INT. J. CONTROL, 1994, VOL. 59, NO. 2, 463–471

Existence of positive-definite and semidefinite solutions of discrete-time algebraic Riccati equations

HARALD K. WIMMER†

The discrete-time algebraic Riccati equation

$$X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XF - H^*H = 0$$

is studied. Necessary and sufficient conditions for the existence of positive-definite and positive-semidefinite solutions are given.

1. Introduction and notation

Let F, G and H be complex matrices of dimensions $n \times n$, $n \times p$ and $q \times n$ respectively. In this note we consider the discrete-time algebraic Riccati equation (DARE)

$$\mathcal{R}(X) = X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XF - H^*H = 0 \quad (1.1)$$

Three results will be proved. Theorem 2.1 provides a necessary and sufficient condition for the existence of a positive-definite solution X of (1.1). In Theorem 3.1 we deal with positive-semidefinite solutions and also with Riccati inequalities. In Theorem 3.2 we compare (1.1) with another DARE given by

$$\tilde{\mathcal{R}}(X) = X - \tilde{F}^*X\tilde{F} + \tilde{F}^*X\tilde{G}(I + \tilde{G}^*X\tilde{G})^{-1}\tilde{G}^*X\tilde{F} - \tilde{H}^*\tilde{H} = 0 \quad (1.2)$$

We will show that the existence of a solution $X \geq 0$ of (1.1) implies the existence of a positive-semidefinite solution of (1.2) provided that

$$\begin{bmatrix} \Delta Q & \Delta F^* \\ \Delta F & -\Delta \Gamma \end{bmatrix} \geq 0$$

where the perturbations are defined by $\Delta Q = H^*H - \tilde{H}^*\tilde{H}$, $\Delta F = F - \tilde{F}$ and $\Delta\Gamma = GG^* - \tilde{G}\tilde{G}^*$.

For some of our results, counterparts are known in the case of the continuous-time algebraic Riccati equation (CARE)

$$F^*X + XF - XGG^*X + H^*H = 0$$

In contrast to Richardson and Kwong (1986) and Geerts (1988) all proofs in this note are strictly algebraic and do not use arguments from linear stability or control theory.

Notation: Let $\mathbb{C}_<$, $\mathbb{C}_\leq$ and $\mathbb{C}_=$ denote the sets of complex numbers with $|\lambda| < 1$, $|\lambda| \leq 1$ and $|\lambda| = 1$ respectively. We define

$$E_\lambda(F) = \operatorname{Ker}(F - \lambda I)^n$$

such that $E_\lambda(F)$ is a generalized eigenspace if $\lambda \in \sigma(F)$, and put

Received 25 February 1991. Revised 8 August 1991. Communicated by Professor V. Kučera.

† Mathematisches Institut, Universität Würzburg, D-8700 Würzburg, Germany.

$$E_<(F) = \bigoplus_{|\lambda|<1} E_\lambda(F)$$

Similarly, we define $E_\leq(F)$ and $E_=(F)$. To the pair (F, G) we associate the reachable subspace

$$R(F, G) = \operatorname{Im}(G, FG, \ldots, F^{n-1}G)$$

and the stabilizable subspace

$$R_<(F, G) = R(F, G) + E_<(F) \tag{1.3}$$

Furthermore we define

$$V(F, H) = \operatorname{Ker}\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

and

$$V_\leq(F, H) = V(F, H) \cap E_\leq(F) \tag{1.4}$$

Put

$$\Gamma = GG^* \quad \text{and} \quad Q = H^*H$$

Then $R_<(F, G) = R_<(F, \Gamma)$ and $V(F, H) = V(F, Q)$. Both $R_<(F, G)$ and $V(F, H)$ are F-invariant subspaces of $\mathbb{C}^n$. Note that the weakly unobservable subspace $V(F, H)$ is the largest F-invariant subspace contained in Ker H. To a solution X of (1.1) we associate the closed loop matrix

$$F_X = F - G(I + G^*XG)^{-1}G^*XF = (I + \Gamma X)^{-1}F \tag{1.5}$$

It is well known that F_X satisfies the equation

$$X = F_X^* X F_X + M^*M + H^*H \tag{1.6}$$

where

$$M = (I + G^*XG)^{-1}G^*XF = G^*XF_X \tag{1.7}$$

2. Auxiliary results, positive-definite solutions

We formulate the main result of this section as a coordinate-free statement.

