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DECOMPOSITION AND PARAMETRIZATION OF SEMIDEFINITE SOLUTIONS OF THE CONTINUOUS-TIME ALGEBRAIC RICCATI EQUATION*

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Abstract. Negative-semidefinite solutions of the ARE $\mathcal{R}(X) = A^*X + XA + XBB^*X - C^*C = 0$ are studied. With respect to an appropriate basis the ARE breaks up into a Lyapunov equation $A_0^*X_0 + X_0A_0 = 0$, where A_0 has only purely imaginary eigenvalues, and an indecomposable Riccati equation $\mathcal{R}_r(X_r) = A_r^*X_r + X_rA_r + X_rB_rB_r^*X_r - C_r^*C_r = 0$ such that each solution $X \leq 0$ is of the form $X = \operatorname{diag}(X_0, X_r)$. The focus is on the solutions $\mathcal{S} = \{X \mid X = \operatorname{diag}(0, X_r), \mathcal{R}_r(X_r) = 0, X_r \leq 0\}$. The set $\mathcal{S}$ has as an order-isomorphic image a well-defined set $\mathcal{N}$ of A-invariant subspaces. The characterization of $\mathcal{N}$ involves the stabilizable and the uncontrollable subspace of (A, B, C).

Key words. algebraic Riccati equation, semidefinite solutions, parametrization by invariant subspaces

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1. Introduction.

We consider the algebraic Riccati equation (ARE)

$$\mathcal{R}(X) = A^*X + XA + XBB^*X - C^*C = 0, \tag{1.1}$$

where A, B, C are complex matrices of sizes $n \times n$, $n \times p$, $q \times n$, respectively. It is the purpose of this paper to give a complete description of the set

$$\mathcal{T} = \{X \mid \mathcal{R}(X) = 0, X \leq 0\}$$

of negative-semidefinite solutions of (1.1).

The following notation will be used. In the partitions

$$\mathbb{C} = \mathbb{C}_{\leq} \cup \mathbb{C}_{>} = C_{<} \cup \mathbb{C}_{=} \cup \mathbb{C}_{>}, \tag{1.2}$$

the subscripts refer to real parts such that $\mathbb{C}_{\leq} = \{\lambda \mid \lambda \in \mathbb{C}, \operatorname{Re} \lambda \leq 0\}$, etc. Put

$$E_\lambda(A) = \operatorname{Ker}(A - \lambda I)^n.$$

To (1.2) correspond the decompositions $\mathbb{C}^n = E_{\leq}(A) \oplus E_{>}(A)$ and

$$\mathbb{C}^n = E_{<}(A) \oplus E_{=}(A) \oplus E_{>}(A), \tag{1.3}$$

where $E_{\leq}(A) = \oplus\{E_\lambda(A), \lambda \in \mathbb{C}_{\leq}\}$, etc. With our choice of notation we also have in mind its use for the discrete-time algebraic Riccati equation [14] where the subscripts refer to $|\lambda| \leq 1$, etc. Let $\operatorname{Inv} A$ denote the lattice of A-invariant subspaces of $\mathbb{C}^n$. To the triple (A, B, C) we associate the controllable subspace

$$R(A, B) = \operatorname{Im}(B, AB, \ldots, A^{n-1}B)$$

and the unobservable subspace

$$V(A, C) = \operatorname{Ker}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

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It will be convenient to define

$$V_{\leq}(A,C) = V(A,C) \cap E_{\leq}(A),$$

and similarly $V_{=}(A,C), V_{<}(A,C)$, etc. Let

$$\sigma(A - sI, B) = \{\lambda \mid \lambda \in \mathbb{C},\ \operatorname{rank}(A - \lambda I, B) < n\}$$

denote the set of uncontrollable eigenvalues of A, and similarly define

$$\sigma\begin{pmatrix} A - sI \\ C \end{pmatrix} = \left\{\lambda \mid \lambda \in \mathbb{C},\ \operatorname{rank}\begin{pmatrix} A - sI \\ C \end{pmatrix} < n\right\}.$$

Then $V_{=}(A,C) = 0$ if and only if

$$\sigma\begin{pmatrix} A - sI \\ C \end{pmatrix} \cap \mathbb{C}_{=} = \emptyset.$$

Note that $R(A,B) + E_{<}(A) = \mathbb{C}^n$ or equivalently $\sigma(A - sI, B) \cap \mathbb{C}_{\geq} = \emptyset$ means that (A,B) is stabilizable.

We know from [5], [7] that $\mathcal{T} \neq \emptyset$ is equivalent to

$$V(A,C) + R(A,B) + E_{<}(A) = \mathbb{C}^n. \tag{1.4}$$

If we write (1.4) as

$$V_{=}(A,C) + V_{>}(A,C) + R(A,B) + E_{<}(A) = \mathbb{C}^n$$

and put

$$U_r = V_{>}(A,C) + R(A,B) + E_{<}(A), \tag{1.5}$$

then (1.4) holds if and only if there exists a subspace U_0 such that $\mathbb{C}^n = U_0 \oplus U_r$ and $U_0 \subseteq V_{=}(A,C)$.

