Existence and uniqueness of unmixed solutions of the discrete-time algebraic Riccati equation

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Abstract: A solution X of a discrete-time algebraic Riccati equation is called unmixed if the corresponding closed-loop matrix $\Phi(X)$ has the property that the common roots of $\det(sI - \Phi(X))$ and $\det(I - s\Phi(X)^*)$ (if any) are on the unit circle. A necessary and sufficient condition is given for existence and uniqueness of an unmixed solution such that the eigenvalues of $\Phi(X)$ lie in a prescribed subset of $\mathbb{C}$.

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1 Introduction

In this paper we consider the discrete-time algebraic Riccati equation (DARE)

$$\mathcal{D}(X) = X - F^*XF + (S+G^*XF)^*(R+G^*XG)^{-1}(S+G^*XF) - Q = 0. \quad (1.1)$$

The matrices in (1.1) are complex, $F \in \mathbb{C}^{n\times n}$, $S \in \mathbb{C}^{m\times n}$, $G \in \mathbb{C}^{n\times m}$, $R = R^* \in \mathbb{C}^{m\times m}$, $Q = Q^* \in \mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is said to be a solution of (1.1) if X is hermitian, $\det(R + G^*XG) \neq 0$ and $\mathcal{D}(X) = 0$.

There is a wide range of problems in systems and control theory which lead to DAREs of the form (1.1). An important example is the discrete-time regulator problem. Suppose the system

$$x(t+1) = Fx(t) + Gu(t),\ x(0) = x_0,$$

is stabilizable and

$$J(x_0, u) = \sum_{t=0}^{\infty} \begin{pmatrix} x^*(t) & u^*(t) \end{pmatrix} \begin{pmatrix} Q & S^* \\ S & R \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$$

is a positive semidefinite cost functional. Then (see e.g. [1]) there exists an optimal control $u(t)$ which minimizes $J(x_0, u)$. The optimal cost is given by $J_{\rm opt} = x_0^* X_{\rm opt} x_0$ where $X_{\rm opt}$ is the smallest positive semidefinite solution of (1.1), and the optimal control is

$$u_{\rm opt}(t) = \big[F - G(R + G^*X_{\rm opt}G)^{-1}(S + G^*X_{\rm opt}F)\big]x(t). \quad (1.2)$$

With a view to (1.2) we define

$$F_X = F - G(R + G^*RG)^{-1}(S + G^*XF).$$

Then the optimal trajectory $x_{\rm opt}(t)$ satisfies the closed-loop equation

$$x(t+1) = F_{X_{\rm opt}}x(t),\ \ x(0) = x_0.$$

We call F_X the *closed-loop matrix* corresponding to X. A solution X is called *stabilizing* if $|\lambda| < 1$ for all eigenvalues λ of F_X. A more general notion is the following. A solution X is said to be *unmixed* if the spectrum $\sigma(F_X)$ has the property that

$$\bar{\lambda}^{-1} \notin \sigma(F_X) \quad \text{if} \quad \lambda \in \sigma(F_X),\ \lambda \neq 0,\ |\lambda| \neq 1.$$

The concept of unmixed solutions goes back to M.A. Shayman [4]. In the case of the continuous-time algebraic Riccati equation

$$-F^*X - XF + XGG^*X - Q = 0 \quad (1.3)$$

a solution X is called unmixed if the corresponding closed-loop matrix $F_X = F - GG^*X$ satisfies $\sigma(F_X) \cap \sigma(-F_X^*) \subseteq i\mathbb{R}$. According to Shayman [4] pairs of opposite unmixed solutions of (1.3) share several properties of the pair of extremal solutions and can be used to parametrize the set of solutions of (1.3). In this note we shall prove an existence and uniqueness theorem (Theorem 1.1) on unmixed solutions of the DARE (1.1) which extends and sharpens a result in [5].

Notation: Let $\mathbb{D}$ be the open unit disc and $\partial\mathbb{D} = \{z;\ |z| = 1\}$ be the unit circle. For $\Lambda \subseteq \mathbb{C}$ define

$$\Lambda^\triangledown = \{\bar{z}^{-1} \mid z \in \Lambda, z \neq 0\}.$$

A set Λ will be called *unmixed* if

$$0 \in \Lambda,\ \ \Lambda \cap \Lambda^\triangledown = \partial\mathbb{D} \ \text{ and } \ \Lambda \cup \Lambda^\triangledown = \mathbb{C}.$$

In particular, the closed unit disc $\overline{\mathbb{D}}$ is an unmixed subset of $\mathbb{C}$. Obviously a solution X of (1.1) is unmixed if and only if $\sigma(F_X) \subseteq \Lambda$ for some unmixed set Λ. We define

$$\sigma(F - zI,\ G) = \{\lambda \in \mathbb{C} \mid \operatorname{rank}(F - \lambda I,\ G) < n\}$$

to be the set of uncontrollable eigenvalues of F.

