Extensions of the bounded real lemma of discrete-time systems

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Abstract: In this paper the bounded real lemma for discrete-time systems is extended in several directions. It is shown that an H_∞-norm bound for a (not necessarily stable) transfer matrix $T(z)$ combined with controllability of unimodular eigenvalues yields the existence of an unmixed solution of the algebraic Riccati equation (ARE) associated with $T(z)$. Conversely, it is proved that the existence of a (not necessarily stabilizing) solution of the associated ARE implies bounded realness of $T(z)$. Inertia results are obtained and a condition for the existence of negative semidefinite solutions of the associated ARE is given.

1 Introduction

The bounded real lemma (Theorem 1.1 below) gives an algebraic condition for a stable linear system

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) + Du(t) \end{aligned}$$

to have less than unit gain. Let $\|\cdot\|$ denote the spectral norm and let the H_∞-norm of the transfer matrix $T(z) = D + H(zI - F)^{-1}G$ be defined by

$$\|T(z)\|_\infty = \sup_{0 \le \phi \le 2\pi} \|T(e^{i\phi})\|. \tag{1.1}$$

Theorem 1.1 : (See e.g. Zhou (1996) or de Souza and Lihua (1992).) *The following statements are equivalent.*

(i) *F is stable and*

$$\|D + H(zI - F)^{-1}G\|_\infty < 1.$$

(ii) *There exists a unique solution of the DARE*

$$\hat{\mathcal{D}}(X) = X - F^*XF + (G^*XF - D^*H)^*(I - D^*D + G^*XG)^{-1}(G^*XF - D^*H) + H^*H = 0 \tag{1.2}$$

such that $X \le 0$, $I - D^*D + G^*XG > 0$, *and the closed loop matrix*

$$F_X = F - G(I - D^*D + G^*XG)^{-1}(G^*XF - D^*H) \tag{1.3}$$

is stable.

It is the purpose of this paper to extend the bounded real lemma. We point out the different aspects of Theorem 1.1. (1) To a given transfer matrix $T(z)$ an algebraic Riccati equation (ARE) is associated, and the existence of a specific solution of the ARE ensures $\|T\|_\infty < 1$. (2) The assumption $\|T\|_\infty < 1$ implies existence and uniqueness of a solution X of the ARE with the property that the spectrum of the corresponding closed loop matrix F_X lies in a prescribed region of $\mathbb{C}$. (3) The inertia of F with respect to the unit circle is related to the inertia of a solution X of the ARE. (4) Properties of the transfer matrix T and the underlying system provide a sufficient condition for the existence of a solution $X \le 0$ of the ARE. In (1) – (4) we have indicated four features of the bounded real lemma which will be investigated separately and which will lead to Theorem 1.2 – Theorem 1.5 below.

Notation: Let $\mathbb{D}$ be the open unit disc. Then $\partial\mathbb{D} = \{z, |z| = 1\}$ is the unit circle and $\mathbb{C} \setminus \mathbb{D} = \{z, |z| \ge 1\}$. For a matrix $F \in \mathbb{C}^{n\times n}$ let $\sigma(F)$ be the spectrum and let Inv$F$ denote the set of its invariant subpaces. If $N \in$ Inv$F$

then $F_{|N}$ is the restriction of F on N. Set $E_\lambda(F) = \mathrm{Ker}(\lambda I - F)^n$ such that $E_\lambda(F)$ is a generalized eigenspace if $\lambda \in \sigma(F)$. For a subset U of $\mathbb{C}$ define

$$E_U(F) = \oplus\{E_\lambda(F),\ \lambda \in U\}.$$

The unobservable subspace of (F, H) is given by

$$V(F,H) = \mathrm{Ker}\begin{pmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{pmatrix}.$$

We define $\pi_d(F), \nu_d(F), \delta_d(F)$, as the number of eigenvalues λ of F with $|\lambda| > 1$, $|\lambda| < 1, |\lambda| = 1$, respectively, and we set

$$\text{d-In}(F) = \big(\pi_d(F), \nu_d(F), \delta_d(F)\big).$$

As usual the inertia of a hermitian $n \times n$ matrix X is the triple

$$\mathrm{In}X = \big(\pi(X), \nu(X), \delta(X)\big)$$

containing the numbers of positive, negative and zero eigenvalues of X. Let $\lambda_{\min}X$ be the smallest eigenvalue of X. For a subset $\Lambda \subseteq \mathbb{C}$ we shall always assume $0 \in \Lambda$. Define $\Lambda^\triangledown = \{\bar{\lambda}^{-1},\ \lambda \neq 0,\ \lambda \in \Lambda\}$. We call the set Λ *unmixed* if

$$\Lambda \cap \partial\mathbb{D} = \emptyset, \quad \Lambda \cap \Lambda^\triangledown = \emptyset \quad \text{and} \quad \Lambda \cup \Lambda^\triangledown \cup \partial\mathbb{D} = \mathbb{C}.$$

