u}$ takes values from the set $\{\alpha, \beta\}$ almost everywhere.

4. Second-order optimality conditions. In general, the problems (P_γ) and (P_0) are not convex. This implies that the first-order necessary optimality conditions are not sufficient for local optimality. In this section, we discuss second-order necessary and sufficient conditions for problem (P_γ) with $\gamma > 0$. It turns out that the additional term $\frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)}$ in the cost function J_γ enables us to obtain a complete analysis of second-order conditions.

Before turning to second-order conditions, let us proof the following useful result.

THEOREM 4.1. *Let $\gamma \geq 0$ hold, and let (u_k) be a sequence of controls $u_k \in \mathbb{K}$ converging strongly in $L^2(\Gamma)$ to u. Let (v_k) be a sequence converging weakly in $L^2(\Gamma)$ to v. Then we have the following:*

1. *$J'_\gamma(u_k)v_k \to J'_\gamma(u)v$ for $k \to \infty$,*
2. *$J''_\gamma(u)v^2 \leq \liminf_{k\to\infty} J''_\gamma(u_k)v_k^2$,*
3. *if $v = 0$ then $\gamma \liminf_{k\to\infty} \|v_k\|^2_{L^2(\Gamma)} \leq \liminf_{k\to\infty} J''_\gamma(u_k)v_k^2$*

Proof. Since $u_k \in \mathbb{K}$ for every k, then (u_k) is uniformly bounded in $L^\infty(\Gamma)$. Hence, we have $u_k \to u$ in $L^{\bar{q}}(\Gamma)$. Then Lemmas 2.6 and 2.7 yield the strong convergence $y_{u_k} \to y_u$ and $\varphi_{u_k} \to \varphi_u$ in $H^1(\Omega) \cap C(\bar{\Omega})$. The convergence $J'_\gamma(u_k)v_k \to J'_\gamma(u)v$ is now an immediate consequence of the adjoint representation of J'_γ in Corollary 2.3.

Lemma 2.8 yields the strong convergence of $z_k := G'(u_k)v_k$ to $z := G'(u)v$. This allows to pass to the limit in the second derivative of J_γ. In the case $\gamma > 0$ we have to use the weakly lower semicontinuity of the norm to deduce *2*.

Let us suppose $v = 0$. Then we have by *2*: $0 \le \liminf_{k\to\infty} J_0''(u_k)v_k^2$. Adding $\gamma \liminf_{k\to\infty} \|v_k\|^2_{L^2(\Gamma)}$ to both sides of the inequality we get *3*. ∎

If $\bar u \in \mathbb{K}$ satisfies (3.1)-(3.3) along with $\bar y$ and $\bar\varphi$, we define the critical cone

$$C_{\gamma,\bar u} := \{v \in L^2(\Gamma) :\ v(x) \ge 0 \text{ if } \bar u(x) = \alpha,\ v(x) \le 0 \text{ if } \bar u(x) = \beta,\ J_\gamma'(\bar u)v = 0\} \quad (4.1)$$

Let us denote $\bar d := \gamma\bar u - \bar y\bar\varphi$. Then, $C_{\gamma,\bar u}$ can be written as follows

$$C_{\gamma,\bar u} = \left\{ v \in L^2(\Gamma) : v(x) = \left\{ \begin{array}{ll} \ge 0 & \text{if } \bar u(x) = \alpha \\ \le 0 & \text{if } \bar u(x) = \beta \\ 0 & \text{if } \bar d(x) \ne 0 \end{array} \right. \text{a.e. in } \Gamma \right\};$$

see [3, pp. 252-253] for the proof. Concerning second-order necessary optimality conditions we have the following result.

THEOREM 4.2. *Let $\bar u$ be locally optimal solution of* (P_γ), $\gamma \ge 0$. *Then it holds*

$$J_\gamma''(\bar u)v^2 \ge 0 \quad \forall v \in C_{\gamma,\bar u}.$$

Proof. For a proof we refer to [3, Theorem 2.2 and pp. 252-253]. ∎

Let us first recall the second-order sufficient optimality conditions for the case $\gamma > 0$. Here we have the following result.

THEOREM 4.3. *Let $\gamma > 0$. Let $\bar u \in \mathbb{K}$ satisfy $J_\gamma'(\bar u)(u - \bar u) \ge 0$ for all $u \in \mathbb{K}$ and $J_\gamma''(\bar u)v^2 > 0$ for all $v \in C_{\gamma,\bar u} \setminus \{0\}$. Then $\bar u$ is locally optimal for* (P_γ). *In particular, there exist positive constants ρ and σ such that*

$$J_\gamma(\bar u) + \frac{\sigma}{2}\|u - \bar u\|^2_{L^2(\Gamma)} \le J_\gamma(u) \quad \forall u \in \mathbb{K} :\ \|u - \bar u\|_{L^2(\Gamma)} < \rho. \quad (4.2)$$

Proof. Remark 2.5 and Theorem 4.1 imply that assumptions (A1) and (A2) of [3] are satisfied. The result follows now from [3, Theorem 2.3]. ∎

Given $\bar u \in \mathbb{K}$ satisfying $J_\gamma'(\bar u)(u - \bar u) \ge 0$ for all $u \in \mathbb{K}$, we define for every $\tau \ge 0$

$$C^\tau_{\gamma,\bar u} := \{v \in L^2(\Gamma) :\ v(x) \ge 0 \text{ if } \bar u(x) = \alpha,\ v(x) \le 0 \text{ if } \bar u(x) = \beta,\\ J_\gamma'(\bar u)v \le \tau\|v\|_{L^2(\Gamma)}\}.$$

Then $C^\tau_{\gamma,\bar u} \supset C^0_{\gamma,\bar u} = C_{\gamma,\bar u}$ holds for all $\tau > 0$.

THEOREM 4.4. *Suppose $\gamma > 0$. Let $\bar u \in \mathbb{K}$ satisfy $J_\gamma(\bar u)(u - \bar u) \ge 0$ for all $u \in \mathbb{K}$. Then the following statements are equivalent:*

1. $J_\gamma''(\bar u)v^2 > 0$ *for all* $v \in C_{\gamma,\bar u} \setminus \{0\}$,
2. $\exists\delta > 0, \tau > 0$ *such that* $J_\gamma''(\bar u)v^2 \ge \delta\|v\|^2_{L^2(\Gamma)}$ *for all* $v \in C^\tau_{\gamma,\bar u}$.
3. $\exists\delta > 0, \tau > 0$ *such that* $J_\gamma''(\bar u)v^2 \ge \delta\left(\|z_v\|^2_{L^2(\Omega)} + \|vz_v\|_{L^1(\Gamma)}\right)$ *for all* $v \in C^\tau_{\gamma,\bar u}$.

