A PARAMETRIZATION OF SOLUTIONS OF THE DISCRETE-TIME ALGEBRAIC RICCATI EQUATION BASED ON PAIRS OF OPPOSITE UNMIXED SOLUTIONS*

HARALD K. WIMMER†

Abstract. The paper describes the set of solutions of the discrete-time algebraic Riccati equation. It is shown that each solution is a combination of a pair of opposite unmixed solutions. There is a one-to-one correspondence between solutions and invariant subspaces of the closed loop matrix of an unmixed solution. The results of the paper provide an extended counterpart of the parametrization theory of continuous-time algebraic Riccati equations by Willems, Coppel, and Shayman.

Key words. discrete-time algebraic Riccati equation, unmixed solutions, opposite solutions, parametrization by invariant subspaces, shorted operators, Schur complements

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1. Introduction. To a large extent the groundwork for the investigation of algebraic Riccati equations by geometric methods was laid by J. C. Willems. In his study of least squares stationary optimal control [24], he gave a complete description of the set of symmetric solutions of the real continuous-time algebraic Riccati equation (CARE)

$$(1.1) \quad -F^T X - XF + XGG^T X - Q = 0.$$

Assuming that (F, G) is controllable and that the associated Hamiltonian matrix

$$H = \begin{bmatrix} F & -GG^T \\ -Q & -F^T \end{bmatrix}$$

has no eigenvalues on the imaginary axis, Willems showed that (1.1) has a greatest solution X^+ and a least solution X^- . The eigenvalues of the corresponding closed loop matrix $F_{X^+} = F - GG^T X^+$ lie in the left half-plane and those of F_{X^-} are in the right half-plane. Moreover, according to [24] there is a one-to-one correspondence of solutions of (1.1) and invariant subspaces of F_{X^+} such that every solution of (1.1) can be expressed as a combination of X^+ and X^- . That result was refined by Coppel [5] and extended further by Shayman [23].

In this paper we consider the complex discrete-time algebraic Riccati equation (DARE)

$$(1.2) \quad X - F^* XF + (G^* XF + S)^*(R + G^* XG)^{-1}(G^* XF + S) - Q = 0.$$

We shall obtain a classification of solutions of (1.2) that corresponds to the Willems–Coppel–Shayman parametrization for (1.1).

Let us first recapitulate the parametrization result for the CARE (1.1). It will be assumed that elementary divisors of H have even degree when they belong to pure

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†Mathematisches Institut, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany (wimmer@mathematik.uni-wuerzburg.de).

imaginary eigenvalues. A solution X of (1.1) is called *unmixed* [23] if the spectrum of $F_X = F - GG^T X$ satisfies

$$(1.3) \quad \sigma(F_X) \cap \sigma(-F_X^T) \subseteq i\mathbb{R}.$$

It is known (see, e.g., [12]) that there exists an unmixed solution of (1.1) if and only if the pair $(F, G) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p}$ is *sign-controllable*, i.e., if $\lambda \in \sigma(F)$ and $\text{rank}[\lambda I - F, G] < n$ imply $\text{rank}[-\bar{\lambda}I - F, G] = n$. Let X_1 be an unmixed solution of (1.1). Then there exists a unique (unmixed) solution X_2 such that F_{X_1} and F_{X_2} have, at most, pure imaginary eigenvalues in common if and only if (F, G) is controllable. In that case (X_1, X_2) is called a *pair of opposite unmixed solutions* [23]. The extremal solutions (X^-, X^+) are an example of such a pair.

Let $\text{Inv}(F)$ denote the set of F -invariant subspaces of $\mathbb{R}^n$. If $i\alpha$ is an eigenvalue of F on the imaginary axis, we set $E_{\pm i\alpha} = \text{Ker}(F^2 + \alpha^2 I)^n$ and we define

$$E_{i\mathbb{R}}(F) = \bigoplus_{\alpha \in \mathbb{R}} E_{\pm i\alpha}.$$

The following theorem can be regarded as an updated version of the Willems–Coppel–Shayman theory. It combines results of [24], [5], and [23] with an approach by Scherer [22] and observations on shorted operators in section 3.

THEOREM 1.1. *Let Γ be the set of real symmetric solutions of (1.1). Assume that (F, G) is controllable. Let (X_1, X_2) be a pair of opposite unmixed solutions of*

$$(1.1) \quad -F^T X - XF + XGG^T X - Q = 0,$$

and let $F_{X_2} = F - GG^T X_2$ be the closed loop matrix corresponding to X_2 . Set $\Delta = X_1 - X_2$. Define

$$\mathcal{N} = \{N \in \text{Inv}(F_{X_2}) \mid E_{i\mathbb{R}}(F_{X_2}) \subseteq N\}.$$

If $N \in \mathcal{N}$, then

$$(\Delta N)^\perp \oplus (N \cap \text{Im } \Delta) = \mathbb{R}^n.$$

Let P_N be the projection on $(\Delta N)^\perp$ along $(N \cap \text{Im } \Delta)$. Define

$$\kappa(N) = X_1 P_N + X_2 (I - P_N).$$

Then $\kappa : \mathcal{N} \rightarrow \Gamma$ is a bijection. If $X \in \Gamma$, then

$$\kappa^{-1}(X) = \text{Ker}(X - X_2).$$

It is not difficult to explain (see also [18]) why direct sum decompositions of the form

$$(1.4) \quad (\Delta N)^\perp \oplus N = \mathbb{R}^n$$

and associated projections P_N should play a role in the theory of the CARE. Consider (1.1) with $Q = 0$, i.e.,

$$(1.5) \quad \mathcal{C}(X) = -F^T X - XF + XGG^T X = 0.$$

If (F, G) is controllable and $\sigma(F) \cap \sigma(-F^T) = \emptyset$, then (1.5) has a unique nonsingular solution Δ (see, e.g., [22, p. 102]). Let $N = \text{Im}(I, O)^T$ be invariant under F and let Δ be partitioned according to F such that

$$F = \begin{bmatrix} F_1 & F_{12} \\ 0 & F_2 \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \Delta_1 & \Delta_{12} \\ \Delta_{12}^T & \Delta_2 \end{bmatrix}.$$

