Arch. Math., Vol. 55, 117–119 (1990)

0003-889X/90/5502-0117 $ 2.10/0

Extreme properties of products of invariant factors

By

HARALD K. WIMMER

In this note we characterize products of invariant factors by a sort of "maximum-minimum" property which extends a result of H. Flanders [2, Thm. 3]. In the sequel L will always be a free module of rank n over a principal ideal domain R. Let E and F be two submodules of L such that $E \supseteqq F$. The *relative weight* of F in E is an ideal denoted by $w(E, F)$, which is defined as follows. Suppose rank E = rank $F = m$ and let $x_1, \dots, x_m$ and $y_1, \dots, y_m$ be bases of E and F respectively. Then $y_i = \sum_j a_{ij} x_j$ and $A = (a_{i_j})$ is a nonsingular matrix. As in [2] we put

$$w(E, F) = (\det A).$$

In the case where rank $E >$ rank F we set $w(E, F) = 0$.

The following notation will be fixed. Let M be a rank r submodule of L, and let $\alpha_1, \dots, \alpha_r$ be the invariant factors of M, ordered such that $\alpha_1 | \alpha_2 \cdots | \alpha_r$. Set $\alpha_{r+1} = \cdots = \alpha_n = 0$.

Theorem 1 (H. Flanders [2, p. 480]). *For each $s = 1, 2, \dots, n$ we have*

$$\alpha_1 \alpha_2 \cdots \alpha_s = \gcd_N \{w(N, N \cap M)\}$$

where N ranges over all ranks s direct summands of L.

Our generalization of the preceding theorem and its proof are closely related to results of A. R. Amir-Moéz [1] on eigenvalues of hermitian matrices. We call $G_1 \subseteqq G_2 \subseteqq \dots \subseteqq G_s$ a *direct summand chain* if each G_ν is a direct summand of L.

Theorem 2. *For any integers k_ν such that $1 \leqq k_1 < k_2 < \cdots < k_s \leqq n$ we have*

$$(1) \qquad \alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_s} = \gcd_{G_1 \subseteqq \dots \subseteqq G_s} \left\{ \operatorname{lcm}_{N = [x_1, \dots, x_s]} w(N, N \cap M) \right\}$$

where $G_1 \subseteqq \dots \subseteqq G_s$ ranges over all direct summand chains with rank $G_\nu = k_\nu$ *and $N = [x_1, \dots, x_s]$ ranges over all rank s direct summands of L with $x_\nu \in G_\nu$.*

For convenience the elements of R appearing in multiplicative equations are fixed modulo units. Accordingly $\gcd(a_1, \dots, a_n)$ and $\operatorname{lcm}(a_1, \dots, a_n)$ denote either the ideals they generate or generators of those ideals. We write $[x_1, \dots, x_s]$ for the submodule spanned *by* $x_1, \dots, x_s$. – For the proof of (1) we need the following facts.

Lemma 1 [2]. *Let E be free over R and of finite rank and let F and G be submodules of E such that $E \supseteqq F \supseteqq G$. Then*

$$w(E, G) = w(E, F)\, w(F, G).$$

Lemma 2 (A. Horn [1]). *Let $G_1 \subseteqq G_2 \subseteqq \cdots \subseteqq G_s$ and $H_s \supseteqq H_{s-1} \supseteqq \cdots \supseteqq H_1$ be two direct summand chains of L such that* $\operatorname{rank} G_\nu + \operatorname{rank} H_\nu = 1 + \operatorname{rank} L$, $\nu = 1, 2, \ldots, s$. *Then there exists a direct summand D of L with a basis $x_1, \ldots, x_s$ such that $x_\nu \in G_\nu$ and with another basis $y_1, \ldots, y_s$ where $y_\nu \in H_\nu$.*

In [1] the preceding result is proved for vector spaces. The proof can easily be carried over to R-modules.

Proof of Theorem 2. Put $p = \alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_s}$ and let $d = \gcd \{\operatorname{lcm} \ldots\}$ denote the right hand side of (1). We call a submodule N *admissible* for a chain $G_1 \subseteqq \cdots \subseteqq G_s$ if N is a rank s direct summand of L and if $N = [x_1, \ldots, x_s]$ such that $x_\nu \in G_\nu$, $\nu = 1, \ldots, s$. Let $b_1, \ldots, b_n$ be a fixed basis of L chosen in such a way that $\alpha_1 b_1, \ldots, \alpha_r b_r$ is a basis of M. We can assume that $k_s \leqq r$.

