Canonical angles of unitary spaces and perturbations of direct complements

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Abstract

Let $\mathbb{C}^n = \mathcal{L} \oplus \mathcal{W}$ be a given direct sum decomposition. We determine the largest number r_i such that all subspaces $\mathcal{M}$ for which the gap θ between $\mathcal{L}$ and $\mathcal{M}$ satisfies $\theta(\mathcal{L}, \mathcal{M}) < r_i$ have the property $\dim(\mathcal{M} \cap \mathcal{W}) < i$. The problem involves angles between subspaces, i.e. singular values of products of projections.

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Running Head: Perturbation of direct complements.

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1 Introduction

Let $\mathcal{L} \oplus \mathcal{W} = \mathbb{C}^n$ be a given direct sum decomposition of $\mathbb{C}^n$. It is well known (see e.g. [2, p. 390]) that all subspaces $\mathcal{M}$ of $\mathbb{C}^n$ which are sufficiently close to $\mathcal{L}$ (with respect to the gap metric) are also direct complements of $\mathcal{W}$. In [1] and [7] the largest ε-neighbourhood $U(\mathcal{L}, \varepsilon)$ of $\mathcal{L}$ has been determined which has the property that $\mathcal{M} \in U(\mathcal{L}, \varepsilon)$ implies $\mathcal{M} \oplus \mathcal{W} = \mathbb{C}^n$. To describe the result of Berkson [1] and Schumacher [7] and a more general result, which is the main theorem of this paper, we use the following notation.

We consider $\mathbb{C}^n$ with the usual inner product $(x, y) = x^*y$. Subspaces of $\mathbb{C}^n$ will be denoted by script capital letters such as $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{M}$, etc. Let $P_{\mathcal{W}}$ denote the orthogonal projection of $\mathbb{C}^n$ on $\mathcal{W}$, and $P(\mathcal{W}, \mathcal{L})$ the projection on $\mathcal{W}$ along $\mathcal{L}$, and set $P(\mathcal{W}, \mathcal{L}; \mathcal{M}) = P(\mathcal{W}, \mathcal{L})_{|\mathcal{M}}$. For a linear map $A \colon \mathcal{Y} \to \mathbb{C}^n$ let $\|A\|$ be the operator norm, i.e. $\|A\| = \sup\{\|Ay\|, y \in \mathcal{Y}, \|y\| = 1\}$. The singular values $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_n(A)$ of A will always be ordered by decreasing magnitude such that $\sigma_1(A) = \|A\|$.

The set of subspaces of $\mathbb{C}^n$ is a metric space (see e.g. [4], [2]) if the distance between $\mathcal{X}$ and $\mathcal{Y}$ is measured by the gap

$$\theta(\mathcal{X}, \mathcal{Y}) = \|P_{\mathcal{X}} - P_{\mathcal{Y}}\|.$$

Note that $\theta(\mathcal{X}, \mathcal{Y}) \leq 1$ and $\theta(\mathcal{X}, \mathcal{Y}) = 1$ if $\dim \mathcal{X} \neq \dim \mathcal{Y}$. If $\mathcal{X}$ and $\mathcal{Y}$ are nontrivial subspaces and $r = \min\{\dim \mathcal{X}, \dim \mathcal{Y}\}$ then we define the canonical angles

$$0 \leq \varphi_1(\mathcal{X}, \mathcal{Y}) \leq \ldots \leq \varphi_r(\mathcal{X}, \mathcal{Y}) \leq \frac{\pi}{2}$$

of $\mathcal{X}$, $\mathcal{Y}$ by

$$\cos \varphi_i(\mathcal{X}, \mathcal{Y}) = \sigma_i(P_{\mathcal{X}} P_{\mathcal{Y}}), \quad i = 1, \ldots, r.$$

If $\dim \mathcal{X} = \dim \mathcal{Y} = r$ then an equivalent definition (see e.g. [8]) is given by

$$\sin \varphi_i(\mathcal{X}, \mathcal{Y}) = \sigma_{r-i+1}(P_{\mathcal{X}} P_{\mathcal{Y}^\perp}), \quad i = 1, \ldots, r.$$

We set

$$\varphi_{\min}(\mathcal{X}, \mathcal{Y}) = \varphi_1(\mathcal{X}, \mathcal{Y}),$$

and if $\dim \mathcal{X} = \dim \mathcal{Y} = r$ we set

$$\varphi_{\max}(\mathcal{X}, \mathcal{Y}) = \varphi_r(\mathcal{X}, \mathcal{Y}).$$

The subsequent theorem relates robustness of complementarity of $\mathcal{L} \oplus \mathcal{W}$ to the minimal angle of $\mathcal{L}$ and $\mathcal{W}$. It gives a sharp bound r_1 such that $\mathbb{C}^n = \mathcal{L} \oplus \mathcal{W}$ and $\theta(\mathcal{L}, \mathcal{M}) < r_1$ imply $\dim(\mathcal{M} \cap \mathcal{W}) = 0$.