DEFINITION 1.1. *We call* $\mathbb{C}^n = U_0 \oplus U_r$ *an LR*-decomposition *of* $\mathbb{C}^n$ *if* $U_0 \subseteq V_{=}(A,C)$. *The subspace* U_r *is the* Riccati component *and* U_0 *is a* Lyapunov complement.

The following decomposition theorem is the main tool for our investigation. It will be proved in §3.

THEOREM 1.2. *Let* $\mathbb{C}^n = U_0 \oplus U_r$ *be an LR-decomposition. Assume*

$$U_0 = \left\{\begin{pmatrix} x_0 \\ 0 \end{pmatrix}\right\} \quad \text{and} \quad U_r = \left\{\begin{pmatrix} 0 \\ x_r \end{pmatrix}, x_r \in \mathbb{C}^{n_r}\right\}. \tag{1.6}$$

Then

$$A = \begin{pmatrix} A_0 & 0 \\ A_{r0} & A_r \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B_r \end{pmatrix}, \quad C = (0, C_r), \tag{1.7}$$

and

$$\sigma(A_0) \subseteq \mathbb{C}_{=}, \tag{1.8}$$

and

$$V_{>}(A_r, C_r) + R(A_r, B_r) + E_{<}(A_r) = \mathbb{C}^{n_r}. \tag{1.9}$$

We have $X \in \mathcal{T}$ *if and only if*

$$X = \operatorname{diag}(X_0, X_r) \tag{1.10}$$

and X_0 *and* X_r *satisfy*

$$\mathcal{L}_0(X_0) = A_0^* X_0 + X_0 A_0 = 0, \qquad X_0 \le 0 \tag{1.11}$$

and

$$\mathcal{R}_r(X_r) = A_r^* X_r + X_r A_r + X_r B_r B_r^* X_r - C_r^* C_r = 0, \qquad X_r \le 0. \tag{1.12}$$

Since (1.11) has the trivial solution $X_0 = 0$, we can associate to each $X = \operatorname{diag}(X_0, X_r) \in \mathcal{T}$ a solution $\rho X \in \mathcal{T}$ given by $\rho X = \operatorname{diag}(0, X_r)$. The basis-free definition of ρ in the theorem below will be proved in §3.

THEOREM 1.3. *Let* $\mathbb{C}^n = U_0 \oplus U_r$ *be an LR-decomposition and let* Π *be the projection on* U_r *along* U_0. *Then we have*

$$(I - \Pi)^* X \Pi = 0 \tag{1.13}$$

for $X \in \mathcal{T}$. *The map* $\rho : \mathcal{T} \to \mathcal{T}$ *defined by* $\rho X = X\Pi$ *is independent of the Lyapunov complement* U_0.

Put $\mathcal{S} = \rho\mathcal{T}$. The partially ordered set $\mathcal{S}$—or rather an order-isomorphic image $\mathcal{N}$ of $\mathcal{S}$—will become the main object of our study. Define

$$\mathcal{N} = \{N \mid N \in \operatorname{Inv} A, V_{\le}(A, C) \subseteq N \subseteq V(A, C), N + R(A, B) + E_{<}(A) = \mathbb{C}^n\}.$$

In §4 we adopt a point of view of [3] and [12] and focus on the kernels of solutions to obtain a parametrization of $\mathcal{S}$. The following result will be proved.

THEOREM 1.4. *The map* $\gamma : \mathcal{S} \to \mathcal{N}$ *given by* $\gamma X = \operatorname{Ker} X$ *is a bijection, and* γ *and* γ^{-1} *are order preserving.*

The remaining part of the paper contains applications of the preceding theorems. Section 5 deals with the existence of negative-definite solutions and §6 with solutions X where the spectrum of the associated closed-loop matrix $A + BB^*$ lies in a prescribed set Λ such that $\mathbb{C}_{\le} \subseteq \Lambda$. Most of what has been known about semidefinite solutions of (1.1) can be found in the survey article [9] of Kučera and in Ando's monograph [1]. In addition we refer to [6], [7], and [12].

2. Auxiliary results.

Put $A_X = A + BB^* X$. Then (1.1) can be written as

$$\mathcal{R}(X) = A_X^* X + X A_X - XBB^*X - C^*C = 0. \tag{2.1}$$

In many instances it will be of advantage to regard (2.1) as a Lyapunov matrix equation

$$A_X^* X + X A_X = XBB^*X + C^*C$$

with semidefinite right-hand side. The following facts are well known.

LEMMA 2.1. *Suppose* X *satisfies* $A^*X + XA = R^*R$ *and* $X \le 0$.

(1) *If* $\sigma(A) \subseteq \mathbb{C}_{=}$, *then* $R = 0$.

(2) *If*

$$\sigma\begin{pmatrix} A - sI \\ R \end{pmatrix} \cap \mathbb{C}_{=} = \emptyset, \tag{2.2}$$

then $\sigma(A) \subseteq \mathbb{C}_<$.