Theorem 2.1: *Let $R_<(F, G)$ be the stabilizable subspace defined as in (1.3) and let $V_\leq(F, H)$ be given by (1.4). Put*

$$T(F, H) = \bigoplus_{|\lambda|=1} \operatorname{Ker}\begin{bmatrix} F - \lambda I \\ H \end{bmatrix}$$

Then

$$\mathcal{R}(X) = X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XF - H^*H = 0 \tag{1.1}$$

has a positive-definite solution if and only if

$$\mathbb{C}^n = R_<(F, G) \oplus V_\leq(F, H) \tag{2.1}$$

and

$$V_{\leqslant}(F, H) = T(F, H) \tag{2.2}$$

If X is a positive-definite solution of (1.1) then $R_{<}(F, G)$ and $V_{\leqslant}(F, H)$ are X-invariant subspaces.

We have a decomposition (2.1) if and only if there exists a non-singular matrix K such that

$$K^{-1}FK = \operatorname{diag}(F_1, F_2), \quad K^{-1}G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \quad HK = (H_1, 0) \tag{2.3}$$

where (F_1, G_1) is stabilizable and

$$\begin{bmatrix} F_1 - \mu I \\ H_1 \end{bmatrix}$$

has full rank for all $\mu \in \mathbb{C}_{\leqslant}$. If (2.1) holds then (2.2) is satisfied if and only if $\sigma(F_2) \subseteq \mathbb{C}_{=}$ and F_2 is diagonalizable. The proof of the preceding theorem is based on the following lemmas.

Lemma 2.2: *Let X be a positive-semidefinite solution of (1.1). Then* Ker X *is an F-invariant subspace contained in* Ker H *and*

$$F = F_X \quad on \quad \operatorname{Ker} X \tag{2.4}$$

In particular we have

$$\operatorname{Ker} X \subseteq V(F, H)$$

Proof: Recall (1.6) and (1.7). For $y \in \operatorname{Ker} X$ we have $y^*Xy = 0$. Then (1.6) and $X \geqslant 0$ yield $Hy = 0$ and

$$XF_Xy = 0 \tag{2.4}$$

From the definition (1.5) of F_X follows $F_Xy = Fy$ and from (2.4) we obtain $XFy = 0$. Hence, we have shown that (2.4) holds and also

$$F \operatorname{Ker} X \subseteq \operatorname{Ker} X \subseteq \operatorname{Ker} H \qquad \square$$

Lemma 2.3: *Let X be a positive-semidefinite solution of (1.1). If $|\mu| < 1$ then*

$$\operatorname{Ker}\begin{bmatrix} F - \mu I \\ H \end{bmatrix} \subseteq \operatorname{Ker} X$$

If $|\mu| = 1$ then

$$w \in \operatorname{Ker}\begin{bmatrix} F - \mu I \\ H \end{bmatrix} \tag{2.5}$$

implies $Xw \in \operatorname{Ker}(F - \mu I, G)^*$. In particular, if (F, G) is stabilizable and (1.1) has a positive-definite solution then

$$\operatorname{rank}\begin{bmatrix} F - \mu I \\ H \end{bmatrix} = n \quad if \quad |\mu| \leqslant 1$$

Proof: Let w be such that (2.5) holds. Then $w^*\mathcal{R}(X)w = 0$ implies

$$w^*Xw + |\mu|^2 w^*XG^*(I + G^*XG)^{-1}G^*Xw = |\mu|^2 w^*Xw$$

If $|\mu| < 1$ then $X \geqslant 0$ yields $Xw = 0$. In the case $|\mu| = 1$ we obtain $G^*Xw = 0$. From $\mathcal{R}(X)w = 0$ follows $(F^*\mu - I)Xw = 0$. □

The following observation on the Stein equation (2.6) is known, see for example Wimmer (1972). In order to make this note self-contained we include a proof.

