La stationnarité forte n'est pas une condition d'optimalité nécessaire pour le problème d'obstacle avec des contrôle sur la frontière

Strong stationarity is not a necessary optimality condition for boundary control of the obstacle problem

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Sommaire. Dans cette note nous investiguons les conditions d'optimalité pour le problème obstacle avec des contrôles à la frontiere. Par un contrexemple nous démontrons que les conditions de stationnarité dites fortes ne sont pas nécessaires, même en cas d'un domaine de dimension une.

Abstract. In this short note, we investigate necessary optimality conditions for boundary control of the obstacle problem. By means of a counterexample, we show that the so-called strong stationarity conditions are not necessary for local optimality, even in the case of a one-dimensional domain.

Keywords. Variational inequalities, optimal control, boundary control, strong stationarity

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1 Introduction

We investigate the following optimal control problem:

$$\min J(y,u) := g(y) + j(u) \tag{1.1}$$

among all $u \in L^2(\Gamma)$ and $y \in K$ satisfying the variational inequality

$$\langle Ay,\, v - y\rangle \geq \int_\Gamma u(v-y)\,\mathrm{d}\gamma \quad \forall\, v \in K \tag{1.2}$$

with the admissible set K given by

$$K = \{v \in V :\ v \leq \psi \text{ in } \Omega\}.$$

Herein, $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\Gamma \subseteq \partial\Omega$ is a part of its boundary with positive boundary measure. Moreover, V is a closed subspace of $H^1(\Omega)$ and the operator $A : V \to V^*$ is assumed to be linear, bounded, and coercive. The obstacle ψ is a given function in V. Moreover, the inequality $v \leq \psi$ in Ω is to be understood in an almost-everywhere sense and the same holds for subsequent inequalities involving functions in Lebesgue spaces unless a different meaning is explicitly mentioned. Furthermore, $g : V \to \mathbb{R}$ and $j : L^2(\Gamma) \to \mathbb{R}$ are continuously Fréchet-differentiable.

Introducing a multiplier $\lambda \in V^*$ for the constraint $y \in K$, the variational inequality can be written equivalently as a system of complementarity conditions

$$\langle Ay,\, v\rangle + \langle \lambda, v\rangle = \int_\Gamma uv\,\mathrm{d}\gamma \quad \forall\, v \in V, \tag{1.3a}$$

$$y \leq \psi, \tag{1.3b}$$

$$\lambda \geq 0, \tag{1.3c}$$

$$\langle \lambda, y - \psi\rangle = 0. \tag{1.3d}$$

Here, the inequality $\lambda \geq 0$ has to be understood as $\langle \lambda, v \rangle \geq 0$ for all $v \in V$ with $v \geq 0$. If V is equal to $H^1(\Omega)$ or $H_0^1(\Omega)$ then the tangent cone to K at $y \in K$ is given by

$$T_K(y) = \{v \in V : \ v \leq 0 \text{ quasi-everywhere on } \mathcal{A}\},$$

see e.g. [1, 13]. Here, $\mathcal{A}$ is the coincidence set defined by

$$\mathcal{A} = \{x \in \Omega : y(x) = \psi(x)\}, \tag{1.4}$$

which is defined up to sets of zero capacity.

The conditions (1.3c)–(1.3d) can be written as $\lambda \in T_K(y)^o$, [13, Prop. 2.5], where, C^o denotes the polar cone of C.

Let us now formally introduce the concept of strong stationarity to the problem under consideration. There the following set will play an important role,

$$S(y, \lambda) := \{v \in V : v \in T_K(y), \ \langle \lambda, v \rangle = 0\} = T_K(y) \cap \{\lambda\}^\perp.$$

Following the exposition in [12, 13], a feasible point (y, u, λ) satisfying (1.3) is called *strong stationary* for the optimal control problem (1.1),(1.3) if there exists multipliers $p \in V$, $\mu \in V^*$ fulfilling the following system of conditions

$$A^* p + \mu = g'(y) \quad \text{in } V^*, \tag{1.5a}$$

$$p \in S(y, \lambda), \tag{1.5b}$$

$$-\mu \in S(y, \lambda)^o, \tag{1.5c}$$

$$j'(u) + p = 0. \tag{1.5d}$$

Conditions (1.5b)(1.5c) allow to *formally* derive sign conditions on p, μ. One obtains

$$p = 0 \text{ on } \{y = \psi, \ \lambda > 0\}, \tag{1.6a}$$

$$p \leq 0, \mu \leq 0 \text{ on } \{y = \psi, \ \lambda = 0\}, \tag{1.6b}$$

$$\mu = 0 \text{ on } \{y < \psi, \ \lambda = 0\}. \tag{1.6c}$$

The aim of this short note is to demonstrate that strong stationarity is not necessary for local optimality of (1.1), (1.2). Let us briefly comment on related results in the literature.

