Diagonal matrix solutions of a discrete-time Lyapunov inequality

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Abstract

Diagonal solutions of a Lyapunov inequality for companion matrices are studied. Such solutions are required if states of a discrete-time linear system are computed with a finite precision arithmetic.

1 Introduction

Let

$$x(i+1) = Ax(i), \quad x(0) = x_0, \tag{1.1}$$

be a discrete-time linear system with $x(i) = \Big(x_1(i), \dots, x_n(i)\Big)^T \in \mathbb{C}^n$. It is well known that the system (1.1) is asymptotically stable if and only if there exists a matrix $P > 0$ (positive-definite) such that

$$A^*PA - P = -Q^*Q \tag{1.2}$$

and

$$(A, Q) \quad \text{is observable.} \tag{1.3}$$

According to [1] diagonal solutions P of (1.2) are required if a finite precision arithmetic is used to calculate the states $x(i)$ of (1.1). Let $g(\cdot)$ be a scalar function which satisfies

$$|g(y)| \le |y| \quad \text{for all} \quad y \in \mathbb{C}, \tag{1.4}$$

and let $\tilde{g}[x(i)] = \Big(g[x_1(i)], \dots, g[x_n(i)]\Big)^T$ be the vector obtained from $x(i)$ using $g(\cdot)$ component-wise as a quantizer operator. It was shown in [1] that the quantized system

$$x(i+1) = A\tilde{x}(i), \quad \tilde{x}(i+1) = \tilde{g}[x(i+1)], \tag{1.5}$$

is asymptotically stable if there exists a diagonal matrix P such that (1.2) and (1.3) hold.

In the case where A is a companion matrix diagonal solutions of the Lyapunov equation (1.2) were studied in [2]. In this note we clarify some issues of [2] and prove the following result.

Theorem 1 *Let*

$$A = \begin{pmatrix} a_1 & \dots & a_{n-1} & a_n \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix} \tag{1.6}$$

be a complex companion matrix. Put

$$s = \sum_{\nu=1}^{n} |a_\nu|.$$

(i) There exists a diagonal matrix $P = \operatorname{diag}(p_1, \dots, p_n)$ *with properties*

$$P \ge 0 \text{ (positive-semidefinite)}, P \ne 0, \tag{1.7}$$

and

$$L(P) = P - A^*PA \ge 0 \tag{1.8}$$

if and only if $s \le 1$.

(ii) There exists a diagonal matrix $P > 0$ (positive-definite) satisfying (1.8) if and only if

$$\text{either } s < 1 \text{ or both } s = 1 \text{ and } a_n \neq 0 \tag{1.9}$$

hold.

In [2] it was shown that $s \leq 1$ is necessary for the existence of a diagonal positive-definite solution P of (1.8). Discrete-time Lyapunov equations (1.2) with a companion matrix A have been investigated by several authors, we refer to [3], [4], [5]. Theorem 1 will be proved in Section 2. A matrix of the form (1.6) which is important for the critical exponent of the row-sum norm [6] will be discussed in Section 3. A counterpart of Theorem 1 for the continuous-time Lyapunov inequality $A^*P + PA \leq 0$ will be derived in Section 4.

2 Explicit solutions of $L(P) \geq 0$

In the following Lemma the matrix inequality (1.8) will be related to a scalar inequality. It will be convenient to allow denominators to be zero. For $\alpha \in \mathbb{R}$ and $\pi = 0$ we set

$$\frac{\alpha^2}{\pi} = \begin{cases} 0 & \text{if } \alpha = 0 \\ \infty & \text{if } \alpha \neq 0 \end{cases}. \tag{2.1}$$

Lemma 2 *Let $a_1, \ldots, a_n \in \mathbb{C}$ and $p_1, \ldots, p_n \in \mathbb{R}$ be given. The matrix $P = \operatorname{diag}(p_1, \ldots, p_n)$ satisfies*

$$P \geq 0, \quad P \neq 0 \tag{2.2}$$

and

$$L(P) = P - A^*PA \geq 0 \tag{2.3}$$

if and only if the conditions

$$p_1 > 0, \quad p_1 \geq p_2 \geq \cdots \geq p_n \geq 0 \tag{2.4}$$

and

$$\frac{1}{p_1} \geq \frac{|a_1|^2}{p_1 - p_2} + \cdots + \frac{|a_{n-1}|^2}{p_{n-1} - p_n} + \frac{|a_n|^2}{p_n} \tag{2.5}$$

hold.