Then Δ_1 is nonsingular [22]. Let $\tilde{\Delta}_2 = \Delta_2 - \Delta_{12}^T \Delta_1^{-1} \Delta_{12}$ be the Schur complement of Δ with respect to N . Set

$$S = \begin{bmatrix} I & -\Delta_1^{-1} \Delta_{12} \\ 0 & I \end{bmatrix}.$$

Then $S^T \Delta S = \text{diag}(\Delta_1, \tilde{\Delta}_2)$ and

$$S^{-1} F S = \begin{bmatrix} F_1 & * \\ 0 & F_2 \end{bmatrix}.$$

It is obvious that

$$\tilde{\Delta} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\Delta}_2 \end{bmatrix}$$

is a solution of (1.5). In the terminology of [2], [16], or [6], the matrix $\tilde{\Delta}$ is a *shorted matrix* of Δ . In section 3 we shall give a basis-free description of $\tilde{\Delta}$ and we shall see that $\tilde{\Delta} = \Delta P_N$, where P_N is the projection corresponding to (1.4).

Notation: Let $\mathbb{D}$ be the open unit disk and $\partial\mathbb{D} = \{z \in \mathbb{C}; |z| = 1\}$ be the unit circle. Let Λ be a set of complex numbers and define

$$\Lambda^\nabla = \{\bar{\lambda}^{-1} \mid \lambda \in \Lambda, \lambda \neq 0\}.$$

To a matrix of complex rational functions of the form $W(s) = D + C(sI - A)^{-1}B$, we associate $W^\nabla(s) = D^* + B^*(s^{-1}I - A^*)C^*$. For $\lambda \in \mathbb{C}$ put

$$E_\lambda(F) = \text{Ker}(\lambda I - F)^n.$$

Then $E_\lambda(F)$ is a generalized eigenspace of F if $\lambda \in \sigma(F)$. In particular, $E_0(F) = \text{Ker } F^n$. We set

$$E_{\partial\mathbb{D}} = \oplus \{E_\eta(F), \eta \in \partial\mathbb{D}\}.$$

A subspace V of $\mathbb{C}^n$ is a *spectral subspace* of F if $V = \oplus \{E_\lambda(F); \lambda \in T\}$ for some $T \subseteq \sigma(F)$. If K is a hermitian matrix, we write $K > 0$ ($K \geq 0$) when K is positive (semi)definite. If F is nonsingular, we set $F^{-*} = (F^*)^{-1}$.

The main result of this paper is Theorem 5.3. It describes a parametrization of the set of hermitian solutions of the DARE (1.2). The proof of that theorem will proceed in several stages. In section 2 we review basic facts on the DARE. We recall that the difference of two solutions of (1.2) satisfies an associated DARE of the form

$$X - F^* X F + F^* X G (R + G^* X G)^{-1} G^* X F = 0.$$

Such equations, where $X = 0$ is a solution, will be considered in section 4. We have indicated before that (1.4) is related to Schur complements and shorted operators. That subject will be touched upon in section 3.

The geometric theory of Willems [24] and Coppel [5] was carried over to the DARE (1.2) first by G. Ruckebusch [20], [21, p. 129] and then by Ran and Trentelman [19]. Those papers give a geometric characterization of the set of hermitian solutions in terms of the pair of extremal solutions.

2. Basic facts of the DARE: Definitions. There is a wide class of problems in systems and control theory that require solutions of the DARE

$$(2.1) \quad \mathcal{D}(X) = X - F^*XF + (G^*XF + S)^*(R + G^*XG)^{-1}(G^*XF + S) - Q = 0.$$

An important example is the discrete-time linear quadratic problem of optimal control. Let the system

$$x(t+1) = Fx(t) + Gu(t), \quad x(0) = x_0$$

be stabilizable, and let

$$J(x_0, u) = \sum_{t=0}^{\infty} [x^*(t) \quad u^*(t)] \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

be a positive semidefinite performance index. Then (see, e.g., [9]) there exists an optimal control $u(t)$ which minimizes $J(x_0, u)$. The optimal cost is given by $J_{\text{opt}} = x_0^* X_{\text{opt}} x_0$, where X_{opt} is the smallest positive semidefinite solution of (2.1), and the optimal control is

$$(2.2) \quad u_{\text{opt}}(t) = -(R + G^* X_{\text{opt}} G)^{-1} (S + G^* X_{\text{opt}} F) x(t).$$

Thus (2.2) gives rise to the closed loop system

$$(2.3) \quad x(t+1) = [F - G(R + G^* X_{\text{opt}} G)^{-1} (S + G^* X_{\text{opt}} F)] x(t).$$

The matrices in (2.1) are assumed to be complex, $F \in \mathbb{C}^{n \times n}$, $G \in \mathbb{C}^{n \times m}$, $S \in \mathbb{C}^{m \times n}$, $R = R^* \in \mathbb{C}^{m \times m}$, and $Q = Q^* \in \mathbb{C}^{n \times n}$. We are concerned with hermitian solutions X of (2.1). In this section we assemble basic facts and concepts related to (2.1). With regard to (2.3) we define

$$F_X = F - G(R + G^* X G)^{-1} (G^* X F + S)$$

as the closed loop matrix corresponding to a solution X . We say that a solution X of the DARE (2.1) is *unmixed* if F_X has the property

$$(2.4) \quad \sigma(F_X) \cap \sigma(F_X)^\nabla \subseteq \partial \mathbb{D},$$

i.e., if $\lambda \in \sigma(F_X)$ and $\lambda \neq 0$ and $|\lambda| \neq 1$, then $\bar{\lambda}^{-1} \notin \sigma(F_X)$. We say that (X_1, X_2) is a *pair of opposite unmixed solutions* if the corresponding closed loop matrices satisfy

$$(2.5) \quad \sigma(F_{X_1}) \cap \sigma(F_{X_2}) \subseteq \partial \mathbb{D} \cup \{0\}.$$

