The generalized Sylvester equation in polynomial matrices

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Abstract

Jordan chains of polynomial matrices are used to characterize the solvability of the generalized Sylvester equation.

1 Introduction and main result

Output regulation with internal stability is one of the problems in control theory which lead to a generalized Sylvester equation

$$AX - YB = C, \tag{1.1}$$

where the matrices involved are polynomial matrices (see e.g. [5], [1], [2], [17], [10]). We also refer to [12], [13], [3] where equation (1.1) is used for the design of control systems, or to [4] for a system theoretic interpretation of (1.1).

In this note the existence of a polynomial solution (X, Y) of (1.1) will be characterized in terms of Jordan chains of A and B. A solvability condition will be derived which extends results of Kučera [11] and Gohberg and Lerer [9].

Let A, B, C be complex polynomial matrices of size $m \times m$, $n \times n$, $m \times n$ respectively and assume $\det A \neq 0$ and $\det B \neq 0$ (zero polynomial). We shall use the following notation. Put

$$N_r = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}_{r \times r} .$$

For $B \in \mathbb{C}^{n \times n}[s]$ define

$$\sigma(B) = \{ \lambda \mid \lambda \in \mathbb{C}, \det B(\lambda) = 0 \} .$$

Consider a characteristic root $\lambda \in \sigma(B)$ with corresponding elementary divisors

$$(s-\lambda)^{l_1}, \ldots, (s-\lambda)^{l_q}, \quad l_1 \geq \ldots \geq l_q \geq 1, \quad l = l_1 + \ldots + l_q,$$

such that $\det B(s) = (s-\lambda)^l c(s)$, $c(\lambda) \neq 0$. We call

$$J = \operatorname{diag}\left(\lambda I - N_{l_1}, \ldots, \lambda I - N_{l_q}\right) \tag{1.2}$$

the *Jordan matrix associated to* λ. It is known (see e.g. [16] or [8]) that there exists matrices $H \in \mathbb{C}^{n \times l}$ and $\hat{H} \in \mathbb{C}^{n \times l}[s]$ such that

$$B(s)H = \hat{H}(s)(sI - J) \tag{1.3}$$

and the columns of $\hat{H}$ are linearly independent over $\mathbb{C}$. We call H a *right Jordan system* (r.J.s.) of $B(s)$ corresponding to the characteristic root λ. Let

$$H = (H_1, \ldots, H_q) \tag{1.4a}$$

be partitioned according to (1.2) with

$$H_j = \left(h_{j0}, \ldots, h_{j,l_j-1}\right), \quad j = 1, \ldots, q. \tag{1.4b}$$

If $B(s) = \sum_{\nu=0}^{r} B_\nu (s-\lambda)^\nu$ then

$$B(s)H(sI - J)^{-1} \in \mathbb{C}^{n\times l}[s] \tag{1.5}$$

implies

$$\begin{aligned} B_0 h_{j0} &= 0 \\ B_1 h_{j0} + B_0 h_{j1} &= 0 \\ &\vdots \\ B_{l_j-1} h_{j0} + \ldots + B_0 h_{j,l_j-1} &= 0, \end{aligned}$$

$j = 1, \ldots, q$. Hence the sequences $h_{j0}, \ldots, h_{j,l_j-1}$, have the properties which characterize Jordan chains of polynomial matrices. In the terminology of [8] a Jordan system H forms a canonical set of Jordan chains of B corresponding to λ. We say that G is a *left Jordan system* (l.J.s.) of $A(s)$ corresponding to $\lambda \in \sigma(A)$ if G^T is a r.J.s. of $A^T(s)$. We call (G, H) a *pair of Jordan systems* of (A, B) associated to $\lambda \in \sigma(A) \cap \sigma(B)$ if G and H are a l.J.s. and a r.J.s. of $A(s)$ and $B(s)$ respectively.