In order to show that $d \mid p$ we introduce the submodules $\tilde{G}_\nu = [b_1, b_2, \ldots, b_{k_\nu}]$, $\nu = 1, \ldots, s$. We have $\operatorname{rank} \tilde{G}_\nu = k_\nu$, and $\tilde{G}_\nu$ is a direct summand of L. The submodule $\tilde{N} = [b_{k_1}, \ldots, b_{k_s}]$ is admissible for the chain $\tilde{G}_1 \subseteqq \cdots \subseteqq \tilde{G}_s$. Note that $\tilde{N} \cap M$ has the basis $\alpha_{k_1} b_{k_1}, \ldots, \alpha_{k_s} b_{k_s}$. Hence $w(\tilde{N}, \tilde{N} \cap M) = p$. Now let $N' = [u_1, \ldots, u_s]$ be an arbitrary admissible submodule for $\tilde{G}_1 \subseteqq \cdots \subseteqq \tilde{G}_s$. Put $Y = [\alpha_{k_1} u_1, \ldots, \alpha_{k_s} u_s]$. Obviously $w(N', Y) = p$. Because of $u_\nu \in \tilde{G}_\nu$ we have $Y \subseteqq N' \cap M$ and from Lemma 1 follows $w(N', Y) = w(N', N' \cap M)\, w(N' \cap M, Y)$. Therefore $w(N', N' \cap M) \mid p$ and we see that $p = \operatorname{lcm} \{w(N', N' \cap M)\}$ where N' ranges over all submodules which are admissible for the special chain $\tilde{G}_1 \subseteqq \cdots \subseteqq \tilde{G}_s$. The definition of d implies $d \mid p$.

To establish the relation $p \mid d$ we consider an arbitrary direct summand chain $G_1 \subseteqq \cdots \subseteqq G_s$ of L where $\operatorname{rank} G_\nu = k_\nu$. For $\nu = 1, \ldots, s$ put $H_\nu = [b_{k_\nu}, b_{k_\nu + 1}, \ldots, b_n]$. According to Lemma 2 there exists a direct summand D which is both admissible for $G_1 \subseteqq \cdots \subseteqq G_s$ and for $H_s \subseteqq \cdots \subseteqq H_1$. Let $y_1, \ldots, y_s$ be a basis of D such that $y_\nu \in H_\nu$, $\nu = 1, \ldots, s$. We have

$$y_\nu = \sum c_{j\nu} b_j, \quad c_{j\nu} = 0 \quad \text{for} \quad j < k_\nu, \tag{2}$$

and as D is a direct summand the gcd of the $s \times s$ minors of $C = (c_{\nu j})$ is 1. Suppose first that $\operatorname{rank} (D \cap M) = s$ and let $z_1, \ldots, z_s$ be a basis of $D \cap M$. We have two representations of z_ν, namely $z_\nu = \sum \alpha_{\sigma\nu} y_\sigma$ and $z_\nu = \sum q_{j\nu} \alpha_j b_j$. The matrices $A = (a_{\sigma\nu})$ and $Q = (q_{j\nu})$ satisfy

$$CA = \operatorname{diag} (\alpha_1, \ldots, \alpha_n)\, Q,$$

and $\det A = w(D, D \cap M)$. Choose rows $m_1 < \cdots < m_s$ of C which yield a submatrix $\hat{C}$ with $\det \hat{C} = 1$. Denote the corresponding submatrix of Q by $\hat{Q}$ such that

$$\hat{C} A = \operatorname{diag} (\alpha_{m_1}, \ldots, \alpha_{m_s})\, \hat{Q}.$$

Then $\det A = \alpha_{m_1} \cdots \alpha_{m_s} \cdot \det \hat{Q}$. Now (2) implies $m_\nu \geqq k_\nu$, $\nu = 1, \ldots, s$. Hence $p \mid \det A$. If $\operatorname{rank} (D \cap M) < s$ then $w(D, D \cap M) = 0$ and trivially $p \mid w(D, D \cap M)$. In the lcm below let $\mathfrak{N}$ be the set of all submodules which are admissible for the given chain $G_1 \subseteqq \ldots \subseteqq G_s$. We have shown that $p \mid w(D, D \cap M) \mid \operatorname{lcm}\{w(N, N \cap M),\ N \in \mathfrak{N}\}$. Hence $p \mid d$, which completes the proof.

Acknowledgement. This work was supported by a NATO Collaborative Research Grant.

Note added in proof.*) I am grateful to Dr. B. Alanced for calling my attention to the paper on “Generalized minimax and interlacing theorems” by D. Carlson and E. M. de Sá (Linear Multilin. Algebra **15**, 77–103 (1984)) which provides an abstract framework for extremum formulas for invariant factors and singular values.

References

[1] A. R. AMIR-MOÉZ, Extreme properties of eigenvalues of a hermitian transformation and singular values of the sum and product of linear transformations. Duke Math. J. **23**, 463–476 (1956).
[2] H. FLANDERS, Finitely generated modules. Duke Math. J. **22**, 477–483 (1955).

Eingegangen am 13. 12. 1988 **)

Anschrift des Autors:

H. K. Wimmer
Mathematisches Institut
der Universität Würzburg
Am Hubland
D-8700 Würzburg

---

*) Eingegangen am 27. 3. 1990.
**) Eine Neufassung ging am 15. 11. 1989 ein.