Proof. Put

$$\Pi = \operatorname{diag}(p_1 - p_2, \ldots, p_{n-1} - p_n, p_n - p_{n+1}), p_{n+1} = 0.$$

As in [2] we note that

$$L(P) = \Pi - p_1(a_1, \dots, a_n)^*(a_1, \dots, a_n)$$

and that (2.3) is equivalent to

$$\Pi \geq p_1(a_1, \dots, a_n)^*(a_1, \dots, a_n). \tag{2.6}$$

If $a_j = |a_j|e^{i\varphi_j}$, $j = 1, \dots, n$, set $D = \operatorname{diag}(e^{-i\varphi_1}, \dots, e^{-i\varphi_n})$ and $a = (|a_1|, \dots, |a_n|)^*$. Then $D^*\Pi D = \Pi$. Hence (2.6) is equivalent to

$$\Pi \geq p_1 a a^*. \tag{2.7}$$

Assume that (2.4) and (2.5) hold. Then (2.1) implies $a_j = 0$ if $p_j = p_{j+1}$. Hence in order to prove (2.7) we can discard the indices j with $p_j - p_{j+1} = 0$ and because of (2.4) assume $\Pi > 0$. Then (2.5) is equivalent to

$$1 \geq p_1 a^* \Pi^{-1} a = p_1 \left(\Pi^{-1/2} a\right)^* \left(\Pi^{-1/2} a\right). \tag{2.8}$$

Note that for a vector $b \in \mathbb{C}^n$ the eigenvalues of the dyadic product bb^* are $b^*b, 0, \dots, 0$. Hence (2.8) is equivalent to

$$I \geq p_1 \Pi^{-1/2} a a^* \Pi^{-1/2},$$

i.e. to (2.7). Starting from (2.7) the converse part of the lemma can be proved along similar lines. ■

We focus on the inequalities (2.4) and (2.5) assuming $s \neq 0$. Set $\hat{p}_{n+1} = 0$ and

$$\hat{p}_\nu = \frac{1}{s}(|a_\nu| + \cdots + |a_n|), \quad \nu = 1, \dots, n. \tag{2.9}$$

Then $\hat{p}_1 = 1$ and

$$\sum_{\nu=1}^{n} \frac{|a_\nu|^2}{\hat{p}_\nu - \hat{p}_{\nu+1}} = \sum_{\nu=1}^{n} |a_\nu| s = s^2.$$

Hence if $0 < s \leq 1$ then (2.4) and (2.5) are satisfied by $p_\nu = \hat{p}_\nu$, $\nu = 1, \dots, n$.

Now consider the case $a_n = 0$ and $s < 1$. Let k be such that

$$a_n = \cdots = a_{k+1} = 0, \quad a_k \neq 0. \tag{2.10}$$

Then $\hat{p}_{k+1} = \cdots = \hat{p}_n = 0$, $\hat{p}_k = \frac{1}{s}|a_k| > 0$. For $\delta \in \mathbb{R}$, $\delta \neq \hat{p}_k$, define

$$f(\delta) = \frac{|a_1|^2}{\hat{p}_1 - \hat{p}_2} + \cdots + \frac{|a_{k-1}|^2}{|\hat{p}_{k-1} - \hat{p}_k} + \frac{|a_k|^2}{\hat{p}_k - \delta}.$$

Then $f(0) = s^2$. By continuity of $f(\delta)$ there exists an R, $R > 0$, such that

$$f(\delta) \leq 1 \quad \text{if} \quad 0 \leq \delta \leq R. \tag{2.11}$$

It can be shown that (2.11) holds for

$$R = \frac{|a_k|(1 - s^2)}{s - s^2(s - |a_k|)} \tag{2.12}$$

and that (2.12) is the best possible bound for (2.11). Hence any $\delta \in [0, R]$ yields a solution

$$(p_1, \ldots, p_n) = (\hat{p}_1, \ldots, \hat{p}_k, \delta, \ldots, \delta) \tag{2.13}$$

of (2.4) and (2.5).

Proof of Theorem 1: (i) Assume $s \leq 1$. If $s = 0$ then $A^*A = \mathrm{diag}(1, \ldots, 1, 0)$. Hence $P = I$ is a solution of (1.8). If $s \neq 0$ we use the numbers $\hat{p}_\nu$ given by (2.9) and set...