Let

$$(2.6) \quad M - sL = \begin{bmatrix} F & 0 & G \\ Q & I & S^* \\ S & 0 & R \end{bmatrix} - s \begin{bmatrix} I & 0 & 0 \\ 0 & F^* & 0 \\ 0 & G^* & 0 \end{bmatrix}$$

be the extended symplectic pencil associated with (2.1), and let

$$(2.7) \quad \Psi(s) = \begin{bmatrix} G^*(s^{-1}I - F^*)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} (sI - F)^{-1}G \\ I \end{bmatrix}$$

be the associated Popov matrix. The following identities and facts can be found in [11], [13], and [17]. If X is a solution of (2.1), then

$$(2.8) \quad \det(M - sL) = \det(R + G^* X G) \det(sI - F_X) \det(I - sF_X^*).$$

It follows from (2.8) that there exists a solution of (2.1) only if $M - sL$ is nonsingular, i.e., if $\det(M - sL)$ is not the zero polynomial. In that case we call

$$\sigma(M - sL) = \{\lambda \in \mathbb{C} \mid \det(M - \lambda L) = 0\}$$

the set of *characteristic roots* of the pencil (2.6). It is obvious from (2.8) that $\sigma(F_X) \subseteq \sigma(M - sL)$. Hence, if 0 is a characteristic root of $M - sL$, then we have $0 \in \sigma(F_X)$ for all solutions X , which accounts for the singleton $\{0\}$ in (2.5). Let X_1 and X_2 be solutions of (2.1) such that (2.5) holds. Then it follows from (2.8) that both X_1 and X_2 are unmixed. Each solution X of (2.1) gives rise to a factorization of the Popov matrix $\Psi(s)$, namely

$$(2.9) \quad \Psi(s) = \Phi_X^\nabla(s)(R + G^* X G)\Phi_X(s)$$

with

$$\Phi_X(s) = I + (R + G^* X G)^{-1}(G^* X F + S)(sI - F)^{-1}G.$$

If $\Psi(s)$ satisfies

$$(\Psi) \quad \Psi(\eta) > 0 \text{ for some } \eta \in \partial\mathbb{D},$$

then (2.9) implies

$$(2.10) \quad R + G^* X G > 0$$

for all solutions X .

Arguments in section 4 show that the conditions (Ψ) and (2.10) are essential for the derivation of Theorem 5.3. Hence it is not within the scope of this paper to deal with DAREs of the form (1.2) where $(R + G^* X G)^{-1}$ is replaced by a generalized inverse $(R + G^* X G)^\#$.

Existence of unmixed solutions was studied in [4], [25], [3]. We note the following result of [3]. Suppose that (Ψ) holds, $\Psi(\eta) \geq 0$ for almost all $\eta \in \partial\mathbb{D}$, and $\text{rank}(F - \lambda I) = n$ for all $\lambda \neq 0$. Then each unmixed set Λ gives rise to a unique solution X such that $\sigma(F_X) \subseteq \Lambda$. In particular there exists a unique pair (X_-, X_+) such that

$$(2.11) \quad \sigma(F_{X_-}) \subseteq \{0\} \cup \{\lambda \in \mathbb{C}; |\lambda| \geq 1\} \text{ and } \sigma(F_{X_+}) \subseteq \overline{\mathbb{D}}.$$

It is known that Cayley transformations allow a passage from continuous-time to discrete-time algebraic Riccati equations [15], [1]. The use of such transformations requires invertibility assumptions (see, e.g., [1, p. 81]) which are not met by the general hypotheses in the present paper. We note that in the case of $0 \in \sigma(M - sL)$ a computational procedure is available [7] to obtain an equivalent DARE of smaller order such that the associated closed loop matrices are nonsingular.

3. Shorted operators and oblique projections. Let N be a subspace of $\mathbb{C}^n$ and Δ be a hermitian $n \times n$ matrix.

LEMMA 3.1. (i) If

$$(3.1) \quad N = \text{Im} \begin{bmatrix} I_t \\ 0 \end{bmatrix}, \quad \text{and if } \Delta = \begin{bmatrix} \Delta_1 & \Delta_{21}^* \\ \Delta_{21} & \Delta_2 \end{bmatrix}$$

is partitioned conformably, then we have

$$(3.2) \quad \mathbb{C}^n = (\Delta N)^\perp \oplus N$$

if and only if Δ_1 is nonsingular.

(ii) Assume (3.2) and let P_N be the projection on $(\Delta N)^\perp$ along N . Then the matrix ΔP_N is hermitian. If N and Δ are given as in (3.1), then

$$(3.3) \quad \Delta P_N = \text{diag}(0, \tilde{\Delta}_2), \quad \tilde{\Delta}_2 = \Delta_2 - \Delta_{21}^* \Delta_1^{-1} \Delta_{21}.$$

Proof. (i) Suppose $\det \Delta_1 = 0$. If $u_1 \in \text{Ker } \Delta_1$, $u_1 \neq 0$, then

$$u = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \in N$$

and

$$u^* \Delta N = [u_1^* \ 0] \begin{bmatrix} \Delta_1 \\ \Delta_{21} \end{bmatrix} = 0.$$