Clearly, the unit disc $\mathbb{D}$ is unmixed. A solution X of (1.2) is *stabilizing* if for $\Lambda = \mathbb{D}$ we have

$$\sigma(F_X) \subseteq \Lambda. \tag{1.4}$$

If the pair (F, G) is stabilizable then

$$\mathrm{rank}(I - \bar{\lambda}F,\ G) = n \ \text{ for all } \ \lambda \in \Lambda \cup \partial\mathbb{D}. \tag{1.5}$$

In our generalization of the bounded real lemma we shall be concerned with matrices F which satisfy (1.5) and with solutions X which satisfy (1.4) for a given unmixed set Λ, which need not be $\Lambda = \mathbb{D}$. For $W(z) = D + C(zI - A)^{-1}B$ define

$$W^\triangledown(z) = D^* + B^*(z^{-1}I - A^*)^{-1}C^*.$$

A pencil $M - zL \in \mathbb{C}^{m\times m}[z]$ is said to be *nonsingular* if its determinant is not the zero polynomial. In that case we call the elements of

$$\sigma(M - zL) = \{\lambda,\ \det(M - zL) = 0\}$$

the *characteristic roots* of $M - zL$.

Only if a transfer matrix $T(z)$ has no poles in $\mathbb{D}^{\nabla} = \{z, \, |z| > 1\}$ then (1.1) can be called its H_∞-norm. We shall use the notation (1.1) also in the case where $T(z)$ is bounded on $\partial\mathbb{D}$ and we call $T(z)$ bounded real if $\|T\|_\infty < 1$.

The first of our four theorems shows how the existence of a solution X of (1.2) implies the condition $\|T\|_\infty < 1$. Let

$$\hat{M} - z\hat{L} = \begin{pmatrix} F & 0 & G \\ -H^*H & I & -H^*D \\ -D^*H & 0 & I - D^*D \end{pmatrix} - z \begin{pmatrix} I & 0 & 0 \\ 0 & F^* & 0 \\ 0 & G^* & 0 \end{pmatrix} \tag{1.6}$$

be the pencil associated to (1.2).
Theorem 1.2 : *Assume*

$$\det(\hat{M} - \eta\hat{L}) \neq 0 \text{ for all } \eta \in \partial\mathbb{D}. \tag{1.7}$$

If there exists a solution of (1.2) *such that* $I - D^*D + G^*XG > 0$ *then we have* $\sigma(F) \cap \partial\mathbb{D} = \emptyset$ *and* $\|T\|_\infty < 1$.

In the second theorem the condition $\|T\|_\infty < 1$ ensures the existence of an unmixed solution of (1.2) without a stability assumption for F.
Theorem 1.3 : *Let* $T(z) = D + H(zI - F)^{-1}G$ *be a transfer matrix with* $\|T\|_\infty < 1$. *Assume* $\det(I - D^*D) \neq 0$ *and*

$$\operatorname{rank}(\eta I - F, \; G) = n \text{ for all } \eta \in \partial\mathbb{D}. \tag{1.8}$$

If Λ *is an unmixed subset of* $\mathbb{C}$ *and*

$$\operatorname{rank}(I - \bar{\lambda}F, \; G) = n \text{ for all } \lambda \in \Lambda, \tag{1.9}$$

then there exists a unique solution X *of* (1.2) *such that* $I - D^*D + G^*XG > 0$ *and the closed loop matrix* F_X *in* (1.3) *satisfies* $\sigma(F_X) \subseteq \Lambda \cup \partial\mathbb{D}$. *If* $\sigma(F) \cap \partial\mathbb{D} = \emptyset$ *then* $\sigma(F_X) \subseteq \Lambda$.

The next two theorems deal with the inertia of X and the d-inertia of F and F_X, and with the existence of negative semidefinite solutions of (1.2). The condition $1 \notin \sigma(F^*)\sigma(F)$ of Theorem 1.4 is crucial for the uniqueness of solutions of discrete-time Lyapunov equations of the form $X - F^*XF = -L^*L$ and for inertia theorems. The condition (1.12) in Theorem 1.5 can be interpreted as stability modulo the unobservable subspace $V(F, H)$.
Theorem 1.4 : *Let* X *be a solution of* (1.2) *with* $I - D^*D + G^*XG > 0$. *Assume* $1 \notin \sigma(F^*)\sigma(F)$. *Then* $\operatorname{Ker}X \subseteq \operatorname{Ker}H$ *and* $\operatorname{Ker}X$ *is invariant under* F *and* F_X, *and*

$$\pi_d(F) = \pi(X) + \pi_d\left(F_{X|_{\operatorname{KerX}}}\right) \tag{1.10}$$

and

$$\nu_d(F) = \nu(X) + \nu_d\left(F_{X|_{\mathrm{KerX}}}\right). \tag{1.11}$$

If the pair (F, H) *is observable then* X *is nonsingular.*

Theorem 1.5 : *Assume* $\|T\|_\infty < 1$ *and* (1.8).

