REGULAR SUBMODULES OF TORSION MODULES OVER A DISCRETE VALUATION DOMAIN

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Abstract. A submodule W of a p-primary module M of bounded order is known to be regular if W and M have simultaneous bases. In this paper we derive necessary and sufficient conditions for regularity of a submodule.

Keywords: regular submodules, modules over discrete valuation domains, Abelian p-groups, simultaneous bases

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1. Introduction

Let R be a discrete valuation domain with maximal ideal Rp, and let M be a torsion module over R and W be a submodule of M. The submodule W is called *regular* [5, p. 65], [6, p. 102] if

$$p^n W \cap p^{n+r} M = p^n (W \cap p^r M) \tag{1.1}$$

holds for all $n \geqslant 0$, $r \geqslant 0$. The regularity condition (1.1) was introduced by Vilenkin [6] in his study of decompositions of topological p-groups. Kaplanski [5] showed that for a module M of bounded order (1.1) is necessary and sufficient for the existence of simultaneous bases of W and M. In this paper we shall identify two conditions which are equivalent to (1.1). One is related to a theorem of Baer [4, p. 4] on the decomposition of elements in Abelian p-groups, the other one was introduced by Ferrer, F. Puerta and X. Puerta [2] to characterize marked invariant subspaces of a linear operator.

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Notation and definitions: The letters $\mathcal{U}, \mathcal{V}, \mathcal{X}, \ldots$ will always denote subsets of M. Let $\langle \mathcal{X} \rangle$ be the submodule spanned by $\mathcal{X}$. We shall use the letters $u, v, x, \ldots$ for elements of the module M, and $\alpha, \beta, \mu, \ldots$ will be elements of the ring R. Using the terminology for Abelian p-groups in [3, p. 4] we say that $x \in M$ has *exponent* k, and we write $\mathrm{e}(x) = k$, if k is the smallest nonnegative integer such that $p^k x = 0$.

An element $x \in M$ is said to have (finite) *height* s if $x \in p^s M$ and $x \notin p^{s+1} M$, and x has *infinite height*, if $x \in p^s M$ for all $s \geqslant 0$. We write $\mathrm{h}(x)$ for the height of x. If $x \in W$ then $\mathrm{h}_W(x)$ will denote the height of x with respect to W. Note that $\mathrm{e}(0) = 0$ and $\mathrm{h}(0) = \infty$. Let R^* be the group of units of R. If $\alpha \in R$ is nonzero and $\alpha = p^s \gamma$, $\gamma \in R^*$, then we set $\mathrm{h}(\alpha) = s$. We put $\mathrm{h}(\alpha) = \infty$ if $\alpha = 0$. We call $x \in M$ an $(s, k; s_1)$-element if $x \neq 0$ and

$$\mathrm{h}(x) = s, \quad \mathrm{e}(x) = k, \quad \mathrm{h}(p^{k-1}x) = (k-1) + s_1.$$

In accordance with a definition of Baer [1] we say that an element x is *regular* if $\mathrm{h}(x) = \infty$ or if $\mathrm{h}(x)$ is finite and

$$\mathrm{h}(p^j x) = j + \mathrm{h}(x), \quad j = 1, \ldots, \mathrm{e}(x) - 1. \tag{1.2}$$

The two concepts of regularity introduced above are consistent. We shall see in Lemma 3.2 that a finite height element $x \in M$ is regular if and only if $\langle x \rangle$ is a regular submodule of M.

For $s \geqslant 0$, $k \geqslant 0$ we define the submodules $M[p^k] = \{x \in M \mid p^k x = 0\}$ and

$$M_k^s = p^s M \cap M[p^k]. \tag{1.3}$$

Then

$$M_k^s = \{x \in M \mid \mathrm{e}(x) \leqslant k, \ \mathrm{h}(x) \geqslant s\}.$$

In particular $M_0^s = 0$.

Our main result will be the following.

Theorem 1.1. *Let M be a torsion module over a discrete valuation domain and let W be a submodule of M. The following conditions are equivalent.*

(K) *W is regular, i.e. if $n \geqslant 0$, $r \geqslant 0$ then*

$$p^n W \cap p^{n+r} M = p^n (W \cap p^r M). \tag{1.4}$$

(B) *If $x \in W$ is nonzero then x can be decomposed as*

$$x = y_{k_1}^{s_1} + \ldots + y_{k_m}^{s_m} \tag{1.5}$$

such that

$$y_{k_i}^{s_i} \in W \text{ is regular}, \quad i = 1, \ldots, m,$$

and

$$\mathrm{h}(y_{k_i}^{s_i}) = s_i, \quad \mathrm{e}(y_{k_i}^{s_i}) = k_i,$$

and

$$k_1 > \ldots > k_m > 0 \quad \text{and} \quad s_1 > \ldots > s_m. \tag{1.6}$$

(FPP) *If* $s \geqslant 0$, $k \geqslant 1$, *then*

$$(W \cap M_k^{s+1}) + (W \cap M_{k-1}^s) = W \cap (M_k^{s+1} + M_{k-1}^s). \tag{1.7}$$