Hence $(\Delta N)^\perp \cap N \neq 0$. Conversely, if Δ_1 is nonsingular, then

$$\Delta N = \text{Im} \begin{bmatrix} \Delta_1 \\ \Delta_{21} \end{bmatrix} = \text{Im} \begin{bmatrix} I_t \\ \Delta_{21} \Delta_1^{-1} \end{bmatrix}$$

and

$$(3.4) \quad (\Delta N)^\perp = \text{Im} \begin{bmatrix} -\Delta_1^{-1} \Delta_{21}^* \\ I_{n-t} \end{bmatrix}$$

imply (3.2). (ii) From $\text{Im } P_N = (\Delta N)^\perp$ and $\Delta \text{Im}(I - P_N) = \Delta N$ follows $(I - P_N)^* \Delta P_N = 0$. Hence, $\Delta P_N = P_N^* \Delta P_N$ is hermitian. Because of (3.4) the projection matrix P_N is given by

$$P_N = \begin{bmatrix} 0 & -\Delta_1^{-1} \Delta_{21}^* \\ 0 & I_{n-t} \end{bmatrix},$$

which yields ΔP_N as a shorted operator in block diagonal form (3.3). $\square$

The following observation is a special case of a formula for the inverse of a block matrix [10, p. 18]. It will be needed in the proof of Theorem 4.5, which deals with a nonsingular solution Δ of a special DARE and its inverse.

LEMMA 3.2. Assume that Δ is nonsingular, set $W = \Delta^{-1}$, and let

$$\Delta = \begin{bmatrix} \Delta_1 & \Delta_{21}^* \\ \Delta_{21} & \Delta_2 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} * & * \\ * & W_2 \end{bmatrix}$$

be partitioned accordingly. (i) Then Δ_1 is nonsingular if and only if W_2 is nonsingular.

(ii) If Δ_1 is nonsingular, then $\tilde{\Delta}_2 = \Delta_2 - \Delta_{21}^* \Delta_1^{-1} \Delta_{21}$ is nonsingular and $\tilde{\Delta}_2^{-1} = W_2$.

Proof. We include a proof which is an application of the preceding lemma. Set $N = \text{Im}[I_t \ O]^T$ and $M = N^\perp$. (i) Assume that Δ_1 is nonsingular, or equivalently (3.2). Because of $(\Delta N)^\perp = \Delta^{-1}N^\perp$ we can write (3.2) as $WM \oplus M^\perp = \mathbb{C}^n$, which in turn is equivalent to $(WM)^\perp \oplus M = \mathbb{C}^n$. Because of $M = \text{Im}[O \ I_{n-t}]^T$ such a decomposition exists if and only if W_2 is nonsingular. (ii) From $\Delta P_N = \text{diag}(0, \tilde{\Delta}_2)$ follows

$$\Delta P_N \Delta^{-1} = \begin{bmatrix} 0 & 0 \\ * & \tilde{\Delta}_2 W_2 \end{bmatrix}.$$

Obviously $\Delta P_N \Delta^{-1}$ is the projection on $N^\perp$ along ΔN . Thus we obtain $\tilde{\Delta}_2 W_2 = I$. $\square$

Let $K = \text{Ker } \Delta$ and $V = \text{Im } \Delta$. We assume $0 < n_c = \dim V < n$ such that the decomposition $\mathbb{C}^n = K \oplus V$ is nontrivial. Let $U \subseteq V$. Then $U^{\perp_c}$ shall denote the orthogonal complement of U with respect to V . Set $\Delta_c = \Delta|_V$. Then $\Delta_c : V \rightarrow V$ is hermitian and nonsingular.

LEMMA 3.3. *Let N be a subspace of $\mathbb{C}^n$ with $K \subseteq N$. Set $N_c = N \cap V$. Then*

$$(3.5) \quad (\Delta_c N_c)^{\perp_c} \oplus N_c = V$$

if and only if

$$(3.6) \quad (\Delta N)^\perp \oplus (N \cap \text{Im } \Delta) = \mathbb{C}^n.$$

Proof. Note that $(\Delta N)^\perp = K \oplus (\Delta_c N_c)^{\perp_c}$ and $N \cap \text{Im } \Delta = N_c$. $\square$

4. A DARE with solution $X = 0$. In this section we deal with the equation

$$(4.1) \quad \mathcal{H}(Y) = Y - F^* Y F + F^* Y G (R + G^* Y G)^{-1} G^* Y F = 0,$$

where R is nonsingular. Set $\Gamma = GR^{-1}G^*$. Then

$$F_Y = F - G(R + G^* Y G)^{-1} G^* Y F = (I + \Gamma Y)^{-1} F,$$

and we have $\mathcal{H}(Y) = Y - F^* Y F_Y = 0$. We first note two results on nonsingular solutions of (4.1).

LEMMA 4.1 (see [26, p. 931]). *Let Y be a hermitian nonsingular $n \times n$ matrix. Assume that F is nonsingular. Then Y is a solution of (4.1) if and only if $W = Y^{-1}$ satisfies the discrete-time Lyapunov equation*

$$(4.2) \quad W - F W F^* = -\Gamma.$$

LEMMA 4.2 (see [26, p. 932]). *Assume that F is nonsingular, (F, G) is controllable, $R > 0$, and $\sigma(F) \cap \sigma(F^{-*}) = \emptyset$. Then (4.1) has a unique nonsingular solution.*

The subsequent result deals with eigenvalues of F in $\{0\} \cup \partial\mathbb{D}$ and corresponding generalized eigenspaces.

LEMMA 4.3 (see [26, p. 933]). *Assume $R > 0$ and $\text{rank}(F - \eta I, G) = n$ for all $\eta \in \partial\mathbb{D}$. If Y is a solution of (4.1), then $E_0(F) + E_{\partial\mathbb{D}}(F) \subseteq \text{Ker } Y$.*

COROLLARY 4.4 (see [26, pp. 923–924]). *Assume $R > 0$, $\text{rank}[F - \lambda I, G] = n$ if $\lambda \neq 0$, and $\sigma(F) \cap \sigma(F)^\nabla \subseteq \partial\mathbb{D}$. Then (4.1) has a unique solution Δ with $\text{Ker } \Delta = E_0(F) + E_{\partial\mathbb{D}}(F)$.*

It is easy to show (see also Proposition 5.2) that $(0, \Delta)$ is a pair of opposite unmixed solutions of $\mathcal{H}(Y) = 0$. Therefore, the following result can already be viewed as a special case of the main theorem.

