ISOLATED SEMIDEFINITE SOLUTIONS OF THE CONTINUOUS-TIME ALGEBRAIC RICCATI EQUATION

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The set of all negative-semidefinite solutions of the CARE $A^*X + XA + XBB^*X - C^*C = 0$ is homeomorphic to a well defined set of A -invariant subspaces provided that the purely imaginary eigenvalues of A are controllable. Based on that homeomorphism isolated n.s.d. solutions of the CARE are characterized by properties of their kernels.

1. INTRODUCTION

In this paper we consider the continuous-time algebraic Riccati equation (CARE)

$$\mathcal{R}(X) = A^*X + XA + XBB^*X - C^*C = 0, \quad (1.1)$$

where A, B and C are complex matrices of dimensions $n \times n$, $n \times p$ and $q \times n$ respectively. We focus on the set

$$\mathcal{T} = \{X \mid \mathcal{R}(X) = 0, X \leq 0\}$$

of negative-semidefinite solutions and we shall characterize those elements X of $\mathcal{T}$ which are isolated (in the topology which $\mathcal{T}$ inherits as a subset of the normed space $\mathbb{C}^{n \times n}$). Such an isolated X has the property that for a sufficiently small ϵ there is no solution $Y \in \mathcal{T}$, $Y \neq X$, with $\|X - Y\| < \epsilon$.

For $\lambda \in \sigma(A)$ let

$$E_\lambda(A) = \text{Ker}(A - \lambda I)^n$$

denote the corresponding generalized eigenspace and let

$$V(A, C) = \text{Ker} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

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be the unobservable subspace. It is the purpose of this paper to prove the following result.

THEOREM 1.1. *Assume that (1.1) has a solution $X \leq 0$. Then $\mathcal{T}$ has an isolated element if and only if all purely imaginary eigenvalues of A are B -controllable. A solution $X \in \mathcal{T}$ is isolated in $\mathcal{T}$ if and only if its kernel has the following property: If $\operatorname{Re} \lambda > 0$ and*

$$\operatorname{rank} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} \leq n - 2$$

then either

$$\operatorname{Ker} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} \cap \operatorname{Ker} X = 0$$

or

$$V(A, C) \cap E_\lambda(A) \subseteq \operatorname{Ker} X.$$

Isolated solutions of quadratic matrix equations were studied for the first time by J. Daughtry [Da]. We note the corresponding result for the ARE

$$A^*X + XA + XBB^*X - Q = 0, \quad (1.2)$$

where $Q = Q^*$.

THEOREM 1.2 [RR]. *Assume that (A, B) is controllable. Then X is isolated in the set of hermitian solutions of (1.2) if and only if each common eigenvalue of $A + BB^*X$ and $-(A + BB^*X)^*$ which is not purely imaginary is an eigenvalue of*

$$H = \begin{pmatrix} A & BB^* \\ Q & -A^* \end{pmatrix}$$

of geometric multiplicity one.

Our study is based on results of [W1] which will be reviewed in Section 2. If A has no uncontrollable modes on the imaginary axis then – according to [W1] – there is a bijection between $\mathcal{T}$ and a well-defined set $\mathcal{N}$ of A -invariant subspaces. Some facts on the gap metric and on isolated invariant subspaces are put together in Section 3. In Section 4 we show that the bijection between $\mathcal{T}$ and $\mathcal{N}$ mentioned above is a homeomorphism. We give a proof of Theorem 1.1 in Section 5.

The following notation will be used. In the partitions

$$\mathcal{C} = \mathcal{C}_{\leq} \cup \mathcal{C}_{>} = \mathcal{C}_{<} \cup \mathcal{C}_{=} \cup \mathcal{C}_{>} \quad (1.3)$$

the subscripts refer to real parts such that $\mathbb{C}_{\leq} = \{\lambda \mid \lambda \in \mathbb{C}, \operatorname{Re} \lambda \leq 0\}$, etc. To (1.3) correspond the decompositions

$$\mathbb{C}^n = E_{\leq}(A) \oplus E_{>}(A) = E_{<}(A) \oplus E_{=}(A) \oplus E_{>}(A)$$