Lemma 2.4: *Let A, X and S be complex $n \times n$ matrices such that X and S are hermitian, $X > 0$ and $S \geqslant 0$. If*

$$X - A^*XA = S \tag{2.6}$$

holds then there exists a non-singular K such that

$$K^{-1}AK = \operatorname{diag}(A_1, A_2) \tag{2.7}$$

$$K^*SK = \operatorname{diag}(S_1, 0) \tag{2.8}$$

and

$$K^*XK = \operatorname{diag}(X_1, X_2) \tag{2.9}$$

Furthermore $\sigma(A_1) \subseteq \mathbb{C}_<$ and $\binom{A_1}{S_1}$ is detectable, A_2 is diagonalizable and $\sigma(A_2) \subseteq \mathbb{C}_=$. If W is another positive-semidefinite matrix which satisfies

$$W - A^*WA = S \tag{2.10}$$

then $K^*WK = \operatorname{diag}(X_1, W_2)$ where X_1 is given by (2.9).

Proof: Let $K = (K_1, K_2)$ be a non-singular matrix such that $\operatorname{Im} K_2 = V(A, S)$. Then we have

$$K^{-1}AK = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}$$

and (2.8), and the pair $\binom{A_1}{S_1}$ is observable. Let

$$K^*XK = \begin{bmatrix} X_1 & X_{12} \\ X_{12}^* & X_2 \end{bmatrix} \tag{2.11}$$

be partitioned conformingly. Put

$$L = \begin{bmatrix} I & 0 \\ -X_2^{-1}X_{12}^* & I \end{bmatrix}$$

Then $L^*XL = \operatorname{diag}(X_1, X_2 - X_{12}X_2^{-1}X_{12}^*)$ and $KL = (K_1 - K_2X_2^{-1}X_{12}^*, K_2)$. Hence, in (2.11) we can assume $X_{12} = 0$ such that $K^*XK = \operatorname{diag}(X_1, X_2)$. Then (2.6) is equivalent to the following set of equations

$$X_2 - A_2^*X_2A_2 = 0 \tag{2.12}$$

$$A_2^*X_2A_{21} = 0 \tag{2.13}$$

$$X_1 - A_1^*X_1A_1 - A_{21}^*X_2A_{21} = S_1 \tag{2.14}$$

Because of $X_2 > 0$ we can define $B = X_2^{1/2}A_2X_2^{1/2}$ and write (2.12) as $I - B^*B = 0$. Therefore, A_2 is similar to the unitary matrix B. Both X_2 and A_2 are non-singular, therefore (2.13) yields $A_{21} = 0$, and (2.14) becomes

$$X_1 - A_1^*X_1A_1 = S_1 \tag{2.15}$$

Clearly the assumptions $X_1 > 0$ and $S_1 \geqslant 0$ in (2.15) imply $\sigma(A_1) \subseteq \mathbb{C}_{\leqslant}$. To show that $\sigma(A_1) \subseteq \mathbb{C}_{<}$ assume $A_1 v = \alpha v$, $|\alpha| = 1$. Then

$$v^*(X_1 - A_1^* X_1 A_1)v = 0 = v^* S_1 v$$

Hence, $S_1 v = 0$. Recall $\binom{A_1}{S_1}$ is observable. Therefore, $v = 0$ and A_1 has no unimodular eigenvalues.