THEOREM 4.5. *The assumptions on the DARE*

$$(4.1) \quad \mathcal{H}(Y) = Y - F^*YF + F^*YG(R + G^*YG)^{-1}G^*YF = 0$$

are the following: $R > 0$, $\sigma(F) \cap \sigma(F)^\nabla \subseteq \partial\mathbb{D}$, and $\text{rank}[F - \lambda I, G] = n$ if $\lambda \neq 0$. Define

$$K = E_0(F) + E_{\partial\mathbb{D}}(F),$$

and

$$(4.3) \quad \mathcal{T} = \{Y \mid \mathcal{H}(Y) = 0\}, \quad \mathcal{N} = \{N \in \text{Inv } F \mid K \subseteq N\}.$$

Let Δ be the solution of (4.1) with $\text{Ker } \Delta = K$. Then the following holds.

- (i) If Y is a solution of (4.1), then $\text{Ker } Y \in \mathcal{N}$.
- (ii) If $N \in \mathcal{N}$, then

$$(4.4) \quad \mathbb{C}^n = (\Delta N)^\perp \oplus (N \cap \text{Im } \Delta),$$

and if P_N is the projection of $\mathbb{C}^n$ on $(\Delta N)^\perp$ along $N \cap \text{Im } \Delta$, then $\tilde{Y} = \Delta P_N$ is the unique solution of (4.1) with $\text{Ker } \tilde{Y} = N$.

- (iii) For $N \in \mathcal{N}$ set $\tilde{\kappa}(N) = \Delta P_N$. Then the map $\tilde{\kappa} : \mathcal{N} \rightarrow \mathcal{T}$ is a bijection, and for $Y \in \mathcal{T}$ we have $\tilde{\kappa}^{-1}(Y) = \text{Ker } Y$.

Proof. Let us first prove the theorem under the stronger assumption that F is nonsingular and $\sigma(F) \cap \sigma(F^{-*}) = \emptyset$. In that case we have $K = 0$ and the solution Δ is nonsingular such that $N \cap \text{Im } \Delta = N$ and $\mathcal{N} = \text{Inv } F$.

- (i) This is obvious since $\mathcal{H}(Y) = Y - F^*YF_Y = 0$ is equivalent to $F_Y^{-*}Y = YF$.
- (ii) Assume $N = \text{Im } [I \ O]^T$ such that

$$(4.5) \quad F = \begin{bmatrix} F_1 & * \\ 0 & F_2 \end{bmatrix}.$$

It is known from Lemma 4.1 that $W = \Delta^{-1}$ satisfies

$$(4.6) \quad W - FW F^* = -GR^{-1}G^*.$$

Let

$$\Delta = \begin{bmatrix} \Delta_1 & \Delta_{21}^* \\ \Delta_{21} & \Delta_2 \end{bmatrix}, \quad G = \begin{bmatrix} * \\ G_2 \end{bmatrix}, \quad \text{and } \Delta^{-1} = W = \begin{bmatrix} * & * \\ * & W_2 \end{bmatrix}$$

be partitioned according to (4.5). To establish a decomposition

$$(4.7) \quad \mathbb{C}^n = (\Delta N)^\perp \oplus N,$$

we recall Lemma 3.1 and the fact that (4.7) holds if and only if Δ_1 is nonsingular. Thus, according to Lemma 3.2 we have to show that W_2 is nonsingular. From (4.6) we obtain

$$(4.8) \quad W_2 - F_2 W_2 F_2^* = -G_2 R^{-1} G_2^*.$$

In the preceding Stein equation the pair (F_2, G_2) is controllable, $\sigma(F_2) \cap \sigma(F_2^{-*}) = \emptyset$, and $R > 0$. Therefore (see, e.g., [14, p. 453]) the matrix W_2 is nonsingular.

Now let us show that $\tilde{Y} = \Delta P_N$ is a solution of (4.1). Since Δ is assumed to be nonsingular we have $\text{Ker } \tilde{Y} = \text{Ker } P_N = N$. Hence it follows from Lemmas 3.1 and 3.2 that $\tilde{Y} = P_N^* \Delta P_N = \text{diag}(0, \tilde{\Delta}_2)$, and $\tilde{\Delta}_2 = \Delta_2 - \Delta_{21} \Delta_1^{-1} \Delta_{21}^*$ is nonsingular and $\tilde{\Delta}_2^{-1} = W_2$. Consider the equation

$$(4.9) \quad \mathcal{H}_2(Y_2) = Y_2 - F_2^* Y_2 F_2 + F_2^* Y_2 G_2 (R + G_2^* Y_2 G_2)^{-1} G_2^* Y_2 F_2.$$

It is easy to check that $\mathcal{H}(\tilde{Y}) = \text{diag}(0, \mathcal{H}_2(\tilde{\Delta}_2))$. Hence $\tilde{Y}$ is a solution of (4.1) if and only if $\mathcal{H}_2(\tilde{\Delta}_2) = 0$. To show that $\tilde{\Delta}_2$ is a solution of (4.9) we use Lemma 4.1 again, which says that $\mathcal{H}_2(\tilde{\Delta}_2) = 0$ is equivalent to

$$(4.10) \quad \tilde{\Delta}_2^{-1} - F_2 \tilde{\Delta}_2^{-1} F_2^* = -G_2 R^{-1} G_2^*.$$

We have seen that the solution Δ gives rise to (4.8). Therefore, because $W_2 = \tilde{\Delta}_2^{-1}$, (4.10) is satisfied.