Theorem 1.1 [1], [7]. *Let $\mathcal{L}$ and $\mathcal{W}$ be nontrivial complementary subspaces of $\mathbb{C}^n$. Then every subspace $\mathcal{M}$ that satisfies*

$$\theta(\mathcal{L},\mathcal{M}) < \sin\varphi_{\min}(\mathcal{W},\mathcal{L})$$

is complementary to $\mathcal{W}$, and

$$\inf\{\theta(\mathcal{L},\mathcal{M}) \mid \mathcal{M}\cap\mathcal{W}\neq 0\} = \sin\varphi_{\min}(\mathcal{W},\mathcal{L}).$$

In this paper we want to extend Theorem 1.1. We shall determine bounds r_k, $1\le k\le r$, such that $\theta(\mathcal{L},\mathcal{M}) < r_k$ implies $\dim(\mathcal{M}\cap\mathcal{W}) < k$. Since the relative position of two subspaces of a unitary space is completely determined by their canonical angles it is not surprising that the bounds r_k can be expressed in terms of angles between $\mathcal{L}$ and $\mathcal{W}$. The following theorem will be proved in Section 2.

Theorem 1.2 *Let $\mathcal{L}$ and $\mathcal{W}$ be subspaces of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathcal{L}\oplus\mathcal{W}$ and $r=\min\{\dim\mathcal{L},\dim\mathcal{W}\}\neq 0$. Assume $1\le k\le r$.*

(i) If a subspace $\mathcal{M}$ satisfies

$$\dim\mathcal{M} = \dim\mathcal{L} \tag{1.1}$$

and

$$\sigma_k[P(\mathcal{W},\mathcal{L};\mathcal{M})] < 1 \tag{1.2}$$

then

$$\dim(\mathcal{M}\cap\mathcal{W}) < k. \tag{1.3}$$

In particular (1.1) and (1.2) are valid if

$$\theta(\mathcal{L},\mathcal{M}) < \sin\varphi_k(\mathcal{W},\mathcal{L}) \tag{1.4}$$

or equivalently if

$$\varphi_{\max}(\mathcal{L},\mathcal{M}) < \varphi_k(\mathcal{W},\mathcal{L}). \tag{1.5}$$

(ii) There exists a subspace $\mathcal{M}$ such that (1.1) and

$$\theta(\mathcal{L},\mathcal{M}) = \sin\varphi_k(\mathcal{W},\mathcal{L}) \tag{1.6}$$

and

$$\dim(\mathcal{M}\cap\mathcal{W}) \ge k \tag{1.7}$$

hold. For such an $\mathcal{M}$ we have

$$\sigma_k[P(\mathcal{W},\mathcal{L};\mathcal{M})] = 1. \tag{1.8}$$

2 Auxiliary results and proof of the main theorem

Assume $\mathbb{C}^n = \mathcal{L} \oplus \mathcal{W}$ with $\dim \mathcal{L} = s$ and $r = \min\{\dim \mathcal{L}, \dim \mathcal{W}\} \neq 0$. A suitable choice of an orthonormal basis of $\mathbb{C}^n$ allows us to work with $\mathcal{L}$ and $\mathcal{W}$ in the form

$$\mathcal{L} = \operatorname{Im} \begin{pmatrix} I_s \\ 0 \end{pmatrix}, \quad \mathcal{W} = \operatorname{Im} \begin{pmatrix} W_{12} \\ I_{n-s} \end{pmatrix}. \tag{2.1}$$

Then $P_{\mathcal{L}} = \operatorname{diag}(I_s, 0)$ and

$$P_{\mathcal{W}} = \begin{pmatrix} W_{12} \\ I \end{pmatrix} (I + W_{12}^* W_{12})^{-1} (W_{12}^*, I)$$

and

$$P(\mathcal{W}, \mathcal{L}) = \begin{pmatrix} 0 & W_{12} \\ 0 & I \end{pmatrix}.$$

Hence

$$\sigma_k^2[P(\mathcal{W}, \mathcal{L})] = \sigma_k(I + W_{12}^* W_{12}). \tag{2.2}$$

From

$$P_{\mathcal{L}} P_{\mathcal{W}} P_{\mathcal{L}} = \operatorname{diag}[W_{12}(I + W_{12}^* W_{12})^{-1} W_{12}^*, 0]$$

we obtain

$$\begin{aligned} \cos^2 \varphi_k(\mathcal{W}, \mathcal{L}) &= \sigma_k^2(P_{\mathcal{W}} P_{\mathcal{L}}) = \sigma_k[W_{12}(I + W_{12}^* W_{12})^{-1} W_{12}^*] = \\ &= \sigma_k[I - (I + W_{12} W_{12}^*)^{-1}] = 1 - \frac{1}{\sigma_k(I + W_{12}^* W_{12})}, \\ & k = 1, \ldots, r, \end{aligned}$$

and we note the following observation.