By a result of Baer [4, p. 4, Lemma 65.4] condition (B) is satisfied for $W = M$. Hence (B) singles out those submodules W where each element $x \in W$ allows a decomposition (1.5) such that the summands $y_{k_i}^{s_i}$ can be chosen from W itself. With regard to condition (FPP) we observe that the inclusion

$$(W \cap M_k^{s+1}) + (W \cap M_{k-1}^s) \subseteq W \cap (M_k^{s+1} + M_{k-1}^s) \tag{1.8}$$

holds for all submodules W.

The proof of the theorem will be split into two parts. In Section 3 we show that (B) and (K) are equivalent and in Section 4 we prove the equivalence of (B) and (FPP).

2. Decomposition of elements

We introduce a condition which will be the link between (B) and (K) on the one hand and between (B) and (FPP) on the other. For a submodule W we define condition (H) as follows.

(H) If $x \in W$ is an $(s, k; s_1)$-element then x can be decomposed as

$$x = y_k^{s_1} + z, \quad y_k^{s_1} \in W, \quad z \in W, \tag{2.1}$$

such that

$$\mathrm{h}(y_k^{s_1}) = s_1, \quad \mathrm{e}(y_k^{s_1}) = k, \quad \text{and} \quad \mathrm{h}(z) = s, \quad \mathrm{e}(z) < k. \tag{2.2}$$

The following technical lemma will be useful in several instances. It implies that the element $y_k^{s_1}$ in (2.1) is regular.

Lemma 2.1. *Let $x \in M$ be an $(s, k; s_1)$-element. Assume*

$$x = y + z, \quad z \in M^s_{k-1}. \tag{2.3}$$

Then $y \neq 0$, $\mathrm{e}(y) = k$, and

$$s \leqslant \mathrm{h}(y) \leqslant s_1. \tag{2.4}$$

The element y is regular if and only if $\mathrm{h}(y) = s_1$. If x is regular then (2.3) implies $\mathrm{h}(y) = s$.

Proof. From (2.3) it follows that $p^{k-1}y = p^{k-1}x \neq 0$, and $\mathrm{e}(y) = k$. Therefore

$$(k-1) + \mathrm{h}(y) \leqslant \mathrm{h}(p^{k-1}y) = \mathrm{h}(p^{k-1}x) = (k-1) + s_1, \tag{2.5}$$

which yields $\mathrm{h}(y) \leqslant s_1$. It is obvious from (2.5) that we have $\mathrm{h}(y) = s_1$ if and only if

$$\mathrm{h}(p^{k-1}y) = (k-1) + \mathrm{h}(y),$$

i.e., if and only if y is regular. If x is regular then $s_1 = s$ and (2.4) yields $\mathrm{h}(y) = s$. □

Lemma 2.2. *For a submodule W the conditions* (B) *and* (H) *are equivalent.*

Proof. There is nothing to prove if x is regular. Thus, in the following we assume that x is a non-regular element of W with $\mathrm{h}(x) = s$ and $\mathrm{e}(x) = k$. In that case we have $k > 1$, $s_1 > s$, and $\mathrm{h}(p^{k-1}x) = (k-1) + s_1$.

(B) $\Rightarrow$ (H): Let x be given as in (1.5), with $m \geqslant 2$. Put $z = y^{s_2}_{k_2} + \ldots + y^{s_m}_{k_m}$. Then (1.6) implies $\mathrm{e}(z) \leqslant k_2 < k$ and $\mathrm{h}(z) = s_m = s$. Hence the decomposition $x = y^{s_1}_{k_1} + z$ is of type (H).

