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***Abstract*—We extend the Eneström-Kakeya theorem and its refinement by Hurwitz to polynomial matrices $H(z)$ with positive semidefinite coefficients. We determine an annular region containing the zeros of $\det H(z)$. A stability result for systems of linear difference equations is given as an application.**

***Index Terms*—Eneström-Kakeya theorem, polynomial matrices, zeros of polynomials, root location, block companion matrix, system of difference equations.**

An Eneström-Kakeya theorem for hermitian polynomial matrices

Gunther Dirr and Harald K. Wimmer

I. INTRODUCTION

The following theorem is known as the Eneström-Kakeya theorem [10, p.4], [3, p.12], [11, p.255].

Theorem 1.1: Let $h(z) = a_m z^m + \cdots + a_1 z + a_0$ be a real polynomial with

$$a_m \geq \cdots \geq a_1 \geq a_0 \geq 0, \;\; a_m > 0. \tag{1}$$

(i) Then all the zeros of $h(z)$ are in the closed unit disc.
(ii) The zeros of $h(z)$ lying on the unit circle are simple.

The Eneström-Kakeya theorem is a powerful tool to determine regions in the complex plane containing zeros of polynomials or eigenvalues of matrices. It has been used to analyze overflow oscillations of discrete-time dynamical systems [8], to investigate properties of orthogonal wavelets [7], to establish the stability of allpass fractional Hilbert transformers [9], and to determine the asymptotic behavior of zeros of the Daubechies filter [12] or the location of zeros of the partial sums of $\cos(z)$ and $\sin(z)$ [13]. For applications to a model of high energy collisions we refer to [4].

It is not difficult to show (see e.g. [3, p.12]) that Theorem 1.1(i) is equivalent to the following.

Theorem 1.2: Let $h(z) = a_0 + a_1 z + \cdots + a_m z^m$ be a real polynomial with positive coefficients. Set

$$\alpha = \min_{0 \leq i \leq m-1} (a_i / a_{i+1}), \;\; \beta = \max_{0 \leq i \leq m-1} (a_i / a_{i+1}).$$

Then the roots λ of $h(z)$ satisfy

$$\alpha \leq |\lambda| \leq \beta. \tag{2}$$

In this note we deal with polynomial matrices

$$H(z) \; = \; A_0 + A_1 z + \cdots + A_m z^m$$

where the coefficients $A_i \in \mathbb{C}^{n \times n}$ are positive semidefinite hermitian matrices. We generalize the preceding two theorems and extend work of [6] und [1] on the sharpness of the Eneström-Kakeya

G. Dirr and H.K. Wimmer are with the Institute of Mathematics, University of Würzburg, Würzburg, D-97074 Germany.

theorem. We apply our results to systems of linear difference equations of the form $\sum A_i x(k+i) = 0$.

We use the following notation. If $M \in \mathbb{C}^{n\times n}$ then $\sigma(M)$ and $\rho(M)$ are the spectrum and the spectral radius of M. For the polyomial matrix $H(z)$ we define $\sigma(H) = \{\lambda \in \mathbb{C};\, \det H(\lambda) = 0\}$ and $\rho(H) = \max\{|\lambda|;\ \lambda \in \sigma(H)\}$.

Let $A, \tilde{A} \in \mathbb{C}^{n\times n}$ be hermitian. We write $A \ge 0$, $\tilde{A} > 0$, if A is positive semidefinite and $\tilde{A}$ is positive definite. Accordingly, $A \ge \tilde{A}$ means that $A - \tilde{A}$ is positive semidefinite. If all the eigenvalues of M are real then $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ shall denote the largest and the smallest eigenvalue of M, respectively. Let $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ be the open unit disc and $\partial\mathbb{D} = \{z \in \mathbb{C}; |z| = 1\}$ the unit circle of the complex plane. The set π_m^+ shall consist of all real polynomials $p(z) = \sum_{i=0}^m a_i z^i$ satisfying (1).

According to [1] it is of interest to know when the inequalities (2) in Theorem 1.2 are sharp, or equivalently when a polynomial $h(z)$ in Theorem 1.1 has no roots with modulus 1. This was first studied by Hurwitz. The following result contains a modified version of Hurwitz's original contribution [6].