Assume that the elementary divisors belonging to $\lambda \in \sigma(A)$ are

$$(s-\lambda)^{k_1}, \ldots, (s-\lambda)^{k_p}, \quad k_1 \geq \ldots k_p \geq 1. \tag{1.6}$$

Let (G, H) be a pair of Jordan systems corresponding to λ, where H is given as in (1.4), (1.5), and

$$G = \begin{pmatrix} G_1 \\ \vdots \\ G_p \end{pmatrix}, \quad G_i = \begin{pmatrix} g_{i0} \\ \vdots \\ g_{i,k_i-1} \end{pmatrix}, \quad i = 1, \ldots, p. \tag{1.7}$$

We say that (G, H) has the property (Σ) if

$$\sum_{\nu+\sigma+\tau=r_{ij}} g_{i\nu} \frac{C^{(\sigma)}(\lambda)}{\sigma!} h_{j\tau} = 0, \tag{1.8a}$$

$$r_{ij} = 0, 1, \ldots, \min\,(k_i, l_j) - 1, \quad i = 1, \ldots, p, \quad j = 1, \ldots, q, \tag{1.8b}$$

holds. Property (Σ) is crucial for the consistency of (1.1). The main result of the paper is the following.

Theorem 1.1: *Assume* $\det A \neq 0$ *and* $\det B \neq 0$. *The following statements are equivalent.*

(1) The equation $AX - YB = C$ *has a solution* (X, Y) *in polynomial matrices.*

(2) For each $\lambda \in \sigma(A) \cap \sigma(B)$ *there exists a pair of Jordan systems* (G, H) *of* (A, B) *with property* (Σ).

(3) For each $\lambda \in \sigma(A) \cap \sigma(B)$ *all associated pairs of Jordan systems of* (A, B) *have property* (Σ).

In the special case where C is constant and the leading coefficient matrices of A and B are nonsingular the preceding theorem yields the result of Gohberg and Lerer [9, p. 172], which was obtained by a reduction of (1.1) to a Sylvester equation

$$PX - XQ = C \tag{1.9}$$

with constant matrices. If we take $A(s) = P - sI$ and $B(s) = Q - sI$ then (1.1) becomes (1.9) and we recover Kučera's [11] solvability criterion for (1.9).

Our approach uses a lemma of Cheng and Pearson [2] where the matrices A and B in (1.1) are transformed into Smith normal form. In [9] and [11] results corresponding to the equivalence of (2) and (3) in Theorem 1.1 have not been mentioned. The fact that the statements (2) and (3) are equivalent is crucial. Its proof will lead to another solvability condition for (1.1) involving polynomial modules [6] associated with A and B.

We note that it is possible to extend Theorem 1.1 and its proof to matrices over a ring $\mathcal{O}(G)$ of complex functions which are holomorphic in a domain G. Since matrices over $\mathcal{O}(G)$ can be transformed into a Smith normal form [15] the tools and concepts of this paper can be adapted from the ring $R = \mathbb{C}[s]$ to $R = \mathcal{O}(G)$. The problem of finding holomorphic solutions X and Y of an operator function equation (1.1) was considered in [7]. Under the assumption $\sigma(A) \cap \sigma(B) = \emptyset$ solutions X, Y of (1.1) can be obtained in closed form using contour integrals [7, p. 125].

2 Auxiliary results

If λ is a characteristic root of both A and B then the corresponding Jordan systems G and H in (1.4) and (1.7) are not uniquely detemined. In this section we want to show that if condition (Σ) holds for a particular choice of (G, H) then it holds for all such Jordan systems. According to [14] there is a connection between Jordan systems of polynomial matrices and their Smith normal form.