(i) *There exists a solution* $X \leq 0$ *of* (1.2) *such that* $I - D^*D + G^*XG > 0$ *if and only if*

$$V(F, H) + E_{\mathbb{D}}(F) = \mathbb{C}^n. \tag{1.12}$$

(ii) *If* $X \leq 0$ *is a solution of* (1.2) *then*

$$E_{\mathbb{C}\setminus\mathbb{D}}(F) \subseteq E_{\mathbb{C}\setminus\mathbb{D}}(F_X). \tag{1.13}$$

The structure of the paper is as follows. Section 2 – Section 5 contain the proofs of Theorem 1.2 – Theorem 1.5. In Section 6 we return to the bounded real lemma and give a proof of Theorem 1.1. We point out that Theorem 3.1, which deals with existence of unmixed solutions, is based on results of Wimmer (1996). In contrast to Wimmer (1996) we now work with unmixed sets instead of unmixed factorizations.

2 From the DARE to bounded realness

Equation (1.2) is of the form

$$\mathcal{D}(X) = X - F^*XF + (S + G^*XF)^*(R + G^*XG)^{-1}(S + G^*XF) - Q = 0. \tag{2.1}$$

Define

$$F_X = F - G(R + G^*XG)^{-1}(S + G^*XF) \tag{2.2}$$

as the closed loop matrix corresponding to a solution X of (2.1). It is standard (see e.g. Lancaster and Rodman (1995)) to associate to the Riccati operator $\mathcal{D}(X)$ the matrix pencil

$$M - zL = \begin{pmatrix} F & 0 & G \\ Q & I & S^* \\ S & 0 & R \end{pmatrix} - z \begin{pmatrix} I & 0 & 0 \\ 0 & F^* & 0 \\ 0 & G^* & 0 \end{pmatrix} \tag{2.3}$$

and the Popov function

$$\Psi(z) = \left(G^*(z^{-1}I - F^*)^{-1},\ I\right) \begin{pmatrix} Q & S^* \\ S & R \end{pmatrix} \begin{pmatrix} (zI - F)^{-1}G \\ I \end{pmatrix}. \tag{2.4}$$

We first collect some facts concerning (2.1).

Lemma 2.1 : *Let* $M - zL$ *and* $\Psi(z)$ *be given by* (2.3) *and* (2.4).

(i) *We have*

$$\det(M - zL) = \det(F - zI)\det(I - zF^*)\det\,\Psi(z). \tag{2.5}$$

(ii) *Let X be a solution of* (2.1)*. Define*

$$\Phi_X(z) = I + (R + G^*XG)^{-1}(S + G^*XF)(zI - F)^{-1}G.$$

Then

$$\Psi(z) = \Phi_X^{\nabla}(z)(R + G^*XG)\Phi_X(z). \tag{2.6}$$

and

$$\det(M - zL) = \det(R + G^*XG)\det(F_X - zI)\det(I - zF_X^*). \tag{2.7}$$

Proof: For (2.5) we refer to Ionescu and Weiss (1992) and for (2.6) to Molinari (1975). To prove (2.7) we set

$$Z = (R + G^*XG)^{-1}(S + G^*XF).$$

Then

$$(zI - F)\big(I + (zI - F)^{-1}GZ\big) = zI - F_X.$$

From

$$\det\big(I + (zI - F)^{-1}GZ\big) = \det\big(I + Z(zI - F)^{-1}G\big)$$

we obtain

$$\det(zI - F)\det\,\Phi_X(z) = \det(zI - F_X),$$

and (2.7) follows from (2.5). ∎

The following lemma (see e.g. Lancaster and Rodman (1995)) is an immediate consequence of Molinari's identity (2.6). It is obvious from (2.7) that the existence of a solution X of (2.1) implies that the pencil $M - zL$ is nonsingular.