Theorem 1.3: Let $h(z) = a_0 + a_1 z + \cdots + a_{m-1}z^{m-1} + a_m z^m$ be a real polynomial with

$$\begin{aligned} 0 \ <\ a_0 = a_1 = \cdots = a_{r_1 - 1} &< \\ a_{r_1} = a_{r_1+1} = \cdots = a_{r_2-1} &< \cdots \\ &< a_{r_s} = a_{r_s+1} = \cdots = a_m. \end{aligned}$$

Set $k = \gcd(m+1, r_1, \dots, r_s)$. We have $\rho(h) = 1$ if and only if $k > 1$. In that case

$$\begin{aligned} 0 < a_0 = \cdots = a_{k-1} \le a_k = \cdots = a_{2k} \le \cdots \\ \le a_{m-k+1} = \cdots = a_m, \end{aligned} \qquad (3)$$

and

$$\begin{aligned} h(z) = (1 + z + \cdots + z^{k-1})p(z^k), \\ p \in \pi_{\ell-1}^+,\ \sigma(p) \cap \partial\mathbb{D} = \emptyset, \end{aligned} \qquad (4)$$

with $\ell = \frac{m+1}{k}$.

II. HERMITIAN POLYNOMIAL MATRICES

In the following let

$$H(z) = A_m z^m + A_{m-1}z^{m-1} + \cdots + A_0 \qquad (5)$$

be a polynomial matrix with hermitian coefficients $A_i \in \mathbb{C}^{n\times n}$, $i = 0, 1, \dots, m$, such that

$$A_m \ge A_{m-1} \ge \cdots \ge A_0 \ge 0,\ \ A_m > 0. \qquad (6)$$

We associate to $u \in \mathbb{C}^n$, $u \ne 0$, the polynomial

$$\begin{aligned} h_u(z) = u^* H(z) u = \sum a_i z^i \\ \text{with} \quad a_i = u^* A_i u,\ i = 0, \dots, m. \end{aligned} \qquad (7)$$

Because of (6) the coefficients of $h_u(z)$ satisfy (1). If $H(\lambda)u = 0$, $u \ne 0$, then $h_u(\lambda) = 0$, and we can apply Theorem 1.1 (i) to $h_u(z)$ and obtain $|\lambda| \le 1$. Thus the following result is obvious.

Theorem 2.1: If the polynomial matrix $H(z)$ satisfies (6) then $\rho(H) \le 1$.

For $k \ge 2$ we define

$$d_k(z) = 1 + z + \cdots + z^{k-1}.$$

Theorem 2.2: Let $|\lambda| = 1$. (i) Suppose

$$\lambda \in \sigma(H) \quad \text{with} \quad H(\lambda)u = 0,\ u \ne 0, \qquad (8)$$

and

$$A_s u \ne 0 \,\text{and}\, A_i u = 0 \text{ if } i < s. \qquad (9)$$

Then $\lambda^k = 1$ for some $k \ge 2$. Set $d(z) = z^s d_k(z)$. Let $U = (u, u_2, \dots, u_n)$ be unitary. Then

$$U^* H(z) U = \begin{pmatrix} d(z)p_{11}(z^k) & d(z)p_{12}(z^k) & \dots & d(z)p_{1n}(z^k) \\ d(z)\overline{p}_{12}(z^k) & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ d(z)\overline{p}_{1n}(z^k) & \cdot & \dots & \cdot \end{pmatrix} \qquad (10)$$

with suitable polynomials $p_{1j}(z^k)$.

(ii) We have $H(\lambda)u = 0$ if and only if $u^* H(\lambda) = 0$.

Proof: Because of (9) we have $h_u(z) = z^s \tilde{h}(z)$, $\tilde{h}(0) \ne 0$. It follows from Theorem 1.3. that there exists a $k > 1$ such that $m + 1 = s + k\ell$ and

$$\begin{aligned} u^* H(z) u = z^s d_k(z) p_{11}(z^k), \\ \text{with } p_{11} \in \pi_{\ell-1}^+,\ \sigma(p_{11}) \cap \partial\mathbb{D} = \emptyset. \end{aligned} \qquad (11)$$

In particular, the coefficients a_i in (7) satisfy

$$\begin{aligned} 0 = a_0 = \cdots = a_{s-1} \le \\ a_s = \cdots = a_{s+k-1} \le \cdots \\ \le a_{m-k+1} = \cdots = a_m. \end{aligned}$$