Proof. The implication (1) ⇒ (2) is a consequence of e.g. [3, Theorem 2.7]. The implication (2) ⇒ (3) follows from the estimate of z_v against v of Lemma 2.4. Let us prove the implication (3) ⇒ (1). Given $v \in C_{\gamma,\bar u} \setminus \{0\}$, we distinguish two cases: either $v\bar y = 0$ a.e. on Γ or $v\bar y \not\equiv 0$ on Γ. In the second case, we have that $z_v \ne 0$ and $v \in C^\tau_{\gamma,\bar u}$, hence (1) follows from (3). In the first case, we observe that $z_v = 0$, then from Corollary 2.3 we deduce that $J_\gamma''(\bar u)v^2 = \gamma\|v\|^2_{L^2(\Gamma)} > 0$, which implies (1) again. ∎

5. Sufficient optimality conditions in the case $\gamma = 0$. In this section we investigate the second-order condition to prove local optimality for problem (P$_0$). Let us observe that the assumption $\gamma > 0$ was crucial in Theorem 4.4 to prove the implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). In the case $\gamma = 0$, the statements (1)–(3) of Theorem 4.4 are not equivalent. The issue is if some of these statements along with the first order conditions is a sufficient condition for local optimality. If $\gamma = 0$ then it is not enough to assume that $J'_\gamma(\bar u)(u-\bar u) \geq 0\ \forall u \in \mathbb{K}$ and $J''_\gamma(\bar u)v^2 > 0\ \forall v \in C_{\gamma,\bar u}\setminus\{0\}$ to deduce that $\bar u$ is a local minimum; see Dunn [6] for a counterexample. At the end of this section, we will prove that (2) cannot be fulfilled if $\gamma = 0$ and $\alpha \not\equiv \bar u(x) \not\equiv \beta$. Hence, the only possibility is the condition (3). The next theorem says that this condition is the correct one.

THEOREM 5.1. *Let $\bar u \in \mathbb{K}$ satisfy $J_0(\bar u)(u-\bar u) \geq 0$ for all $u \in \mathbb{K}$. Let us assume that there exist $\delta > 0, \tau > 0$ such that*

$$J''_0(\bar u)v^2 \geq \delta\left(\|z_v\|^2_{L^2(\Omega)} + \|vz_v\|_{L^1(\Gamma)}\right)\ \ \forall v \in C^\tau_{0,\bar u}. \tag{5.1}$$

Then, there exist $\rho, \sigma > 0$ such that

$$J_0(\bar u) + \frac{\sigma}{2}\left(\|z_{u-\bar u}\|^2_{L^2(\Omega)} + \|(u-\bar u)z_{u-\bar u}\|_{L^1(\Gamma)}\right) \leq J_0(u) \quad \forall u \in \mathbb{K}\cap B_\rho(\bar u), \tag{5.2}$$

where $B_\rho(\bar u) = \{u \in L^2(\Gamma) : \|u-\bar u\|_{L^2(\Gamma)} < \rho\}$.

Proof. We argue by contradiction. Then there is a sequence $(u_k) \subset \mathbb{K}$ such that $u_k \to \bar u$ in $L^2(\Gamma)$ and

$$\frac{1}{k}\left(\|z_{u_k-\bar u}\|^2_{L^2(\Omega)} + \|(u_k-\bar u)z_{u_k-\bar u}\|_{L^1(\Gamma)}\right) > J_0(u_k) - J_0(\bar u).$$

Due to the pointwise control constraints, we have $u_k \to \bar u$ in $L^{\bar q}(\Gamma)$. Lemmas 2.6 and 2.7 imply the convergence $y_{u_k} \to \bar y$ and $\varphi_{u_k} \to \bar\varphi$ in $H^1(\Omega)\cap C(\bar\Omega)$. Let us set $v_k := \frac{u_k-\bar u}{\|u_k-\bar u\|_{L^2(\Gamma)}}$. Then it holds after extracting subsequences if necessary $v_k \rightharpoonup v$ in $L^2(\Gamma)$. Lemma 2.4 implies $z_{v_k} := G'(\bar u)v_k \to G'(\bar u)v := z_v$ in $H^1(\Omega)$. By Taylor expansion, we find

$$\frac{1}{k}\left(\|z_{u_k-\bar u}\|^2_{L^2(\Omega)} + \|(u_k-\bar u)z_{u_k-\bar u}\|_{L^1(\Gamma)}\right) > J'_0(\bar u)(u_k-\bar u) + \frac{1}{2}J''_0(u_{k,\theta})(u_k-\bar u)^2$$

with $u_{k,\theta} = \bar u + \theta_k(u_k-\bar u)$, $\theta_k \in (0,1)$. This inequality implies

$$\frac{1}{k}\left(\|z_{v_k}\|_{L^2(\Omega)}\|z_{u_k-\bar u}\|_{L^2(\Omega)} + \|(u_k-\bar u)z_{v_k}\|_{L^1(\Gamma)}\right)$$
$$> J'_0(\bar u)v_k + \frac{1}{2}J''_0(u_{k,\theta})v_k^2\|u_k-\bar u\|_{L^2(\Gamma)}. \tag{5.3}$$

Since $u_{k,\theta} \to \bar u$ in $L^{\bar q}(\Gamma)$ we have also $y_{k,\theta} \to \bar y$ and $\varphi_{k,\theta} \to \bar\varphi$ in $H^1(\Omega)\cap C(\bar\Omega)$, and $z_{v_k,\theta} \to z_v$ in $H^1(\Omega)$, where $z_{v_k,\theta} := G'(u_{k,\theta})v_k$, $y_{k,\theta} = G(u_{k,\theta})$ and $\varphi_{k,\theta}$ is the adjoint state associated to $u_{k,\theta}$; see lemmas 2.6, 2.7 and 2.8. Thus we find

$$J''_0(u_{k,\theta})v_k^2 = \int_\Omega\left(1 - \frac{\partial^2 a}{\partial y^2}(x, y_{u_{k,\theta}})\varphi_{u_{k,\theta}}\right)z^2_{v_k,\theta}dx - \int_\Gamma 2v_k z_{v_k,\theta}\varphi_{u_{k,\theta}}dx$$
$$\to \int_\Omega\left(1 - \frac{\partial^2 a}{\partial y^2}(x,\bar y)\bar\varphi\right)z_v^2dx - \int_\Gamma 2vz_v\bar\varphi dx = J''_0(\bar u)v^2.$$