Lemma 2.1 *If* $\mathbb{C}^n = \mathcal{L} \oplus \mathcal{W}$ *and* $r = \min\{\dim \mathcal{L}, \dim \mathcal{W}\} \neq 0$ *then*

$$\sin \varphi_k(\mathcal{W}, \mathcal{L}) = \frac{1}{\sigma_k[P(\mathcal{W}, \mathcal{L})]}, \quad k = 1, \ldots, r. \tag{2.3}$$

In the case $k = 1$ we can write (2.3) as

$$\|P(\mathcal{W}, \mathcal{L})\|^2 = \frac{1}{1 - \|P_{\mathcal{L}} P_{\mathcal{W}}\|^2}. \tag{2.4}$$

The identity (2.4) is due to Ljance [5] (see also [6]).

If $\dim \mathcal{M} = s$ and $\mathcal{M}$ is represented as

$$\mathcal{M} = \operatorname{Im} \begin{pmatrix} M_1 \\ M_{21} \end{pmatrix}, \quad M_1^* M_1 + M_{21}^* M_{21} = I_s \tag{2.5}$$

then

$$\sigma_k^2[P(\mathcal{W}, \mathcal{L}; \mathcal{M})] = \sigma_k[M_{21}^*(I + W_{12}^* W_{12}) M_{21}]. \tag{2.6}$$

The following result on the gap can be found in [8]. In our set-up the proof is straightforward.

Lemma 2.2 *If* $\dim \mathcal{L} = \dim \mathcal{M} = s$*,* $s \neq 0$*, then*

$$\theta(\mathcal{L}, \mathcal{M}) = \sin \varphi_{\max}(\mathcal{L}, \mathcal{M}). \tag{2.7}$$

Proof. Let $\mathcal{L}$ and $\mathcal{M}$ be given as in (2.1) and (2.5). Then

$$(P_{\mathcal{L}} - P_{\mathcal{M}})^2 = \operatorname{diag}(I - M_1 M_1^*, M_{21} M_{21}^*),$$

and $\|I - M_1 M_1^*\| = \|M_{21} M_{21}^*\|$ yields

$$\theta(\mathcal{L}, \mathcal{M}) = \|M_{21}\|.$$

Hence (2.7) follows from

$$\begin{aligned} &\cos^2 \varphi_{\max}(\mathcal{L}, \mathcal{M}) = \sigma_s(P_{\mathcal{L}} P_{\mathcal{M}} P_{\mathcal{L}}) = \sigma_s(M_1 M_1^*) = \\ &= \sigma_s(I - M_{21}^* M_{21}) = 1 - \|M_{21}\|^2 = 1 - \theta^2(\mathcal{L}, \mathcal{M}). \end{aligned}$$

■

We shall need an estimate for the singular values of $P(\mathcal{W}, \mathcal{L}; \mathcal{M})$.

Lemma 2.3 *Assume* $\mathbb{C}^n = \mathcal{L} \oplus \mathcal{W}$*,* $r = \min\{\dim \mathcal{L}, \dim \mathcal{W}\} \neq 0$*. If* $\mathcal{M}$ *is a subspace with* $\dim \mathcal{M} = \dim \mathcal{L}$ *then*

$$\sigma_k[P(\mathcal{W}, \mathcal{L}; \mathcal{M})] \leq \sigma_k[P(\mathcal{W}, \mathcal{L})] \theta(\mathcal{L}, \mathcal{M}), \quad k = 1, \ldots, r. \tag{2.8}$$

Proof. Using an inequality for singular values of products [3, p.178] we obtain

$$\begin{aligned} &\sigma_k[P(\mathcal{W}, \mathcal{L}; \mathcal{M})] \leq \sigma_k[(I + W_{12}^* W_{12})^{1/2} M_{21}] \leq \\ &\leq \sigma_k[(I + W_{12}^* W_{12})^{1/2}] \sigma_1(M_{21}) = \sigma_k[P(\mathcal{W}, \mathcal{L})] \theta(\mathcal{L}, \mathcal{M}). \end{aligned}$$