Lemma 2.2 : *If there exists a solution X of* (2.1) *such that*

$$R + G^*XG > 0, \tag{2.8}$$

then we have $\Psi(\eta) > 0$ for almost all $\eta \in \partial\mathbb{D}$. Conversely if $\Psi(\eta) > 0$ for some $\eta \in \partial\mathbb{D}$ then (2.8) *holds for all solutions X.*

We now turn to the transfer matrix $T(z) = D + H(zI - F)^{-1}G$ and to the DARE (1.2) with the corresponding Popov function

$$\hat{\Psi}(z) = \big(G^*(z^{-1}I - F^*)^{-1}, \quad I\big)\begin{pmatrix} -H^*H & -H^*D \\ -D^*H & I - D^*D \end{pmatrix}\begin{pmatrix} (zI - F)^{-1}G \\ I \end{pmatrix}. \tag{2.9}$$

Note that

$$I - T^{\nabla}(z)T(z) = \hat{\Psi}(z).$$

If $\eta \in \partial\mathbb{D}$ then $T^{\nabla}(\eta) = T(\eta)^*$. Thus

$$I - T(\eta)^*T(\eta) = \hat{\Psi}(\eta) \tag{2.10}$$

implies

$$1 - \|T(\eta)\|^2 = \lambda_{\min}\hat{\Psi}(\eta). \tag{2.11}$$

Lemma 2.3 : *We have $\|T(z)\|_\infty < 1$ if and only if the matrix $\hat{\Psi}(z)$ has no poles on $\partial\mathbb{D}$ and*

$$\hat{\Psi}(\eta) > 0 \quad \textit{for all} \quad \eta \in \partial\mathbb{D}. \tag{2.12}$$

Proof: Suppose there are no poles of $T(z)$ and (equivalently) of $\hat{\Psi}(z)$ on the unit circle. Then $\|T(\eta)\|$ and $\lambda_{\min}\hat{\Psi}(\eta)$ are continuous on $\partial\mathbb{D}$. Hence $\|T\|_\infty < 1$ is equivalent to

$$T(\eta)^*T(\eta) < I \quad \text{for all} \quad \eta \in \partial\mathbb{D}, \tag{2.13}$$

and (2.12) is equivalent to

$$\inf\{\lambda_{\min}\hat{\Psi}(\eta), \eta \in \partial\mathbb{D}\} > 0.$$

Now (2.10) completes the proof. ■

The next lemma deals with unimodular eigenvalues of F.

Lemma 2.4 : *Let X be a solution of* (1.2) *which satisfies $I - D^*D + G^*XG > 0$ and let $\eta \in \partial\mathbb{D}$ be such that*

$$\operatorname{rank}(F - \eta I,\ G) = n.$$

Then we have $E_\eta(F) \subseteq \operatorname{Ker}H$ and $E_\eta(F) \subseteq \operatorname{Ker}X$, and

$$E_\eta(F) \subseteq E_\eta(F_X) \quad \textit{and} \quad F = F_X \quad \textit{on} \quad E_\eta(F). \tag{2.14}$$

Proof: Set $y_0 = 0$, and let $y_1, \ldots, y_k$, be a Jordan chain of F belonging to η such that $y_1 \neq 0$, $(F - \eta I)y_{j+1} = y_j$, $j = 0, \ldots, k-1$. If we write (1.2) as a Lyapunov equation $X - F^*XF = -L^*L$ then it is not difficult to prove by induction that $Hy_j = 0$, and $Xy_j = 0$, $j = 1, \ldots, k$. Hence (2.14) follows from (1.3) ■

Proof of Theorem 1.2: To put Lemma 2.4 to work we note that the assumption (1.7), namely $\sigma(\hat{M} - z\hat{L}) \cap \partial\mathbb{D} = \emptyset$, implies $\operatorname{rank}(F - \eta I, G) = n$ for all $\eta \in \partial\mathbb{D}$. Recall the identities of Lemma 2.1. From

$$\det(\hat{M} - z\hat{L}) = c\det(F_X - zI)\det(I - zF_X^*),\ c \neq 0, \tag{2.15}$$

we obtain $\sigma(F_X) \cap \partial\mathbb{D} = \emptyset$. Then (2.14) yields $\sigma(F) \cap \partial\mathbb{D} = \emptyset$. Hence $\hat{\Psi}(z)$ has no poles on $\partial\mathbb{D}$, and

$$\det(\hat{M} - z\hat{L}) = \det(F - zI)\det(I - zF^*)\det\hat{\Psi}(z)$$

shows that $\det\hat{\Psi}(\eta) \neq 0$ on $\partial\mathbb{D}$. Taking $I - D^*D + G^*XG > 0$ into account we deduce from Lemma 2.2 that $\hat{\Psi}(\eta) > 0$ for all $\eta \in \partial\mathbb{D}$. According to Lemma 2.3 this is equivalent to $\|T\|_\infty < 1$. ■

3 Unmixed solutions

Again let us first consider the DARE (2.1),

$$\mathcal{D}(X) = X - F^*XF + (S + G^*XF)^*(R + G^*XG)^{-1}(S + G^*XF) - Q = 0,$$

but now with the additional assumptions that R is nonsingular and that (1.7), i.e. $\sigma(M - zL) \cap \partial\mathbb{D} = \emptyset$, holds. If R^{-1} exists then is well known that (2.1) can be transformed into a DARE with $S = 0$. Define

$$\tilde{F} = F - GR^{-1}S,\ \tilde{Q} = Q - S^*R^{-1}S,\ \tilde{F}_X = \tilde{F} - G(R + G^*XG)^{-1}G^*X\tilde{F},$$

and

$$\tilde{\mathcal{D}}(X) = X - \tilde{F}^*X\tilde{F} + \tilde{F}^*XG(R + G^*XG)^{-1}G^*X\tilde{F} - \tilde{Q} = 0. \tag{3.1}$$