From $u^* A_m u = \cdots = u^* A_{m-k+1} u$ and $A_m \ge \cdots \ge A_{m-k+1}$ we obtain $u^* A_m = \cdots = u^* A_{m-k+1} = (\alpha_{m1}, \dots, \alpha_{mn})$. Similarly, $u^* A_{m-k} = \cdots = u^* A_{m-2k+1} = (\alpha_{m-k,1}, \dots, \alpha_{m-k,n})$, $\dots$, $u^* A_{s+k-1} = \cdots =$

$u^*A_s = (\alpha_{s+k-1,1}, \dots, \alpha_{s+k-1,n})$, $u^*A_{s-1} = \cdots = u^*A_0 = (0, \dots, 0)$. Hence

$$\begin{aligned}
&u^*H(z)\\
&\quad = (z^m + \cdots + z^{m-k+1})(\alpha_{m1}, \dots, \alpha_{mn}) +\\
&(z^{m-k} + \cdots + z^{m-2k+1})(\alpha_{m-k,1}, \dots, \alpha_{m-k,n}) +\\
&\cdots + (z^{s+k-1} + \cdots + z^{s})(\alpha_{s+k-1,1}, \dots, \alpha_{s+k-1,n})\\
&\quad = d(z)\Big(z^{m-k+1-s}(\alpha_{m1}, \dots, \alpha_{mn}) +\\
&\qquad z^{m-2k+1-s}(\alpha_{m-k,1}, \dots, \alpha_{m-k,n}) +\\
&\qquad\quad \cdots + (\alpha_{s+k-1,1}, \dots, \alpha_{s+k-1,n})\Big)\\
&\quad = d(z)\Big((z^k)^{\ell-1}\alpha_{m1} + (z^k)^{\ell-2}\alpha_{m-k,1} + \cdots\\
&\quad + \alpha_{s+k-1,1}, \dots, (z^k)^{\ell-1}\alpha_{mn} + \cdots + \alpha_{s+k-1,n}\Big)\\
&\quad = d(z)\Big(p_{11}(z^k), \dots, p_{1n}(z^k)\Big),
\end{aligned}$$

and we obtain (10) since all A_i are hermitian.
(ii) Suppose $H(\lambda)u = 0$. Then $h_u(\lambda) = 0$, and therefore $|\lambda| = 1$ and (11) imply $d(\lambda) = 0$. Hence (10) shows that $u^*H(\lambda) = 0$. Conversely, we can use the same line of argument if $u^*H(\lambda) = 0$. ☐

Corollary 2.3: Let $H(z) = \Big(h_{\mu\nu}(z)\Big)$. If $|\lambda| = 1$ and $\lambda \in \sigma(h_{ii})$ for some i, then $\lambda \in \sigma(H)$.

Proof: Let e_i be the i-th unit vector of $\mathbb{C}^n$. If $\lambda \in \partial\mathbb{D}$ and $e_i^* H(\lambda) e_i = h_{ii}(\lambda) = 0$ then we have $H(\lambda)e_i = 0$, and therefore $\lambda \in \sigma(H)$. ☐

Theorem 2.4: If $\lambda \in \sigma(H)$ and $|\lambda| = 1$ then the corresponding elementary divisors of $H(z)$ have degree 1. Moreover, $\lambda \neq 1$, and $\lambda^k = 1$ for some $k \in \mathbb{N}$, $k \leq m+1$. If $A_0 > 0$ then $\lambda^{m+1} = 1$.

Proof: If $\lambda \in \sigma(H)$ has an elementary divisor of degree at least 2 then (see e.g. [2, p. 342]) there exist nonzero vectors u, w such that $H(\lambda)u = 0$ and $H'(\lambda)u + H(\lambda)w = 0$. Suppose $|\lambda| = 1$. Then $u^*H(\lambda) = 0$, and therefore $h_u'(\lambda) = u^*H'(\lambda)u = 0$. Then $\lambda \in \partial\mathbb{D}$ would be a multiple zero of $h_u(z)$, which is a contradiction to Theorem 1.1 (ii).