From the assumptions of the theorem we have that $0 \le J_0'(\bar u)v_k$ for every k. Hence, with (5.3) we find $J_0'(\bar u)v_k \to 0$. Thus, there exists $k_\tau > 0$ such that $J_0'(\bar u)v_k \le \tau$ for every $k \ge k_\tau$, or equivalently $J_0'(\bar u)(u_k - \bar u) \le \tau\|u_k - \bar u\|_{L^2(\Gamma)}$ for every $k \ge k_\tau$. Hence, $u_k - \bar u \in C^\tau_{\bar u}$ for $k \ge k_\tau$. Invoking the second-order condition and using again that $J_0'(\bar u)(u_k - \bar u) \ge 0$, we get

$$\frac{1}{k}\Big(\|z_k\|^2_{L^2(\Omega)} + \|(u_k - \bar u)z_k\|_{L^1(\Gamma)}\Big) > \frac{1}{2}J_0''(\bar u)(u_k - \bar u)^2 + \frac{1}{2}\left(J_0''(u_{k,\theta}) - J_0''(\bar u)\right)(u_k - \bar u)^2$$
$$\ge \frac{\delta}{2}\Big(\|z_k\|^2_{L^2(\Omega)} + \|(u_k - \bar u)z_k\|_{L^1(\Gamma)}\Big) + \frac{1}{2}\left(J_0''(u_{k,\theta}) - J_0''(\bar u)\right)(u_k - \bar u)^2, \quad (5.4)$$

where we set $z_k := G'(\bar u)(u_k - \bar u)$. Below we will prove that there exists $k_\delta \ge k_\tau$ such that

$$\left|\left(J_0''(u_{k,\theta}) - J_0''(\bar u)\right)(u_k - \bar u)^2\right| \le \frac{\delta}{2}\Big(\|z_k\|^2_{L^2(\Omega)} + \|(u_k - \bar u)z_k\|_{L^1(\Gamma)}\Big) \ \forall k \ge k_\delta. \quad (5.5)$$

Inequalities (5.4) and (5.5) lead to

$$\frac{1}{k}\Big(\|z_k\|^2_{L^2(\Omega)} + \|(u_k - \bar u)z_k\|_{L^1(\Gamma)}\Big) > \frac{\delta}{4}\Big(\|z_k\|^2_{L^2(\Omega)} + \|(u_k - \bar u)z_k\|_{L^1(\Gamma)}\Big) \quad \forall k \ge k_\delta,$$

which is a contradiction. □

Now, we prove (5.5). We maintain the notation z_k, $u_{k,\theta}$, $y_{k,\theta}$ and $\varphi_{k,\theta}$ introduced above. We also set $y_k = G(u_k)$ and $z_{k,\theta} = G'(u_{k,\theta})(u_k - \bar u)$. Now, we establish two lemmas.

LEMMA 5.2. *Let $1 < p < 2$ and $q > 2$ if $n = 3$, and $p = q = 2$ if $n = 2$. Then, there exist constants $C_p > 0$ and $C_q > 0$ independent of k such that for any $k > 0$*

$$\|y_{k,\theta} - \bar y\|_{L^2(\Omega)} + \|y_{k,\theta} - \bar y\|_{L^p(\Gamma)} \le C_p\|u_k - \bar u\|_{L^1(\Gamma)}, \quad (5.6)$$
$$\|y_{k,\theta} - \bar y\|_{C(\bar\Omega)} \le C_q\|u_k - \bar u\|_{L^q(\Gamma)}. \quad (5.7)$$

Proof. Subtracting the states equations satisfied by $y_{k,\theta}$ and $\bar y$, and using the mean value theorem we get

$$\begin{cases} A(y_{k,\theta} - \bar y) + \dfrac{\partial a}{\partial y}(x, \hat y_k)(y_{k,\theta} - \bar y) = 0 & \text{in } \Omega, \\ \partial_{\nu_A}(y_{k,\theta} - \bar y) + \bar u(y_{k,\theta} - \bar y) + \theta(u_k - \bar u)y_{k,\theta} = 0 & \text{on } \Gamma, \end{cases} \quad (5.8)$$

where $\hat y_k(x) = \bar y(x) + \vartheta_k(x)(y_{k,\theta}(x) - \bar y(x))$, for some measurable function $0 \le \vartheta_k(x) \le 1$. Now, it is well known that for any $r < n/(n-1)$ (5.8) implies the estimate

$$\|y_{k,\theta} - \bar y\|_{W^{1,r}(\Omega)} \le C_r\|\theta_k(u_k - \bar u)\bar y\|_{L^1(\Gamma)}.$$

Since $W^{1,r}(\Omega) \subset L^2(\Omega)$ and the traces of functions of $W^{1,r}(\Omega)$ belong to $L^p(\Gamma)$ if r is close enough to $n/(n-1)$, then (5.6) follows from the above estimate. The estimate (5.7) can be obtained in a standard way from (5.8) by Stampacchia's method [8, Section 4]. □

LEMMA 5.3. *Under the notation of Lemma 5.2, there exist constants $M_p > 0$ and $K_p > 0$ independent of k such that*

$$\|z_{k,\theta} - z_k\|_{L^2(\Omega)} + \|z_{k,\theta} - z_k\|_{L^p(\Gamma)} \le M_p\Big(\|z_k\|^2_{L^2(\Omega)} + \|(u_k - \bar u)z_k\|_{L^1(\Gamma)}\Big) \quad (5.9)$$

and

$$\|z_{k,\theta}\|^2_{L^2(\Omega)} \le K_p\left(\|z_k\|^2_{L^2(\Omega)} + \|(u_k-\bar u)z_k\|_{L^1(\Gamma)}\right). \tag{5.10}$$

Proof. Let us set $z_{u_{k,\theta}-\bar u} = G'(\bar u)(u_{k,\theta}-\bar u)$. Observe that $u_{k,\theta}-\bar u = \theta_k(u_k-\bar u)$, therefore $z_{u_{k,\theta}-\bar u} = \theta_k z_k$. Now, subtracting (5.8) and the equation satisfied by $z_{u_{k,\theta}-\bar u}$, and applying the mean value theorem we get for $e_k = y_{k,\theta}-\bar y-\theta_k z_k$

$$\left\{\begin{aligned} Ae_k + \frac{\partial a}{\partial y}(x,\bar y)e_k + \frac12\frac{\partial^2 a}{\partial y^2}(x,\tilde y_k)(y_{k,\theta}-\bar y)^2 &= 0 \quad \text{in } \Omega,\\ \partial_{\nu_A} e_k + \bar u e_k + \theta_k(u_k-\bar u)(y_{k,\theta}-\bar y) &= 0 \quad \text{on } \Gamma,\end{aligned}\right. \tag{5.11}$$

where $\tilde y_k(x) = \bar y(x) + \eta_k(x)(y_{k,\theta}(x)-\bar y(x))$, with $0\le \eta_k(x)\le 1$ measurable, satisfies

$$a(x,y_{k,\theta}(x)) = a(x,\bar y(x)) + \frac{\partial a}{\partial y}(x,\bar y(x))(y_{k,\theta}-\bar y(x)) + \frac12\frac{\partial^2 a}{\partial y^2}(x,\tilde y_k(x))(y_{k,\theta}-\bar y(x))^2.$$