(3) *If* $A^*X + XA = 0$ *and* $X < 0$, *then* $\sigma(A) \subseteq \mathbb{C}_=$ *and* A *is diagonalizable.*

For matrices A_i, B_i, C_i that will appear in partitions of A, B, C we define Riccati operators

$$\mathcal{R}_i(X) = A_i^*X + XA_i + XB_iB_i^*X - C_i^*C_i$$

and Lyapunov operators

$$\mathcal{L}_i(X) = A_i^*X + XA_i.$$

In what follows we encounter more than once a subspace $N \in \operatorname{Inv}A$ such that $N \subseteq \operatorname{Ker}C$. Therefore we fix the following set-up. It is understood that in a given context matrices and vectors shall be partitioned in a conforming manner. Let a basis of $\mathbb{C}^n$ be given such that

$$N = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, x_1 \in \mathbb{C}^{n-n_2} \right\}. \tag{2.3}$$

Then

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad C = (0, C_2). \tag{2.4}$$

Assume

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{2.5}$$

If

$$X = \begin{pmatrix} X_1 & X_{12} \\ X_{12}^* & X_2 \end{pmatrix}$$

is a solution of (1.1), then we have

$$A_1^*X_1 + X_1A_1 = -(X_1 \quad X_{12})BB^*(X_1 \quad X_{12})^*. \tag{2.6}$$

In the case where X is of the form $X = \operatorname{diag}(X_1, X_2)$, the block X_2 satisfies $\mathcal{R}_2(X_2) = 0$. The following observations are well known and easy to prove.

LEMMA 2.2. *Let* N *and* A, B, C *be as in* (2.3)–(2.5). *Then*

$$V_{\leq}(A, C) \subseteq N \tag{2.7}$$

is equivalent to

$$\sigma\begin{pmatrix} A_2 - sI \\ C_2 \end{pmatrix} \cap \mathbb{C}_{\leq} = \emptyset, \tag{2.8}$$

and

$$N + R(A, B) + E_<(A) = \mathbb{C}^n \tag{2.9}$$

is equivalent to

$$\sigma(A_2 - sI, B_2) \cap \mathbb{C}_{\geq} = \emptyset, \tag{2.10}$$

i.e., to stabilizability of (A_2, B_2).

The conditions (2.8) and (2.10) have implications for definiteness of solutions.

THEOREM 2.3 (see, e.g., [8]). *There exists a unique solution* $W_2 < 0$ *of* $\mathcal{R}_2(X_2) = 0$ *if and only if both* (2.8) *and* (2.10) *are satisfied. The matrix* W_2 *is the least solution, i.e.,* $\mathcal{R}_2(X_2) = 0$ *implies* $W_2 \leq X_2$.

Apart from the use of Theorem 2.3 we want the proof of the parametrization theorem in §4 to be self-contained. For that purpose we include already at this stage an existence result that appears in much more general form in Theorem 1.4.

COROLLARY 2.4 [5], [7]. *If* $V(A,C) + R(A,B) + E_<(A) = \mathbb{C}^n$ *holds, then* $\mathcal{T} \neq \emptyset$.

Proof. Take $N = V(A,C)$ in (2.3). Then according to Lemma 2.2, the matrices A_2, B_2, C_2 in (2.4) and (2.5) satisfy (2.8) and (2.10). Hence there exists a solution $X_2 < 0$ of $\mathcal{R}_2(X_2) = 0$, and $X = \operatorname{diag}(0, X_2) \leq 0$ satisfies $\mathcal{R}(X) = 0$. □

Let

$$H = \begin{pmatrix} A & BB^* \\ C^*C & -A^* \end{pmatrix}$$

be the Hamiltonian matrix associated to (1.1). There is a link between $V_=(A,C)$ and the space $E_=(H)$.

LEMMA 2.5. *We have*

$$\dim E_=(H) = 2 \dim V_=(A,C)$$

if and only if

$$E_=(A) \subseteq V(A,C) + R(A,B). \tag{2.11}$$

Proof. Note that (2.11) is equivalent to

$$E_=(A) \subseteq V_=(A,C) + R(A,B). \tag{2.12}$$

Assume

$$V_=(A,C) = \operatorname{Im} \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix}$$

or equivalently (2.4) with $\sigma(A_1) \subseteq \mathbb{C}_=$ and

$$\sigma \begin{pmatrix} A_2 - sI \\ C_2 \end{pmatrix} \cap \mathbb{C}_= = \emptyset. \tag{2.13}$$

If B is given by (2.5), then (2.12) means $E_=(A_2) \subseteq R(A_2, B_2)$, i.e.,

$$\sigma(A_2 - sI, B_2) \cap \mathbb{C}_= = \emptyset. \tag{2.14}$$

On the other hand (2.4) yields

$$H = \begin{pmatrix} A_1 & A_{12} & * & * \\ 0 & A_2 & * & B_2B_2^* \\ 0 & 0 & -A_1^* & 0 \\ 0 & C_2^*C_2 & -A_{12}^* & -A_2^* \end{pmatrix}.$$