It is easy to see that $1 \notin \sigma(H)$. Suppose (8). Then (11) implies $d_k(\lambda) = (\lambda^k - 1)(\lambda - 1)^{-1} = 0$, and we obtain $\lambda^k = 1$. If $A_0 > 0$ then $s = 0$ in (11). Therefore $m + 1 = k\ell$, and $\lambda^{m+1} = 1$. ☐

Furuta and Nakamura [5] extended Theorem 1.1 (i) to polynomials $\sum A_i z^i$ with positive definite operator coefficients A_i. The approach in [5] is different, its relies on a power inequality for the numerical radius of an operator acting on a Hilbert space.

Note that Theorem 1.3 completely characterizes those polynomials $h(z) \in \pi_m^+$ with $a_0 > 0$, which satisfy $\rho(h) < 1$. In the case of polynomial matrices $H(z)$ we can only give sufficient conditions for $\rho(H) < 1$.

Theorem 2.5: Assume

$$0 < A_0 \leq A_1 \leq \cdots \leq A_{r_1-1} < A_{r_1} \leq A_{r_1+1} \leq \cdots \leq A_{r_2-1} < \cdots < A_{r_s} \leq A_{r_s+1} \leq \cdots \leq A_m. \tag{12}$$

(i) If $\gcd(m+1, r_1, \dots, r_s) = 1$ then $\rho(H) < 1$.
(ii) In particular, we have $\rho(H) < 1$ if $0 < A_0 < A_1 < \cdots < A_m$.

Proof: Let $h_u(z)$ be defined as in (7). Then (12) implies $a_{r_j-1} < a_{r_j}$, $j = 1, \dots, s$, and Theorem 1.3 yields $\rho(h) < 1$. Hence we have $u^*H(\lambda)u \neq 0$ for all $\lambda \in \partial\mathbb{D}$, and therefore $\rho(H) < 1$. ☐

We now turn to Theorem 1.2 and its generalization. Suppose $A, \tilde{A} \in \mathbb{C}^{n\times n}$ are positive semidefinite. Let $A^\sharp$ denote the Moore-Penrose generalized inverse of A. It is obvious that there exists a number $b \in \mathbb{R}$ such that $bA \geq \tilde{A}$, if and only if

$$\operatorname{Ker} A \subseteq \operatorname{Ker} \tilde{A}. \tag{13}$$

It is not difficult to verify that under the hypothesis (13) we have

$$\min\{b \in \mathbb{R};\, bA \geq \tilde{A}\} = \lambda_{\max}(\tilde{A}A^\sharp).$$

If $\tilde{A} > 0$ then (13) implies $A > 0$. In that case

$$\max\Big\{a \in \mathbb{R};\, aA \leq \tilde{A}\Big\} = \lambda_{\min}(\tilde{A}A^{-1}).$$

Theorem 2.6: Let $H(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a polynomial matrix with $A_i \geq 0$, $i = 0, \dots, m-1$, and $A_m > 0$. Suppose

$$\operatorname{Ker} A_{m-1} \subseteq \cdots \subseteq \operatorname{Ker} A_0. \tag{14}$$

Define

$$\beta = \max\Big\{\lambda_{\max}(A_i\, A_{i+1}^\sharp);\, i = 0, \dots, m-1\Big\},$$

and either $\alpha = 0$ if $\det A_0 = 0$, or

$$\alpha = \min\Big\{\lambda_{\min}(A_i\, A_{i+1}^{-1});\, i = 0, \dots, m-1\Big\} \tag{15}$$

if $A_0 > 0$. Then

$$\alpha \leq |\lambda| \leq \beta \quad \text{for all} \quad \lambda \in \sigma(H). \tag{16}$$

Proof: Suppose $A_{m-1} \neq 0$. Then $\beta > 0$. We have $A_{i+1}\beta \geq A_i$, and thus $A_{i+1}\beta^{i+1} \geq A_i\beta^i$,

$i = 0, \dots, m-1$. Moreover, $A_m \beta^m > 0$. Therefore we can apply Theorem 2.1 to the polynomial matrix

$$H(\beta z) = A_0 + A_1 \beta z + \cdots + A_m \beta^m z^m,$$

and we obtain $|\lambda| \le \beta$ for all $\lambda \in \sigma(H)$. If $A_{m-1} = 0$ then (14) implies $H(z) = A_m z^m$. Thus $\sigma(H) = \{0\}$ and $\beta = 0$.