To prove uniqueness, take a solution Y of (4.1) with $\text{Ker } Y = N$. Then $Y = \text{diag}(0, Y_2)$, and Y_2 is nonsingular satisfying (4.9). According to Lemma 4.2 there exists a unique nonsingular solution of (4.9), namely, $\tilde{\Delta}_2$. Hence $Y = \text{diag}(0, \tilde{\Delta}_2) = \tilde{Y}$. Part (iii) is an obvious consequence of (i) and (ii).

We now discard the assumptions about F made at the beginning and consider the case $K = E_0(F) + E_{\partial\mathbb{D}}(F) \neq 0$. Let $V = \oplus \{E_\lambda(F) \mid \lambda \neq 0, \lambda \notin \partial\mathbb{D}\}$ be the F -invariant complement of K such that $\mathbb{C}^n = K \oplus V$. Suppose

$$(4.11) \quad K = \text{Im} \begin{bmatrix} I_{n_a} \\ 0 \end{bmatrix} \text{ and } V = \text{Im} \begin{bmatrix} 0 \\ I_{n_c} \end{bmatrix}$$

such that

$$F = \begin{bmatrix} F_a & 0 \\ 0 & F_c \end{bmatrix}, \quad G = \begin{bmatrix} * \\ G_c \end{bmatrix}.$$

Then $\sigma(F_a) \subseteq \{0\} \cup \partial\mathbb{D}$, F_c is nonsingular, and $\sigma(F_c) \cap \sigma(F_c^{-*}) = \emptyset$. It follows from Lemma 4.3 that Y is a solution of (4.1) if and only if $Y = \text{diag}(0, Y_c)$ and Y_c is a solution of

$$(4.12) \quad \mathcal{H}_c(Y_c) = Y_c - F_c^* Y_c F_c + F_c^* Y_c G_c (R + G_c^* Y_c G_c)^{-1} G_c^* Y_c F_c = 0.$$

Set $\mathcal{T}_c = \{Y_c \mid \mathcal{H}_c(Y_c) = 0\}$ and let $\iota : \mathcal{T}_c \rightarrow \mathcal{T}$ be the bijection given by $\iota(Y_c) = \text{diag}(0, Y_c)$. Let Δ_c be the unique nonsingular solution of (4.12). Then $\Delta = \text{diag}(0, \Delta_c)$ is the unique solution of (4.1) with $\text{Ker } \Delta = K$.

Let $\hat{N}_c \in \text{Inv } F_c$. We embed $\hat{N}_c$ into the space

$$N_c = \left\{ \begin{bmatrix} 0 \\ x_c \end{bmatrix} \in \mathbb{C}^n, x_c \in \hat{N}_c \right\}$$

and define $\tau(\hat{N}_c) = K \oplus N_c$. Then $\tau : \text{Inv } F_c \rightarrow \mathcal{N}$ is a bijection. In accordance with (4.7) we have

$$(4.13) \quad \mathbb{C}^{n_c} = (\Delta_c \hat{N}_c)^\perp \oplus \hat{N}_c,$$

which is equivalent to $V = (\Delta_c N_c)^{\perp_c} \oplus N_c$. Hence, it follows from Lemma 3.3 that $N = \tau(\hat{N}_c) \in \mathcal{N}$ satisfies (4.4). Let $P_{\hat{N}_c}$ be the projection corresponding to (4.13) and set $\tilde{\kappa}_c(\hat{N}_c) = \Delta_c P_{\hat{N}_c}$. We know that $\tilde{\kappa}_c : \text{Inv } F_c \rightarrow \mathcal{T}_c$ is a bijection. Define $\mu_c = \tilde{\kappa}_c^{-1}$, i.e., $\mu_c(Y_c) = \text{Ker } Y_c$, $Y_c \in \mathcal{T}_c$. The maps $\tilde{\kappa}_c$ and μ_c can now be extended to bijections between $\mathcal{T}$ and $\mathcal{N}$. For $Y \in \mathcal{T}$ define $\mu(Y) = \text{Ker } Y$. Then $\mu = \tau\mu_c\iota^{-1} : \mathcal{T} \rightarrow \mathcal{N}$, and μ is bijective. Let P_N be the projection arising from (4.4). For $N \in \mathcal{N}$ define $\tilde{\kappa}(N) = \Delta P_N$. In the setting of (4.11) we have $P_N = \text{diag}(I, P_{\hat{N}_c})$. Therefore

$$\tilde{\kappa}(N) = \text{diag}(0, \Delta_c P_{\hat{N}_c}) = \iota(\Delta_c P_{\hat{N}_c}) = \iota\tilde{\kappa}_c(\hat{N}_c) = \iota\tilde{\kappa}_c\tau^{-1}(N).$$

Hence $\tilde{\kappa} : \mathcal{N} \rightarrow \mathcal{T}$ and $\tilde{\kappa} = \mu^{-1}$. $\square$

In order to characterize unmixed solutions of $\mathcal{D}(X) = 0$, we need a discrete-time counterpart of a result of Scherer [22, p. 106]. Let $A \in \mathbb{C}^{n \times n}$ and $M \in \text{Inv } A$; let $\bar{A}$ denote the endomorphism of $\mathbb{C}^n/M$ induced by A ; and let $A|_M$ be the restriction of A to M . We define $\sigma_i(A, M) = \sigma(A|_M)$ and $\sigma_o(A, M) = \sigma(\bar{A})$. Assumptions and definitions in the lemma below are those of Theorem 4.5 and its proof.

LEMMA 4.6. *For $N \in \mathcal{N}$ let $Y = \Delta P_N$ be the corresponding solution of (4.1). (i) Then $F = F_Y$ on N and $N \in \text{Inv } F_Y$, and we have*

$$(4.14) \quad \sigma_i(F_Y, N) = \sigma_i(F, N) \text{ and } \sigma_o(F_Y, N) = \sigma_o(F, N)^\nabla.$$

(ii) *The solution Y is unmixed if and only if $N \in \mathcal{N}$ is a spectral subspace of F .*

Proof. Let us assume for simplicity that F is nonsingular and $\sigma(F) \cap \sigma(F^{-*}) = \emptyset$.