Hence we have $\dim E_=(H) = 2n_1$ if and only if

$$\sigma(H_2) \cap \mathbb{C}_= = \emptyset, \tag{2.15}$$

where

$$H_2 = \begin{pmatrix} A_2 & B_2B_2^* \\ C_2^*C_2 & -A_2^* \end{pmatrix}.$$

It is well known that

$$\sigma(H) \cap \mathbb{C}_= = [\sigma(A - sI, B) \cap \mathbb{C}_=] \cup \left[\sigma\begin{pmatrix} A - sI \\ C \end{pmatrix} \cap \mathbb{C}_=\right].$$

Because of (2.13) the property (2.15) is equivalent to (2.14), which completes the proof. □

3. The decomposition theorem. Recall U_r in (1.5). The space

$$U_1 = V_=(A, C) \cap U_r = V(A, C) \cap R(A, B) \cap E_=(A)$$

is crucial for the proof of Theorem 1.2. We shall see that

$$U_1 \subseteq \operatorname{Ker} X \quad \text{if } X \in \mathcal{T}. \tag{3.1}$$

The following statement is equivalent to (3.1).

LEMMA 3.1. *Assume*

$$U_r = V_>(A, C) + R(A, B) + E_<(A) = \mathbb{C}^n. \tag{3.2}$$

Then we have

$$V_=(A, C) \subseteq \operatorname{Ker} X \tag{3.3}$$

for all $X \in \mathcal{T}$.

Proof. Because of (1.3), condition (3.2) implies $E_=(A) \subseteq R(A, B)$. Hence we have $V_=(A, C) \subseteq R(A, B)$, or equivalently

$$R(A, B)^\perp = V(A^*, B^*) \subseteq V_=(A, C)^\perp. \tag{3.4}$$

Now assume $N = V_=(A, C)$ in (2.3) such that $\sigma(A_1) \subseteq \mathbb{C}_=$. From (2.6) and Lemma 2.1 we obtain

$$B^* \begin{pmatrix} X_1 \\ X_{12}^* \end{pmatrix} = 0, \tag{3.5}$$

which implies

$$A^* \begin{pmatrix} X_1 \\ X_{12}^* \end{pmatrix} + \begin{pmatrix} X_1 \\ X_{12}^* \end{pmatrix} A_1 = 0. \tag{3.6}$$

Hence (3.5) and (3.6) yield

$$B^*(A^*)^i \begin{pmatrix} X_1 \\ X_{12}^* \end{pmatrix} = 0, \quad i = 0, 1, \ldots, n-1,$$

or equivalently

$$\operatorname{Im} \begin{pmatrix} X_1 \\ X_{12}^* \end{pmatrix} \subseteq V(A^*, B^*).$$

From (3.4) and $V_=(A,C)^{\perp} = \operatorname{Im}\binom{0}{I}$ follows $X_1 = 0$. And because of $X \leq 0$ we have $X = \operatorname{diag}(0, X_2)$, which proves (3.3). □

Proof of Theorem 1.2. Let $\mathbb{C}^n = U_0 \oplus U_r$ be a given *LR*-decomposition and assume (1.6). Then it is obvious that (1.7)–(1.9) hold. If

$$X = \begin{pmatrix} X_0 & X_{0r} \\ X_{0r}^* & X_r \end{pmatrix} \in \mathcal{T}$$

is partitioned accordingly, then the block X_r satisfies (1.12). From Lemma 3.1 we obtain

$$V_=(A_r, C_r) \subseteq \operatorname{Ker} X_r. \tag{3.7}$$

Note that

$$V_=(A,C) = U_0 \oplus [V_=(A,C) \cap U_r] = U_0 \oplus U_1.$$

Let U_2 be such that $U_r = U_1 \oplus U_2$. If we choose an appropriate basis and take $U_0 \oplus U_1 \in \operatorname{Inv} A$ into account then we have

$$A = \begin{pmatrix} A_0 & 0 & 0 \\ A_{10} & A_1 & A_{12} \\ 0 & 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ B_1 \\ B_2 \end{pmatrix}, \qquad C = (0, 0, C_2),$$

where

$$\sigma(A_0) \cup \sigma(A_1) \subseteq \mathbb{C}_= \tag{3.8}$$

and

$$\sigma\begin{pmatrix} A_2 - sI \\ C_2 \end{pmatrix} \cap \mathbb{C}_= = \emptyset. \tag{3.9}$$

From

$$U_1 = \{(0, x_1^T, 0)^T, (x_1^T, 0)^T \in V_=(A_r, C_r)\} \tag{3.10}$$

and (3.7) follows (3.1), i.e., $U_1 \subseteq \operatorname{Ker} X$. Hence

$$X = \begin{pmatrix} X_0 & 0 & X_{02} \\ 0 & 0 & 0 \\ X_{02}^* & 0 & X_2 \end{pmatrix}, \tag{3.11}$$

and (1.1) is equivalent to the following set of three equations: $\mathcal{L}_0(X_0) = A_0^* X_0 + X_0 A_0 = 0$,

$$A_0^* X_{02} + X_{02}(A_2 + B_2 B_2^* X_2) = 0, \tag{3.12}$$

and

$$\mathcal{R}_2(X_2) = A_2^* X_2 + X_2 A_2 + X_2 B_2 B_2^* X_2 - C_2^* C_2 = 0. \tag{3.13}$$