Arguing in a similar way to the proof of Lemma 5.2, we get

$$\|e_k\|_{L^2(\Omega)} + \|e_k\|_{L^p(\Gamma)}$$

$$\le c_p\left(\frac12\|\frac{\partial^2 a}{\partial y^2}(x,\tilde y_k)(y_{k,\theta}-\bar y)^2\|_{L^1(\Omega)} + \|\theta_k(u_k-\bar u)(y_{k,\theta}-\bar y)\|_{L^1(\Gamma)}\right)$$

$$\le c_p\left(C\left(\|e_k\|_{L^2(\Omega)} + \|\theta_k z_k\|_{L^2(\Omega)}\right)^2 + \|(u_k-\bar u)e_k\|_{L^1(\Gamma)} + \|(u_k-\bar u)\theta_k z_k\|_{L^1(\Gamma)}\right)$$

$$\le c_p\left(2C\left(\|e_k\|^2_{L^2(\Omega)} + \|z_k\|^2_{L^2(\Omega)}\right) + \|u_k-\bar u\|_{L^{p'}(\Gamma)}\|e_k\|_{L^p(\Gamma)} + \|(u_k-\bar u)z_k\|_{L^1(\Gamma)}\right).$$

Taking into account that $\|e_k\|_{L^2(\Omega)} + \|u_k-\bar u\|_{L^{p'}(\Gamma)} \to 0$, we infer from the above inequality

$$\|e_k\|_{L^2(\Omega)} + \|e_k\|_{L^p(\Gamma)} \le c'_p\left(\|z_k\|^2_{L^2(\Omega)} + \|(u_k-\bar u)z_k\|_{L^1(\Gamma)}\right). \tag{5.12}$$

Now, we subtract the equations satisfied by $z_{k,\theta}$ and z_k and we obtain

$$\left\{\begin{aligned} A(z_{k,\theta}-z_k) + \frac{\partial a}{\partial y}(x,\bar y)(z_{k,\theta}-z_k) + \frac{\partial^2 a}{\partial y^2}(x,\check y_k)(y_{k,\theta}-\bar y)z_{k,\theta} &= 0 \quad \text{in } \Omega,\\ \partial_{\nu_A}(z_{k,\theta}-z_k) + u_{k,\theta}(z_{k,\theta}-z_k) + (u_k-\bar u)e_k &= 0 \quad \text{on } \Gamma,\end{aligned}\right.$$

where $\check y_k(x) = \bar y(x) + \iota_k(x)(y_{k,\theta}(x)-\bar y(x))$, with $0\le\iota_k(x)\le 1$ measurable, satisfies

$$\frac{\partial a}{\partial y}(x,y_{k,\theta}(x)) - \frac{\partial a}{\partial y}(x,\bar y(x)) = \frac{\partial^2 a}{\partial y^2}(x,\check y_k(x))(y_{k,\theta}(x)-\bar y(x)).$$

Once again, arguing as above we get

$$\|z_{k,\theta}-z_k\|_{L^2(\Omega)} + \|z_{k,\theta}-z_k\|_{L^p(\Gamma)}$$

$$\leq c_p\left(\|\frac{\partial^2 a}{\partial y^2}(x,\check{y}_k)(y_{k,\theta}-\bar{y})z_{k,\theta}\|_{L^1(\Omega)}+\|(u_k-\bar{u})e_k\|_{L^1(\Gamma)}\right)$$

$$\leq c_p\left[C\|(y_{k,\theta}-\bar{y})(z_{k,\theta}-z_k)+(e_k+\theta_k z_k)z_k\|_{L^1(\Omega)}+\|u_k-\bar{u}\|_{L^{p'}(\Gamma)}\|e_k\|_{L^p(\Gamma)}\right]$$

$$\leq c_pC\left(\|y_{k,\theta}-\bar{y}\|_{L^2(\Omega)}\|z_{k,\theta}-z_k\|_{L^2(\Omega)}+\|e_k\|_{L^2(\Omega)}\|z_k\|_{L^2(\Omega)}+\|z_k\|^2_{L^2(\Omega)}\right)$$

$$+c_p\|u_k-\bar{u}\|_{L^{p'}(\Gamma)}\|e_k\|_{L^p(\Gamma)}.$$

Now, taking into account that $\|y_{k,\theta}-\bar{y}\|_{L^2(\Omega)}\to 0$, (5.9) follows from this inequality and (5.12).

Finally, to prove (5.10) we proceed as follows

$$\|z_{k,\theta}\|^2_{L^2(\Omega)}\leq\left(\|z_{k,\theta}-z_k\|_{L^2(\Omega)}+\|z_k\|_{L^2(\Omega)}\right)^2\leq 2\left(\|z_{k,\theta}-z_k\|^2_{L^2(\Omega)}+\|z_k\|^2_{L^2(\Omega)}\right)$$

$$\leq\|z_{k,\theta}-z_k\|_{L^2(\Omega)}M_p\left(\|z_k\|^2_{L^2(\Omega)}+\|(u_k-\bar{u})z_k\|_{L^1(\Gamma)}\right)+2\|z_k\|^2_{L^2(\Omega)},$$

which implies (5.10). □

Proof of (5.5). From Corollary 2.3 we obtain

$$\begin{aligned}
&\left|\left(J_0''(u_{k,\theta})-J_0''(\bar{u})\right)(u_k-\bar{u})^2\right|\\
&\quad\leq\int_\Omega\left|[1-\varphi_{k,\theta}\frac{\partial^2 a}{\partial y^2}(x,y_{k,\theta})]z^2_{k,\theta}-[1-\bar{\varphi}\frac{\partial^2 a}{\partial y^2}(x,\bar{y})]z_k^2\right|dx\\
&\qquad+2\int_\Gamma|(u_k-\bar{u})z_{k,\theta}\varphi_{k,\theta}-(u_k-\bar{u})z_k\bar{\varphi}|dx\\
&\quad\leq\int_\Omega|z^2_{k,\theta}-z_k^2|\,dx+\int_\Omega|\bar{\varphi}-\varphi_{k,\theta}||\frac{\partial^2 a}{\partial y^2}(x,y_{k,\theta})|z^2_{k,\theta}\,dx\\
&\qquad+\int_\Omega|\bar{\varphi}||\frac{\partial^2 a}{\partial y^2}(x,y_{k,\theta})-\frac{\partial^2 a}{\partial y^2}(x,\bar{y})|z^2_{k,\theta}\,dx+\int_\Omega|\bar{\varphi}\frac{\partial^2 a}{\partial y^2}(x,\bar{y})||z^2_{k,\theta}-z^2_k|\,dx\\
&\qquad+2\int_\Gamma|\bar{\varphi}-\varphi_{k,\theta}||(u_k-\bar{u})z_k|\,dx+2\int_\Gamma|\bar{\varphi}_{k,\theta}||(u_k-\bar{u})||z_{k,\theta}-z_k|\,dx.
\end{aligned}$$

Let us estimate all these terms. For the first term we use (5.9) as follows

$$\int_\Omega|z^2_{k,\theta}-z^2_k|\,dx\leq\|z_{k,\theta}+z_k\|_{L^2(\Omega)}\|z_{k,\theta}-z_k\|_{L^2(\Omega)}$$

$$\leq M_p\|z_{k,\theta}+z_k\|_{L^2(\Omega)}\left(\|z_k\|^2_{L^2(\Omega)}+\|(u_k-\bar{u})z_k\|_{L^1(\Gamma)}\right),$$

with $\|z_{k,\theta}+z_k\|_{L^2(\Omega)}\leq\|z_{k,\theta}\|_{L^2(\Omega)}+\|z_k\|_{L^2(\Omega)}\to 0$ when $k\to\infty$.