where $E_{\leq}(A) = \oplus\{E_{\lambda}(A), \lambda \in \mathbb{C}_{\leq}\}$, etc. Let $\operatorname{Inv} A$ denote the lattice of A -invariant subspaces of $\mathbb{C}^n$. To the triple (A, B, C) we associate the controllable subspace

$$R(A, B) = \operatorname{Im}(B, AB, \dots, A^{n-1}B)$$

and the unobservable subspace $V(A, C)$. It will be convenient to define

$$V_{\leq}(A, C) = V(A, C) \cap E_{\leq}(A)$$

and similarly $V_{=}(A, C)$, $V_{>}(A, C)$, etc. With this notation $V_{\geq}(A, C)$ is the undetectable and $R(A, B) + E_{<}(A)$ is the stabilizable subspace.

2. DECOMPOSITION AND PARAMETRIZATION OF SOLUTIONS

The subsequent theorems which describe the structure of the solution set $\mathcal{T}$ are taken from [W1]. According to [G], [GH] there exists a solution $X \leq 0$ of (1.1) if and only if

$$V(A, C) + R(A, B) + E_{<}(A) = \mathbb{C}^n. \quad (2.1)$$

Because of $V_{<}(A, C) \subseteq E_{<}(A)$ the preceding condition (2.1) can be written as

$$V_{=}(A, C) + [V_{>}(A, C) + R(A, B) + E_{<}(A)] = \mathbb{C}^n. \quad (2.2)$$

Put

$$U_r = V_{>}(A, C) + R(A, B) + E_{<}(A). \quad (2.3)$$

Then (2.2) is equivalent to

$$\mathbb{C}^n = U_0 \oplus U_r \quad (2.4)$$

for some subspace $U_0 \subseteq V_{=}(A, C)$. We call (2.4) an LR -decomposition with Riccati part U_r and a Lyapunov complement U_0 . If $\mathbb{C}^n$ admits a decomposition (2.4) with nontrivial summands then (1.1) breaks up into a Lyapunov matrix equation and an irreducible Riccati equation.

THEOREM 2.1 [W1]. *(1) Let $\mathbb{C}^n = U_0 \oplus U_r$ be an LR -decomposition. If $S = (S_0, S_r)$ is nonsingular such that $\operatorname{Im} S_0 = U_0$, $\operatorname{Im} S_r = U_r$, $\dim U_r = n_r$, then*

$$S^{-1}AS \begin{pmatrix} A_0 & 0 \\ A_{r0} & A_r \end{pmatrix}, \quad S^{-1}B = \begin{pmatrix} 0 \\ B_r \end{pmatrix}, \quad CS = (0, C_r) \quad (2.5)$$

and

$$\sigma(A_0) \subseteq \mathbb{C}_= \quad (2.6)$$

and

$$V_{>}(A_r, C_r) + R(A_r, B_r) + E_{<}(A_r) = \mathbb{C}^{n_r}. \quad (2.7)$$

Assume (2.5) – (2.7). Then we have $X \in \mathcal{T}$ if and only if

$$X = (S^{-1})^* \begin{pmatrix} X_0 & 0 \\ 0 & X_r \end{pmatrix} S^{-1} \quad (2.8)$$

and $X_0 \leq 0$ satisfies the Lyapunov equation

$$\mathcal{L}_0(X_0) = A_0^* X_0 + X_0 A_0 = 0$$

and X_r is a solution of

$$\mathcal{R}_r(X_r) = A_r^* X_r + X_r A_r + X_r B_r B_r^* X_r - C_r^* C_r = 0.$$

(2) Let $\Pi : \mathbb{C}^n \rightarrow U_r$ be the projection on U_r along U_0 . Put $\rho(X) = X\Pi$. Then $\rho(X) \in \mathcal{T}$ and $(I - \Pi)^* X\Pi = 0$. If X is given as in (2.8) then

$$\rho(X) = (S^{-1})^* \begin{pmatrix} 0 & 0 \\ 0 & X_r \end{pmatrix} S^{-1}.$$

Put $\mathcal{S} = \rho(\mathcal{T})$ such that $\mathcal{S}$ contains all solutions of the form (2.8) with Lyapunov part $X_0 = 0$. Whether a Lyapunov complement U_0 does appear or not depends on purely imaginary eigenvalues of A .