$$\hat{P} = \mathrm{diag}(\hat{p}_1, \ldots, \hat{p}_n). \tag{2.14}$$

Then Lemma 2 implies that $P = \hat{P}$ has the properties (1.7) and (1.8). Note that $\hat{P} > 0$ if $a_n \neq 0$. To show that $s \leq 1$ is necessary for (1.7) and (1.8) consider (1.8) in the equivalent form (2.7). Put $e = (1, \ldots, 1)^T$ then (2.7) yields

$$p_1 = e^*\Pi e \geq p_1(e^*a)^2 = p_1 s^2.$$

Because of (2.4) we have $p_1 > 0$ and therefore $1 \geq s^2$.
(ii) To show that (1.8) has a positive-definite solution if (1.9) holds consider the three cases $s = 0$ and $0 < s \leq 1$ with $a_n \neq 0$, and

$$0 < s < 1, \quad a_n = 0. \tag{2.15}$$

In the first two cases we know that a solution $P > 0$ of (1.8) is given by $P = I$ and $P = \hat{P}$ respectively. In the third case (2.15) we assume (2.10) and choose $0 < \delta \leq R$ with R as in (2.12). Using the n-tuple (2.13) we obtain the solution

$$P = \mathrm{diag}(\hat{p}_1, \ldots, \hat{p}_n, \delta, \ldots, \delta) > 0. \tag{2.16}$$

To complete the proof we now assume that (1.8) holds for some $P > 0$. Then $s \leq 1$ and we have to exclude the case

$$s = 1, \quad a_n = 0. \tag{2.17}$$

As before the assumption $a_\nu = |a_\nu|$, $\nu = 1, \ldots, n$, is no loss of generality. Put $g = (1, \ldots, 1, 0)^T$. Then (2.17) implies $Ag = e$ and $g^T L(P) g = -p_n$. From $L(P) \geq 0$ follows

$p_n \leq 0$ which is incompatible with $P > 0$. ∎

Let $\rho(A)$ denote the spectral radius of A. Clearly $P > 0$ and $L(P) \geq 0$ imply $\rho(A) \leq 1$, and as it was mentioned in Section 1 an observability condition (1.3) ensures $\rho(A) < 1$. In the case of a companion matrix A and a diagonal P we note the following result.

Corollary 3 *There exists a diagonal matrix* $P = \operatorname{diag}(p_1, \ldots, p_n)$ *such that*

$$P > 0 \text{ and } L(P) = P - A^*PA \geq 0 \tag{2.18}$$

and the pair

$$(A, L(P)) \text{ is observable} \tag{2.19}$$

if and only if

$$\text{either } s < 1 \text{ or both } s = 1 \text{ and } 0 < |a_n| < 1 \tag{2.20}$$

hold. From (2.20) follows $\rho(A) < 1$.

Proof. If A is of the form (1.6) and $H = (h^{(1)}, \ldots, h^{(n)}) \in \mathbb{C}^{m\times n}$, then the pair (A, H) is observable if and only if $h^{(n)} \neq 0$.

Recall

$$L(P) = \operatorname{diag}(p_1 - p_2, \ldots, p_{n-1} - p_n, p_n) - p_1(a_1, \ldots, a_n)^*(a_1, \ldots, a_n).$$

If $L(P) \geq 0$ then the last column of $L(P)$ is nonzero if and only if

$$p_n - p_1|a_n|^2 > 0. \tag{2.21}$$

Hence (2.18) and (2.19) implies (1.9) and (2.21). From (2.4) and (2.21) we deduce $|a_n| < 1$ which leads to (2.20). Now assume (2.20) to establish the existence of a diagonal P satisfying (2.18) and (2.19) three cases will be considered, namely $s = 0$, and

$$0 < s < 1, \quad a_n = 0, \tag{2.22}$$

and

$$0 < s \leq 1, \quad 0 < |a_n| < 1. \tag{2.23}$$

In the case $s = 0$ again take $P = I$. Then $L(P) = \operatorname{diag}(0, \ldots, 0, 1)$, and (2.18) and (2.19) are satisfied. In the case (2.22) take P as in (2.16). Then $p_n = \delta > p_1|a_n|^2 = 0$ and (2.21) and equivalently (2.19) hold. In the case (2.23) take the matrix $P = \hat{P}$ of (2.14). Then $p_n = \frac{1}{s}|a_n|$, $p_1 = 1$, and

$$p_n - p_1|a_n|^2 = \frac{1}{s}|a_n|(1 - s|a_n|) > 0.$$

■

We call a matrix $A \in \mathbb{C}^{n\times n}$ *diagonally d-stable* if there exists a diagonal matrix P such that $P > 0$ and $L(P) = P - A^*PA > 0$. For diagonal stability of discrete- and of continuous-time Lyapunov equations we refer to [7]. A slight modification of the proof of Theorem 1 yields the following result.