Now, assume that $W > 0$ and (2.10) holds. For the blocks of

$$K^*WK = \begin{bmatrix} W_1 & W_{12} \\ W_{12}^* & W_2 \end{bmatrix}$$

we note

$$W_1 - A_1^* W_1 A_1 = S_1 \tag{2.16}$$

and $W_{12} - A_1^* W_{12} A_2 = 0$. Because of $\sigma(A_1) \subseteq \mathbb{C}_{<}$ the solution W_1 of (2.16) is unique. Therefore, $W_1 = X_1$. Similarly $1 \notin \sigma(A_1^*)\sigma(A_2)$ implies $W_{12} = 0$. □

In the following we do not mention a similarity K explicitly as in (2.3) and (2.9), we shall simply say that with respect to an appropriate basis the matrices F, G, H and X have the form given by the right-hand sides in (2.3) and (2.9). Let us also agree that to a partitioning $F = (F_{ij})$, $i, j = 1, 2$, always correspond matrices G and Γ partitioned as

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad \Gamma = (\Gamma_{ij}), \quad i = 1, 2$$

and similarly $H = (H_1, H_2)$, $Q = (Q_{ij})$.

We indicated earlier that Theorem 2.1 can be stated in the following equivalent form. The result for the CARE which corresponds to Theorem 2.5 is due to Richardson and Kwong (1986, p. 103) and in part to Martenson (1971).

Theorem 2.5: *The Riccati equation (1.1) has a positive-definite solution if and only if, with respect to an appropriate basis, the matrices F, G and H are of the form*

$$F = \operatorname{diag}(F_1, F_2),\ F_1 \in \mathbb{C}^{n_1 \times n_1},\ G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$

and $H = (H_1, 0)$ such that the following conditions are satisfied

$$(F_1, G_1) \text{ is stabilizable} \tag{2.17}$$

$$\operatorname{rank} \begin{bmatrix} F_1 - \mu I \\ H_1 \end{bmatrix} = n_1 \quad \text{if} \quad |\mu| \leqslant 1 \tag{2.18}$$

F_2 is diagonalizable and $\sigma(F_2) \subseteq \mathbb{C}_{=}$. Furthermore, if $X > 0$ is a solution of (1.1) then

$$X = \operatorname{diag}(X_1, X_2) \tag{2.19}$$

Proof:

Sufficiency. It is known, see for example de Souza *et al.* (1986) that under the assumptions (2.17) and (2.18) there exists a unique positive-definite solution X_1 of the equation

$$\mathcal{R}_1(X_1) = X_1 - F_1^* X_1 F_1 + F_1^* X_1 G_1 (I + G_1^* X_1 G_1)^{-1} G_1^* X_1 F_1 - H_1^* H_1$$
$$= 0 \tag{2.20}$$

Now, consider the equation

$$X_2 - F_2^* X_2 F_2 = 0 \tag{2.21}$$

where $F_2 = \operatorname{diag}(\alpha_1, \ldots, \alpha_{n_2})$ and $\sigma(F_2) \subseteq \mathbb{C}_=$. Then $X_2 = \operatorname{diag}(p_1, \ldots, p_{n_2})$ is a positive-definite solution of (2.21) if $p_i \in \mathbb{R}$ and $p_i > 0$. Hence, we have a solution $X_1 > 0$ of (2.20) as well as a solution $X_2 > 0$ of (2.21). Therefore, $X = \operatorname{diag}(X_1, X_2)$ is a positive-definite solution of (1.1).

Necessity. Let X be a solution of (1.1) and $X > 0$. Put $A = F_X$ and

$$S = M^*M + H^*H \tag{2.22}$$

where M is given by (1.7). Then (1.6) can be written as $X - A^*XA = S$, and the hypotheses of Lemma 2.4 are satisfied. Hence, an appropriate basis yields $X = \operatorname{diag}(X_1, X_2)$, $A = \operatorname{diag}(A_1, A_2)$ and $S = \operatorname{diag}(S_1, 0)$. Then (2.22) implies $H = (H_1, 0)$ and

$$M = (G_1^*, G_2^*) \operatorname{diag}(X_1 A_1, X_2 A_2) = (M_1, 0)$$

As $X_2 A_2$ is non-singular we obtain $G_2 = 0$ and

$$G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$

From $F = (I + GG^*X)A$ follows $F = \operatorname{diag}(F_1, F_2)$ where $F_1 = (I + G_1 G_1^* X_1) A_1$ and $F_2 = A_2$. The properties of the matrix A_2 are those described in Lemma 2.4. The pair (F_1, G_1) is stabilizable since $\sigma(A_1) \subseteq \mathbb{C}_<$ and

$$A_1 = (I + \Gamma_1 X_1)^{-1} F_1 = F_1 - G_1 (I + G_1^* X_1 G_1)^{-1} G_1^* X_1 F_1$$