(i) As before, suppose $N = \text{Im}[I \ O]^T$ and let F be given by (4.5). Then $\sigma_i(F, N) = \sigma(F_1)$ and $\sigma_o(F, N) = \sigma(F_2)$. From $Y = \Delta P_N = \text{diag}(0, \tilde{\Delta}_2)$ follows

$$F_Y = (I + \Gamma Y)^{-1} F = \begin{bmatrix} F_1 & * \\ 0 & A_2 \end{bmatrix}$$

with $A_2 = (I + \Gamma_2 \tilde{\Delta}_2)^{-1} F_2$. Thus $\sigma_i(F_Y, N) = \sigma(F_1)$. On the other hand, $Y - F^* Y F_Y = 0$ implies $A_2 = \tilde{\Delta}_2^{-1} F_2^{-*} \tilde{\Delta}_2$, and we obtain $\sigma(A_2) = \sigma(F_2)^\nabla$.

(ii) N is a spectral subspace of F if and only if

$$(4.15) \quad \sigma(F_1) \cap \sigma(F_2) = \emptyset.$$

The assumption $\sigma(F) \cap \sigma(F^{-*}) = \emptyset$ together with (4.14) imply

$$\begin{aligned} \sigma(F_Y) \cap \sigma(F_Y^{-*}) &= [\sigma(F_1) \cup \sigma(F_2^{-*})] \cap [\sigma(F_1^{-*}) \cup \sigma(F_2)] = \\ &= [\sigma(F_2^{-*}) \cap \sigma(F_1^{-*})] \cup [\sigma(F_1) \cap \sigma(F_2)] = [\sigma(F_1) \cap \sigma(F_2)]^\nabla \cup [\sigma(F_1) \cap \sigma(F_2)]. \end{aligned}$$

Hence, the property that Y is unmixed, i.e., $\sigma(F_Y) \cap \sigma(F_Y^{-*}) = \emptyset$, is equivalent to (4.15). $\square$

5. The main result. The passage from the general DARE

$$(5.1) \quad \mathcal{D}(X) = X - F^* X F + (G^* X F + S)^* (R + G^* X G)^{-1} (G^* X F + S) - Q = 0$$

to an equation of the form $\mathcal{H}(Y) = 0$ in (4.1) is a crucial step in the derivation of our main theorem. It is based on the following lemma for which we refer to [19] and [8, Lemma 5.2] or [1, Lemma 6.8.9].

LEMMA 5.1. (i) Let X_2 be a solution of $\mathcal{D}(X) = 0$. Then X is a solution of $\mathcal{D}(X) = 0$ if and only if $Y = X - X_2$ is a solution of

$$(5.2) \quad \mathcal{H}_2(Y) = Y - F_{X_2}^* Y F_{X_2} + F_{X_2}^* Y G [(R + G^* X_2 G) + G^* Y G]^{-1} G^* Y F_{X_2} = 0.$$

(ii) Let X_2 and X be solutions of $\mathcal{D}(X) = 0$. Set $Y = X - X_2$. Then

$$(5.3) \quad F_X = F_{X_2} - G(R + G^* X_2 G)^{-1} G^* Y F_{X_2}.$$

Thus, when a solution X_2 of (5.1) is at our disposal we can pass from (5.1) to (5.2) and apply the results of section 4.

PROPOSITION 5.2. Assume (Ψ) and

$$(5.4) \quad \text{rank}[F - \lambda I, G] = n \text{ if } \lambda \neq 0.$$

Let X_2 be an unmixed solution of $\mathcal{D}(X) = 0$. Set $K = E_0(F_{X_2}) + E_{\partial\mathbb{D}}(F_{X_2})$. Then there exists a unique solution Δ of (5.2) with $\text{Ker } \Delta = K$. Moreover (X, X_2) is a pair of opposite unmixed solutions of $\mathcal{D}(X) = 0$ if and only if $X = X_2 + \Delta$.

Proof. The assumption (Ψ) implies $R + G^* X_2 G > 0$ in (5.2). Clearly (5.4) implies the corresponding condition for $F = F_{X_2}$. The assumption that X_2 should be an unmixed solution of (5.1) is equivalent to

$$(5.5) \quad \sigma(F_{X_2}) \cap \sigma(F_{X_2})^\nabla \subseteq \partial\mathbb{D}.$$

Hence it follows from Corollary 4.4 that (5.2) has a unique solution Δ with $\text{Ker } \Delta = K$. Again it is no loss of generality to assume

$$K = \text{Im} \begin{bmatrix} I_{n-n_c} \\ 0 \end{bmatrix}$$

and $F_{X_2} = \text{diag}(F_a, F_c)$. Then $\Delta = \text{diag}(0, \Delta_c)$, $\det \Delta_c \neq 0$. Obviously (5.5) implies that F_c is nonsingular and

$$(5.6) \quad \sigma(F_c) \cap \sigma(F_c^{-*}) = \emptyset.$$

Let X be a solution of (5.1). Set $Y = X - X_2$. Then $\mathcal{H}_2(Y) = 0$, or equivalently

$$(5.7) \quad Y - F_{X_2}^* Y F_X = 0.$$

Because of Lemma 4.3 we have $K \subseteq \text{Ker } Y$. Hence $Y = \text{diag}(0, Y_c)$. The identity (5.3) yields

$$(5.8) \quad F_X = \begin{bmatrix} F_a & * \\ 0 & B \end{bmatrix}.$$

From (5.7) we obtain

$$(5.9) \quad Y_c - F_c^* Y_c B = 0.$$

Let us consider the solution $X = X_2 + \Delta$ and let F_X be as in (5.8). In this case we have $Y = \Delta = \text{diag}(0, \Delta_c)$. Thus (5.9) becomes $\Delta_c - F_c^* \Delta_c B = 0$, where F_c and Δ_c are nonsingular. Hence $B = \Delta_c^{-1} F_c^{-*} \Delta_c$, $\sigma(B) = \sigma(F_c^{-*})$, and

$$\sigma(F_X) = \sigma(F_a) \cup \sigma(B) = \sigma(F_a) \cup \sigma(F_c^{-*}).$$

Thus (5.6) implies $\sigma(F_X) \cap \sigma(F_{X_2}) = \sigma(F_a) \subseteq \{0\} \cup \partial\mathbb{D}$, and therefore (X, X_2) is a pair of opposite unmixed solutions.