The second term is estimated with (5.10)

$$\int_\Omega|\bar{\varphi}-\varphi_{k,\theta}||\frac{\partial^2 a}{\partial y^2}(x,y_{k,\theta})|z^2_{k,\theta}\,dx\leq\|\bar{\varphi}-\varphi_{k,\theta}\|_{L^\infty(\Omega)}\|\frac{\partial^2 a}{\partial y^2}(x,y_{k,\theta})\|_{L^\infty(\Omega)}\|z_{k,\theta}\|^2_{L^2(\Omega)}$$

$$\leq C\|\bar\varphi - \varphi_{k,\theta}\|_{L^\infty(\Omega)} K_p \left(\|z_k\|^2_{L^2(\Omega)} + \|(u_k - \bar u) z_k\|_{L^1(\Gamma)}\right),$$

with $\|\bar\varphi - \varphi_{k,\theta}\|_{L^\infty(\Omega)} \to 0$ when $k \to \infty$. For the next term we use again (5.10) and obtain

$$\int_\Omega |\bar\varphi| |\frac{\partial^2 a}{\partial y^2}(x, y_{k,\theta}) - \frac{\partial^2 a}{\partial y^2}(x, \bar y)| z^2_{k,\theta}\, dx$$

$$\leq \|\bar\varphi\|_{L^\infty(\Omega)} \|\frac{\partial^2 a}{\partial y^2}(x, y_{k,\theta}) - \frac{\partial^2 a}{\partial y^2}(x, \bar y)\|_{L^\infty(\Omega)} K_p \left(\|z_k\|^2_{L^2(\Omega)} + \|(u_k - \bar u) z_k\|_{L^1(\Gamma)}\right),$$

and (2.2) implies

$$\|\frac{\partial^2 a}{\partial y^2}(x, y_{k,\theta}) - \frac{\partial^2 a}{\partial y^2}(x, \bar y)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } k \to \infty.$$

Recalling that $\bar\varphi \frac{\partial^2 a}{\partial y^2}(x, \bar y) \in L^\infty(\Omega)$, we can estimate the fourth term as the first one. Let us estimate the following term

$$\int_\Gamma |\bar\varphi - \varphi_{k,\theta}| |(u_k - \bar u) z_k|\, dx \leq \|\bar\varphi - \varphi_{k,\theta}\|_{L^\infty(\Omega)} \|(u_k - \bar u) z_k\|_{L^1(\Gamma)}.$$

Finally, we obtain for the last term with (5.9)

$$\int_\Gamma |\bar\varphi_{k,\theta}| |(u_k - \bar u)| |z_{k,\theta} - z_k|\, dx \leq \|\bar\varphi_{k,\theta}\|_{L^\infty(\Omega)} \|u_k - \bar u\|_{L^{p'}(\Gamma)} \|z_{k,\theta} - z_k\|_{L^p(\Gamma)}$$

$$\leq C\|u_k - \bar u\|_{L^{p'}(\Gamma)} M_p \left(\|z_k\|^2_{L^2(\Omega)} + \|(u_k - \bar u) z_k\|_{L^1(\Gamma)}\right),$$

with $\|u_k - \bar u\|_{L^{p'}(\Gamma)} \to 0$ when $k \to \infty$. All these estimates prove the existence of some $k_\delta \geq k_\tau$ such that (5.5) holds. ∎

Corollary 5.4. *Under the assumptions of Theorem 5.1, there exist $\rho, \kappa > 0$ such that*

$$J_0(\bar u) + \frac{\kappa}{2}\|y_u - \bar y\|^2_{L^2(\Omega)} \leq J_0(u) \quad \forall u \in \mathbb{K} \cap B_\rho(\bar u). \tag{5.13}$$

Proof. Let ρ and τ as in Theorem 5.1. We proceed as in the proof of Lemma 5.2. Given $u \in \mathbb{K} \cap B_\rho(\bar u)$, we set $y_u = G(u)$, $z_{u-\bar u} = G'(\bar u)(u - \bar u)$ and $e = y_u - \bar y - z_{u-\bar u}$, then we have

$$\begin{cases} Ae + \dfrac{\partial a}{\partial y}(x, \bar y) e + \dfrac{1}{2}\dfrac{\partial^2 a}{\partial y^2}(x, \tilde y)(y_u - \bar y)^2 = 0 & \text{in } \Omega, \\ \partial_{\nu_A} e + \bar u e + (u - \bar u)(y_u - \bar y) = 0 & \text{on } \Gamma, \end{cases}$$

where $\tilde y(x) = \bar y(x) + \eta(x)(y_u(x) - \bar y(x))$, with $0 \leq \eta(x) \leq 1$ measurable. Now, as in the proof of (5.12), we get

$$\|e\|_{L^2(\Omega)} \leq C\left(\|z_{u-\bar u}\|^2_{L^2(\Omega)} + \|(u - \bar u) z_{u-\bar u}\|_{L^1(\Gamma)}\right)$$

for all $u \in \mathbb{K} \cap B_{\rho'}(\bar u)$ with $\rho' > 0$. Let $\rho > 0$ be given by Theorem 4.1 and take $0 < \hat\rho \le \min(\rho, \rho)$ such that

$$\|z_{u-\bar u}\|^2_{L^2(\Omega)} + \|(u-\bar u)z_{u-\bar u}\|_{L^1(\Gamma)} \le 1 \quad \forall u \in \mathbb{K} \cap B_{\hat\rho}(\bar u).$$

Hence,

$$\begin{aligned}\|y_u - \bar y\|^2_{L^2(\Omega)} &\le \big(\|e\|_{L^2(\Omega)} + \|z_{u-\bar u}\|_{L^2(\Omega)}\big)^2 \\ &\le \Big\{C\Big(\|z_{u-\bar u}\|^2_{L^2(\Omega)} + \|(u-\bar u)z_{u-\bar u}\|_{L^1(\Gamma)}\Big) + \|z_{u-\bar u}\|_{L^2(\Omega)}\Big\}^2 \\ &\le C'\Big(\|z_{u-\bar u}\|^2_{L^2(\Omega)} + \|(u-\bar u)z_{u-\bar u}\|_{L^1(\Gamma)}\Big).\end{aligned}$$