LEMMA 2.2 [W1]. Assume $\mathcal{T} \neq \emptyset$. Then we have $\mathcal{S} = \mathcal{T}$ if and only if

$$\text{rank}(A - \lambda I, B) = n \quad \text{for all } \lambda \in \mathbb{C}_=. \quad (2.9)$$

In the case where (A, B) is stabilizable the conditions (2.1) and (2.9) are satisfied and we have $\mathcal{T} \neq \emptyset$ and $\mathcal{T} = \mathcal{S}$. The following observation will be useful.

LEMMA 2.3. For $X \in \mathcal{T}$ we have

$$E_=(A + BB^* X) = V_=(A, C).$$

PROOF. Put

$$A_X = A + BB^*X.$$

Then $\mathcal{R}(X) = 0$ is equivalent to

$$A_X^*X + XA_X = XB^*BX + C^*C. \quad (2.10)$$

We recall the well known fact that a Lyapunov equation

$$F^*Y + YF = D^*D$$

where $\sigma(F) \subseteq \mathbb{C}_=$ and Y is semidefinite implies $D = 0$. Assume

$$E_=(A_X) = \text{Im} \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Then

$$A_X = \begin{pmatrix} \hat{A}_1 & \hat{A}_{12} \\ 0 & \hat{A}_2 \end{pmatrix} \quad (2.11)$$

and $\sigma(\hat{A}_1) \subseteq \mathbb{C}_=$. Let

$$X = \begin{pmatrix} X_1 & X_{21}^* \\ X_{21} & X_2 \end{pmatrix} \quad (2.12)$$

and $C = (C_1 \ C_2)$ be partitioned accordingly. Then (2.10) yields

$$\hat{A}_1^*X_1 + X_1\hat{A}_1 = (X_1 \ X_{21}^*)BB^* \begin{pmatrix} X_1 \\ X_{21} \end{pmatrix} + C_1^*C_1.$$

Hence we obtain

$$BB^* \begin{pmatrix} X_1 \\ X_{21} \end{pmatrix} = 0 \quad (2.13)$$

and $C_1 = 0$ and

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \quad (2.14)$$

with $A_1 = \hat{A}_1$. Therefore we have $E_=(A_X) \subseteq V_=(A, C)$. To prove the converse inclusion

$$V_=(A, C) \subseteq E_=(A_X) \quad (2.15)$$

assume now that a basis has been chosen such that A and C are in the form (2.14) and $C = (0, C_2)$. By a slight abuse of notation take X as in (2.12). Then

$$A_1^* X_1 + X_1 A_1 = -(X_1 \ X_{21}^*) B B^* \begin{pmatrix} X_1 \\ X_{21} \end{pmatrix}. \quad (2.16)$$

As before we can regard (2.16) as a Lyapunov equation and conclude that (2.13) holds. From (2.14) we obtain A_X in the form (2.11) with $\hat{A}_1 = A_1$ which yields (2.15). □

There is an order isomorphism between $\mathcal{S}$ and the following system $\mathcal{N}$ of A -invariant subspaces of $\mathbb{C}^n$. Define

$$\begin{aligned} \mathcal{N} = \{ N \mid N \in \text{Inv} A, V_{\leq}(A, C) \subseteq N \subseteq V(A, C), N + \\ + R(A, B) + E_{<}(A) = \mathbb{C}^n \}. \end{aligned} \quad (2.17)$$

THEOREM 2.4 [W1]. (1) *The map $\gamma : \mathcal{S} \rightarrow \mathcal{N}$ given by $\gamma(X) = \text{Ker } X$ is a bijection, and both γ and γ^{-1} are order preserving, i.e. for $X, Y \in \mathcal{S}$ and $M, N \in \mathcal{N}$ the relations $X \leq Y$ and $M \subseteq N$ imply $\gamma(X) \subseteq \gamma(Y)$ and $\gamma^{-1}(M) \leq \gamma^{-1}(N)$.*