Now consider a solution X such that $Y = X - X_2 \neq \Delta$. Then $Y = \text{diag}(0, Y_c)$ and Y_c is singular. Take $w \in \text{Ker } Y_c$, $w \neq 0$. Recall that F_c in (5.9) is nonsingular. Hence $F_c B w = 0$, and $\text{Ker } Y_c$ is invariant under B . Therefore we can assume that w is an eigenvector of B , say $Bw = \lambda w$. Interchanging the role of X and X_2 in (5.3), we obtain

$$F_{X_2} = \text{diag}(F_a, F_c) = F_X - G(R + G^* X_2 G)^{-1} G^* (-Y) F_X.$$

Hence $F_c = B - W Y_c B$ with some $n_c \times n_c$ matrix W . Thus $F_c w = Bw$. Therefore $\lambda \in \sigma(F_c)$, and $\lambda \notin (\{0\} \cup \partial\mathbb{D})$. On the other hand $\lambda \in \sigma(F_X) \cap \sigma(F_{X_2})$. Hence (X, X_2) is not a pair of opposite unmixed solutions. $\square$

At this point the pieces can be put together.

THEOREM 5.3. *Let (X_1, X_2) be a pair of opposite unmixed solutions of the DARE $\mathcal{D}(X) = 0$ in (5.1). Set $\Delta = X_1 - X_2$ and*

$$F_{X_2} = F - G(R + G^* X_2 G)^{-1} (G^* X_2 F + S).$$

Let $\mathcal{S}$ be the set of hermitian solutions of $\mathcal{D}(X) = 0$, and define

$$(5.10) \quad \mathcal{N} = \{N \in \text{Inv } F_{X_2} \mid E_0(F_{X_2}) + E_{\partial\mathbb{D}}(F_{X_2}) \subseteq N\}.$$

Assume (Ψ) and $\text{rank}[F - \lambda I, G] = n$ if $\lambda \neq 0$. If $N \in \mathcal{N}$, then

$$(5.11) \quad (\Delta N)^\perp \oplus (N \cap \text{Im } \Delta) = \mathbb{C}^n.$$

Let P_N be the projection on $(\Delta N)^\perp$ along $N \cap \text{Im } \Delta$. Define

$$(5.12) \quad \kappa(N) = X_1 P_N + X_2 (I - P_N).$$

Then $\kappa : \mathcal{N} \rightarrow \mathcal{S}$ is a bijection. If $X \in \mathcal{S}$, then

$$\kappa^{-1}(X) = \text{Ker}(X - X_2).$$

Proof. Let $\mathcal{T}_2$ denote the set of hermitian solutions of $\mathcal{H}_2(Y) = 0$ in (5.2). By Lemma 5.1 we have $X \in \mathcal{S}$, if and only if $Y = X - X_2 \in \mathcal{T}_2$. Hence $\mathcal{S} = X_2 + \mathcal{T}_2$. If (X_1, X_2) is a pair of opposite unmixed solutions, then it follows from Proposition 5.2 that $\Delta = X_1 - X_2$ is the solution of (5.2) with $\text{Ker } \Delta = E_0(F_{X_2}) + E_{\partial\mathbb{D}}(F_{X_2})$. Hence we can parametrize the set $\mathcal{T}_2$ according to Theorem 4.5. If we set $\tilde{\kappa}(N) = \Delta P_N$, then $\tilde{\kappa} : \mathcal{N} \rightarrow \mathcal{T}_2$ is a bijection with $\tilde{\kappa}^{-1}(Y) = \text{Ker } Y$. Thus $\mathcal{S} = X_2 + \mathcal{T}_2$ gives rise to a bijection $\kappa : \mathcal{N} \rightarrow \mathcal{S}$ defined by $\kappa(N) = X_2 + \tilde{\kappa}(N)$. Then $\kappa(N) = X_2 + (X_1 - X_2) P_N$ yields (5.12). Finally, for $X \in \mathcal{S}$ we have $\kappa^{-1}(X) = \tilde{\kappa}^{-1}(X - X_2) = \text{Ker}(X - X_2)$. $\square$

COROLLARY 5.4. *A solution X of (5.1) is unmixed if and only if $\text{Ker}(X - X_2)$ is a spectral subspace of F_{X_2} .*

Proof. Let X be a solution of (5.1) and set $Y = X - X_2$ such that Y is a solution of $\mathcal{H}_2(Y) = 0$ in (5.2). We claim that X is an unmixed solution of (5.1) if and only if the corresponding matrix Y is an unmixed solution of (5.2). Let A be the closed loop matrix associated with Y (with respect to $\mathcal{H}_2 = 0$), i.e.,

$$A = F_{X_2} - G[(R + G^* X_2 G) + G^* Y G]^{-1} G^* Y F_{X_2}.$$

Now the identity (5.3) shows that $A = F_X$. We can apply Lemma 4.6, which tells us that Y is an unmixed solution of (5.2) if and only if $N = \tilde{\kappa}^{-1}(Y) = \text{Ker } Y$ is a spectral subspace of F_{X_2} . $\square$

The pair of opposite solutions (X_-, X_+) satisfying (2.11) consists of the smallest and the greatest solution of (5.1), i.e., for each $X \in \mathcal{S}$ we have $X_- \leq X \leq X_+$. In that particular case, Theorem 5.3 can be obtained from a result on intervals of solutions in [27].

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