(H) $\Rightarrow$ (B): Let x be an $(s, k; s_1)$-element of W and assume that x is decomposed according to (H) as

$$x = y^{s_1}_k + z \tag{2.6}$$

such that (2.2) holds. We know from Lemma 2.1 that $y^{s_1}_k$ is regular. Consider x with $s_1 > s$, $k > 1$. Assume as an induction hypothesis that condition (H) ensures a decomposition of type (B) for all $w \in W$ with $\mathrm{e}(w) < k$. Thus we have

$$z = z^{t_2}_{l_2} + \ldots + z^{t_m}_{l_m}, \quad m \geqslant 2,$$

with properties in accordance with (B). Thus $\mathrm{h}(p^{l_2-1}z) = (l_2-1)+t_2$, $t_2 \geqslant s$, and $t_2 > \ldots > t_m = s = \mathrm{h}(z)$, and $k > \mathrm{e}(z) = l_2 > \ldots > l_m > 0$. If $s_1 > t_2$ then we already have the desired decomposition. Now suppose $t_2 \geqslant s_1$. Let j be such that

$$t_2 > \ldots t_j \geqslant s_1 > t_{j+1}. \tag{2.7}$$

Note that $t_m \geqslant s_1$ can not occur because of $t_m = s$ and $s_1 > s$. Set

$$v = y_k^{s_1} + (z_{l_2}^{t_2} + \ldots + z_{l_j}^{t_j}).$$

Then $k > l_2$ yields $\mathrm{e}(v) = k$. Since $y_k^{s_1}$ is regular we see that $p^{k-1}v = p^{k-1}y_k^{s_1}$ implies $(k-1)+s_1 = \mathrm{h}(p^{k-1}v)$. Hence $\mathrm{h}(v) \leqslant s_1$. On the other hand it follows from (2.7) that $\mathrm{h}(v) \geqslant s_1$. Therefore $\mathrm{h}(v) = s_1$, and v is regular. If we rewrite (2.6) in the form

$$x = v + z_{l_{j+1}}^{t_{j+1}} + \ldots + z_{l_m}^{t_m},$$

then we have a decomposition with $\mathrm{h}(v) = s_1$ and $s_1 > t_{j+1} > \ldots > t_m = s$ and $\mathrm{e}(v) = k > l_{j+1} > \ldots > l_m > 0$. □

It is not difficult to check that the following observation characterizes the numbers m, k_i and s_i in (1.5). For a nonzero element $x \in M$ with $\mathrm{e}(x) = k$ define $\mathrm{g}(x) = \mathrm{h}(x) + \mathrm{e}(x)$.

Lemma 2.3. *Let $x \in M$ be decomposed as*

$$x = y_{k_1}^{s_1} + \ldots + y_{k_m}^{s_m} \tag{2.8}$$

such that

$$\mathrm{h}(y_{k_i}^{s_i}) = s_i, \quad \mathrm{e}(y_{k_i}^{s_i}) = k_i, \quad \text{and} \quad y_{k_i}^{s_i} \text{ is regular}, \quad i = 1, \ldots, m,$$

and

$$k_1 > \ldots > k_m > 0 \quad \text{and} \quad s_1 > \ldots > s_m.$$

Set $K = \{k_1, \ldots, k_m\}$. Then $j \in \{1, \ldots, k-1\}$ is in K if and only if $g(p^j x) > g(p^{j-1}x)$. Moreover

$$\mathrm{h}(p^{k_j-1}x) = (k_j - 1) + s_j, \quad j = 1, 2, \ldots m.$$

In particular, we have $\mathrm{e}(x) = k_1$ and $\mathrm{h}(x) = s_m$.

3. Equivalence of (K) and (B)

Condition (K) can be reformulated in a more convenient form.

Lemma 3.1. *We have*

$$p^nW \cap p^{n+r}M = p^n(W \cap p^rM), \quad n \geqslant 0, \ r \geqslant 0, \tag{3.1}$$

if and only if for each $w \in W$ *with* $\mathrm{h}(p^nw) = n + r$ *there exists an element* $\tilde{w} \in W$ *such that*

$$p^nw = p^n\tilde{w} \quad \text{and} \quad \mathrm{h}(\tilde{w}) = r. \tag{3.2}$$

Proof. Obviously (3.1) is equivalent to

$$p^nW \cap p^{n+r}M \subseteq p^n(W \cap p^rM), \quad n \geqslant 0, \ r \geqslant 0. \tag{3.3}$$

Now (3.3) holds if and only if

$$x \in p^nW, \quad x \in p^{n+r}M \quad \text{and} \quad x \notin p^{n+r+1}M$$

imply $x \in p^n(W \cap p^rM)$. That implication means the following. If $x = p^nw$ and $w \in W$ and $\mathrm{h}(x) = n + r$, then $x = p^n\tilde{w}$ for some $\tilde{w} \in W$ with $\mathrm{h}(\tilde{w}) \geqslant r$. Because of $\mathrm{h}(p^n\tilde{w}) = n + r$ the inequality $\mathrm{h}(\tilde{w}) \geqslant r$ is equivalent to $\mathrm{h}(\tilde{w}) = r$. □