Now, taking $0 < \kappa \le \sigma/C'$, we deduce (5.13) from Theorem 5.1. □

We finish this section by proving that the condition *2* of Theorem 4.4 does not hold if $\gamma = 0$, and $\bar u \not\equiv \alpha$ and $\bar u \not\equiv \beta$. To this end we proceed by contradiction. Let us assume that *2* holds. Then, arguing as in the proof of Theorem 5.1, we deduce the existence of $\sigma, \rho > 0$ such that

$$J_0(\bar u) + \frac{\sigma}{2}\|u - \bar u\|^2_{L^2(\Gamma)} \le J_0(u) \quad \forall u \in \mathbb{K} \cap B_\rho(\bar u). \tag{5.14}$$

We consider the control problem

$$\text{(Q)} \quad \min_{\alpha \le u \le \beta} I(u) = J_0(u) - \frac{\sigma}{2}\|u - \bar u\|^2_{L^2(\Gamma)}.$$

From (5.14) we deduce that $\bar u$ is a local minimum of problem (Q). The Hamiltonian of this control problem is $H : \Gamma \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$H(x, y, \varphi, u) = -\frac{\sigma}{2}(u - \bar u(x))^2 - \varphi y u.$$

Pontryagin's principle says that

$$H(x, \bar y(x), \bar\varphi(x), \bar u(x)) = \min_{t \in [\alpha,\beta]} H(x, \bar y(x), \bar\varphi(x), t) \quad \text{for a.a. } x \in \Gamma. \tag{5.15}$$

Let us prove that (5.15) is false. This contradiction implies that *2* cannot hold. First, we observe that the set of points Γ_0 of Γ where $\bar\varphi\bar y$ vanishes has a zero measure. Indeed, if $\bar\varphi(x)\bar y(x) = 0$, then

$$H(x, \bar y(x), \bar\varphi(x), \bar u(x)) = 0 > \min_{t \in [\alpha,\beta]} -\frac{\sigma}{2}(t - \bar u(x))^2 = \min_{t \in [\alpha,\beta]} H(x, \bar y(x), \bar\varphi(x), t),$$

which contradicts (5.15) except if the measure of Γ_0 is zero.

On the other hand, (3.5), $\bar u \not\equiv \alpha$ and $\bar u \not\equiv \beta$ imply that the sign of $\bar\varphi\bar y$ is not constant on Γ. Since the functions $\bar\varphi$ and $\bar y$ are continuous, we deduce that $\bar\varphi\bar y$ vanishes at least at one point of Γ. Consequently, for every $\varepsilon > 0$ there exists an open set $\Gamma_\varepsilon \subset \Gamma$, with a strict positive measure, such that $0 < |\bar\varphi(x)\bar y(x)| < \varepsilon$ for every $x \in \Gamma_\varepsilon$. Let us take $\varepsilon = \sigma(\beta - \alpha)/2$. Given $x \in \Gamma_\varepsilon$ we distinguish two cases.

Case I.- $\bar\varphi(x)\bar y(x) > 0$. Then, (3.5) shows that $\bar u(x) = \beta$. Using that $0 < \bar\varphi(x)\bar y(x) < \sigma(\beta - \alpha)/2$, it is easy to check

$$H(x, \bar y(x), \bar\varphi(x), \alpha) < H(x, \bar y(x), \bar\varphi(x), \beta) = H(x, \bar y(x), \bar\varphi(x), \bar u(x)),$$

which contradicts (5.15).

Case II.- $\bar\varphi(x)\bar y(x) < 0$. Now, (3.5) implies that $\bar u(x) = \alpha$. Using that $-\sigma(\beta - \alpha)/2 < \bar\varphi(x)\bar y(x) < 0$, we deduce

$$H(x, \bar y(x), \bar\varphi(x), \beta) < H(x, \bar y(x), \bar\varphi(x), \alpha) = H(x, \bar y(x), \bar\varphi(x), \bar u(x)),$$

which again contradicts (5.15).

6. Stability analysis with respect to $\gamma \searrow 0$. In this section we investigate the stability of solutions of (P_γ) for $\gamma \searrow 0$.

THEOREM 6.1. *Let $(u_\gamma)_{\gamma>0}$ be a family of global solutions of (P_γ) such that $u_\gamma \rightharpoonup u_0$ for $\gamma \searrow 0$ in $L^2(\Gamma)$. Then u_0 is a global solution of (P_0) and $\|u_\gamma - u_0\|_{L^2(\Gamma)} \to 0$. Moreover, the following identity holds*

$$\|u_0\|_{L^2(\Gamma)} = \min\{\|u\|_{L^2(\Gamma)} : \ u \text{ is a global solution of } (\mathrm{P}_0)\}. \tag{6.1}$$

Proof. Let $u \in \mathbb{K}$ be given. Then it holds $J_\gamma(u_\gamma) \le J_\gamma(u)$ for all $\gamma > 0$. Since the family (u_γ) is uniformly bounded in $L^\infty(\Gamma)$, we obtain $u_\gamma \rightharpoonup u_0$ in $L^{\bar q}(\Gamma)$. Then, Lemma 2.6 implies

$$J_0(u_0) = \lim_{\gamma\searrow 0} J_0(u_\gamma) = \lim_{\gamma\searrow 0} J_\gamma(u_\gamma) \le \lim_{\gamma\searrow 0} J_\gamma(u) = J_0(u).$$

Since $u \in \mathbb{K}$ was arbitrary, it follows that u_0 is a global solution of (P_0). Let us prove the strong convergence $u_\gamma \to u_0$. On one side, we have that

$$\|u_0\|_{L^2(\Gamma)} \le \liminf_{\gamma\to 0} \|u_\gamma\|_{L^2(\Gamma)} \le \limsup_{\gamma\to 0} \|u_\gamma\|_{L^2(\Gamma)}. \tag{6.2}$$

On the other hand, using that u_0 is a global solution of (P_0), we have

$$J_0(u_\gamma)+\gamma\|u_\gamma\|_{L^2(\Gamma)} = J_\gamma(u_\gamma) \le J_\gamma(u_0) = J_0(u_0)+\gamma\|u_0\|_{L^2(\Gamma)} \le J_0(u_\gamma)+\gamma\|u_0\|_{L^2(\Gamma)},$$

which implies that $\|u_\gamma\|_{L^2(\Gamma)} \le \|u_0\|_{L^2(\Gamma)}$ for every γ. This inequality and (6.2) proves that $\|u_\gamma\|_{L^2(\Gamma)} \to \|u_0\|_{L^2(\Gamma)}$. This convergence and the weak convergence of $u_k \rightharpoonup u_0$ imply the strong convergence.