(2) *For $X \in \mathcal{S}$ we have*

$$\text{Ker } X = V_{\leq}(A, C) \oplus E_{>}(A_X). \quad (2.18)$$

3. THE GAP METRIC AND ISOLATED INVARIANT SUBSPACES

For $k \in \mathbb{N}$ let $\mathbb{C}^k$ be endowed with the usual scalar product $(x, y) = x^* y$ and with the corresponding norm $\|x\| = (x^* x)^{1/2}$. The norm of a matrix $G \in \mathbb{C}^{n \times m}$ (or of a linear map $G : \mathbb{C}^m \rightarrow \mathbb{C}^n$) is defined accordingly as operator norm, $\|G\| = \max\{\|Gx\|, x \in \mathbb{C}^m, \|x\| = 1\}$. Let P_M denote the orthogonal projection of $\mathbb{C}^n$ onto a subspace M of $\mathbb{C}^n$. The gap $\Theta(L, M)$ between two subspaces L and M of $\mathbb{C}^n$ is defined as

$$\Theta(L, M) = \|P_L - P_M\|.$$

It is a metric in the set of all subspaces of $\mathbb{C}^n$. The following lemma can be used to compute the gap.

LEMMA 3.1. *Let L and M be subspaces of $\mathbb{C}^n$ such that $1 \leq \dim L = \dim M = r < n$.*

Assume

$$L = \text{Im} \begin{pmatrix} I_r \\ 0 \end{pmatrix}$$

and let

$$U = \begin{pmatrix} U_1 & U_{12} \\ U_{21} & U_2 \end{pmatrix}$$

be unitary such that

$$M = \text{Im} \begin{pmatrix} U_1 \\ U_{21} \end{pmatrix} \quad \text{and} \quad M^\perp = \text{Im} \begin{pmatrix} U_{12} \\ U_2 \end{pmatrix}.$$

Then

$$\Theta(L, M) = \Theta(L^\perp, M^\perp) = \|U_{21}\| = \|U_{12}\|.$$

PROOF. We have

$$P_L = \begin{pmatrix} I \\ 0 \end{pmatrix} (I, 0) \quad \text{and} \quad P_M = \begin{pmatrix} U_1 \\ U_{21} \end{pmatrix} (U_1^*, U_{21}^*).$$

From

$$\begin{aligned} (P_L - P_M)^2 &= P_L - P_L P_M - P_M P_L + P_M = \\ &= \begin{pmatrix} I - U_1 U_1^* & 0 \\ 0 & U_{21} U_{21}^* \end{pmatrix} \end{aligned}$$

and $\|I - U_1 U_1^*\| = \|I - U_1^* U_1\| = \|U_{21}^* U_{21}\|$ follows $\Theta(L, M) = \|U_{21}\|$. Similarly we have $\Theta(L^\perp, M^\perp) = \|U_{12}\|$, and $P_L - P_M = (I - P_M) - (I - P_L) = P_{M^\perp} - P_{L^\perp}$ yields $\Theta(L, M) = \Theta(L^\perp, M^\perp)$. □

For the proofs of the subsequent results we refer to [GLR].

LEMMA 3.2. *Let L, M and V be subspaces of $\mathbb{C}^n$.*

(i) *If $\Theta(L, M) < 1$ then $\dim L = \dim M$.*

(ii) *Assume $L + V = \mathbb{C}^n$. If $\Theta(L, M)$ is sufficiently small, then $M + V = \mathbb{C}^n$.*

LEMMA 3.3. *Let $X \in \mathbb{C}^{n \times n}$ be given. Then there exists a constant $\alpha > 0$ such that for all $Y \in \mathbb{C}^{n \times n}$ with*

$$\dim \text{Ker } X = \dim \text{Ker } Y \tag{3.1}$$

we have

$$\Theta(\text{Ker } X, \text{Ker } Y) \leq \alpha \|X - Y\|. \tag{3.2}$$

Let $A \in \mathbb{C}^{n \times n}$ be given. A subspace $M \in \text{Inv} A$ is called isolated if there is an $\epsilon > 0$ such that the only subspace $N \in \text{Inv} A$ satisfying $\Theta(M, N) < \epsilon$ is M itself.