From finite-dimensional optimization, it is well-known, that the Karush-Kuhn-Tucker (KKT) conditions associated with an optimization problem with complementarity constraints are equivalent to strong stationarity. Under suitable constraint qualifications one can show that strong stationarity is a necessary condition, see e.g. [2, 10].

Much less general results are known for infinite-dimensional optimal control problems. In the case of distributed control with control acting on the whole domain, Mignot [7] and Mignot and Puel [8] showed that strong stationarity is necessary for local optimality. Outrata, Jarušek, and Stará [9] proved M-stationarity for distributed control and control constraints for the one-dimensional case $n = 1$, where the analysis relied on the embedding of $H^1(\Omega)$ into $C(\bar{\Omega})$. In a recent contribution, Gerd Wachsmuth [13] investigated strong stationarity for distributed control problems with control constraints. Under suitable conditions on the control constraints and active sets, strong stationarity was proven to be necessary. In the context of optimal control of static plasticity, strong stationarity was obtained by Herzog, Meyer, and Gerd Wachsmuth [3]. We mention also the results of Gerd Wachsmuth [12], who investigated programming problems with complementarity constraints in Banach spaces.

A lot of work was devoted to the study of regularization or smoothing schemes for the optimal control problem (1.1), (1.3). Convergence of solutions of regularized problems were studied, and their limits were investigated. Let us mention only [4, 5, 6, 11]. In the analysis, it turns out that the most challenging condition in proving strong stationarity is condition (1.6b). The investigated limiting procedures can at best only provide that the product of p and μ is non-negative.

To the best of our knowledge strong stationarity conditions for boundary control of the obstacle problem have not been investigated so far. At the end of the fundamental work [8] the authors raised the question whether strong stationarity is also necessary for local optimality if the set of admissible controls is not the whole space $L^2(\Omega)$. With this short note we aim to show that this is indeed not the case if the controls act on the boundary.

2 A counterexample to strong stationarity for boundary control of the obstacle problem

Let us now demonstrate that the strong stationarity system (1.3)–(1.5) is not necessary for local optimality for the optimal control problem (1.1)–(1.2). To this end, we consider the following one dimensional optimal control problem:

$$
\left\{
\begin{aligned}
&\min \quad J(y,u) := \frac{1}{2}\int_0^{\frac{1}{2}} (y - y_d)^2 \,\mathrm{d}x - \int_{\frac{1}{2}}^1 y \,\mathrm{d}x + \frac{1}{2}\,u^2 \\
&\text{s.t.} \quad \int_0^1 y'(v' - y') \,\mathrm{d}x \geq u\big(v(1) - y(1)\big) \\
&\qquad\qquad \forall\, v \in K := \{v \in H^1(0,1) : v(0) = 0, v \leq 0 \text{ in } \Omega\} \\
&\qquad y \in K,\ u \in \mathbb{R}.
\end{aligned}
\right.
\tag{2.1}
$$

Herein we choose

$$
y_d(x) := \begin{cases} -2, & x \leq \frac{1}{2}, \\ 0, & x > \frac{1}{2}. \end{cases}
$$

With

$$
V := \{v \in H^1(0,1) : v(0) = 0\} \quad \text{and} \quad \langle Ay,\, v\rangle := \int_0^1 y'(x)\, v'(x) \,\mathrm{d}x,
$$

problem (2.1) fits into the general setting of (1.1)–(1.2). By introducing a slack variable $\lambda \in V^*$ the VI in (2.1) is equivalent to

$$
\begin{aligned}
\langle Ay + \lambda, v\rangle &= \int_\Gamma uv \,\mathrm{d}\gamma \quad \forall v \in V \\
y \le 0,\ \lambda &\ge 0,\ \langle y\,,\,\lambda\rangle = 0.
\end{aligned}
\tag{2.2}
$$