With the rank property (2.18), which follows from Lemma 2.3, the proof is complete. □

Corollary 2.6: *Let X and W be positive-definite solutions of (1.1). Then $E_=(F_X) = T(F, H)$ and $F_X = F$ on $T(F, H)$. Furthermore $X = W$ on $R_<(F, G)$.*

We add an observation on the transmission polynomials (= invariant factors) of

$$S(z) = \begin{bmatrix} F - zI & G \\ H & 0 \end{bmatrix}$$

Corollary 2.7: *If the matrices F, G and H satisfy the conditions of Theorem 2.5 then the non-trivial transmission polynomials of $S(z)$ are the invariant factors of $F_2 - \lambda I$. Hence, the transmission zeros of $S(z)$ lie on the unit circle and the corresponding elementary divisors are linear.*

3. Positive-semidefinite solutions, a comparison result

Condition (3.1) below is due to Geerts who proved the equivalence of (1) and (2) of Theorem 3.1 in the case of the CARE using control theory arguments (Geerts 1988, Geerts and Hautus 1990).

Theorem 3.1: *The following statements are equivalent.*

(1) *There exists a positive-semidefinite solution of the Riccati equation*

$$\mathcal{R}(X) = X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XF - H^*H = 0 \quad (1.1)$$

(2) *There exists a positive-semidefinite solution of the Riccati inequality* $\mathcal{R}(X) \geq 0$

(3) $$\mathbb{C}^n = R_<(F, G) + V(F, H) \quad (3.1)$$

Proof:

(3) $\Rightarrow$ (1). Since $V(F, H)$ is an F-invariant subspace we can assume that F and H are given as

$$F = \begin{bmatrix} F_1 & 0 \\ F_{21} & F_2 \end{bmatrix} \quad \text{and} \quad H = (H_1, 0) \quad (3.2)$$

where

$$\begin{bmatrix} F_1 \\ H_1 \end{bmatrix}$$

is observable. If $G^T = (G_1^T, G_2^T)$ then (3.1) holds if and only if the pair (F_1, G_1) is stabilizable (Geerts and Hautus 1990). Obviously $X = \text{diag}(X_1, 0)$ is a solution of $\mathcal{R}(X) = 0$ if X_1 satisfies the equation $\mathcal{R}_1(X_1) = 0$ given by (2.20). Again, the assumptions are such that they guarantee the existence of a positive-definite solution X_1^+ of $\mathcal{R}_1(X_1) = 0$. Hence $X^+ = \text{diag}(X_1^+, 0) \geq 0$ is a solution of (1.1).

(1) $\Rightarrow$ (3). Let $X \geq 0$ be a solution of (1.1). Choose a basis of $\mathbb{C}^n$ such that

$$N = \text{Ker}\, X = \left\{ \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, x_2 \in \mathbb{C}^{m_2} \right\}$$

and

$$X = \text{diag}(X_1, 0), \; X_1 \in \mathbb{C}^{m_1 \times m_1}, \; X_1 > 0$$

Then Lemma 2.2 yields (3.2). We identify $\bar{N} = \mathbb{C}^n/N$ with $\mathbb{C}^{m_1}$ and define a projection $\pi\colon \mathbb{C}^n \to \bar{N}$ by

$$\pi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1$$

such that F_1 is a matrix of the mapping induced by F. Clearly $\pi E_\lambda(F) = E_\lambda(F_1)$ and it is easy to see that $\pi R(F, G) = R(F_1, G_1)$. Because of $N \subseteq \text{Ker}\, H$ we have $\pi V(F, H) = V(F_1, H_1)$. Hence

$$\pi[R_<(F, G) + V(F, H)] = R_<(F_1, G_1) + V(F_1, H_1)$$

Since X_1 is a positive-definite solution of $\mathcal{R}_1(X_1) = 0$ if follows from Theorem 2.1 that

$$\bar{N} = \mathbb{C}^{m_1} = R_<(F_1, G_1) \oplus T(F_1, H_1) = R_<(F_1, G_1) + V(F_1, H_1)$$