THEOREM 3.4. (1) *A subspace $M \in \text{Inv} A$ is isolated if and only if for each $\lambda \in \sigma(A)$ the primary component $M \cap E_\lambda(A)$ is isolated as an $A|_{E_\lambda(A)}$ -invariant subspace.*
(2) *A subspace $M \in \text{Inv}(A|_{E_\lambda(A)})$ is isolated if and only either $\dim \text{Ker}(A - \lambda I) = 1$ or $\dim \text{Ker}(A - \lambda I) \geq 2$ and $M = 0$ or $M = E_\lambda(A)$.*

4. A HOMEOMORPHISM

In this section it will be shown that the map $\gamma : \mathcal{S} \rightarrow \mathcal{N}$ of Theorem 2.4 and its inverse are continuous. A technical lemma and a theorem on parameter dependence of least solutions will be needed.

LEMMA 4.1. *Let*

$$U = \begin{pmatrix} U_1 & U_{12} \\ U_{21} & U_2 \end{pmatrix}$$

be a unitary $n \times n$ matrix and let

$$X = \begin{pmatrix} 0 & 0 \\ 0 & X_2 \end{pmatrix}, \quad X_2 < 0,$$

and

$$Y = \begin{pmatrix} Y_1 & Y_{12} \\ Y_{12}^* & Y_2 \end{pmatrix} \leq 0, \tag{4.1}$$

be hermitian $n \times n$ matrices which are partitioned conformingly. Assume

$$\text{Ker } Y = \text{Im} \begin{pmatrix} U_1 \\ U_{21} \end{pmatrix}. \tag{4.2}$$

Then

$$\Theta(\text{Ker } X, \text{Ker } Y) = \|U_{21}\| = \|U_{12}\|. \tag{4.3}$$

Assume furthermore $\|U_{12}\| < \frac{1}{2}$. Then U_2 is nonsingular. Put

$$T = (U_{12}U_2^{-1})^*. \tag{4.4}$$

Then

$$Y = \begin{pmatrix} T^* \\ I \end{pmatrix} Y_2 (T \ I), \quad Y_2 < 0, \tag{4.5}$$

and we have

$$\|U_{12}\| \leq \|T\| \leq \sqrt{2}\|U_{12}\| \quad (4.6)$$

and

$$\|Y - X\| \leq (\|T\|^2 + \|T\|)\|Y_2\| + \|Y_2 - X_2\|. \quad (4.7)$$

PROOF. The identity (4.3) is obvious from Lemma 3.1. Now let μ denote the smallest eigenvalue of $U_2^*U_2$. Then

$$U_{12}^*U_{12} + U_2^*U_2 = I \quad (4.8)$$

implies $\mu = 1 - \|U_{12}\|^2 > \frac{3}{4}$. Hence U_2 is nonsingular and

$$\|U_2^{-1}\|^2 = \frac{1}{\mu} < \frac{4}{3} < 2,$$

which implies $\|T\| \leq \|U_{12}\|\|U_2^{-1}\| \leq \sqrt{2}\|U_{12}\|$. On the other hand we see from (4.8) that $\|U_2\| \leq 1$ and conclude that $\|U_{12}\| \leq \|T\|\|U_2\| \leq \|T\|$. From (4.2) follows $U^*YU = \text{diag}(0, \Lambda)$ for some nonsingular Λ . Hence

$$Y = \begin{pmatrix} U_{12} \\ U_2 \end{pmatrix} \Lambda (U_{12}^* \ U_2^*),$$

and in particular $Y_2 = U_2\Lambda U_2^*$, which yields (4.5). The estimate (4.7) is obtained from

$$Y - X = \text{diag}(T^*Y_2T, 0) + \begin{pmatrix} 0 & T^*Y_2 \\ Y_2T & 0 \end{pmatrix} + \text{diag}(0, Y_2 - X_2).$$

□

THEOREM 4.2 [De],[R]. *Let W be the set of all ordered triples (A, D, Q) of complex $n \times n$ matrices with the following properties:*

(i) $D \geq 0$, $Q = Q^*$, (A, D) is stabilizable.

