A variational characterization of canonical angles between subspaces

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Abstract

Canonical angles between subspaces of a unitary space are characterized by a min-max property which involves inner products.

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Let $\mathcal{X}$ and $\mathcal{Y}$ be two nonzero subspaces of an n-dimensional unitary space $\mathcal{V}$. Angles between $\mathcal{X}$ and $\mathcal{Y}$ can be defined in several equivalent ways. The starting point for our note is a recursive definition (see e.g. [1],[2]) based on the inner product in $\mathcal{V}$. Let $S_{\mathcal{X}} = \{x \in \mathcal{X} : |x| = 1\}$ and $S_{\mathcal{Y}}$ be the unit spheres of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Set $r = \min\{\dim \mathcal{X}, \dim \mathcal{Y}\}$. The smallest angle $\phi_1 \in [0, \frac{\pi}{2}]$ between $\mathcal{X}$ and $\mathcal{Y}$ is defined by

$$\cos \phi_1 = \max_{x \in S_{\mathcal{X}},\, y \in S_{\mathcal{Y}}} |(x, y)|. \tag{1}$$

Let the maximum in (1) be attained at $x_1 \in S_{\mathcal{X}}$ and $y_1 \in S_{\mathcal{Y}}$. Then $\phi_2 \geq \phi_1$ can be defined as the smallest angle between the orthogonal complements of x_1 in $\mathcal{X}$ and y_1 in $\mathcal{Y}$. Thus, starting from $x_0 = y_0 = 0$ one can construct recursively pairs of vectors $(x_i, y_i) \in S_{\mathcal{X}} \times S_{\mathcal{Y}}$ and a set of angles $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \frac{\pi}{2}$ given by

$$\cos\phi_k = \max_{\substack{x\in S_{\mathcal{X}}\ ,\ y\in S_{\mathcal{Y}} \\ x\perp x_i, y\perp y_i, i=0,\ldots,k-1}} |(x,y)|. \tag{2}$$

Theorem 1 below shows that the angles ϕ_k are well defined by (2), independent of the choice of the vectors x_i, y_i.

Let $P_{\mathcal{X}}$ denote the orthogonal projection of $\mathcal{V}$ on $\mathcal{X}$. If $A : \mathcal{Y} \to \mathcal{X}$ is a linear map then we assume that the singular values $\sigma_i(A)$ are ordered by decreasing magnitude such that $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_r(A)$.

Theorem 1 [1, p. 382], [4, p. 43] *Let $\phi_1, \ldots, \phi_r$ be defined as in* (2). *Then*

$$\cos\phi_k = \sigma_k(P_{\mathcal{X}} P_{\mathcal{Y}}),\ \ k = 1, \ldots, r. \tag{3}$$

The numbers $\phi_1, \ldots, \phi_r$ given by (2) or (3) are called the *canonical* (or *principal*) *angles* between $\mathcal{X}$ and $\mathcal{Y}$. It is the purpose of this note to derive a min-max characterization of canonical angles.

We use the following notation. Let $\mathcal{R}(A)$ be the range of A. If $A^\sharp$ is the Moore-Penrose inverse of A then $AA^\sharp = P_{\mathcal{R}(A)}$ (see e.g. [4, p. 106]). The matrix $(AA^*)^{1/2}$ is the positive semidefinite square root of AA^*, so that the first r eigenvalues of $(AA^*)^{1/2}$ are the singular values of A. For a subspace $\mathcal{U} \subseteq \mathcal{X}$ let $\mathcal{U}^\perp$ denote the orthogonal complement of $\mathcal{U}$ in $\mathcal{X}$.