Finally, we prove (6.1). Let u be a global solution of (P_0), then

$$J_0(u_\gamma) + \frac{\gamma}{2}\|u_\gamma\|^2_{L^2(\Gamma)} = J_\gamma(u_\gamma) \le J_\gamma(u) = J_0(u) + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)}$$

$$= J_0(u_0) + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \le J_0(u_\gamma) + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)}.$$

Hence, $\|u_\gamma\|_{L^2(\Gamma)} \le \|u\|_{L^2(\Gamma)}$, and

$$\|u_0\|_{L^2(\Gamma)} = \lim_{\rho\to 0} \|u_\gamma\|_{L^2(\Gamma)} \le \|u\|_{L^2(\Gamma)},$$

which concludes the proof. ◻

THEOREM 6.2. *Let $\bar u$ be a strict local solution of (P_0). Then there exist $\rho > 0$ and a family $(u_\gamma)_{\gamma\in(0,\bar\gamma)}$ of local solutions of (P_γ) such that $u_\gamma \to \bar u$ in $L^2(\Gamma)$ and every u_γ is a global minimum of J_γ in $\mathbb{K}_\rho := \mathbb{K} \cap \{v \in L^2(\Gamma) : \ \|v - \bar u\|_{L^2(\Gamma)} \le \rho\}$.*

Proof. Let $\rho > 0$ be such that $\bar u$ is the unique global minimum of J_0 in the set $\mathbb{K}_\rho$. Let us investigate the following auxiliary problem: minimize $J_\gamma(u)$ for $u \in \mathbb{K}_\rho$. For every $\gamma > 0$ let $u_{\rho,\gamma}$ be a global solution of this auxiliary problem. By construction, the family $(u_{\rho,\gamma})$ is uniformly bounded in $L^\infty(\Gamma)$. Hence we find a sequence $\gamma_k \to 0$ such that $u_{\rho,\gamma_k} \rightharpoonup u_0$ in $L^2(\Gamma)$. Arguing as in the proof of Theorem 6.1, it follows that u_0 is a global minimum of J_0 on $\mathbb{K}_\rho$ and $\|u_{\rho,\gamma_k} - u_0\|_{L^2(\Gamma)} \to 0$. Hence it follows $u_0 = \bar{u}$, and $\lim_{\gamma\searrow 0} u_{\rho,\gamma} = \bar{u}$ strongly in $L^2(\Gamma)$ for the whole family $(u_{\rho,\gamma})$. This implies that there is $\bar{\gamma}$ such that $\|u_{\rho,\gamma} - \bar{u}\|_{L^2(\Gamma)} < \rho$ for all $\gamma < \bar{\gamma}$. Thus, the controls $u_{\rho,\gamma}$ are local and global minima of J_γ on $\mathbb{K}$ and $\mathbb{K}_\rho$, respectively, for all $\gamma < \bar{\gamma}$. □

Let us observe the following property. If $\hat{u}_0$ is global minimum of (P$_0$) such that $\|\hat{u}_0\|_{L^2(\Gamma)} > \|u_0\|_{L^2(\Gamma)}$, then (6.1) implies that there is no a sequence (u_γ) of global minima of problems (P$_\gamma$) converging to $\hat{u}_0$. However, Theorem 6.2 proves that $\hat{u}_0$ can be approximated by local minima of problems (P$_\gamma$).

Given a local minimum u_γ of (P$_\gamma$), we set $y_\gamma = y_{u_\gamma}$. Now we estimate $y_\gamma - \bar{y}$.

THEOREM 6.3. *Let $\bar{u}$ and $(u_\gamma)_{\gamma\in(0,\bar{\gamma})}$ be as in Theorem 6.2. Let us assume that* (5.1) *is satisfied. Then, the following identity holds*

$$\lim_{\gamma\to 0} \frac{\|y_\gamma - \bar{y}\|_{L^2(\Omega)}}{\sqrt{\gamma}} = 0. \tag{6.3}$$

Proof. Using (5.13) and the fact that $J_\gamma(u_\gamma) \le J_\gamma(\bar{u})$ we get

$$J_0(\bar{u}) + \frac{\kappa}{2}\|y_\gamma - \bar{y}\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u_\gamma\|_{L^2(\Gamma)} \le J_0(u_\gamma) + \frac{\gamma}{2}\|u_\gamma\|_{L^2(\Gamma)} = J_\gamma(u_\gamma)$$

$$\le J_\gamma(\bar{u}) = J_0(\bar{u}) + \frac{\gamma}{2}\|\bar{u}\|^2_{L^2(\Gamma)}.$$

As in the proof of Theorem 6.1, we have that $\|u_\gamma\|_{L^2(\Gamma)} \le \|\bar{u}\|_{L^2(\Gamma)}$. Using this in the above inequality we get

$$\frac{\kappa}{2}\|y_\gamma - \bar{y}\|^2_{L^2(\Omega)} \le \frac{\gamma}{2}\left(\|\bar{u}\|^2_{L^2(\Gamma)} - \|u_\gamma\|^2_{L^2(\Gamma)}\right) \le \gamma\|\bar{u}\||_{L^2(\Gamma)}\|u_\gamma - \bar{u}\|_{L^2(\Gamma)}.$$

Hence

$$\frac{\|y_\gamma - \bar{y}\|_{L^2(\Omega)}}{\sqrt{\gamma}} \le \left(\frac{2}{\kappa}\|\bar{u}\|_{L^2(\Gamma)}\|u_\gamma - \bar{u}\|_{L^2(\Gamma)}\right)^{1/2} \longrightarrow 0 \ \text{ as } \gamma \to 0,$$

which concludes the proof. □

Remark 6.4. *Let us remark that it is an open question to prove convergence rates for the error in the controls. In [9, 10], a strengthened complementarity condition was used to prove such error estimates. However, the analysis heavily relied on the fact that the controls enter linearly into the equation. Transferred to the problem under consideration, this condition would read: there exist constants $C > 0$ and $0 < \sigma \le 1$ such that*

$$\text{meas}\{x \in \Gamma : \ |\bar{y}(x)\bar{\varphi}(x)| < \epsilon\} \le C\epsilon^\sigma \quad \forall \epsilon > 0. \tag{6.4}$$

This condition implies that the set $\{x \in \Gamma : \ \bar{y}(x)\bar{\varphi}(x) = 0\}$ has zero measure. Hence by (3.3)*, the control $\bar{u}$ is of bang-bang type, i.e. $\bar{u}(x) \in \{\alpha, \beta\}$ almost everywhere. Arguing as in the proof of Theorem 6.3 above and using the growth condition of Theorem 5.1, one finds the estimate*

$$\frac{\kappa}{2}\left(\|y_\gamma - \bar{y}\|^2_{L^2(\Omega)} + \|z_{u_\gamma-\bar{u}}\|^2_{L^2(\Omega)} + \|(u_\gamma - \bar{u})z_{u_\gamma-\bar{u}}\|_{L^1(\Gamma)}\right) + \frac{\gamma}{2}\|u_\gamma - \bar{u}\|^2_{L^2(\Gamma)} \le \int_\Gamma \bar{u}(u_\gamma - \bar{u})\,dx.$$