Lemma 3.2. *Let* x *be an element of finite height with* $\mathrm{e}(x) = k$. *Then* x *is regular if and only if the submodule* $\langle x \rangle$ *is regular , i.e.*

$$p^r\langle x \rangle \cap p^{n+r}M = p^r(\langle x \rangle \cap p^nM), \quad n \geqslant 0, \ r \geqslant 0. \tag{3.4}$$

Proof. Assume (3.4). We want to show that $\mathrm{h}(p^{k-1}x) = (k-1) + s_1$ implies $s_1 = \mathrm{h}(x)$. According to Lemma 3.1 there exists an element $\tilde{x} \in \langle x \rangle$ with properties corresponding to (3.2), i.e. $\tilde{x} = \gamma p^tx$, $\gamma \in R^*$, and $p^{k-1}x = p^{k-1}(\gamma p^tx)$ and $\mathrm{h}(p^tx) = s_1$. Then we have $t = 0$, and $\mathrm{h}(x) = s_1$. It is easy to check that (3.4) holds if x is regular. □

Proof of Theorem 1.1. Part I: (B) ⇔ (K). (B) ⇒ (K): We want to show that condition (B) implies (K) in the equivalent form of Lemma 3.1. Let $w \in W$ be such that $\mathrm{h}(p^nw) = n + r$, and $h(w) = s$, $\mathrm{e}(w) = k_1$. Then $s \leqslant r$ and $k_1 > n$. Hence (B) yields a decomposition

$$w = y_{k_1}^{s_1} + \ldots + y_{k_m}^{s_m}$$

where the elements $y_i^{s_i} \in W$ are regular, $h(y_i^{s_i}) = s_i$, and

$$s_1 > \ldots > s_m = s = \mathrm{h}(w)$$

and $\mathrm{e}(w) = k_1 > \ldots > k_m > 0$. Let t be such that $k_t > n \geqslant k_{t+1}$. Then

$$n + r = \mathrm{h}(p^n w) = \mathrm{h}(p^n y_{k_1}^{s_1} + \ldots + p^n y_{k_t}^{s_t}),$$

and $\mathrm{h}(p^n w) = \mathrm{h}(p^n y_{k_t}^{s_t}) = n + s_t$. Hence $s_t = r$. Set $\tilde{w} = y_{k_1}^{s_1} + \ldots + y_{k_t}^{s_t}$. Then $\tilde{w} \in W$ and $\mathrm{h}(\tilde{w}) = r$ and $p^n w = p^n \tilde{w}$.

(K) $\Rightarrow$ (B): Because of Lemma 2.2 it suffices to show that (K) implies (H). Let $x \in W$ be an $(s, k; s_1)$-element. Set $w = p^{k-1} x$. Then (K), resp. Lemma 3.1, imply that there exists an $\tilde{x} \in W$ such that

$$p^{k-1} x = p^{k-1} \tilde{x} \tag{3.5}$$

and $\mathrm{h}(\tilde{x}) = s_1$. From (3.5) it follows that $\mathrm{e}(\tilde{x}) = k$ and $\mathrm{h}(p^{k-1}\tilde{x}) = (k-1) + s_1$. Now set $z = x - \tilde{x}$. Then (3.5) yields $\mathrm{e}(z) < k$. Hence $x = \tilde{x} + z$ is a decomposition of type (H). □

As (K) holds for $W = M$ we can write each nonzero element x of M according to (H) in the form (2.1). Similarly we can decompose x according to (B) as a sum of the form (1.5). In that case we recover the result of Baer [4, p. 4, Lemma 65.4] mentioned in Section 1.

4. Equivalence of (B) and (FPP)

In [2] J. Ferrer, and F. and X. Puerta studied marked invariant subspaces of an endomorphism A of $\mathbb{C}^n$. Their investigation is based on subspaces of the form $\mathrm{Im}(\lambda I - A)^s \cap \mathrm{Ker}(\lambda I - A)^k$. Thus the submodules M_k^s in (1.3) are a generalization of those subspaces. The next lemma is adapted from [2]. It characterizes regular elements in terms of M_k^s. Note that $M_k^s \subseteq M_{k_1}^{s_1}$ if $s_1 \leqslant s$ and $k \leqslant k_1$. Hence $M_k^{s+1} + M_{k-1}^s \subseteq M_k^s$.

