

# Pairs of Modules over a Principal Ideal Domain

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## Abstract

We study pairs of finitely generated modules over a principal ideal domain and their corresponding matrix representations. We introduce equivalence relations for such pairs and determine invariants and canonical forms.

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## 1 Introduction

In this note we study pairs of submodules of  $R^n$ , where  $R$  is a principal ideal domain. We first review some problems and results on isometries of  $\mathbb{C}^n$  for which we seek analogies in  $R^n$ . Let  $\mathcal{X}_1$  and  $\mathcal{X}_2$  be subspaces of  $\mathbb{C}^n$ . How are  $\mathcal{X}_1$  and  $\mathcal{X}_2$  situated with respect to each other? The following theorem, which can be traced back to Jordan [2], shows that there exists a suitable orthonormal basis of  $\mathbb{C}^n$  which displays the respective position of  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . To illustrate our purposes it is enough to review the case where  $\dim \mathcal{X}_1 = \dim \mathcal{X}_2 = m$  and  $2m = n$ . For a subspace  $\mathcal{X}$  of  $\mathbb{C}^n$  let  $P_{\mathcal{X}}$  denote the orthogonal projection on  $\mathcal{X}$ .

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**Theorem 1.1.** [4] *Assume*

$$\mathcal{X}_i = X_i \mathbb{C}^m, X_i \in \mathbb{C}^{n \times m}, X_i^* X_i = I_m, i = 1, 2, \text{ and } 2m = n. \quad (1.1)$$

*Then there are unitary matrices  $Q, V_1, V_2$ , such that*

$$QX_1V_1 = \begin{pmatrix} I_m \\ 0 \end{pmatrix}, \quad QX_2V_2 = \begin{pmatrix} \Gamma \\ \Sigma \end{pmatrix},$$

*where*

$$\Gamma = \text{diag}(\gamma_1, \dots, \gamma_m) \quad \text{and} \quad \Sigma = \text{diag}(\sigma_1, \dots, \sigma_m),$$

*and*

$$0 \leq \gamma_1 \leq \dots \leq \gamma_m, \quad \sigma_1 \geq \dots \geq \sigma_m \geq 0,$$

*and*

$$\gamma_\mu^2 + \sigma_\mu^2 = 1, \quad \mu = 1, \dots, m.$$

*The singular values of  $P_{\mathcal{X}_1}(I - P_{\mathcal{X}_2})$  are  $\sigma_1, \sigma_2, \dots, \sigma_m, 0, \dots, 0$ , such that the numbers  $\gamma_\mu, \sigma_\mu$  are uniquely determined by  $\mathcal{X}_1$  and  $\mathcal{X}_2$ .*

If  $\mathcal{X}_i, \tilde{\mathcal{X}}_i, i = 1, 2$ , are subspaces of  $\mathbb{C}^n$  we set

$$(\mathcal{X}_1, \mathcal{X}_2) \sim (\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2) \quad (1.2)$$

and we call the two pairs *isometrically equivalent* if there exists an isometry  $\alpha : \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that

$$(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2) = (\alpha \mathcal{X}_1, \alpha \mathcal{X}_2). \quad (1.3)$$

Assume (1.1) and  $\tilde{\mathcal{X}}_i = \tilde{X}_i \mathbb{C}^m, \tilde{X}_i^* \tilde{X}_i = I_m, i = 1, 2$ . Then it is obvious that we have (1.2) if and only if  $(X_1, X_2)$  and  $(\tilde{X}_1, \tilde{X}_2)$  can be transformed into the same canonical form

$$\left( \begin{pmatrix} I \\ 0 \end{pmatrix}, \begin{pmatrix} \Gamma \\ \Sigma \end{pmatrix} \right).$$

That means that one can identify a complete set of invariants under the equivalence (1.2).