Then

$$\mathcal{D}(X) = \tilde{\mathcal{D}}(X) \quad \text{and} \quad F_X = \tilde{F}_X. \tag{3.2}$$

In particular, if $R = (I - D^*D) > 0$ and $Q = -H^*H$ and $S = -D^*H$ then we have $\tilde{Q} = -H^*(I - DD^*)^{-1}H \leq 0$. It is convenient to associate with the Riccati operator $\tilde{\mathcal{D}}(X)$ in (3.1) the (compressed) pencil

$$\tilde{M} - z\tilde{L} = \begin{pmatrix} \tilde{F} & 0 \\ -\tilde{Q} & I \end{pmatrix} - z\begin{pmatrix} I & GR^{-1}G^* \\ 0 & \tilde{F}^* \end{pmatrix}. \tag{3.3}$$

The corresponding Popov function is

$$\tilde{\Psi}(z) = G^*(z^{-1}I - \tilde{F}^*)^{-1}\tilde{Q}(zI - \tilde{F})^{-1}G + R. \tag{3.4}$$

The following identities can be found in Lancaster and Rodman (1995). If

$$K_1 = \begin{pmatrix} I & 0 & -G^*R^{-1} \\ 0 & -I & S^*R^{-1} \\ 0 & 0 & I \end{pmatrix} \quad \text{and} \quad K_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ -R^{-1}S & 0 & I \end{pmatrix}$$

then

$$K_1(M - zL)K_2 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\tilde{M} - z\tilde{L}$}} & 0 \\ & & 0 \\ \hline 0 & zG^* & R \end{array}\right). \tag{3.5}$$

Hence the pencils $M - zL$ and $\tilde{M} - z\tilde{L}$ have the same elementary divisors. The Popov functions $\tilde{\Psi}(z)$ and $\Psi(z)$ are related by

$$\Psi(z) = \left(I + G^*(z^{-1}I - F^*)^{-1}S^*R^{-1}\right)\tilde{\Psi}(z)\left(I + R^{-1}S(zI - F)^{-1}G\right). \qquad (3.6)$$

According to de Souza *et al.* (1986) the characteristic roots λ, $\lambda \notin \partial\mathbb{D}$, and the corresponding elementary divisors of $\tilde{M} - z\tilde{L}$ are symmetric with respect to $\partial\mathbb{D}$. In particular if $\lambda \neq 0$, $|\lambda| \neq 1$, then we have

$$\det(M - zL) = (z - \lambda)^k(1 - \bar{\lambda}z)^k g(z)$$

where $g(\lambda)g(\bar{\lambda}^{-1}) \neq 0$. Hence if Λ is an unmixed subset of $\mathbb{C}$ then there is a unique factorization

$$\det(M - zL) = c\Pi_{\nu=1}^{n}(z - \lambda_\nu)\Pi_{\nu=1}^{n}(1 - z\bar{\lambda}_\nu),\ c \neq 0,\ \{\lambda_1, \ldots, \lambda_n\} \subseteq \Lambda. \qquad (3.7)$$

We call (3.7) the *unmixed factorization* of $\det(M - zL)$ corresponding to Λ. Similarly we say that X is an *unmixed solution* of (2.1) if the corresponding closed loop matrix satisfies $\sigma(F_X) \subseteq \Lambda$ for some unmixed Λ. Clearly, a stabilizing solution is unmixed. From (2.7) we see that an unmixed solution X gives rise to an unmixed factorization (3.7).

Let us digress briefly to the continuous-time algebraic Riccati equation

$$F^*X + XF - XGR^{-1}G^*X + Q = 0. \qquad (3.8)$$

For the CARE (3.8) unmixed solutions (in a slightly more general form) were introduced by Shayman (1983). In the case of (3.8) the closed loop matrix associated to a solution X is

$$F_X = F - GR^{-1}G^*X. \qquad (3.9)$$

Suppose the Hamiltonian matrix

$$H = \begin{pmatrix} F & -GR^{-1}G^* \\ -Q & -F^* \end{pmatrix}$$

corresponding to (3.8) has no eigenvalues on the imaginary axis. Then X is said to be unmixed if F_X and $-F_X^*$ have no eigenvalues in common. We know from Shayman (1983) that pairs of opposite unmixed solutions of (3.8) are important as they share properties of the pair (X_+, X_-) of extremal solutions. We also refer to Kučera (1991) for a discussion of unmixed solutions of (3.8).