Lemma 2.1 a: *Assume* $\det B \neq 0$ *and* $\lambda \in \sigma(B)$. *Let* U_B *and*

$$V_B = (v_1, \ldots, v_n) \tag{2.1}$$

be unimodular matrices such that

$$U_B B V_B = S_B = \text{diag}\,(b_1, \ldots, b_n)\,, \quad b_n \mid \ldots \mid b_1. \tag{2.2}$$

Let

$$(s-\lambda)^{l_1}, \ldots, (s-\lambda)^{l_q}, \quad l_1 \geq \ldots \geq l_q \geq 1, \tag{2.3}$$

be the elementary divisors of B *corresponding to* λ, *and let* v_j, $j = 1, \ldots, q$, *be expanded as*

$$v_j = h_{j0} + \cdots + h_{j,l_j-1}(s-\lambda)^{l_j-1} + (s-\lambda)^{l_j}\, t_j(s). \tag{2.4}$$

Put

$$H_j = \left(h_{j0}, \ldots, h_{j,l_j-1}\right). \tag{2.5a}$$

Then

$$H = (H_1, \ldots, H_q) \tag{2.5b}$$

is a right Jordan system of $B(s)$ *corresponding to* λ.

Proof: Because of (2.3) the invariant factors b_j, $j = 1, \ldots, q$, in (2.2) are of the form $b_j(s) = (s-\lambda)^{l_j}\, r_j(s)$. Then $BV_B = U_B^{-1} S_B$ implies $B(s)v(s) = (s-\lambda)^{l_j}\, \hat{v}_j(s)$ for some $\hat{v}_j \in \mathbb{C}^n[s]$. Hence

$$B(s)\left[h_{j0}(s-\lambda)^{-l_j} + \cdots + h_{j,l_j-1}(s-\lambda)^{-1}\right] = \hat{h}_{j,l_j}(s) \in \mathbb{C}^n[s],$$

and we obtain

$$B(s)\left(h_{j0}, \ldots, h_{j,l_j-1}\right) \begin{pmatrix} (s-\lambda)^{-1} & \cdots & (s-\lambda)^{-l_j} \\ & \ddots & \vdots \\ 0 & & (s-\lambda)^{-1} \end{pmatrix}$$

$$= B(s)H_j\left[sI - \left(\lambda I - N_{l_j}\right)\right]^{-1} = \hat{H}_j(s) \in \mathbb{C}^{n \times l_j}[s].$$

Put $\hat{H} = (\hat{H}_1, \ldots, \hat{H}_q)$ and let J be the Jordan matrix associated to $\lambda \in \sigma(B)$. Then $B(s)H = \hat{H}(s)(sI - J)$. Since the columns of $H(sI-J)^{-1}$ are linearly independent over $\mathbb{C}$ if and only if the pair (J, H) is observable we have to check whether

$$\rho = \text{rank}\begin{pmatrix} J - \lambda I \\ H \end{pmatrix}$$

is maximal, i.e.,

$$\rho = l_1 + \cdots + l_q. \tag{2.6}$$

It is easy to see that

$$\rho = (l_1 - 1) + \cdots + (l_q - 1) + \text{rank}\ (h_{10}, \ldots, h_{q0})\,.$$

Since $V_B(s)$ is unimodular the matrix

$$V_B(\lambda) = (h_{10}, \ldots, h_{q0}, \ldots)$$

is nonsingular and we obtain (2.6).

□

We note the analogous result for left Jordan systems of $A(s)$.

Lemma 2.1 b: *Let U_A and V_A be unimodular matrices which transform A into Smith form,*

$$U_A A V_A = S_A = \text{diag}\ (a_1, \ldots, a_m)\,, \quad a_m \mid \ldots \mid a_1. \tag{2.7}$$

For $\lambda \in \sigma(A)$ let

$$(s-\lambda)^{k_1}, \ldots, (s-\lambda)^{k_p}, \quad k_1 \geq \ldots \geq k_p \geq 1, \tag{2.8}$$

be the corresponding elementary divisors of A. If

$$U_A = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \tag{2.9}$$

and

$$u_i = g_{i0} + \cdots + g_{i,k_i-1}(s-\lambda)^{k_i-1} + (s-\lambda)^{k_i}\, d_i(s), \quad i = 1, \ldots, p\,, \tag{2.10}$$

then

$$G = \begin{pmatrix} G_1 \\ \vdots \\ G_p \end{pmatrix}, \quad G_i = \begin{pmatrix} g_{i0} \\ \vdots \\ g_{i,k_i-1} \end{pmatrix}, \quad i = 1, \ldots, p\,,$$

is a left Jordan system of A.