Recall $N = \text{Ker}\, X \subseteq V(F, H)$. Hence

$$\mathbb{C}^n = \pi^{-1}\bar{N} + N = R_<(F, G) + V(F, H)$$

(2) $\Rightarrow$ (1). Assume now that there exists a matrix X such that $X \geqslant 0$ and $\mathcal{R}(X) \geqslant 0$. Then, X satisfies a Riccati equation $\mathcal{R}(X) = P$ for some $P \geqslant 0$. Put $\hat{Q} = Q + P$. Then we have

$$X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XF - \hat{Q} = 0 \tag{3.3}$$

From $\hat{Q} \geqslant Q \geqslant 0$ follows $\operatorname{Ker}\hat{Q} \subseteq \operatorname{Ker} Q$ and

$$V(F, \hat{Q}) \subseteq V(F, Q) \tag{3.4}$$

The existence of a positive-semidefinite solution of (3.3) is equivalent to

$$\mathbb{C}^n = R_<(F, G) + V(F, \hat{Q})$$

Thus, (3.4) yields (3.1). □

Let $\tilde{\mathcal{R}}(X)$ be another Riccati operator given by

$$\tilde{\mathcal{R}}(X) = X - \tilde{F}^*X\tilde{F} + \tilde{F}^*X\tilde{G}(I + \tilde{G}^*X\tilde{G})^{-1}\tilde{G}^*X\tilde{F} - \tilde{H}^*\tilde{H}$$

and put $\tilde{\Gamma} = \tilde{G}\tilde{G}^*$ and $\tilde{Q} = \tilde{H}^*\tilde{H}$.

Theorem 3.2: *Assume that the Riccati equation $\mathcal{R}(X) = 0$ has a positive-semidefinite solution. If*

$$\begin{bmatrix} Q & F^* \\ F & -\Gamma \end{bmatrix} \geqslant \begin{bmatrix} \tilde{Q} & \tilde{F}^* \\ \tilde{F} & -\tilde{\Gamma} \end{bmatrix} \tag{3.5}$$

then there exists a positive-semidefinite solution of $\tilde{\mathcal{R}}(X) = 0$.

Proof: By Theorem 3.1 the equation $\mathcal{R}(X) = 0$ has a solution $X \geqslant 0$ if and only if the Geerts condition

$$\mathbb{C}^n = R_<(F, \Gamma) + V(F, Q) \tag{3.6}$$

is satisfied. Let us suppose for the moment that

$$V(\tilde{F}, \tilde{Q}) = V(F, Q) = 0 \tag{3.7}$$

We want to show that $(\tilde{F}, \tilde{G})$ is stabilizable if (3.5) holds and if (F, G) is already a stabilizable pair. With respect to an appropriate basis we have

$$\tilde{F} = \begin{bmatrix} \tilde{F}_1 & \tilde{F}_{12} \\ 0 & \tilde{F}_2 \end{bmatrix}, \tilde{G} = \begin{bmatrix} \tilde{G}_1 \\ 0 \end{bmatrix}, \tilde{\Gamma} = \operatorname{diag}(\tilde{\Gamma}_1, 0)$$

where $(\tilde{F}_1, \tilde{G}_1)$ is controllable. Then, stabilizability of $(\tilde{F}, \tilde{G})$ means

$$\sigma(\tilde{F}_2) \subseteq \mathbb{C}_< \tag{3.8}$$

Note that $\tilde{\Gamma} \geqslant \Gamma \geqslant 0$ implies $\Gamma = \operatorname{diag}(\Gamma_1, 0)$ and

$$G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \tag{3.9}$$

Now consider the matrix

$$F = \begin{bmatrix} \cdot & \cdot \\ F_{21} & F_2 \end{bmatrix}$$

From

$$\begin{bmatrix} Q - \tilde{Q} & F^* - \tilde{F}^* \\ F - \tilde{F} & \tilde{\Gamma} - \Gamma \end{bmatrix} \geqslant 0 \tag{3.10}$$

and $\tilde{\Gamma} - \Gamma = \operatorname{diag}(\tilde{\Gamma}_1 - \Gamma_1, 0)$ we obtain $F_2 = \tilde{F}_2$ and $F_{21} = 0$. Hence

$$F = \begin{bmatrix} F_1 & F_{12} \\ 0 & F_2 \end{bmatrix}, \; F_2 = \tilde{F}_2$$