(ii) There exists a solution $X = X^*$ of

$$A^*X + XA + XDX - Q = 0. \quad (4.9)$$

Then (4.9) has a least solution X_- , and X_- is a continuous function of $(A, D, Q) \in W$.

We adapt the preceding theorem for our purposes. Consider the assumption

(iii) $Q \geq 0$, $(-A, Q)$ is detectable.

It is well known (see e.g. [K]) that (i) together with (iii) implies (ii). In that case the least solution X_- is the unique negative-definite solution of (4.9). Theorem 4.2 remains valid if we replace (ii) by (iii). A proof that X_- depends continuously on the parameters of (4.9) could be given along the lines described in [GL, p. 1465] using the implicit function theorem.

THEOREM 4.3. *The map $\gamma : \mathcal{S} \rightarrow \mathcal{N}$ given by $\gamma(X) = \text{Ker } X$ is a homeomorphism.*

PROOF. It is not difficult to show that the map $X \mapsto \gamma(X) = \text{Ker } X$ is continuous. We fix a solution $X \in \mathcal{S}$. According to Lemma 2.3 we have $E_=(A_X) = E_=(A_Y)$ for all $Y \in \mathcal{S}$. Hence if Y is sufficiently close to X then $\dim E_>(A_X) = \dim E_>(A_Y)$ and (2.18) yields $\dim \text{Ker } X = \dim \text{Ker } Y$. Condition (3.1) of Lemma 3.3 is satisfied and (3.2) implies continuity of γ . In order to prove that the map $\gamma^{-1} : \mathcal{N} \rightarrow \mathcal{S}$ is continuous we fix a subspace $N \in \mathcal{N}$ and choose an orthonormal basis of $\mathbb{C}^n$ such that

$$N = \text{Im} \begin{pmatrix} I \\ 0 \end{pmatrix}. \quad (4.10)$$

Then the solution $X \in \mathcal{S}$ with $\gamma(X) = \text{Ker } X = N$ is of the form $X = \text{diag}(0, X_2)$. Note that $N = \text{Ker } X \in \text{Inv } A$ and $N \subseteq V(A, C)$ imply

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = (0 \ C_2). \quad (4.11)$$

Furthermore $N + R(A, B) + E_<(A) = \mathbb{C}^n$ is equivalent to stabilizability of (A_2, B_2) , whereas $V_<(A, C) \subseteq N$ means that $(-A_2, C_2)$ is detectable. The matrix X_2 is the unique negative-definite (and thus the least) solution of

$$\mathcal{R}_2(X_2) = A_2^* X_2 + X_2 A_2 + X_2 B_2 B_2^* X_2 - C_2^* C_2 = 0.$$

Now let $P \in \mathcal{N}$ and $Y \in \mathcal{S}$ be such that $\Theta = \Theta(P, N) < \frac{1}{2}$ and $\text{Ker } Y = P$. Then Lemma 3.2 implies $\dim \text{Ker } X = \dim \text{Ker } Y$. Hence we can apply Lemma 4.1 and assume that Y and $\text{Ker } Y$ are given by (4.1) and (4.2). In particular we have $\Theta = \|U_{21}\|$. From $\mathcal{R}(Y) = 0$ and (4.11) we obtain

$$\begin{aligned} & (A_{12}^* \ A_2^*) \begin{pmatrix} Y_{21}^* \\ Y_2 \end{pmatrix} + (Y_{21} \ Y_2) \begin{pmatrix} A_{12} \\ A_2 \end{pmatrix} + \\ & + (Y_{21} \ Y_2) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} (B_1^* \ B_2^*) \begin{pmatrix} Y_{21}^* \\ Y_2 \end{pmatrix} - C_2^* C_2 = 0. \end{aligned} \quad (4.12)$$

If $T = (U_{12}U_2^{-1})^*$ is defined as in (4.4) then (4.5) yields

$$(Y_{21} \ Y_2) = Y_2(T \ I)$$

such that (4.12) can be written as

$$\begin{aligned} & (A_2 + TA_{12})^*Y_2 + Y_2(A_2 + TA_{12}) + \\ & + Y_2(B_2B_2^* + TB_2B_1^* + B_1B_2^*T^* + TB_1B_1^*T^*)Y_2 - C_2^*C_2 = 0. \end{aligned}$$