Put $\hat{A}_2 = A_2 + B_2 B_2^* X_2$ such that (3.13) can be written as $\hat{A}_2^* X_2 + X_2 \hat{A}_2 = R_2^* R_2$ with $R_2^* = (X_2 B_2, C_2^*)$. Since (3.9) implies

$$\sigma\begin{pmatrix} \hat{A}_2 - sI \\ R_2 \end{pmatrix} \cap \mathbb{C}_= = \emptyset,$$

Lemma 2.1 yields $\sigma(\hat{A}_2) \cap \mathbb{C}_= = \emptyset$. Because of (3.8), (3.12) has only the trivial solution $X_{02} = 0$. Hence the matrix (3.11) is further reduced to $X = \operatorname{diag}(X_0, 0, X_2)$, which is a decomposition of the form (1.10).

Conversely if $X = \operatorname{diag}(X_0, X_r)$ is such that (1.11) and (1.12) hold, then Lemma 3.1 yields $X_r = \operatorname{diag}(0, X_2)$ and X is a solution of $\mathcal{R}(X) = 0$. □

Proof of Theorem 1.3. Fix U_2 such that $U_r = U_1 \oplus U_2$. Let a basis of $\mathbb{C}^n$ be given such that (1.6), (3.10), and $U_2 = \{(0, 0, x_2^T)^T\}$ hold. Then we have

$$X = \operatorname{diag}(X_0, X_r) \tag{3.14}$$

and

$$X_r = \operatorname{diag}(0, X_2) \tag{3.15}$$

for $X \in \mathcal{T}$. Therefore $\Pi = \operatorname{diag}(0, I)$; hence (3.14) an equivalent to (1.13). Let Ψ be the projection onto U_2 along $V_=(A, C) = U_0 \oplus U_1$. Obviously (3.15) implies $X\Psi = X\Pi$, which shows that $\rho X = X\Pi$ is independent of the choice of the Lyapunov component U_0. □

The map ρ has the properties of a closure operator [2] on the partially ordered set $\mathcal{T}$, namely: (1) $X \leq \rho X$, (2) $X \leq Y \Rightarrow \rho X \leq \rho Y$, (3) $\rho^2 X = \rho X$ for all $X, Y \in \mathcal{T}$.

In the next section we are concerned with the set $\mathcal{S} = \rho\mathcal{T}$. It is obvious that $X \in \mathcal{T}$ is in $\mathcal{S}$ if and only if $V_=(A, C) \subseteq \operatorname{Ker} X$. Similarly, we have $\mathcal{T} \neq \emptyset$ together with $\mathcal{T} = \mathcal{S}$ if and only if

$$V_>(A, C) + R(A, B) + E_<(A) = \mathbb{C}^n. \tag{3.16}$$

LEMMA 3.2. *Assume $\mathcal{T} \neq \emptyset$. Then $\mathcal{T} = \mathcal{S}$ is equivalent to*

$$E_=(A) \subseteq R(A, B) \tag{3.17}$$

and also to

$$V_=(A, C) \subseteq R(A, B). \tag{3.18}$$

Proof. If we intersect both sides of (3.18) with $E_=(A)$ we obtain (3.17) in the form $R(A, B) \cap E_=(A) = E_=(A)$. Assume now (3.18). Then $\mathcal{T} \neq \emptyset$, i.e., $\mathbb{C}^n = V(A, C) + R(A, B) + E_<(A) = V_>(A, C) + [V_=(A, C) + R(A, B)] + [V_<(A, C) + E_<(A)]$ implies (3.16). □

4. Kernels of solutions. Scherer's approach in [12], which is based on the map $\gamma : X \mapsto \operatorname{Ker} X$, can be adapted to our analysis and leads to a description of the solution set $\mathcal{S}$.

LEMMA 4.1. (1) *If X is a solution of* (1.1) *then* Ker X *is A-invariant and satisfies*

$$\operatorname{Ker} X \subseteq V(A, C) \tag{4.1}$$

(2) *For $X \in \mathcal{T}$ we have*

$$V_<(A, C) \subseteq \operatorname{Ker} X. \tag{4.2}$$

(3) *A solution $X \in \mathcal{S}$ satisfies*

$$V_\leq(A, C) \subseteq \operatorname{Ker} X \tag{4.3}$$

and

$$\operatorname{Ker} X + R(A, B) + E_<(A) = \mathbb{C}^n. \tag{4.4}$$

Proof. (1) If $Xy = 0$, then $y^*\mathcal{R}(X)y = -y^*C^*Cy = 0$ yields $\operatorname{Ker} X \subseteq \operatorname{Ker} C$. From $Cy = 0$ and $\mathcal{R}(X)y = 0$ we obtain $XAy = 0$. Now (4.1) follows from the fact that $V(A, C)$ is the largest A-invariant subspace in $\operatorname{Ker} C$.