If we define $A_\lambda(s) = A(s-\lambda)$, and accordingly, B_λ, C_λ, then it is obvious that the matrices G and H are Jordan systems of A and B corresponding to λ if and only if they are Jordan systems of A_λ and B_λ corresponding to the charactristic value 0. Therefore we can express (1.8) as a condition for $\lambda = 0 \in \sigma(A_\lambda) \cap \sigma(B_\lambda)$ if we use C_λ instead of C. Hence it is no loss of generality to assume from now on $0 \in \sigma(A) \cap \sigma(B)$ and to focus on the characteristic root $\lambda = 0$. We write C as $C(s) = C_0 + C_1 s + \cdots C_t s^t$.

For a matrix (or a vector)

$$M(s) = M_{-r}s^{-r} + \cdots + M_0 + M_1 s + \cdots + M_k s^k$$

with elements in $\mathbb{C}\,[s, s^{-1}]$ we define

$$\pi_-\, M = M_{-r}s^{-r} + \cdot + M_{-1}s^{-1},$$

and if $M_{-r} \neq 0$ we put

$$\eta(M) = r.$$

Clearly, if L and M are of the same size, then

$$\eta(L + M) \leq \max \left\{ \eta(L), \eta(M) \right\}. \tag{2.11}$$

Lemma 2.2: *Let*

$$g^T(s) = g_0^T s^{-k} + \cdots + g_{k-1}^T s^{-1} \in s^{-1}\mathbb{C}^m[s^{-1}], \quad g_0 \neq 0, \tag{2.12}$$

and

$$h(s) = h_0 s^{-l} + \cdots + h_{l-1}s^{-1} \in s^{-1}\mathbb{C}^n[s^{-1}], \quad h_0 \neq 0, \tag{2.13}$$

be given. Put

$$g^{[\alpha]}(s) = g_0 s^{-\alpha} + \cdots + g_{\alpha-1}s^{-1}, \quad \alpha = 1, \ldots, k\ ,$$

and

$$h^{[\beta]}(s) = h_0 s^{-\beta} + \cdots + h_{\beta-1}s^{-1}, \quad \beta = 1, \ldots, l\ ,$$

such that $g^{[k]} = g$ *and* $h^{[l]} = h$. *The following statements are equivalent*

(1)

$$\sum_{\nu+\sigma+\tau=r} g_\nu C_\sigma h_\tau = 0, \quad r = 0, 1, \ldots, \min(k, l) - 1. \tag{2.14}$$

(2)

$$\eta\,[gCh] \leq \max(k, l). \tag{2.15}$$

(3)

$$\eta\left(g^{[\alpha]}Ch^{[\beta]}\right) \leq \max(\alpha, \beta), \quad \alpha = 1, \ldots, k\ , \quad \beta = 1, \ldots, l.$$

Proof: (1) $\Leftrightarrow$ (2) The coefficients ξ_r of $g(s)C(s)h(s) = \sum \xi_r s^{-k-l+r}$ are given by $\xi_r = \displaystyle\sum_{\nu+\sigma+\tau=r} g_\nu C_\sigma h_\tau$. Hence (2.14) is equivalent to

$$g(s)C(s)h(s) = s^{-(k+l)+\min(k,l)} f(s)$$

for some $f \in \mathbb{C}\,[s]$.

(1) $\Leftrightarrow$ (2) For $1 \leq \alpha \leq k$, $1 \leq \beta \leq l$ we have

$$g^{[\alpha]}(s)C(s)h^{[\beta]}(s) = \left[g(s)s^{k-\alpha} + a(s)\right] C(s) \left[h(s)s^{l-\beta} + b(s)\right] = s^{-\mu}d(s),$$

where a and b are polynomial vectors and $d \in \mathbb{C}[s]$. From (2.11) we obtain $\mu \leq \max\{\max(k,l) - [(k-\alpha)+(l-\beta)], \alpha, \beta\} = \max\{\alpha, \beta\}$.