Now set

$$\tilde{A}_2 = A_2 + TA_{12}, \quad \tilde{B}_2\tilde{B}_2^* = B_2B_2^* + (TB_1B_2^* + B_2B_1^*T^* + TB_1B_1^*T^*)$$

such that $Y_2 < 0$ is a solution of

$$\tilde{\mathcal{R}}_2(W_2) = \tilde{A}_2^*W_2 + W_2\tilde{A}_2 + W_2\tilde{B}_2\tilde{B}_2^*W_2 - C_2^*C_2 = 0. \quad (4.13)$$

Recall (4.6) which implies that $\|U_{12}\| \rightarrow 0$ is equivalent to $\|T\| \rightarrow 0$. Hence if $\Theta = \|U_{12}\|$ is sufficiently small then $\tilde{A}_2$ and $\tilde{B}_2\tilde{B}_2^*$ are close to A_2 and $B_2B_2^*$, respectively. Hence Y_2 is the least solution of (4.13) and according to Theorem 4.2 the solution $Y_2 = Y_2(T)$ is a continuous function of T . Therefore

$$\lim_{\|T\| \rightarrow 0} Y_2(T) = X_2.$$

We conclude from (4.7) that $\lim \|\gamma^{-1}(N) - \gamma^{-1}(P)\| = 0$ if $\Theta(N, P) \rightarrow 0$.

□

5. ISOLATED SOLUTIONS

For $N \in \text{Inv}A$ define $N_\lambda = N \cap E_\lambda(A)$ such that $N_\lambda \in \text{Inv}A$ and $N = \oplus\{N_\lambda \mid \lambda \in \sigma(A)\}$. In particular

$$V_\lambda(A, C) = V(A, C) \cap E_\lambda(A).$$

Furthermore define

$$\begin{aligned} \mathcal{N}_\lambda &= \{N \mid N \in \text{Inv}A, N \subseteq V_\lambda(A, C), \\ & N + [R(A, B) \cap E_\lambda(A)] = E_\lambda(A)\}. \end{aligned}$$

Then $\mathcal{N}_\lambda \neq \emptyset$ is equivalent to $V_\lambda(A, C) \in \mathcal{N}_\lambda$. It is not difficult to characterize an element $N \in \mathcal{N}$ by its components N_λ .

LEMMA 5.1 [W2]. *Assume $\mathcal{N} \neq \emptyset$. Then $N \in \text{Inv}A$ is in $\mathcal{N}$ if and only if the subspaces N_λ satisfy*

$$N_\lambda = V_\lambda(A, C) \quad \text{for all } \lambda \in \mathbb{C}_\leq$$

and

$$N_\lambda \in \mathcal{N}_\lambda \quad \text{for all } \lambda \in \mathbb{C}_{>}.$$

For the proof of Theorem 1.1 we have to determine the isolated elements of $\mathcal{N}$.

LEMMA 5.2. *For a subspace $N \in \mathcal{N}$ the following statements are equivalent: (i) N is isolated in $\mathcal{N}$. (ii) N is isolated in $\text{Inv}(A|_{V(A,C)})$. (iii) If $\text{Re } \lambda > 0$ and*

$$\text{rank} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} \leq n - 2 \quad (5.1)$$

then either

$$\text{Ker} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} \cap N = 0 \quad (5.2)$$

or

$$V(A, C) \cap E_\lambda(A) \subseteq N. \quad (5.3)$$

PROOF. (i) $\Rightarrow$ (ii). Put $\tilde{A} = A|_{V(A,C)}$ and $\tilde{A}_\lambda = A|_{V_\lambda(A,C)}$. Suppose N is not isolated in $\text{Inv} \tilde{A}$. Consider the decomposition $N = \oplus N_\lambda$. According to Theorem 3.4 there exists an $\alpha \in \sigma(A)$ such that N_α is not isolated in $\text{Inv} \tilde{A}_\alpha$. We have $\alpha \in \mathbb{C}_{>}$ since $N_\lambda = V_\lambda(A, C)$ if $\lambda \in \mathbb{C}_{\leq}$. Now replace the component N_α in N by a subspace $M_\alpha \in \text{Inv} \tilde{A}_\alpha$, $M_\alpha \neq N_\alpha$, and set