(2) Take $N = \operatorname{Ker} X$ in (2.3) such that $X = \operatorname{diag}(0, X_2)$, $X_2 < 0$. Put $Y = X_2^{-1}$ and $\tilde{A}_2 = A_2 - YC_2^*C_2$. Then $\mathcal{R}_2(X_2) = 0$ can be written as

$$Y\tilde{A}_2^* + \tilde{A}_2Y = -(B_2B_2^* + YC_2^*C_2Y).$$

Obviously $Y < 0$ implies $\sigma(\tilde{A}_2) \cap \mathbb{C}_< = \emptyset$, which is equivalent to

$$\sigma\begin{pmatrix} A_2 - sI \\ C_2 \end{pmatrix} \cap \mathbb{C}_< = \emptyset$$

or to (4.2).

(3) For $X \in \mathcal{S}$ we have $V_=(A, C) \subseteq \operatorname{Ker} X$, which establishes (4.3). If we take $N = \operatorname{Ker} X$ as above then we conclude that

$$V_=(A_2, C_2) = 0. \tag{4.5}$$

Now put $\hat{A}_2 = A_2 + B_2B_2^*X_2$. Again write $\mathcal{R}_2(X_2) = 0$ as

$$\hat{A}_2^*X_2 + X_2\hat{A}_2 = X_2B_2B_2^*X_2 + C_2^*C_2.$$

Then $X_2 < 0$ implies $\sigma(\hat{A}_2) \subseteq \mathbb{C}_\leq$. Suppose $\hat{A}_2y = \lambda y$ and $\lambda \in \mathbb{C}_=$. Then the preceding Lyapunov equation yields $B_2^*X_2y = 0$ and $C_2y = 0$. Hence $A_2y = \lambda y$, and $y \in V_=(A_2, C_2)$. From (4.5) follows $y = 0$. Hence

$$\sigma(\hat{A}_2) \subseteq \mathbb{C}_<, \tag{4.6}$$

which means (A_2, B_2) is stabilizable, i.e., we have $R(A_2, B_2) + E_<(A_2) = \mathbb{C}^{n_2}$, which in turn proves (4.4). □

COROLLARY 4.2. *For $X \in \mathcal{S}$ we have*

$$\operatorname{Ker} X = V_\leq(A, C) \oplus E_>(A_X). \tag{4.7}$$

Furthermore $E_>(A_X) \in \operatorname{Inv} A$ and $A = A_X$ on $E_>(A_X)$.

Proof. As before assume $X = \operatorname{diag}(0, X_2), X_2 < 0$. Then

$$A_X = \begin{pmatrix} A_1 & \hat{A}_{12} \\ 0 & \hat{A}_2 \end{pmatrix} \quad \text{and} \quad C = (0, C_2).$$

Hence (4.6) and (4.3) imply

$$\operatorname{Ker} X = \{(x_1^T, 0)^T, x_1 \in E_\leq(A_1) \oplus E_>(A_1)\} = V_\leq(A, C) \oplus E_>(A_X).$$

The remaining statements are easy to verify. □

With (4.1), (4.3), and (4.4) we have the properties which characterize the set $\{\operatorname{Ker} X \mid X \in \mathcal{S}\}$. Recall the definition

$$\mathcal{N} = \{N \mid N \in \operatorname{Inv} A, V_\leq(A, C) \subseteq N \subseteq V(A, C), N + R(A, B) + E_<(A) = \mathbb{C}^n\}.$$

Proof of Theorem 1.4. From Lemma 4.1 it is clear that $X \in \mathcal{S}$ implies $\gamma X = \operatorname{Ker} X \in \mathcal{N}$. To show that $\gamma : \mathcal{S} \to \mathcal{N}$ is bijection, we fix a subspace $N \in \mathcal{N}$. We want to show that there exists a unique $Y \in \mathcal{T}$ with $\operatorname{Ker} Y = N$. Because of $V_=(A,C) \subseteq N$ such a Y is necessarily in $\mathcal{S}$. As usual we work in the set-up (2.3)–(2.5). Note that $\operatorname{Ker} X = N$ together with $X \leq 0$ is equivalent to $X = \operatorname{diag}(0, X_2), X_2 < 0$. Furthermore X is a solution of $\mathcal{R}(X) = 0$ if and only if X_2 satisfies $\mathcal{R}_2(X_2) = 0$. According to Lemma 2.2 the properties (2.7) and (2.9) of N can be expressed by (2.8) and (2.10). Hence Theorem 2.3 yields a unique solution $Y_2 < 0$ of $\mathcal{R}_2(X_2) = 0$. Then $Y = \operatorname{diag}(0, Y_2)$ is the uniquely determined solution in $\mathcal{T}$ with $\operatorname{Ker} Y = N$. It is obvious that γ is order preserving, since $X \leq Y \leq 0$ implies $\operatorname{Ker} X \subseteq \operatorname{Ker} Y$. Now assume that $N, M \in \mathcal{N}$ satisfy $N \subseteq M$. Let N be given as in (2.3). Put $W = \gamma^{-1}(N), Y = \gamma^{-1}(M)$. Then $\operatorname{Ker} W = N \subseteq \operatorname{Ker} Y$ implies $W = \operatorname{diag}(0, W_2), W_2 < 0$, and $Y = \operatorname{diag}(0, Y_2), Y_2 \leq 0$. Both W_2 and Y_2 are solutions of $\mathcal{R}_2(X_2) = 0$. According to Theorem 2.3 the definite matrix W_2 is the least solution of $\mathcal{R}_2(X_2) = 0$. Hence $W_2 \leq Y_2$ and $W \leq Y$, which shows that also γ^{-1} is order preserving. □