Lemma 4.1. *An element $x \in M$ satisfies*

$$x \in M_k^s \quad \text{and} \quad x \notin M_k s + 1 + M_{k-1}^s \tag{4.1}$$

if and only of

$$x \text{ is regular} \quad \text{and} \quad \mathrm{h}(x) = s \quad \text{and} \quad \mathrm{e}(x) = k. \tag{4.2}$$

Proof. "$\Rightarrow$": Assume that x satisfies (4.1). Recall that $x \in M_k^s$ if and only if both $\mathrm{h}(s) \geqslant s$ and $\mathrm{e}(x) \leqslant k$. Hence

$$x \notin M_k^{s+1} + M_{k-1}^s \tag{4.3}$$

implies $\mathrm{h}(x) = s$ and $\mathrm{e}(x) = k$. Assume $\mathrm{h}(p^{k-1}x) = (k-1) + s_1$. If we decompose x according to (H) then $x = y_k^{s_1} + z \in M_k^{s_1} + M_{k-1}^s$. Hence (4.3) implies $s_1 = s$, and x is regular.

"$\Leftarrow$": Consider an element x with properties (4.2). Then $x \in M_k^s$, $x \notin M_k^{s+1}$ and $x \notin M_{k-1}^s$. If

$$x = y + z, \quad y \neq 0, \quad z \in M_{k-1}^s,$$

and x is regular then it follows from Lemma 2.1 that $\mathrm{h}(y) = s$. Hence we have $y \notin M_k^{s+1}$ and $x \notin M_k^{s+1} + M_{k-1}^s$. □

P r o o f of Theorem 1.1, Part II: (H) $\Leftrightarrow$ (FPP). (H) $\Rightarrow$ (FPP): Because of the inclusion (1.8) the identity (1.7) in (FPP) is equivalent to

$$W \cap (M_k^{s+1} + M_{k-1}^s) \subseteq (W \cap M_k^{s+1}) + (W \cap M_{k-1}^s). \tag{4.4}$$

We want to show that condition (H) implies (4.4) for all $s \geqslant 0$, $k \leqslant 1$. Take an element

$$x \in W \cap (M_k^{s+1} + M_{k-1}^s). \tag{4.5}$$

Then $x \in M_k^s$ and therefore $\mathrm{h}(x) \geqslant s$ and $\mathrm{e}(x) \leqslant k$. To prove that

$$x \in (W \cap M_k^{s+1}) + (W \cap M_{k-1}^s) \tag{4.6}$$

we consider three cases. First, let $\mathrm{h}(x) \geqslant s+1$ then $x \in W \cap M_k^{s+1}$ and (4.6) is obvious. Secondly, let $\mathrm{e}(x) \leqslant k-1$. In that case $x \in W \cap M_{k-1}^s$. Now assume $\mathrm{h}(x) = s$ and $\mathrm{e}(x) = k$. By Lemma 4.1 it follows from (4.5) that x is not regular. Hence $\mathrm{h}(p^{k-1}x) = (k-1) + s_1$ and $s_1 > s$. According to (H) we have $x = y_k^{s_1} + z$ with $y_k^{s_1} \in W \cap M_k^{s_1}$ and $z \in W \cap M_{k-1}^s$, which yields (4.6).

(FPP) $\Rightarrow$ (H): Let x be an $(s, k; s_1)$-element. If $s_1 = s$ then x is regular and we have (2.1) with $z = 0$. Suppose now that x is not regular , i.e. $s_1 \geqslant s+1$. Then Lemma 4.1 implies $x \in W \cap (M_k^{s+1} + M_{k-1}^s)$. From (FPP) we obtain

$$x = y + z, \quad y \in W \cap M_k^{s+1}, \quad z \in W \cap M_{k-1}^s. \tag{4.7}$$

Then $y \neq 0$, $\mathrm{e}(y) = k$ and $\mathrm{h}(y) \geqslant s+1$. Let y in (4.7) be such that $\mathrm{h}(y)$ is maximal. We shall show that such a choice of y implies $\mathrm{h}(y) = s_1$, and in that case (4.7) is a decomposition of type (H). Now suppose that $\mathrm{h}(y) = \tilde{s} < s_1$. Then, by Lemma 2.1, the element $y \in W$ is not regular. Applying Lemma 4.1 to $y \in W \cap M_k^{\tilde{s}}$ we obtain $y \in W \cap (M_k^{\tilde{s}+1} + M_{k-1}^{\tilde{s}})$. Thus (FPP) yields

$$y = \tilde{y} + z_2, \quad \tilde{y} \in W \cap M_k^{\tilde{s}+1}, \quad \tilde{y} \neq 0, \quad z_2 \in W \cap M_{k-1}^{\tilde{s}}.$$

Hence $x = \tilde{y} + (z + z_2)$, and we have another decomposition of the form (4.7), but now with $h(\tilde{y}) > \tilde{s}$, which contradics the maximality of $\tilde{s}$. □

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