Suppose $A_0 > 0$. Then (14) implies $A_i > 0$ for all i. Hence α in (15) is well defined. The argument concerning $H(\beta z)$ can be adapted for $z^m H(\alpha/z)$. It yields the lower bound in (16). In the case $\det A_0 = 0$ we have $0 \in \sigma(H)$, and $\alpha = 0$ is the trivial bound lower bound in (16). $\square$

We remark that in the case of $\det A_0 = 0$ the lower bound $\alpha = 0$ is rather conservative. A more detailed investigation of the role of the characteristic root $0 \in \sigma(H)$ would yield a sharper estimate for the nonzero elements of $\sigma(H)$.

III. A DIFFERENCE EQUATION

We now apply Theorem 2.1 and Theorem 2.4 to a linear system of difference equations of the form

$$A_0 x(t) + \cdots + A_{m-1} x(t+m-1) + A_m x(t+m) = 0,$$
$$x(0) = x_0, \dots, x(m-1) = x_{m-1}. \quad (17)$$

Assuming

$$0 \le A_0 \le \cdots \le A_{m-1} \le A_m, \; 0 < A_m, \quad (18)$$

we shall prove that in general each solution $\big(x(t)\big)$ can be decomposed into a norm-decreasing part $\big(y(t)\big)$ and a periodic part $\big(w(t)\big)$.

Theorem 3.1: Suppose (18), and let $\big(x(t)\big)$ be a solution of (17). Then

$$x(t) = y(t) + w(t), \; t \ge 0,$$

such that $\lim_{t\to\infty} y(t) = 0$ and $\big(w(t)\big)$ is periodic, i.e. there exists a $p \ge 2$ such that $w(t+p) = w(t)$ for all $t \ge 0$. In particular, if $A_0 > 0$ then $p = m+1$ is a period of $\big(w(t)\big)$.

Proof: Because of

$$A_m^{-1/2} H(z) A_m^{-1/2} = I z^m + \cdots + A_m^{-1/2} A_0 A_m^{-1/2}$$

we can assume $A_m = I$. Set

$$v(t) = (x(t)^T, x(t+1)^T, \cdots, x(t+m-1)^T)^T$$

and define v_0 conforming to (17). Let

$$A = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ -A_0 & -A_1 & -A_2 & \dots & -A_{m-1} \end{pmatrix}$$

be the block companion matrix associated with $H(z) = \sum A_i z^i$. Then (17) is equivalent to

$$v(t+1) = A v(t), \; v(0) = v_0. \quad (19)$$

Thus $\sigma(H) = \sigma(A) \subseteq \overline{\mathbb{D}}$. Suppose $\rho(H) = \rho(A) = 1$. Let Y and W be A-invariant subspaces such that $\mathbb{C}^n = Y \oplus W$ and $\sigma(A_{|Y}) \subseteq \mathbb{D}$ and $\sigma(A_{|W}) \subseteq \partial\mathbb{D}$. Let the initial value v_0 in (19) be decomposed correspondingly as $v_0 = y_0 + w_0$, $y_0 \in Y$, $w_0 \in W$. If $\big(y(t)\big)$ and $\big(w(t)\big)$ are the solutions of

$$y(t+1) = Ay(t), \, y(0) = y_0,$$
$$\text{and } w(t+1) = Aw(t), \, w(0) = w_0,$$

respectively, then we have $v(t) = y(t) + w(t)$. From $\rho(A_{|Y}) < 1$ follows $\lim_{t\to\infty} y(t) = 0$. If $\lambda \in \sigma(H) \cap \partial\mathbb{D}$ is a zero of multiplicity r of $\det H(z)$ then Theorem 2.4 implies $A_{|\operatorname{Ker}(A-\lambda I)} = \lambda I_r$ and $\lambda^k = 1$ for some $k \ge 2$, depending on λ. Let p be the least common multiple of all those k. If $\dim W = q$ then $\big(A_{|W}\big)^p = I_q$. Therefore, $w(t+p) = A^p w(k) = w(t)$ for all $t \ge 0$. If $A_0 > 0$ then $p | (m+1)$ by Theorem 2.4. $\square$

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