Remark 4.3. The following statements are equivalent:

$$\begin{aligned} &(1)\ \mathcal{N} \neq \emptyset, \\ (4.8)\qquad &(2)\ V(A,C) + R(A,B) + E_<(A) = \mathbb{C}^n, \\ &(3)\ V(A,C) \in \mathcal{N}, \\ (4.9)\qquad &(4)\ E_>(A) \subseteq V(A,C) + R(A,B), \end{aligned}$$

and

$$\dim E_=(H) = 2 \dim V_=(A,C). \tag{4.10}$$

Proof. It is easy to see that the definition of $\mathcal{N}$ implies the equivalence of (1), (2), and (3). Because of (1.3) we can state (4.8) as

$$E_>(A) + E_=(A) \subseteq V(A,C) + R(A,B) \tag{4.11}$$

or equivalently as a pair of two inclusions, namely (4.9) together with

$$E_=(A) \subseteq V(A,C) + R(A,B). \tag{4.12}$$

According to Lemma 2.5 the conditions (4.12) and (4.10) are equivalent, which implies the equivalence of (2) and (4). □

Since $\mathcal{T} \neq \emptyset$ if and only if $\mathcal{S} \neq \emptyset$ (i.e., $\mathcal{N} \neq \emptyset$) the preceding remark yields the known necessary and sufficient conditions for the existence of a solution $X \leq 0$ of (1.1). Condition (2) is contained in [5], [7] whereas (4) can be found in [10]. The fact $\mathcal{T}$ has a greatest element [5], [7] is another immediate consequence of Theorem 1.4. Since $X \leq \rho X$ and $V = V(A,C) = \sup \mathcal{N}$, we see that $\gamma^{-1}(V)$ is greatest negative-semidefinite solution of (1.1).

In [13], [8], [4], [1] solutions of (1.1) are parametrized under more restrictive hypotheses such as controllability or stabilizability, and the parametrization is based on the subspaces $E_>(A_X)$. The subsequent observation makes a connection to those results. From (4.7) follows that a solution $X \in \mathcal{S}$ is uniquely determined by $E_>(A_X)$. Define

$$\mathcal{G} = \{G \mid G \in \operatorname{Inv} A, G \subseteq V_>(A,C), G + V_\leq(A,C) + R(A,B) + E_<(A) = \mathbb{C}^n\}.$$

Then the map $\mu : \mathcal{S} \to \mathcal{G}$ given by $\mu X = E_>(A_X)$ is a bijection, and μ and μ^{-1} are order preserving. □

5. Negative-definite solutions. From Theorem 1.2 we obtain a condition for the existence of negative-definite solutions of (1.1). The following result is attributed to Richardson and Kwong.

THEOREM 5.1 [11]. *The* ARE (1.1) *has a solution* $W < 0$ *if and only if with respect to an appropriate basis the matrices* A, B, C *take the form*

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_r \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ B_r \end{pmatrix}, \qquad C = (0, C_r), \tag{5.1}$$

where

$$\sigma(A_0) \subseteq \mathbb{C}_= \quad \textit{and } A_0 \textit{ is diagonalizable}, \tag{5.2}$$

and

$$\sigma(A_r - sI, B_r) \cap \mathbb{C}_{\geq} = \emptyset, \qquad \sigma\begin{pmatrix} A_r - sI \\ C_r \end{pmatrix} \cap \mathbb{C}_{\leq} = \emptyset. \tag{5.3}$$

Assume (5.1)–(5.3). *Then* $W < 0$ *is a solution of* (1.1) *if and only if*

$$W = \operatorname{diag}(W_0, W_r), \tag{5.4}$$

and $W_0 < 0$, $\mathcal{L}_0(W_0) = 0$, *and* W_r *is the unique negative-definite solution of* $\mathcal{R}_r(X_r) = 0$.

The concept of LR-decomposition yields a more specific existence condition.

THEOREM 5.2. *The* ARE (1.1) *has a negative-definite solution if and only if the following conditions are satisfied*:

$$(1)\ \mathbb{C}^n = V_=(A, C) \oplus U_r, \tag{5.5}$$

$$(2)\ V_<(A, C) = 0, \tag{5.6}$$

(3) $A_{|V_=(A,C)}$ *is diagonalizable.*

If

$$V_=(A, C) = \left\{ \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \right\} \quad \textit{and} \quad U_r = \left\{ \begin{pmatrix} 0 \\ x_r \end{pmatrix} \right\}, \tag{5.7}$$

then the matrices A, B, C *take the form* (5.1) *and have the properties* (5.2) *and* (5.3).