■

Proof of Theorem 1.2. (i) Let $\mathcal{W}$ and $\mathcal{M}$ be given as in (2.1) and (2.5). Put

$$T = \begin{pmatrix} M_1 & W_{12} \\ M_{21} & I \end{pmatrix}$$

and

$$Q = M_1 - W_{12}M_{21}.$$

Then

$$\operatorname{Im} T = \operatorname{Im} \begin{pmatrix} Q & W_{12} \\ 0 & I \end{pmatrix} = \mathcal{M} + \mathcal{W}$$

and

$$\dim(\mathcal{M} \cap \mathcal{W}) = \dim \operatorname{Ker} Q. \tag{2.9}$$

With regard to (2.6) we define

$$C = M_{21}^*(I + W_{12}^* W_{12})M_{21}.$$

Note that

$$C = I - M_1^* M_1 + M_{21}^* W_{12}^* W_{12} M_{21} = I + Q^*Q - M_1^*Q - Q^*M_1. \tag{2.10}$$

Suppose $\dim(\mathcal{M} \cap \mathcal{W}) = \dim \operatorname{Ker} Q \geq k$. Then (2.10) implies $\dim \operatorname{Ker}(C - I) \geq k$ and because of (2.6) at least k singular values of $P(\mathcal{W}, \mathcal{L}; \mathcal{M})$ are equal to 1. Therefore $\sigma_k(C) \geq 1$ and we have shown that (1.1) and (1.2) imply (1.3).

Now assume (1.4). Then $\theta(\mathcal{L}, \mathcal{M}) < 1$ and $\dim \mathcal{M} = \dim \mathcal{L}$. Because of (2.3) we see that (1.4) is equivalent to

$$\sigma_k[P(\mathcal{W}, \mathcal{L})]\theta(\mathcal{L}, \mathcal{M}) < 1,$$

and (1.2) follows from (2.8). Because of (2.7) it is obvious that (1.4) and (1.5) are equivalent.
(ii) Let W_{12} have the singular value decomposition

$$W_{12} = G \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} H, \quad \Sigma = \operatorname{diag}(\omega_1, \ldots, \omega_r), \ \omega_1 \geq \ldots \geq \omega_r \geq 0.$$

Then Lemma 2.1 and (2.2) yield

$$\sin^2 \varphi_k(\mathcal{W}, \mathcal{L}) = \frac{1}{1 + \omega_k^2}.$$

Choose

$$\mathcal{M} = \operatorname{Im} \begin{pmatrix} M_1 \\ M_{21} \end{pmatrix}$$

such that

$$M_1 = G \operatorname{diag} \left(\frac{\omega_1}{\sqrt{1 + \omega_1^2}}, \ldots, \frac{\omega_k}{\sqrt{1 + \omega_k^2}}, 1, \ldots, 1 \right)_{s \times s}$$

and

$$M_{21} = H^{-1} \begin{pmatrix} \frac{1}{\sqrt{1+\omega_1^2}} & \dots & 0 & 0 & \dots & 0 \\ 0 & & \vdots & \vdots & & \vdots \\ \vdots & & \frac{1}{\sqrt{1+\omega_k^2}} & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}_{(n-s)\times s} .$$

Then $\dim \mathcal{M} = s$ and

$$\theta^2(\mathcal{L}, \mathcal{M}) = \|M_{21}\|^2 = \frac{1}{1+w_k^2}.$$

Hence $\mathcal{M}$ has the properties (1.1) and (1.6). Furthermore

$$Q = G \operatorname{diag}(\underbrace{0, \dots, 0}_{k-\text{times}}, 1, \dots, 1)$$

and (2.9) imply (1.7).

Suppose now $\mathcal{M}$ satisfies (1.1), (1.6) and (1.7). From part (i) and Lemma 2.3 we conclude that

$$1 \le \sigma_k[P(\mathcal{W}, \mathcal{L}; \mathcal{M})] \le \sigma_k[P(\mathcal{W}, \mathcal{L})]\theta(\mathcal{L}, \mathcal{M}) = 1,$$

which yields (1.8) and completes the proof. ■

References

[1] E. Berkson. Some metrics on the subspaces of Banach space. *Pacific J. Math.*, 13:7–22, 1963.

[2] I. Gohberg, P. Lancaster, and L. Rodman. *Invariant Subspaces of Matrices with Applications.* Wiley, New York, 1986.

[3] R.A. Horn and Ch.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[4] T. Kato. *Perturbation Theory for Linear Operators.* Springer, Berlin, 1996.

[5] V.E. Ljance. Some properties of idempotent operators. *Teor. i. Prikl. Mat. L'vov*, 1:16–22, 1959 (in Russian).

[6] V. Pták. Extremal operators and oblique projections. *Časopis Pěst. Mat.*, 110:343–350, 1985.

[7] J.M. Schumacher. A pointwise criterion for controller robustness. *Systems Control Lett.*, 18:1-8, 1992.

[8] G. W. Stewart and J. Sun. *Matrix Perturbation Theory.* Academic Press, San Diego, 1990.