The following result generalizes a theorem of Molinari (1975) from $\Lambda = \mathbb{D}$ to an arbitrary unmixed set Λ.

Proposition 3.1 : *Assume that the Popov function* $\Psi(z)$ *in* (2.4) *satifies* $\Psi(\eta) > 0$ *for some* $\eta \in \partial\mathbb{D}$ *and that the pencil* $M - zL$ *in* (2.3) *is such that*

$$\det(M - \eta L) \neq 0 \quad \textit{for all} \quad \eta \in \partial\mathbb{D}.$$

If $\Lambda \subseteq \mathbb{C}$ *is unmixed and*

$$\operatorname{rank}(I - \bar{\lambda} F, \; G) = n \quad \textit{for all} \quad \lambda \in \Lambda, \tag{3.10}$$

then there exists a unique solution X *of* (2.1) *such that* $\sigma(F_X) \subseteq \Lambda$.

Proof: We recall the links between (2.1) and the equivalent DARE $\tilde{\mathcal{D}}(X) = 0$ in (3.1) and their associated matrix pencils and Popov functions. We see that $\tilde{\Psi}(\eta) > 0$ for some $\eta \in \partial\mathbb{D}$, and $\sigma(\tilde{M} - z\tilde{L}) \cap \partial\mathbb{D} = \emptyset$. Let (3.7) be the unmixed factorization corresponding to Λ of $\det(M - \eta L)$ and (neglecting a constant factor) also of $\det(\tilde{M} - z\tilde{L})$. Then (3.10) and $\tilde{F} = F - GR^{-1}S$ imply

$$\operatorname{rank}(I - \bar{\lambda}_\nu \tilde{F}, \; G) = n, \; \nu = 1, \ldots, n.$$

Now it follows immediately from Theorem 1.1 of Wimmer (1996) that there exists a unique solution X of $\tilde{\mathcal{D}}(X) = 0$ which satisfies $\sigma(\tilde{F}_X) \subseteq \Lambda$. Because of (3.2) the proof is complete. ■

Lemma 3.2 : *If* $\|T\|_\infty < 1$ *and* $\eta \in \partial\mathbb{D}$ *and*

$$\operatorname{rank}(F - \eta I, \; G) = n \tag{3.11}$$

then

$$E_\eta(F) \subseteq \operatorname{Ker} H. \tag{3.12}$$

Proof: A suitable change of basis yields $F = \operatorname{diag}(F_1, \; F_2)$, $\sigma(F_2) = \{\eta\}$ and $\eta \notin \sigma(F_1)$. If $G^T = (G_1^T, \; G_2^T)$ and $H = (H_1, \; H_2)$ are partitioned accordingly then

$$T(z) = D + H_1(zI - F_1)^{-1}G_1 + H_2(zI - F_2)^{-1}G_2.$$

Since $T(z)$ has no poles on $\partial\mathbb{D}$ we have $H_2(zI - F_2)^{-1}G_2 = 0$. From (3.11) follows that the pair (F_2, G_2) is controllable. Hence $H_2 = 0$, which proves (3.12). ■

Let W be an F-invariant subspace such that

$$W \subseteq V(F, H). \tag{3.13}$$

If

$$W = \operatorname{Im}\begin{pmatrix} 0 \\ I_{n-n_1} \end{pmatrix} \tag{3.14}$$

then

$$F = \begin{pmatrix} F_1 & 0 \\ F_{21} & F_2 \end{pmatrix}, \; H = (H_1, \; 0), \; G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}. \tag{3.15}$$

Define $T_1(z) = D + H_1(zI - F_1)^{-1}G_1$, and in accordance with (1.6) and (2.9) let $\hat{M}_1 - z\hat{L}_1$ and $\hat{\Psi}_1(z)$ be the pencil and the Popov function associated with $T_1(z)$. Then

$$T(z) = T_1(z) \quad \text{and} \quad \hat{\Psi}(z) = \hat{\Psi}_1(z). \tag{3.16}$$

Let

$$\begin{aligned}\hat{\mathcal{D}}_1(X) = X - F_1^* X F_1 \\ - (G_1^* X F_1 - D^* H_1)^* (I - D^*D + G_1^* X G_1)^{-1} (G_1^* X F_1 - D^* H_1) \\ + H_1^* H_1 = 0 \quad (3.17)\end{aligned}$$

be the corresponding DARE. Then X is a solution of $\hat{\mathcal{D}}(X) = 0$ with $W \subseteq \mathrm{Ker}X$ if and only if $X = \mathrm{diag}(X_1, 0)$ and $\hat{\mathcal{D}}_1(X_1) = 0$. For such an X we have

$$F_X = \begin{pmatrix} F_{1_{X_1}} & 0 \\ * & F_2 \end{pmatrix} \tag{3.18}$$

with

$$F_{1_{X_1}} = F_1 - G_1(I - D^*D + G_1^* X_1 G_1)^{-1}(G_1^* X_1 F_1 - D^* H_1).$$

Proof of Theorem 1.3: If we take $W = E_{\partial\mathbb{D}}(F)$ then Lemma 3.2 yields (3.13). Hence W satisfies (3.13) and we can assume W as in (3.14). Then we have (F, G, H) as in (3.15) and

$$\sigma(F_2) \subseteq \partial\mathbb{D}, \ \sigma(F_1) \cap \partial\mathbb{D} = \emptyset. \tag{3.19}$$