**Theorem 1.2.** [4] *Two pairs  $(\mathcal{X}_1, \mathcal{X}_2)$  and  $(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2)$  are isometrically equivalent if and only if*

$$\dim \mathcal{X}_i = \dim \tilde{\mathcal{X}}_i, \quad i = 1, 2,$$

*and the singular values of  $P_{\mathcal{X}_1}(I - P_{\mathcal{X}_2})$  and  $P_{\tilde{\mathcal{X}}_1}(I - P_{\tilde{\mathcal{X}}_2})$  are the same.*

Now let  $\mathcal{X}_i, \tilde{\mathcal{X}}_i, i = 1, 2$ , be submodules of  $R^n$ . We say that the pairs  $(\mathcal{X}_1, \mathcal{X}_2)$  and  $(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2)$  are  *$R$ -unimodular equivalent*, and we write

$$(\mathcal{X}_1, \mathcal{X}_2) \stackrel{R}{\sim} (\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2),$$

if (1.3) holds for some  $R$ -automorphism  $\alpha : R^n \rightarrow R^n$ . Given two pairs of submodules of  $R^n$  how can one decide whether they are  $R$ -unimodular

equivalent? In the case where  $\mathcal{X}_1 + \mathcal{X}_2$  and  $\tilde{\mathcal{X}}_1 + \tilde{\mathcal{X}}_2$  are direct summands of  $R^n$  we shall obtain a criterion given in Theorem 1.3 below. It is known (see e.g. [1]) that for a PID one can characterize a *closed* (or *pure*) submodule  $\mathcal{X}$  of  $R^n$  by the property that  $\mathcal{X}$  is a direct summand of  $R^n$ . If  $\mathcal{X} = XR^m$  for some  $X \in R^{n \times m}$  then  $\mathcal{X}$  is closed if and only if all invariant factors of  $X$  are 1.

**Theorem 1.3.** *Let  $\mathcal{X}_i, \tilde{\mathcal{X}}_i, i = 1, 2$ , be submodules of  $R^n$ . Assume that  $\mathcal{X}_1 + \mathcal{X}_2$  and  $\tilde{\mathcal{X}}_1 + \tilde{\mathcal{X}}_2$  are closed in  $R^n$ . Then the pairs  $(\mathcal{X}_1, \mathcal{X}_2)$  and  $(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2)$  are  $R$ -unimodular equivalent if and only if*

$$\text{rank}(\mathcal{X}_1 + \mathcal{X}_2) = \text{rank}(\tilde{\mathcal{X}}_1 + \tilde{\mathcal{X}}_2) \quad (1.4)$$

and

$$\mathcal{X}_i / (\mathcal{X}_1 \cap \mathcal{X}_2) \cong \tilde{\mathcal{X}}_i / \tilde{\mathcal{X}}_1 \cap \tilde{\mathcal{X}}_2, \quad i = 1, 2, \quad (1.5)$$

hold.

Let  $(X_1, X_2) \in R^{n \times m_1} \times R^{n \times m_2}$  and  $(\tilde{X}_1, \tilde{X}_2) \in R^{n \times \tilde{m}_1} \times R^{n \times \tilde{m}_2}$  be two pairs of matrices such that  $X_i, \tilde{X}_i, i = 1, 2$ , have full column rank. We set

$$(X_1, X_2) \stackrel{u}{\sim} (\tilde{X}_1, \tilde{X}_2)$$

if

$$(\tilde{X}_1, \tilde{X}_2) = (QX_1V_1, QX_2V_2) \quad (1.6)$$

for some unimodular matrices  $Q, V_1, V_2$ . If, for  $i = 1, 2$ , the columns of  $X_i$  and  $\tilde{X}_i$  are bases of  $\mathcal{X}_i$  and  $\tilde{\mathcal{X}}_i$ , respectively, then we have  $(\mathcal{X}_1, \mathcal{X}_2) \stackrel{R}{\sim} (\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2)$  if and only if  $(X_1, X_2) \stackrel{u}{\sim} (\tilde{X}_1, \tilde{X}_2)$ . A ‘canonical form’ of a pair  $(X_1, X_2)$  under the transformation (1.6) will be needed for the proof of Theorem 1.3. The following result is a counterpart of Theorem 1.1.