$$M = \oplus \{N_\lambda, \lambda \neq \alpha\} \oplus M_\alpha.$$

The proof of Theorem 3.4 (1) in [GLR, p. 429] shows that for M_α sufficiently close to N_α we have an estimate

$$\Theta(N, M) \leq \kappa \Theta(N_\alpha, M_\alpha) \quad (5.4)$$

where κ is independent of M_α . Recall that $N_\alpha \in \mathcal{N}_\alpha$ has the property

$$N_\alpha + [R(A, B) \cap E_\alpha(A)] = E_\alpha(A).$$

Then Lemma 3.2 implies

$$M_\alpha + [R(A, B) \cap E_\alpha(A)] = E_\alpha(A),$$

which yields $M_\alpha \in \mathcal{N}_\alpha$ and $M \in \mathcal{N}$. From (5.4) we conclude that we can find an $M \in \mathcal{N}$, $M \neq N$, arbitrarily close to N . Hence N is not isolated in $\mathcal{N}$. Because of $\mathcal{N} \subseteq \text{Inv} \tilde{A}$ the implication (ii) $\Rightarrow$ (i) is obvious.

(ii) $\Leftrightarrow$ (iii). Lemma 5.1 and Theorem 3.4 imply that (ii) holds if and only if for all $\lambda \in \mathbb{C}_>$ the subspace N_λ is isolated in $\text{Inv} \tilde{A}_\lambda$. Because of

$$\dim \text{Ker}(\tilde{A}_\lambda - \lambda I) = \dim \text{Ker} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

only those eigenvalues λ have to be considered for which (5.1) holds. If $\dim \text{Ker}(\tilde{A}_\lambda - \lambda I) \geq 2$ then according to Theorem 3.4 the subspace N_λ is isolated in $\text{Inv} \tilde{A}_\lambda$ if and only if $N_\lambda = 0$ or $N_\lambda = V_\lambda(A, C)$. The first case is equivalent to (5.2) the second one to (5.3). $\square$

PROOF OF THEOREM 1.1. Assume $\mathcal{T} \neq \emptyset$. If the controllability condition (2.9) is not satisfied then according to Lemma 2.2 the subspace U_r given by (2.3) is not the whole $\mathbb{C}^n$. Let $X \leq 0$ be a solution of (1.1) and let S be as in Theorem 2.1 such that $X = (S^{-1})^* \text{diag}(X_0, X_r) S^{-1}$ and

$$\mathcal{L}_0(X_0) = A_0^* X_0 + X_0 A_0 = 0,$$

and $\sigma(A_0) \subseteq \mathbb{C}_=$. Put $Y_0 = X_0(1 + \epsilon)$. Then $\mathcal{L}_0(Y_0) = 0$. If ϵ is small then $Y = (S^{-1})^* \text{diag}(Y_0, X_r) S^{-1}$ is a solution of (1.1) which is close to X . Hence X is not isolated in $\mathcal{T}$.

Now assume $\mathcal{T} \neq \emptyset$ and (2.9). Then $\mathcal{T} = \mathcal{S}$ and $\mathcal{N}$ are homeomorphic. Hence a solution X is isolated in $\mathcal{T}$ if and only if $N = \text{Ker} X$ is isolated in $\mathcal{N}$, and we can apply Lemma 5.2.

Note that (2.1) is equivalent to $V(A, C) \in \mathcal{N}$. If $\mathcal{N} \neq \emptyset$ then $N = V(A, C)$ is the greatest element of $\mathcal{N}$ and we have (5.3) for all $\lambda \in \mathbb{C}$. Then $\hat{X} = \gamma^{-1}(V(A, C))$ is the greatest negative-semidefinite solution of (1.1) and if (2.9) holds $\hat{X}$ is isolated in $\mathcal{T}$, which completes the proof. $\square$

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