□

A modul theoretic approach leads to another characterization of Jordan systems. Assume $0 \in \sigma(B)$ and $\det B(s) = s^l f(s)$, $f(0) \neq 0$. Define

$$K_B = \left\{ w \mid w \in s^{-1}\mathbb{C}^n[s^{-1}],\ Bw \in \mathbb{C}^n[s] \right\}.$$

Then K_B is a vector space over $\mathbb{C}$ and (see Fuhrmann [6]) we have $\dim K_B = l$. For $p \in \mathbb{C}[s]$ and $w \in K_B$ let a scalar product be defined by $p \cdot w = \pi_- pw$ and put $Sw = s \cdot w$. Then K_B is a $\mathbb{C}[s]$-module and the shift S is a (nilpotent) linear operator on K_B.

Lemma 2.3: *Let J be the $l \times l$ Jordan matrix associated to $0 \in \sigma(B)$. The matrix $H \in \mathbb{C}^{n\times l}$ is a right Jordan system of $B(s)$ corresponding to 0 if and only if the columns of $H(sI-J)^{-1}$ form a basis of K_B.*

Proof: If $H \in \mathbb{C}^{n\times l}$ is a r.J.s. then (1.3) implies

$$BH(sI-J)^{-1} = \hat{H}(s) \in \mathbb{C}^{n\times l}[s]. \tag{2.16}$$

Hence the columns of $H(sI-J)^{-1}$ are in K_B. Since they are free over $\mathbb{C}$ and $l = \dim K_B$ they form a basis of K_B. Conversely if the columns of $H(sI-J)^{-1}$ form a basis of K_B then they are linearly independent over $\mathbb{C}$ and we have (2.16) for some polynomial matrix $\hat{H}$.

□

We note that

$$\pi_-\, sH(sI-J)^{-1} = H(sI-J)^{-1}J,$$

which shows that J is the matrix representation of S with respect to the basis $H(sI-J)^{-1}$ of K_B.

Lemma 2.4: *Let (G,H) be a pair of Jordan systems of (A,B) corresponding to $\lambda = 0$ and let G and H be given as in (1.7) and (1.4). Then (G,H) has the property (Σ), i.e.*

$$\sum_{\nu+\sigma+\tau=r_{ij}} g_{i\nu} C_\sigma h_{j\tau} = 0, \tag{2.17}$$

$$r_{ij} = 0,1,\dots,\min(k_i,l_j)-1, \quad i = 1,\dots,p, \quad j = 1,\dots,q,$$

if and only if

$$\eta(xCy) \leq \max[\eta(x),\eta(y)] \quad \textit{for all} \quad x^T \in K_{A^T} \quad \textit{and} \quad y \in K_B. \tag{2.18}$$

Proof: We are going to apply Lemma 2.2 and put

$$g_i^{[\alpha_i]}(s) = g_{i0}s^{-\alpha_i} + \cdots + g_{i,\alpha_i-1}s^{-1}, \quad \alpha_i = 1, \ldots, k_i, \quad i = 1, \ldots, p\,,$$

and

$$h_j^{[\beta_j]}(s) = h_{j0}s^{-\beta_j} + \cdots + h_{j,\beta_j-1}s^{-1}, \quad \beta_j = 1, \ldots, l_j, \quad j = 1, \ldots, q\,.$$

Define $H_j^{\square} = \left(h_j^{[1]}, \ldots, h_j^{[l_j]}\right)$ and $H^{\square} = \left(H_1^{\square}, \ldots, H_p^{\square}\right)$. Then

$$H_j^{\square} = \left(h_{j0}, \ldots, h_{j,l_j-1}\right)\left(sI - N_{l_j}\right)^{-1} \quad \text{and} \quad H^{\square} = H(sI - J)^{-1}$$

where $J = \operatorname{diag}\left(N_{l_1}, \ldots, N_{l_q}\right)$. It follows from Lemma 2.3 that the columns of $H^{\square}$ are a basis of K_B, more precisely, a Jordan basis for the shift S with