Proof. Suppose (1.1) has a solution $W < 0$. Let $\mathbb{C}^n = U_0 \oplus U_r$ be an LR-decomposition such that $U_0 \subseteq V_=(A, C)$ and $U_r = V_>(A, C) + R(A, B) + E_<(A)$. We know that $U_1 = V_=(A, C) \cap U_r \subseteq \operatorname{Ker} W$. Therefore $V_=(A, C) \cap U_r = 0$, and from $V_=(A, C) + U_r = \mathbb{C}^n$ follows (5.5). From (4.2) we obtain (5.6). Let a basis of $\mathbb{C}^n$ be chosen such that (5.7) holds. Then (5.1) is obvious, and (5.4) follows from Theorem 1.2. Since $W_0 < 0$ satisfies $A_0^* W_0 + W_0 A_0 = 0$, we obtain (5.2) from Lemma 2.1. The matrix $W_r < 0$ is a solution of $\mathcal{R}_r(X_r) = 0$ and because of $V_=(A_r, C_r) = 0$, it is the only negative-definite solution. Hence Theorem 2.3 yields (5.3). The sufficiency part of the theorem is obvious. □

6. Location of $\sigma(A_X)$. In addition to definiteness or semidefiniteness of solutions, the location of $\sigma(A_X)$ is of interest. For a subset $\Lambda \subseteq \mathbb{C}$ put $E_\Lambda(A) = \oplus\{E_\lambda(A), \lambda \in \Lambda\}$ and $V_\Lambda(A) = V(A, C) \cap E_\Lambda(A)$. In the important case where $\Lambda = \mathbb{C}_{\leq}$, the equivalence of (1) and (2) of the subsequent theorem can be found in [4].

THEOREM 6.1. *Let* $\Lambda \subseteq \mathbb{C}$, *be given, and assume* $\mathbb{C}_{\leq} \subseteq \Lambda$. *Then the following statements are equivalent*:

(1) *There exists an* $X \in \mathcal{T}$ *such that*

$$\sigma(A_X) \subseteq \Lambda. \tag{6.1}$$

(2) *Both*

$$V(A,C) + R(A,B) + E_<(A) = \mathbb{C}^n \tag{6.2}$$

and

$$R(A,B) + E_\Lambda(A) = \mathbb{C}^n \tag{6.3}$$

hold.

$$(3)\ V_\Lambda(A,C) + R(A,B) + E_<(A) = \mathbb{C}^n. \tag{6.4}$$

Proof. (1) $\Rightarrow$ (2). Because of $\sigma(A - sI, B) = \sigma(A_X - sI, B)$, the inclusion (6.1) implies $\sigma(A - sI, B) \subseteq \Lambda$, which is equivalent to (6.3).

(2) $\Rightarrow$ (3). Put $K = \mathbb{C} \backslash \Lambda$. Because of

$$\begin{aligned} R(A,B) + E_\Lambda(A) &= [R(A,B) \cap E_\Lambda(A)] + [R(A,B) \cap E_K(A)] + E_\Lambda(A) \\ &= [R(A,B) \cap E_K(A)] + E_\Lambda(A), \end{aligned}$$

condition (6.3) is equivalent to $R(A,B) \cap E_K(A) = E_K(A)$, i.e., to $E_K(A) \subseteq R(A,B)$. If we write (6.2) as

$$V_\Lambda(A,C) + [V_K(A,C) + R(A,B)] + E_<(A) = \mathbb{C}^n,$$

then $V_K(A,C) \subseteq E_K(A) \subseteq R(A,B)$ yields (6.4).

(3) $\Rightarrow$ (1). Clearly (6.1) is equivalent to $E_\Lambda(A_X) = \mathbb{C}^n$. Put $M = \Lambda \backslash \mathbb{C}_\leq$ such that $E_\Lambda(A_X) = E_\leq(A_X) \oplus E_M(A_X)$. Then (6.1) takes the equivalent form

$$E_>(A_X) = E_M(A_X). \tag{6.5}$$

Note that $\mathbb{C}_\leq \subseteq \Lambda$ yields $V_\leq(A,C) \subseteq V_\Lambda(A,C)$. Hence (6.4) implies $V_\Lambda(A,C) \in \mathcal{N}$, and there exists a solution $X \in \mathcal{S}$ such that $\operatorname{Ker} X = V_\Lambda(A,C) = V_\leq(A,C) \oplus V_M(A,C)$. From (4.7), i.e., $\operatorname{Ker} X = V_\leq(A,C) \oplus E_>(A_X)$, we obtain $E_>(A_X) = V_M(A,C)$. Therefore we have $\sigma(A_X) \cap \mathbb{C}_> \subseteq M$ or $E_>(A_X) \subseteq E_M(A_X)$, which yields (6.5). □

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