Let

$$\Psi(z) = \Big(G^*(z^{-1}I - F^*)^{-1},\ I\Big)\begin{pmatrix} Q & S^* \\ S & R \end{pmatrix}\begin{pmatrix} (zI - F)^{-1}G \\ I \end{pmatrix}, \tag{1.4}$$

be the Popov function associated with (1.1). We write $\Psi(\eta) \geq 0$ a.e. in $\partial\mathbb{D}$ if $\Psi(\eta)$ is positive semidefinite in $\partial\mathbb{D}$ with the possible exception of a finite number of points. Our main result is the following theorem. The proof will be given in Section 3 based on the auxiliary material of the next section.

Theorem 1.1. *Let $\Lambda \subseteq \mathbb{C}$ be an unmixed set. Assume*

$$\Psi(\eta) > 0 \ \textit{for some}\ \eta \in \partial\mathbb{D},\ \textit{and}\ \ \Psi(\eta) \geq 0 \ \textit{a.e. in}\ \partial\mathbb{D}. \tag{1.5}$$

Then there exists a unique solution X of (1.1) such that $\sigma(F_X) \subseteq \Lambda$ if and only if

$$\sigma(F - zI,\ G) \subseteq \Lambda \setminus \partial\mathbb{D}. \tag{1.6}$$

2 Auxiliary results

In contrast to Theorem 1.1 of [5, p.142] we shall not assume that the matrix R is nonsingular. It will be crucial however that the matrix $(G^*, R) \in \mathbb{C}^{m\times(n+m)}$ has full rank.

Lemma 2.1. (i) *If* (1.5) *holds then we have* $\operatorname{rank}(G^*, R) = m$.
(ii) *There exists a hermitian matrix P such that $R+G^*PG$ is nonsingular if and only if* $\operatorname{rank}(G^*, R) = m$.

Proof. (i) Suppose $\Psi(\eta) \geq 0$ a.e. in $\partial\mathbb{D}$ and $\operatorname{rank}(G^*, R) < m$. Let w be nonzero such that $w^*G = 0$ and $w^*R = 0$. Then $w^*\Psi(\eta)w = 0$ for all $\eta \in \partial\mathbb{D}$, which is incompatible with the first part of (1.5).
(ii) Assume $\operatorname{rank}(G^*, R) = m$. It is not difficult to see that one can choose nonsingular matrices K and M such that

$$\tilde{R} = K^*RK = \begin{pmatrix} R_1 & 0 \\ 0 & 0_{m_2} \end{pmatrix}, \ \det R_1 \neq 0, \ \ \tilde{G} = MGK = \begin{pmatrix} G_{11} & 0 \\ 0 & I_{m_2} \end{pmatrix}.$$

(Here 0_{m_2} and I_{m_2} denote $m_2 \times m_2$ matrices.) Let $P_2 \in \mathbb{C}^{m_2\times m_2}$ be nonsingular. Put

$$\tilde{P} = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}.$$

Then $\tilde{R} + \tilde{G}^*\tilde{P}\tilde{G} = \operatorname{diag}(R_1, P_2)$. Hence, if $P = M^*\tilde{P}M$ then $R + G^*PG$ is nonsingular. ■

The following lemma relates closed-loop matrices and Popov functions of different DAREs. To have a more explicit notation at our disposal we replace $\mathcal{D}(X)$ in (1.1) by $\mathcal{D}(X; F, G, R, S, Q)$ and $\Psi(z)$ in (1.4) by $\Psi(z; F, G, R, S, Q)$. Instead of F_X we write

$$\Phi(X, F, G, R, S) = F - G(R + G^*XG)^{-1}(S + G^*XF).$$

Lemma 2.2. *Let P be hermitian and set*

$$\widetilde{R} = R + G^*PG, \ \ \widetilde{S} = S + G^*PF, \ \ \widetilde{Q} = Q + F^*PF - P. \tag{2.1}$$

Then

$$\mathcal{D}(X; F, G, R, S, Q) = \mathcal{D}(X - P; F, G, \widetilde{R}, \widetilde{S}, \widetilde{Q}) \tag{2.2}$$

and

$$\Psi(z; F, G, R, S, Q) = \Psi(z; F, G, \widetilde{R}, \widetilde{S}, \widetilde{Q}), \tag{2.3}$$

and

$$\Phi(X, F, G, R, S) = \Phi(X - P, F, G, \widetilde{R}, \widetilde{S}). \tag{2.4}$$