Applying (6.4) *one finds [10, Corollary 3.2, Lemma 3.3],*

$$\frac{\kappa}{2}\left(\|y_\gamma-\bar y\|^2_{L^2(\Omega)}+\|z_{u_\gamma-\bar u}\|^2_{L^2(\Omega)}+\|(u_\gamma-\bar u)z_{u_\gamma-\bar u}\|_{L^1(\Gamma)}\right)+\frac{\gamma}{2}\|u_\gamma-\bar u\|^2_{L^2(\Gamma)} \le c\,\gamma\left(\gamma^\sigma+\|\bar y\bar\varphi-y_\gamma\varphi_\gamma\|^\sigma_{L^\infty(\Gamma)}\right)$$

Here, one needs to estimate $\|\bar y-y_\gamma\|_{L^\infty(\Gamma)}$ *and* $\|\bar\varphi-\varphi_\gamma\|_{L^\infty(\Gamma)}$ *against* $\|y_\gamma-\bar y\|_{L^2(\Omega)}$, $\|z_{u-\bar u}\|_{L^2(\Omega)}$, *and* $\|(u-\bar u)z_{u-\bar u}\|_{L^1(\Gamma)}$ *to be able to conclude by Young's inequality. Such an estimate is not available, unfortunately.*

7. Extension to a distributed control problem. Let us briefly discuss the possible extension to a problem with distributed control, which is given as

$$(\mathrm{P}_\gamma) \qquad \min_{\alpha\le u\le\beta} J_\gamma(u)=\frac{1}{2}\|y_u-y_d\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\|u\|^2_{L^2(\Omega)}, \tag{7.1}$$

where $0<\alpha<\beta<+\infty$, $\gamma\ge 0$, $y_d\in L^2(\Omega)$, and y_u is the solution of the semilinear state equation

$$\left\{\begin{aligned} Ay+a(x,y)+uy=0 &\quad \text{in } \Omega,\\ \partial_{\nu_A}y=g &\quad \text{on } \Gamma.\end{aligned}\right. \tag{7.2}$$

Under the assumption in Section 2, the equation (7.2) admits a unique solution y_u for every admissible control u. Moreover, it can be proven that the optimal control problem (7.1) is solvable. Let $\bar u$ be a local solution of (P_γ) with associated state $\bar y:=y_{\bar u}$. Then there exists $\bar\varphi:=\varphi_{\bar u}$ such that the following system is satisfied:

$$\left\{\begin{aligned} A\bar y+a(x,\bar y)+\bar u\bar y=0 &\quad \text{in } \Omega,\\ \partial_{\nu_A}\bar y=g &\quad \text{on } \Gamma,\end{aligned}\right. \tag{7.3}$$

$$\left\{\begin{aligned} A^*\bar\varphi+\frac{\partial a}{\partial y}(x,\bar y)\bar\varphi+\bar u\bar\varphi=\bar y-y_d &\quad \text{in } \Omega,\\ \partial_{\nu_{A^*}}\bar\varphi=0 \qquad\quad &\quad \text{on } \Gamma,\end{aligned}\right. \tag{7.4}$$

$$\int_\Omega(\gamma\bar u-\bar y\bar\varphi)(u-\bar u)dx\ge 0 \quad \forall u\in\mathbb{K}. \tag{7.5}$$

The second-order sufficient condition for the case $\gamma=0$ reads as follows. Let $\bar u$ be an admissible control satisfying

$$J_0'(\bar u)(u-\bar u)=\int_\Omega -\bar y\bar\varphi(u-\bar u)dx\ge 0 \quad \forall u\in\mathbb{K}$$

together with

$$J_0''(\bar u)v^2=\int_\Omega\left\{\left(1-\frac{\partial^2 a}{\partial y^2}(x,\bar y)\bar\varphi\right)z_v-2vz_v\bar\varphi\right\}dx\ge\delta(\|z_v\|^2_{L^2(\Omega)}+\|vz_v\|_{L^1(\Omega)})$$

for all $v\in C^\tau_{0,\bar u}$,

$$C^\tau_{0,\bar u}:=\{v\in L^2(\Omega):\ v(x)\ge 0 \text{ if } \bar u(x)=\alpha,\ v(x)\le 0 \text{ if } \bar u(x)=\beta,\ J_0'(\bar u)v\le\tau\|v\|_{L^2(\Omega)}\},$$

where δ and τ are positive numbers. Then one can prove (compare Theorem 5.1 and Corollary 5.4) that $\bar{u}$ is locally optimal for (P_0), and it holds

$$J_0(u) - J_0(\bar{u}) \geq \frac{\kappa}{2}\|y_u - \bar{y}\|^2_{L^2(\Omega)}$$

for all $u \in \mathbb{K}$ in a $L^2(\Omega)$-neighborhood of $\bar{u}$.

REFERENCES

[1] J.F. Bonnans. Second-order analysis for control constrained optimal control problems of semilinear elliptic systems. *Appl. Math. Optim.*, 38:303–325, 1998.

[2] E. Casas. Second order analysis for bang-bang control problems of PDEs. *SIAM J. Control Optim.*, 50(4):2355–2372, 2012.

[3] E. Casas and F. Tröltzsch. Second order analysis for optimal control problems: improving results expected from abstract theory. *SIAM J. Optim.*, 22(1):261–279, 2012.

[4] E. Casas and F. Tröltzsch. Second-order and stability analysis for state-constrained elliptic optimal control problems with sparse controls. *SIAM J. Control Optim.*, In press.

[5] K. Deckelnick and M. Hinze. A note on the approximation of elliptic control problems with bang-bang controls. *Comput. Optim. Appl.*, 51(2):931–939, 2012.

[6] J. C. Dunn. Second-order optimality conditions in sets of L^∞ functions with range in a polyhedron. *SIAM J. Control Optim.*, 33(5):1603–1635, 1995.

[7] R. Haller-Dintelmann, C. Meyer, J. Rehberg, and A. Schiela. Hölder continuity and optimal control for nonsmooth elliptic problems. *Appl. Math. Optim.*, 60(3):397–428, 2009.

[8] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15:189–258, 1965.

[9] G. Wachsmuth and D. Wachsmuth. Convergence and regularization results for optimal control problems with sparsity functional. *ESAIM Control Optim. Calc. Var.*, 17(3):858–886, 2011.

[10] G. Wachsmuth and D. Wachsmuth. On the regularization of optimization problems with inequality constraints. *Control and Cybernetics*, 4:1125–1154, 2011.