The strong stationarity conditions (1.3)–(1.5) read as follows for this example:

$$y \in K, \quad \lambda \in V^*, \quad p \in V, \quad \mu \in V^* \tag{2.3a}$$

$$\langle Ay + \lambda, v\rangle = \int_\Gamma uv \,\mathrm{d}\gamma \quad \forall v \in V \tag{2.3b}$$

$$y \le 0,\ \lambda \ge 0,\ \langle y\,,\,\lambda\rangle = 0 \tag{2.3c}$$

$$\int_0^1 p'v' \,\mathrm{d}x = \int_0^{\frac{1}{2}} (y - y_d)v \,\mathrm{d}x - \int_{\frac{1}{2}}^1 v \,\mathrm{d}x - \langle \mu\,,\, v\rangle \quad \forall v \in V \tag{2.3d}$$

$$p(1) + u = 0 \tag{2.3e}$$

$$\langle p\,,\,\lambda\rangle = 0 \tag{2.3f}$$

$$p(x) \le 0 \quad \forall x \in \mathcal{A} \tag{2.3g}$$

$$\langle \mu\,,\, v\rangle \le 0 \quad \forall v \in V : \langle \lambda\,,\, v\rangle = 0,\, v(x) \ge 0\, \forall x \in \mathcal{A} \tag{2.3h}$$

with the active set $\mathcal{A}$ as given in (1.4). Due to the embedding $H^1(0,1) \hookrightarrow C([0,1])$, the set $\mathcal{A}$ can be defined by using the continuous representative of y, and thus the sign conditions in (2.3g) and (2.3h) hold for all $x \in \mathcal{A}$.

At first, we show that $(u^*, y^*, \lambda^*) = (0,0,0)$ is locally optimal for (2.1). If $u > 0$ on the right hand side of (2.2), then the maximum principle implies $y \equiv 0$ giving in turn

$$J(y,u) = \frac{1}{2} \int_0^{\frac{1}{2}} |y_d|^2 + \frac{1}{2}\, u^2 > J(0,0).$$

If on the other hand $u \le 0$, then the solution of (2.2) is given by $\lambda = 0$ and $y(x) = ux$. Hence we obtain

$$
\begin{aligned}
J(y,u) &= \frac{1}{2} \int_0^{\frac{1}{2}} (ux + 2)^2 \,\mathrm{d}x - \int_{\frac{1}{2}}^1 ux \,\mathrm{d}x + \frac{1}{2}\, u^2 \\
&= \int_0^{\frac{1}{2}} |y_d|^2 \,\mathrm{d}x + \left(\frac{1}{48} + \frac{1}{2}\right) u^2 + \left(\frac{3}{8} - \frac{1}{4}\right)(-u),
\end{aligned}
$$

which proves $J(y,u) \ge J(0,0)$. Thus, in both cases, we have $J(y,u) \ge J(0,0) = J(y^*, u^*)$, which proves that $(0,0,0)$ is a local minimum as claimed.

Therefore, if the strong stationarity system (2.3) would be necessary for local optimality, there would exist $p^* \in V$ and $\mu^* \in V^*$ so that (2.3) is satisfied for $(u^*, y^*, \lambda^*) = (0,0,0)$. Since $\mathcal{A} = [0,1]$, the equations (2.3d)–(2.3h) read in this case

$$\int_0^1 (p^*)'v'\,\mathrm{d}x = -\int_0^{\frac{1}{2}} y_d\, v\,\mathrm{d}x - \int_{\frac{1}{2}}^1 v\,\mathrm{d}x - \langle \mu^*\,,\, v\rangle \quad \forall\, v \in V \tag{2.4a}$$

$$p^*(1) = 0 \tag{2.4b}$$

$$p^*(x) \le 0 \quad \forall\, x \in [0,1] \tag{2.4c}$$

$$\langle \mu^*\,,\, v\rangle \le 0 \quad \forall\, v \in V : v(x) \ge 0\,\forall\, x \in [0,1]. \tag{2.4d}$$

To show that this assertion is wrong, we argue by contradiction, and assume that there exists $p^* \in V$ and $\mu^* \in V^*$ such that (2.4) is satisfied.