**Theorem 1.4.** *Let  $X_1 \in R^{n \times m_1}$  and  $X_2 \in R^{n \times m_2}$  have full column rank and let all invariant factors of the matrix  $(X_1 \ X_2) \in R^{n \times (m_1 + m_2)}$  be equal to 1. Then we have  $(X_1, X_2) \stackrel{u}{\sim} (Y_1, Y_2)$  where*

$$Y_1 = \begin{pmatrix} A & 0 \\ 0 & I_{m_1-t} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \text{diag}(\alpha_1, \dots, \alpha_t), \quad \alpha_t | \dots | \alpha_1, \quad (1.7)$$

and

$$Y_2 = \begin{pmatrix} B & 0 \\ 0 & 0 \\ 0 & I_{m_2-t} \\ 0 & 0 \end{pmatrix}, \quad B = \text{diag}(\beta_1, \dots, \beta_t), \quad \beta_1 | \dots | \beta_t, \quad (1.8)$$

and  $0 \leq t \leq \min\{m_1, m_2\}$ , and

$$(\alpha_\tau, \beta_\tau) = 1, \quad \tau = 1, \dots, t. \quad (1.9)$$

The integer  $t$  and the elements  $\alpha_\tau, \beta_\tau$ , are uniquely determined by  $X_1$  and  $X_2$  (up to multiplication by units).

The proofs of Theorem 1.3 and Theorem 1.4 will be given in the next section. We would like to point out that Theorem 1.4 and the canonical form of pairs  $(X_1, X_2)$  under unimodular equivalence are the main contributions of our paper. For a general theory of pairs of modules over an arbitrary ring we refer to [3]. In particular, Theorem 1.6 of [3, p.69] on direct sums of projective rank-1 modules over a commutative ring goes far beyond our Theorem 1.3.

## 2 Proofs

Let  $\mathcal{M}$  and  $\mathcal{N}$  be submodules of  $R^n$  and  $\mathcal{M} \subseteq \mathcal{N}$ . The *closure* of  $\mathcal{M}$  in  $\mathcal{N}$  is the submodule

$$\text{cl}(\mathcal{M}, \mathcal{N}) = \{x \in \mathcal{N}; \alpha x \in \mathcal{M} \text{ for some } \alpha \in R, \alpha \neq 0\}.$$

If  $\mathcal{N} = R^n$  we denote the closure by  $\overline{\mathcal{M}}$ . The following facts on the closure are known and easy to prove. For two submodules  $\mathcal{M}_1, \mathcal{M}_2$  of  $R^n$  we have

$$\overline{\mathcal{M}_1 \cap \mathcal{M}_2} = \overline{\mathcal{M}_1} \cap \overline{\mathcal{M}_2} \quad (2.10)$$

and

$$\overline{\mathcal{M}_1 + \mathcal{M}_2} \supseteq \overline{\mathcal{M}_1} + \overline{\mathcal{M}_2}. \quad (2.11)$$

A submodule  $\mathcal{M}$  is closed in  $R^n$  if and only if  $\overline{\mathcal{M}} = \mathcal{M}$ . From (2.11) follows

$$\mathcal{M}_1 + \mathcal{M}_2 = \overline{\mathcal{M}_1} + \overline{\mathcal{M}_2}, \quad (2.12)$$

if  $\mathcal{M}_1 + \mathcal{M}_2$  is closed.

**Lemma 2.1.** *Assume that  $\mathcal{X}_1 + \mathcal{X}_2$  is a closed submodule of  $R^n$ . Define  $\mathcal{D} = \mathcal{X}_1 \cap \mathcal{X}_2$ . Then*

$$\mathcal{X}_1 + \mathcal{X}_2 = \overline{\mathcal{D}} \oplus \mathcal{K}_1 \oplus \mathcal{K}_2 \quad (2.13)$$

with  $\mathcal{K}_i = \overline{\mathcal{K}_i} \subseteq \mathcal{X}_i$ ,  $i = 1, 2$ , and

$$\mathcal{X}_i = (\overline{\mathcal{D}} \cap \mathcal{X}_i) \oplus \mathcal{K}_i, \quad i = 1, 2. \quad (2.14)$$