Proof. The identity (2.2) is easy to verify (see e.g. [2]). To prove (2.3) we have to show that

$$T(z) = \big(G^*(z^{-1} - F^*)^{-1},\, I\big) \begin{pmatrix} F^*PF - P & F^*PG \\ G^*PF & G^*PG \end{pmatrix} \begin{pmatrix} (zI - F)^{-1}G \\ I \end{pmatrix} = 0.$$

Note that

$$T(z) = G^*[-(z^{-1} - F^*)^{-1}P(zI - F)^{-1}]G + \\ + G^*\big(z^{-1} - F^*)^{-1},\ I\big) \begin{pmatrix} F^* \\ I \end{pmatrix} P(F,\ I) \begin{pmatrix} (zI - F)^{-1}G \\ I \end{pmatrix} G.$$

Because of $F(zI - F)^{-1} + I = z(zI - F)^{-1}$ it is obvious that $T(z) = 0$. ■

The following lemma is well known (see e.g. [3]).

Lemma 2.3. *Let X_2 be a solution of $\mathcal{D}(X) = 0$ and let Δ be a solution of*

$$\mathcal{H}(Y) = Y - F_{X_2}^* Y F_{X_2} + F_{X_2}^* Y G\left[(R + G^*X_2G) + G^*YG\right]^{-1} G^*YF_{X_2} = 0. \tag{2.5}$$

Then $\widetilde{X} = X_2 + \Delta$ is a solution of $\mathcal{D}(X) = 0$.

3 Proof of Theorem 1.1

"⇒" Suppose (1.6) is not satisfied. Then there exists a $\lambda \in \sigma(F - zI,\, G)$ such that $\lambda \notin \Lambda$ or otherwise there exists an uncontrollable eigenvalue $\lambda \in \Lambda \cap \partial\mathbb{D}$. It is easy to see that

$$\sigma(F - zI,\, G) \subseteq \sigma(F_X).$$

Hence in the first case there is no solution X with $\sigma(F_X) \subseteq \Lambda$. In the second case, i.e. with $\lambda \in \Lambda \cap \partial\mathbb{D} \cap \sigma(F - zI,\, G)$, suppose there exists a solution X_2 with $\sigma(F_{X_2}) \subseteq \Lambda$. We want to show that such a solution is not unique. Because of $\lambda \in \sigma(F - zI,\, G)$ we can assume (after a suitable change of coordinates)

$$F = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ * & * & \dots & * \\ & & & \\ * & * & \dots & * \end{pmatrix},\quad G = \begin{pmatrix} 0 & \dots & 0 \\ * & \dots & * \\ & & \\ * & \dots & * \end{pmatrix}.$$

Then the first row of F_{X_2} is $(\lambda,\, 0,\, \dots\, 0)$. Set $\Delta = \mathrm{diag}(1, 0, \dots, 0)$. Then $G^*\Delta = 0$ and it follows from $\bar{\lambda}\lambda = 1$ that Δ is a solution of $\mathcal{H}(Y) = 0$ in

(2.5). Thus Lemma 2.3 implies that $\hat{X} = X_2 + \Delta$ is a solution of $\mathcal{D}(X) = 0$. It is easy to see that $G^*\Delta = 0$ yields $F_{\widetilde{X}} = F_{X_2}$.

"$\Leftarrow$" Note that the assumptions on $\Psi(z)$ imply rank $(G^*,\ R) = m$. Hence, according to Lemma 2.1, we can choose a hermitian matrix P such that $R + G^*PG$ is nonsingular. Let $\widetilde{R}$, $\widetilde{S}$ and $\widetilde{Q}$ be defined by (2.1). Consider the equation

$$\mathcal{D}(W; F, G, \widetilde{R}, \widetilde{S}, \widetilde{Q}) = 0. \tag{3.1}$$

Because of Lemma 2.2 the Popov function remains invariant if (R, S, Q) is replaced by $(\widetilde{R}, \widetilde{S}, \widetilde{Q})$. According to Theorem 1.1 of [5, p.142] the assumptions (1.5) - (1.6) imply that there exists a unique solution W of (3.1) with $\sigma(F_W) \subseteq \Lambda$. It follows from (2.2) and (2.4) that $X = W + P$ is the unique solution of $\mathcal{D}(X; F, G, R, S, Q) = 0$ with $\sigma(F_X) \subseteq \Lambda$. ∎

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