Corollary 4 *The companion matrix A in (1.6) is diagonally d-stable if and only if $s < 1$.*

3 The critical exponent of the row-sum norm

For a matrix $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ put

$$||B|| = \max_i \left\{ \sum_{j=1}^{n} |b_{ij}| \right\}.$$

Mařík and Pták [6] define the critical exponent of the row-sum norm $||\cdot||$ on $\mathbb{C}^{n\times n}$ as the positive integer κ which has the property that

$$\rho(B) = 1 \quad \text{if} \quad ||B|| = ||B^2|| = \cdots ||B^\kappa|| = 1, \tag{3.1}$$

and for which there exists a matrix C with

$$\rho(C) < 1 \quad \text{and} \quad ||C|| = \cdots = ||C^{\kappa-1}|| = 1; \tag{3.2}$$

i.e. κ is the smallest number with property (3.1). To prove that $\kappa = n^2 - n + 1$ Mařík and Pták use the following matrix C which (up to a permutation of rows and columns) is of the form (1.6) and satisfies (2.20). Let τ be real, $0 < \tau < 1$, and put

$$C = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ -\tau & \tau - 1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}.$$

Then Corollary 3 with $a_n = -\tau$ and $a_{n-1} = \tau - 1$ implies $\rho(C) < 1$. Note that $C^n = -[(1-\tau)A + \tau I]$ and

$$C^{n^2-n} = (-1)^{n-1}\left[(1-\tau)A + \tau I\right]^{n-1}. \tag{3.3}$$

Again set $e = (1,\dots,1)^T$. Then

$$Ce = (1,\dots,1,-1),\dots,C^{n-1}e = (1,-1,\dots,-1),$$

and (3.3) implies

$$C^{n^2-n}e = (-1)^{n-1}\left(\sum_{j=0}^{n-1}\binom{n-1}{j}\tau^{n-1-j}(1-\tau)^j,\dots\right) = (-1)^{n-1}(1,\dots).$$

Hence the first row of C^{n^2-n} has norm 1 and from $1 = \|C\| \le \|C^2\| \le \dots$ follows

$$\|C\| = \dots = \|C^{n^2-n}\| = 1, \tag{3.4}$$

which shows that $\kappa \ge n^2 - n + 1$. A different proof of (4.4) using a graph theoretical approach is contained in [8].

4 The continuous-time Lyapunov inequality

For a continuous-time n-dimensional linear system $\dot{x}(t) = Ax(t)$ the Lyapunov inequality corresponding to (1.8) is

$$S(P) = A^*P + PA \le 0. \tag{4.1}$$

In contrast to the discrete-time inequality (1.8) a matrix A in companion form is hardly an advantage in (4.1) if diagonal solutions P are required.

Theorem 5 *Let $A \in \mathbb{C}^{n\times n}$, $n \ge 2$, be a companion matrix of the form (1.6). (i) Then there exists a diagonal matrix P such that $P \ne 0$, $P \ge 0$ and $S(P) \le 0$ hold if and only if*

$$\det(\lambda I - \lambda) = \lambda^{n-2}(\lambda^2 - a_1\lambda - a_2) \tag{4.2}$$

and

$$\bar{a}_1 + a_1 \le 0 \text{ and } a_2 \in \mathbb{R},\ a_2 \le 0. \tag{4.3}$$

(ii) If (4.2) and (4.3) hold then a diagonal matrix P satisfies $P \ge 0$ and $S(P) \le 0$ if and only if

$$P = \mathrm{diag}(p_1, -a_2p_1, 0, \dots, 0), \quad p_1 \ge 0.$$

Proof. (i) If $P = \mathrm{diag}(p_1,\dots,p_n)$ has real entries then

$$S(P) = \begin{pmatrix} (\bar{a}_1 + a_1)p_1 & a_2p_1 + p_2 & a_3p & \dots & a_np_1 \\ \bar{a}_2p_1 + p_2 & 0 & p_3 & \dots & 0 \\ \bar{a}_3p_1 & p_3 & \ddots & \ddots & \\ \vdots & \vdots & \ddots & \ddots & p_n \\ \bar{a}_np_1 & 0 & & p_n & 0 \end{pmatrix} \tag{4.4}$$

Assume $S(P) \leq 0$ and $P \neq 0$. Then $p_3 = \cdots = p_n = 0$ and $p_1 \neq 0$, and we obtain $a_3 = \cdots = a_n = 0$ which is (4.2). Clearly

$$\begin{pmatrix} (\bar{a}_1 + a_1)p_1 & a_2 p_1 + p_2 \\ \bar{a}_2 p_1 + p_2 & 0 \end{pmatrix} \leq 0, \quad p_1 > 0, \; p_2 \geq 0$$

implies (4.3). The converse part of (i) and also of (ii) is obvious. ■

In accordance with the discrete time concept we call a matrix $A \in \mathbb{C}^{n \times n}$ *diagonally c-stable* if $S(P) = A^*P + PA < 0$ for some positive-definite diagonal matrix P. In mathematical biology [9, p. 199] this property is known as Volterra-Lotka stability. From (4.4) it is clear that a companion matrix can never be diagonally c-stable if $n \geq 2$.

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