*Proof.* Define  $\mathcal{H}_1 = \mathcal{X}_1 \cap \overline{\mathcal{X}_2}$  and  $\mathcal{H}_2 = \mathcal{X}_2 \cap \overline{\mathcal{X}_1}$ . Let us show first that  $\mathcal{H}_i$  is closed in  $\mathcal{X}_i$ . Take  $i = 1$ . If  $x$  is in  $\text{cl}(\mathcal{H}_1, \mathcal{X}_1)$  then  $\alpha x \in \mathcal{H}_1$  for some nonzero  $\alpha$ . Hence  $x \in \overline{\mathcal{X}_2}$  and  $x \in \mathcal{H}_1$ . Now consider  $\mathcal{S} = \mathcal{H}_1 + \mathcal{H}_2$ . Obviously we have  $\mathcal{S} \subseteq \overline{\mathcal{D}}$ . To prove the reverse inclusion take  $d \in \overline{\mathcal{D}}$ ,  $d = x_1 + x_2$ ,  $x_i \in \mathcal{X}_i$ , such that  $\alpha d \in \mathcal{D}$  for some  $\alpha \neq 0$ . Then  $\alpha x_2 \in \mathcal{X}_1$  and  $x_2 \in \overline{\mathcal{X}_2}$  and we obtain  $x_2 \in \mathcal{H}_2$ , similarly  $x_1 \in \mathcal{H}_1$ , which proves  $\overline{\mathcal{D}} \subseteq \mathcal{S}$ . Hence we have

$$\mathcal{H}_1 + \mathcal{H}_2 = \overline{\mathcal{D}} \quad (2.15)$$

Since, for  $i = 1, 2$ , the submodule  $\mathcal{H}_i$  is closed in  $\mathcal{X}_i$  there exists a  $\mathcal{K}_i$  such that  $\mathcal{X}_i = \mathcal{H}_i \oplus \mathcal{K}_i$ . From (2.15) follows

$$\mathcal{X}_1 + \mathcal{X}_2 = \overline{\mathcal{D}} + \mathcal{K}_1 + \mathcal{K}_2 \quad (2.16)$$

We shall prove the sum in (2.16) is direct. Take  $y \in \overline{\mathcal{D}} \cap (\mathcal{K}_1 + \mathcal{K}_2)$ ,  $y = k_1 + k_2$ ,  $k_i \in \mathcal{K}_i$ . From  $\alpha y \in \mathcal{D}$ ,  $\alpha \neq 0$ , we obtain  $\alpha k_1 \in \mathcal{X}_1$ , which yields  $k_1 \in \mathcal{H}_1$ . But  $\mathcal{H}_1 \cap \mathcal{K}_1 = 0$ . Hence  $k_1 = 0$ , and similarly  $k_2 = 0$ . Therefore  $y = 0$  and

$$\mathcal{X}_1 + \mathcal{X}_2 = \overline{\mathcal{D}} \oplus (\mathcal{K}_1 + \mathcal{K}_2). \quad (2.17)$$

It is easy to see that  $\mathcal{K}_1 \cap \mathcal{K}_2 = 0$ , which combined with (2.17) yields (2.13). To show that  $\mathcal{K}_i = \overline{\mathcal{K}}_i$  it suffices to prove that  $\mathcal{K}_i$  is closed in  $\mathcal{X}_1 + \mathcal{X}_2$ . Take  $i = 1$  and  $x \in \overline{\mathcal{K}}_1$ . If  $\alpha x \in \mathcal{K}_1$ ,  $\alpha x \neq 0$ , then  $x = d + k_1 + k_2$ ,  $d \in \overline{\mathcal{D}}$ ,  $k_i \in \mathcal{K}_i$ , and (2.13) imply  $\alpha x = \alpha k_1$ . Hence  $x = k_1$ , and  $\mathcal{K}_1$  is closed. ■

**Proof of Theorem 1.4** Let us first consider the case where  $X_1$  and  $X_2$  are nonsingular  $n \times n$  matrices. The assumption that all invariant factors of  $(X_1 \ X_2)$  are units in  $R$  implies  $X_1 S_1 + X_2 S_2 = I_n$  for some  $S_1, S_2 \in R^{n \times n}$ . Hence  $X_1$  and  $X_2$  are left coprime. Put  $T = X_1^{-1} X_2$ . Then there are unimodular matrices  $V_1$  and  $V_2$  which transform  $T$  into Smith-McMillan form such that

$$T = V_1^{-1} \text{diag}(\beta_1/\alpha_1, \dots, \beta_n/\alpha_n) V_2$$

and

$$\alpha_n | \dots | \alpha_1, \quad \beta_1 | \dots | \beta_n, \quad \text{and} \quad (\alpha_\nu, \beta_\nu) = 1, \quad \nu = 1, \dots, n.$$

Define

$$\hat{X}_1 = \text{diag}(\alpha_1, \dots, \alpha_n) V_1, \quad \hat{X}_2 = \text{diag}(\beta_1, \dots, \beta_n) V_2.$$

Then  $\hat{X}_1$  and  $\hat{X}_2$  are left coprime and  $T = \hat{X}_1^{-1} \hat{X}_2$ . Thus we have two left coprime factorizations of  $T$ . Hence (see e.g. [5])  $X_1 = Q \hat{X}_1$ ,  $X_2 = Q \hat{X}_2$  for some unimodular  $Q$ .