$$h_j^{[\beta_j]} \in \text{Ker } S^{\beta_j} \setminus \text{Ker } S^{\beta_j - 1}$$

and $Sh_j^{[\beta_j]} = h_j^{[\beta_j-1]}$. Now consider $y \in K_B$ and $x^T \in K_{A^T}$. If $\eta(y) = N$ then $y \in \text{Ker } S^N \setminus \text{Ker } S^{N-1}$ and the Jordan basis $H^{\square}$ yields a representation

$$y = \sum_{j=1}^{q} \sum_{\beta_j \le N} \gamma_{j,\beta_j} h_j^{[\beta_j]} \tag{2.19}$$

where $\sum_{j=1}^{q} \gamma_{jN} h_j^{[N]} \neq 0$. Similarly if $\eta(x) = M$ then

$$x = \sum_{i=1}^{p} \sum_{\alpha_i \le M} \theta_{i,\alpha_i} g_i^{[\alpha_i]} \tag{2.20}$$

where $\sum_{i=1}^{p} \theta_{iM} g_i^{[M]} \neq 0$. From Lemma 2.2 we know that (2.17) is equivalent to

$$\eta\left(g_i^{[\alpha_i]} C h_j^{[\beta_j]}\right) \le \max\left(\alpha_i, \beta_j\right), \qquad i = 1, \ldots, p\,, \quad \alpha_i = 1, \ldots, k_i\,, \quad j = 1, \ldots, q\,, \quad \beta_j = 1, \ldots, l_j. \tag{2.21}$$

Taking inequalities of the form (2.11) into account it follows from (2.19) and (2.20) that (2.21) is equivalent to (2.18).

□

Corollary 2.5: *If one pair (G, H) of Jordan systems corresponding to λ has the property (Σ) then all such pairs have property (Σ).*

3 Proof of Theorem 1.1

Let U_B, V_B, U_A, V_A be unimodular matrices as in (2.1) and (2.9) which transform A and B into Smith normal form such that (2.2), (2.3), (2.7) and (2.8) hold. Then (1.1) is equivalent to

$$S_A\tilde{X} - \tilde{Y}S_B = \tilde{C} \tag{3.1}$$

where $\tilde{X} = (\tilde{x}_{ij}) = V_A^{-1}XV_B$, $\tilde{Y} = (\tilde{y}_{ij}) = U_AYU_B^{-1}$ and $\tilde{C} = (\tilde{c}_{ij}) = (u_iCv_j) = U_ACV_B$. In (3.1) we have a set of scalar equations

$$a_i\tilde{x}_{ij} - \tilde{y}_{ij}b_j = \tilde{c}_{ij}, \quad i = 1, \ldots, m\ , \quad j = 1, \ldots, n\ . \tag{3.2}$$

Since the ideal (a_i, b_j) is generated by $d_{ij} = \gcd(a_i, b_j)$ we see that (3.2) holds for some polynomials $\tilde{x}_{ij}$, $\tilde{y}_{ij}$ if and only if

$$d_{ij} \mid \tilde{c}_{ij}\ , \tag{3.3}$$

which is the solvability condition of Cheng and Pearson [2]. Clearly (3.3) is satisfied if and only if we have

$$(s-\lambda)^{\min(k_i,l_j)} \mid u_iCv_j \tag{3.4}$$

for each $\lambda \in \sigma(a_i) \cap \sigma(b_j)$, where λ is a root of multiplicity k_i of $a_i(s)$ and of multiplicity l_j of $b_j(s)$. Let u_i and v_j be given as in (2.10) and (2.4) and put $g_i(s) = g_{i0}s^{-k_i} + \cdot + g_{i,k_i-1}s^{-1}$ and $h_j(s) = h_{j0}s^{-l_j} + \cdot + h_{j,l_j-1}s^{-1}$. Then (3.4) is equivalent to

$$g_i(s)C(s)h_j(s) = (s-\lambda)^{\max(k_i,l_j)}f(s) \tag{3.5}$$

for some $f \in \mathbb{C}[s]$. From Lemma 2.2 we deduce that (3.5) is equivalent to

$$\sum_{\nu+\sigma+\tau=r} g_{i\nu}\,\frac{C^{(\sigma)}(\lambda)}{\sigma!}\,h_{j\tau}, \quad r = 0, 1, \ldots, \min(k_i, l_j) - 1.$$