Let us now consider the following auxiliary problem:

$$\begin{cases} \quad \min \quad J(y,u) \\ \quad \text{s.t.} \quad \langle Ay + \lambda,\, v\rangle = \displaystyle\int_\Gamma uv\ \mathrm{d}\gamma \quad \forall\, v \in V \\ \qquad\qquad y(x) \le 0\ \forall\, x \in [0,1],\ \lambda \ge 0 \\ \qquad\qquad y \in V,\ u \in \mathbb{R},\ \lambda \in V^*. \end{cases} \tag{2.5}$$

In the context of programming problems with complementarity constraints this system constitutes the so-called relaxed NLP associated the original problem, see [10]. The KKT-conditions for this problem are given by

$$p \in V, \quad \mu \in V^* \tag{2.6a}$$

$$\int_0^1 p'v'\,\mathrm{d}x = \int_0^{\frac{1}{2}} (y - y_d)v\,\mathrm{d}x - \int_{\frac{1}{2}}^1 v\,\mathrm{d}x - \langle \mu\,,\, v\rangle \quad \forall\, v \in V \tag{2.6b}$$

$$p(1) + u = 0 \tag{2.6c}$$

$$p \le 0, \quad \langle \lambda\,,\, p\rangle = 0 \tag{2.6d}$$

$$\mu \le 0, \quad \langle \mu\,,\, y\rangle = 0. \tag{2.6e}$$

Note that we do not claim these conditions to be necessary for local optimality for (2.5). In fact, this is a delicate question, since it cannot be proven that the Slater condition holds due to the homogeneous boundary condition at $x = 0$. However, since (2.5) is a convex problem, classical arguments of convex optimization give that every feasible point satisfying (2.6) is a global optimum of (2.5), i.e., the sufficiency of (2.6).

Let us now show that $(u^*, y^*, \lambda^*, p^*, \mu^*)$, which is assumed to satisfy (2.3), is a KKT-point of (2.5). Clearly $(u^*, y^*, \lambda^*) = (0, 0, 0)$ is feasible for (2.5). Moreover, the KKT conditions (2.6) are equivalent to (2.4). Hence $(u^*, y^*, \lambda^*, p^*, \mu^*)$ is a KKT-point of (2.5). As (2.5) is a convex problem with strictly convex functional J, (u^*, y^*, λ^*) is therefore a global solution of (2.5). This however leads to a contradiction as we will see in the following. For this purpose, consider the point

$$\hat{y}(x) := \begin{cases} -\beta\, x, & x \le \frac{1}{2} \\ -\beta\,(1-x), & x > \frac{1}{2}, \end{cases}$$

$$\hat{u} := \hat{y}'(1) = \beta, \quad \hat{\lambda} := 2\beta \cdot \delta_{\frac{1}{2}} \in C([0,1])^* \hookrightarrow V^*,$$

where $\delta_{\frac{1}{2}}$ denotes the Dirac-measure concentrated at $x = \frac{1}{2}$ and $\beta > 0$ will be chosen afterwards. It is easily seen that $(\hat{u}, \hat{y}, \hat{\lambda})$ is feasible for (2.5). For the corresponding objective one finds

$$\begin{aligned} J(\hat{y}, \hat{u}) &= \frac{1}{2} \int_0^{\frac{1}{2}} (-\beta\, x + 2)^2 \,\mathrm{d}x + \int_{\frac{1}{2}}^1 \beta(1-x)\,\mathrm{d}x + \frac{\beta^2}{2} \\ &= J(0,0) + \frac{\beta^2}{48} - \frac{\beta}{4} + \frac{\beta}{8} + \frac{\beta^2}{2}. \end{aligned}$$

Thus, it holds $J(\hat{y}, \hat{u}) < J(0,0) = J(u^*, y^*)$ provided that $\beta(\frac{1}{48} + \frac{1}{2}) - \frac{1}{8} < 0$. This condition is for instance fulfilled for $\beta = \frac{1}{8}$. Hence (u^*, y^*, λ^*) is not globally optimal for (2.5), giving the desired contradiction. This means, our assumptions were wrong, and the strong stationarity system (2.3) is not necessary for local optimality.

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