We now deal with the general case and put  $\mathcal{X}_i = X_i R^{m_i}$ ,  $i = 1, 2$ ,  $\mathcal{D} = \mathcal{X}_1 \cap \mathcal{X}_2$ , and  $t = \text{rank } \mathcal{D}$ . Let  $S$  be unimodular such that the columns of

$$S \begin{pmatrix} I_t & 0 & 0 \\ 0 & I_{m_1-t} & 0 \\ 0 & 0 & I_{m_2-t} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.18)$$

are a basis of  $\mathcal{X}_1 + \mathcal{X}_2$  which corresponds to the direct sum in (2.13). In the following we shall assume

$$\text{row rank}(X_1 \ X_2) = n$$

or equivalently  $\mathcal{X}_1 + \mathcal{X}_2 = R^n$ , which allows us to discard the bottom row of zero blocks in (2.18). Because of (2.14) a basis of  $\mathcal{X}_1$  is given by the columns

of a matrix

$$S \begin{pmatrix} A_1 & 0 \\ 0 & I_{m_1-t} \\ 0 & 0 \end{pmatrix}$$

for some nonsingular  $A_1 \in R^{t \times t}$ . Similarly a basis of  $\mathcal{X}_2$  is of the form

$$S \begin{pmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & I_{m_2-t} \end{pmatrix}$$

where  $B_1$  is nonsingular. Since  $\mathcal{X}_1 + \mathcal{X}_2$  is pure we have  $A_1G + B_1H = I_t$  for some  $G, H$ . We know already that there are unimodular matrices  $Q_c, W_1, W_2$ , such that  $Q_c A_1 W_1 = A$  and  $Q_c B_1 W_2 = B$  with  $A, B$  as in (1.7) and (1.8), which yields the desired canonical form.

To prove uniqueness of  $A$  and  $B$  in (1.7) and (1.8) we note that (1.9) implies that the columns of  $(AB \ 0 \ 0 \ 0)^T$  are a basis of  $Q(\mathcal{X}_1 \cap \mathcal{X}_2)$ . Therefore

$$Q\mathcal{X}_1/Q(\mathcal{X}_1 \cap \mathcal{X}_2) \cong \mathcal{X}_1/(\mathcal{X}_1 \cap \mathcal{X}_2) \cong R/\beta_1 R \oplus \cdots \oplus R/\beta_t R, \quad (2.19)$$

and similarly

$$\mathcal{X}_2/(\mathcal{X}_1 \cap \mathcal{X}_2) \cong R/\alpha_1 R \oplus \cdots \oplus R/\alpha_t R, \quad (2.20)$$

which characterizes the entries of  $A$  and  $B$  in terms of the pairs  $(\mathcal{X}_1, \mathcal{X}_2) = (X_1 R^{m_1}, X_2 R^{m_2})$ . ■

**Proof of Theorem 1.3** It is obvious that  $(\mathcal{X}_1, \mathcal{X}_2) \stackrel{R}{\sim} (\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2)$  implies (1.4) and (1.5). To prove the converse let  $X_i, \tilde{X}_i, i = 1, 2$ , be basis matrices of  $\mathcal{X}_i, \tilde{\mathcal{X}}_i, i = 1, 2$ , respectively. Then, according to Theorem 1.4 and (2.19) and (2.20), the pairs  $(X_1, X_2)$  and  $(\tilde{X}_1, \tilde{X}_2)$  have the same normal form, which implies  $(X_1, X_2) \stackrel{u}{\sim} (\tilde{X}_1, \tilde{X}_2)$  and  $(\mathcal{X}_1, \mathcal{X}_2) \stackrel{R}{\sim} (\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2)$ . ■

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