We focus on the DARE $\hat{\mathcal{D}}_1(X) = 0$ in (3.17). Note that the rank condition (1.9) implies $\mathrm{rank}(I - \bar{\lambda}F_1, \ G_1) = n_1$ for all $\lambda \in \Lambda$. Because of (3.16) we have $\|T_1\|_\infty < 1$. Then Lemma 2.3 yields $\hat{\Psi}_1(\eta) > 0$ for all $\eta \in \partial\mathbb{D}$. From

$$\det(\hat{M}_1 - z\hat{L}_1) = \det(F_1 - zI)\det(I - zF_1^*)\hat{\Psi}_1(z)$$

and (3.19) we obtain $\sigma(\hat{M}_1 - z\hat{L}_1) \cap \partial\mathbb{D} = \emptyset$. We apply Proposition 3.1 and conclude that there exists a unique solution X_1 of $\hat{\mathcal{D}}_1(X) = 0$ such that $I - D^*D + G_1^* X_1 G_1 > 0$ and $\sigma(F_{1_{X_1}}) \subseteq \Lambda$. Then $X = \mathrm{diag}(X_1, 0)$ is a solution of (1.2) which satisfies $\sigma(F_X) \subseteq \Lambda \cup \partial\mathbb{D}$ and $I - D^*D + G^*XG > 0$. From Lemma 2.4 we see that all solutions of (1.2) are of the form $X = \mathrm{diag}(X_1, 0)$ as above, which proves uniqueness. If $W = E_{\partial\mathbb{D}} = 0$, that is if $\sigma(F) \cap \partial\mathbb{D} = \emptyset$, then we have $X = X_1$ and $\sigma(F_X) \subseteq \Lambda$. ■

4 Inertia results

Lemma 4.1 : *Assume*

$$1 \notin \sigma(F^*)\sigma(F). \tag{4.1}$$

If X is the solution of $X - F^*XF = -L^*L$ *then*

$$\mathrm{Ker}X = V(F, L). \tag{4.2}$$

Furthermore

$$\pi_d(F) = \pi(X) + \pi_d\left(F_{|_{V(F,H)}}\right) \quad \textit{and} \quad \nu_d(F) = \nu(X) + \nu_d\left(F_{|_{V(F,H)}}\right). \quad (4.3)$$

Proof: We can assume

$$V(F,L) = \operatorname{Im}\begin{pmatrix} 0 \\ I \end{pmatrix}, \; F = \begin{pmatrix} F_1 & 0 \\ F_{21} & F_2 \end{pmatrix}, \quad L = (L_1, \; 0),$$

and $(F_1, \; L_1)$ observable. Let

$$X = \begin{pmatrix} X_1 & X_{21}^* \\ X_{21} & X_2 \end{pmatrix}$$

be partitioned conformally. Then $X_2 - F_2^* X_2 F_2 = 0$, and (4.1) implies $X_2 = 0$. Hence X_{21} satisfies the equation $X_{21} - F_2^* X_{21} F_1 = 0$ and (4.1) yields $X_{21} = 0$. It is well known (see e.g. Bittanti *et al.* (1986)) that

$$X_1 - F_1^* X_1 F_1 = -L_1^* L_1$$

implies $\operatorname{In}(X_1) = \text{d-In}(F_1)$, if $1 \notin \sigma(F_1^*)\sigma(F_1)$ and (F_1, L_1) is observable. Hence X_1 is nonsingular and we have (4.2). Then (4.3) follows from $F_{|_{V(F,H)}} = F_2$. ■

Proof of Theorem 1.4: We regard (1.2) as a discrete-time Lyapunov equation $X - F^*XF = -L^*L$ with

$$L^*L = H^*H + (G^*XF - D^*H)^*(I - D^*D + G^*XG)^{-1}(G^*XF - D^*H). \quad (4.4)$$

From Lemma 4.1 we obtain

$$\operatorname{Ker}X = V(F,L) \subseteq V(F,H) \quad (4.5)$$

and $\operatorname{Ker}X \in \operatorname{Inv}F$. Furthermore $\operatorname{Ker}X \in \operatorname{Inv}F_X$ and $F = F_X$ on $\operatorname{Ker}X$. Take $W = \operatorname{Ker}X$ and assume (3.14). Then F and F_X are the form (3.15) and (3.18), respectively. Hence (4.3) yields the inertia relations (1.10) and (1.11). From (4.5) follows $\operatorname{Ker}X = 0$ if (F,H) is observable. ■

5 Negative semidefinite solutions

If F is stable then it is known that $\|T\|_\infty < 1$ implies $D^*D < I$ (see e.g. p. 548 in Zhou *et al.* (1996)). We include a proof which gives a sharper estimate.