According to Lemma 2.1 the collections $H = (\ldots, h_{j,\beta_j-1}, \ldots)$ and $G^T = (\ldots, g^T_{i,\alpha_i-1}, \ldots)$ are right Jordan systems of B and A^T. We have proved so far that (1.1) is solvable if and only if the special pair (G, H) of Jordan systems which we obtained from the Smith forms S_B and S_A has the property (Σ). But Corollary 2.5 tells us that property (Σ) is independent of the choice of the pair (G, H), which completes the proof of Theorem 1.1.

□

4 An example

Consider (1.1) with

$$A(s) = \begin{pmatrix} s^4 - 2s^3 + 2s^2 & s & s^2 \\ s^2 & s & s^2 \\ s^3 + 2s^2 & s^2 + 2s & s^3 + 2s^2 + 1 \end{pmatrix},$$

$$B(s) = \begin{pmatrix} 2s^3 - s^2 + s & s + 1 \\ s^2 & 1 \end{pmatrix},$$

$$C(s) = \begin{pmatrix} s^5 - 2s^4 + 2s^3 + 3s^2 + s & 2 + s \\ s^2 + s & 0 \\ -s^3 + 4s^2 + 3s - 1 & -2s + 1 \end{pmatrix}.$$

It is not difficult to determine the elementary divisors of $A(s)$ to be s^2, s, $(s-1)^2$, and those of $B(s)$ to be s, $(s-1)^2$. In the case of the characteristic root $\lambda = 0$ we have $r_{11} = r_{21} = 0$ in (1.8b). Furthermore $A(0) = \mathrm{diag}\,(0, 0, 1)$,

$$B(0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C(0) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

From $g_{i0}A(0) = 0$ and $B(0)h_{10} = 0$ we obtain $g_{10} = (1, 0, 0)$, $g_{20} = (0, 1, 0)$ and $h_{10} = (1, 0)^T$, and for $i = 1, 2$ condition (1.8a), i.e. $g_{i0}C(0)h_{10} = 0$ is satisfied. In the case of $\lambda = 1$ the following matrices are needed:

$$A(1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 3 & 4 \end{pmatrix}, \quad A'(1) = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 7 & 4 & 7 \end{pmatrix},$$

$$B(1) = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \quad B'(1) = \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix},$$

$$C(1) = \begin{pmatrix} 5 & 3 \\ 2 & 0 \\ 5 & -1 \end{pmatrix}, \quad C'(1) = \begin{pmatrix} 10 & 1 \\ 3 & 0 \\ 8 & -2 \end{pmatrix}.$$

From $g_{10}A(1) = 0$, $g_{10}A'(1) + g_{11}A(1) = 0$ and $B(1)h_{10} = 0$, $B'(1)h_{10} + B(1)h_{11} = 0$ we compute the Jordan chains $g_{10} = (1, -1, 0)$, $g_{11} = (0, 0, 0)$ and $h_{10} = (-1, 1)^T$, $h_{11} = (1, 1)^T$. Since both $g_{10}C(1)h_{10} = 0$ and $g_{11}C(1)h_{10} + g_{10}C'(1)h_{10} + g_{10}C(1)h_{11} = 0$ are satisfied it follows that the given equation has property (Σ) and thus admits a polynomial solution. Indeed a particular solution of (1.1) is given by

$$X(s) = \begin{pmatrix} s & 0 \\ -2s^2 + 2s + 1 & 0 \\ s - 1 & 0 \end{pmatrix}, \quad Y(s) = -\begin{pmatrix} 0 & 2 + s \\ 0 & 0 \\ 0 & -2s + 1 \end{pmatrix}.$$

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