Lemma 2 *The singular values of a linear map $A : \mathcal{Y} \to \mathcal{X}$ are given by*

$$\sigma_k(A) = \min_{\substack{\mathcal{U}\subseteq\mathcal{X} \\ \dim\mathcal{U}=k-1}} \max_{\substack{x\in\mathcal{U}^\perp\cap S_{\mathcal{X}} \\ y\in S_{\mathcal{Y}}}} |(x, Ay)|,\ \ k = 1, \ldots, r. \tag{4}$$

Proof. It is known (see e.g. [3, p. 148]) that

$$\begin{aligned} \sigma_k(A) = & \min_{\substack{\mathcal{U}\subseteq\mathcal{X} \\ \dim\mathcal{U}=k-1}} \max_{x\in\mathcal{U}^\perp\cap S_{\mathcal{X}}} \left(x, (AA^*)^{\frac{1}{2}} x\right) \\ & = \min_{\substack{\mathcal{U}\subseteq\mathcal{X} \\ \dim\mathcal{U}=k-1}} \max_{x\in\mathcal{U}^\perp\cap S_{\mathcal{X}}} |A^* x|. \end{aligned} \tag{5}$$

Because of $\mathcal{R}[(AA^*)^{1/2}] = \mathcal{R}(A)$ we have

$$(AA^*)^{1/2}\left[(AA^*)^{1/2}\right]^\sharp = P_{\mathcal{R}(A)}.$$

Hence, if $\tilde{y} = A^*\left[(AA^*)^{1/2}\right]^\sharp x$ then $(\tilde{y},\tilde{y}) = \big(P_{\mathcal{R}(A)}x, P_{\mathcal{R}(A)}x\big)$ and therefore $|\tilde{y}| \le |x|$. Set

$$\tau_k = \min_{\substack{\mathcal{U}\subseteq\mathcal{X}\\ \dim\mathcal{U}=k-1}} \max_{\substack{x\in\mathcal{U}^\perp\cap S_\mathcal{X}\\ y\in S_\mathcal{Y}}} |(x,Ay)|. \tag{6}$$

In (6) we can replace the unit sphere $S_\mathcal{Y}$ by the closed unit ball $\{y \in \mathcal{Y} : |y| \le 1\}$. Then

$$\begin{aligned}\tau_k \ge & \min_{\substack{\mathcal{U}\subseteq\mathcal{X}\\ \dim\mathcal{U}=k-1}} \max_{\substack{x\in\mathcal{U}^\perp\cap S_\mathcal{X}\\ y=A^*\left[(AA^*)^{1/2}\right]^\sharp x}} |(x,Ay)| \\ & = \min_{\substack{\mathcal{U}\subseteq\mathcal{X}\\ \dim\mathcal{U}=k-1}} \max_{x\in\mathcal{U}^\perp\cap S_\mathcal{X}} \big(x,(AA^*)^{1/2}x\big) = \sigma_k(A).\end{aligned}$$

On the other hand, if $|y| = 1$ then $|(x,Ay)| \le |A^*x|$. Hence (5) yields $\tau_k \le \sigma_k(A)$, which completes the proof. □

The case $k = 1$ in (4), i.e.

$$\sigma_1(A) = \max_{x\in S_\mathcal{X},\, y\in S_\mathcal{Y}} |(x,Ay)|$$

can be found in [3, p.155]. Note that $k = 1$ in (7) below yields ϕ_1 as in (1).

Theorem 3 *The canonical angles* $0 \le \phi_1 \le \phi_2 \le \cdots \le \phi_r \le \frac{\pi}{2}$ *between* $\mathcal{X}$ *and* $\mathcal{Y}$ *satisfy*

$$\cos\phi_k = \min_{\substack{\mathcal{U}\subseteq\mathcal{X}\\ \dim\mathcal{U}=k-1}} \max_{\substack{x\in\mathcal{U}^\perp\cap S_\mathcal{X}\\ y\in S_\mathcal{Y}}} |(x,y)|, \quad k = 1,\ldots,r. \tag{7}$$

Proof. We apply Lemma 2 to $A = P_\mathcal{X}P_\mathcal{Y}$. For $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ we have $(x, P_\mathcal{X}P_\mathcal{Y}y) = (P_\mathcal{X}x, P_\mathcal{Y}y) = (x,y)$. Thus (7) is an immediate consequence of (4) and (3). □

References

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