Lemma 5.1 : *Let F be stable and assume*

$$\|T(z)\|_\infty = \|D + H(zI - F)^{-1}G\|_\infty < 1. \quad (5.1)$$

Define

$$P = \sum_{\nu=0}^{\infty} (F^*)^\nu H^* H F^\nu.$$

Then $P \geq 0$ *and* $D^*D + G^*PG < I$.

Proof: The matrix P is the (unique) solution of the discrete-time Lyapunov equation $X - F^*XF = H^*H$. According to Wimmer and Ziebur (1972) we have

$$P = \frac{1}{2\pi} \int_0^{2\pi} (e^{-i\phi}I - F^*)^{-1} H^* H (e^{i\phi}I - F)^{-1} d\phi.$$

Condition (5.1) is equivalent to

$$0 < I - T(e^{i\phi})^* T(e^{i\phi}) = (I - D^*D) - D^*H(e^{i\phi}I - F)^{-1}G - G^*(e^{-i\phi}I - F^*)^{-1}H^*D \\ - G^* \left[(e^{-i\phi}I - F^*)^{-1} H^* H (e^{i\phi}I - F)^{-1} \right] G.$$

Because of

$$\int_0^{2\pi} (e^{i\phi}I - F)^{-1} d\phi = 0$$

we obtain

$$0 < \frac{1}{2\pi} \int_0^{2\pi} \left[I - T(e^{i\phi})^* T(e^{i\phi}) \right] d\phi = I - D^*D - G^*PG.$$

■

Proof of Theorem 1.5: Let us assume first that (1.2) has a solution $X \leq 0$. It is easy to see that $\mathrm{Ker}X \in \mathrm{Inv}F$ and $\mathrm{Ker}X \subseteq \mathrm{Ker}H$. Now take $W = \mathrm{Ker}X$ and assume (3.14). Then

$$X = \mathrm{diag}(X_1,\, 0), \;\; X_1 < 0,$$

and we have (F, G, H) as in (3.15), and X_1 is a solution of $\hat{\mathcal{D}}_1(X_1) = 0$ in (3.17). It is obvious that the eigenvalues of F_1 are in the closed unit disc. Because of

$$\mathrm{rank}(F_1 - \eta I,\; G_1) = n_1 \text{ for all } \eta \in \partial\mathbb{D}$$

we obtain $\sigma(F_1) \subseteq \mathbb{D}$, which proves (1.12). Note that those eigenvalues of F which are not in $\mathbb{D}$ must be in $\sigma(F_2)$ and that F_2 is a diagonal block of F_X in (3.18). Hence we have (1.13).

Now assume (1.12). Taking $W = V(F, H) = \mathrm{Im}(0\ I)^T$ we are again in the setting (3.15). We have $\sigma(F_1) \subseteq \mathbb{D}$ and the pair $(F_1,\, H_1)$ is observable. Because of $T(z) = T_1(z) = D + H_1(zI - F_1)^{-1}G$ we have $\|T_1\|_\infty < 1$. Since F_1 is stable Lemma 5.1 implies $I - D^*D > 0$. Then Theorem 1.3 yields a stabilizing solution of X_1 of (1.2). From (1.11) follows $X_1 < 0$. Thus $X = \mathrm{diag}(X_1, 0) \leq 0$ is the desired solution. ■

6 The bounded real lemma

It is easy to see that the bounded real lemma follows immediately from the results of the previous sections.

Proof of Theorem 1.1: (i) $\Rightarrow$ (ii) If F is stable then each solution of (1.2) which satisfies

$$I - D^*D + G^*XG > 0 \tag{6.1}$$

is negative semidefinite. According to Lemma 5.1 the condition $\|T\|_\infty < 1$ implies $I - D^*D > 0$. Now take $\Lambda = \mathbb{D}$. Then it follows from Theorem 1.3. that there exists a unique solution X of (1.2) such that (6.1) and $\sigma(F_X) \subseteq \mathbb{D}$ hold.
(ii) $\Leftarrow$ (i) From $\sigma(F_X) \subseteq \mathbb{D}$ and (2.15) we obtain $\det(\hat{M} - \eta\hat{L}) \neq 0$ for all $\eta \in \partial\mathbb{D}$. Hence Theorem 1.2 yields $\|T\|_\infty < 1$. That F is stable is a consequence of (1.13). ∎

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