Since (F, G) is assumed to be stabilizable and G has the form (3.9) we have $\sigma(F_2) = \sigma(\tilde{F}_2) \subseteq \mathbb{C}_<$, which is (3.8).

The restrictive hypothesis (3.7) will now be removed. Our goal is to prove

$$\mathbb{C}^n = R_<(\tilde{F}, \tilde{\Gamma}) + V(\tilde{F}, \tilde{Q}) \tag{3.11}$$

Choose a basis such that

$$V(F, Q) = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\}$$

Then $Q = \operatorname{diag}(0, Q_2)$ and

$$F = \begin{bmatrix} F_1 & F_{12} \\ 0 & F_2 \end{bmatrix}$$

Condition (3.6) holds if and only if (F_2, G_2) is stabilizable. Because of $Q \geq \tilde{Q} \geq 0$ we have $\tilde{Q} = \operatorname{diag}(0, \tilde{Q}_2)$. For

$$\tilde{F} = \begin{bmatrix} \tilde{F}_1 & \cdot \\ \tilde{F}_{21} & \cdot \end{bmatrix}$$

we conclude from $Q - \tilde{Q} = \operatorname{diag}(0, Q_2 - \tilde{Q}_2)$ and (3.5) that $\tilde{F}_1 = F_1$ and $\tilde{F}_{21} = 0$. Therefore

$$V(\tilde{F}, \tilde{Q}) \supseteq V(F, Q) \tag{3.12}$$

From the first part of the proof we know that $(\tilde{F}_2, \tilde{G}_2)$ inherits the stabilizability from the pair (F_2, G_2). Hence, (3.6) and (3.12) yield (3.11). □

Using a result of Wimmer (1992) a shorter proof could be given. Assume that $Y \geq 0$ is a solution of the Riccati inequality $\mathcal{R}(X) \geq 0$. Then, a monotonicity theorem implies $\tilde{\mathcal{R}}(Y) \geq \mathcal{R}(Y)$. Hence, Y is a positive-semidefinite solution of $\tilde{\mathcal{R}} \geq 0$ which, according to Theorem 3.1, implies the existence of a positive-semidefinite solution of $\tilde{\mathcal{R}}(X) = 0$.

References

De Souza, C. E., Gevers, M. R., and Goodwin, G. C., 1986, Riccati equations in optimal filtering of nonstabilizable systems having singular state transition matrices. *IEEE Transactions on Automatic Control*, **31**, 831–838.

Geerts, A. H. W., 1988, A necessary and sufficient condition for solvability of the linear-quadratic control problem without stability. *Systems and Control Letters*, **11**, 47–51.

Geerts, A. H. W., and Hautus, M. L. J., 1990, The output-stabilizable subspace and linear optimal control. *Proceedings of the International Symposium MTNS-1989*, Vol. II, edited by M. A. Kaashoek *et al.* (Boston, Mass: Birkhäuser), pp. 113–120.

Martenson, K., 1971, On the matrix Riccati equation. *Information Science*, **3**, 17–49.

Richardson, T. J., and Kwong, R. H., 1986, On positive definite solutions to the algebraic Riccati equation. *Systems and Control Letters*, **7**, 99–104.

Wimmer, H. K., 1972, Über Matrizen mit semidefinitem Realteil. *Monatshefte für Mathematik*, **76**, 276–281; 1992, Monotonicity and maximality of solutions of discrete-time algebraic Riccati equations. *Journal of Mathematical Systems, Estimation, and Control*, **2**, 219–235.