ISOLATED SEMIDEFINITE SOLUTIONS OF THE CONTINUOUS-TIME ALGEBRAIC RICCATI EQUATION

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The set of all negative-semidefinite solutions of the CARE $A^*X + XA + XBB^*X - C^*C = 0$ is homeomorphic to a well defined set of A-invariant subspaces provided that the purely imaginary eigenvalues of A are controllable. Based on that homeomorphism isolated n.s.d. solutions of the CARE are characterized by properties of their kernels.

1. INTRODUCTION

In this paper we consider the continuous-time algebraic Riccati equation (CARE)

$$\mathcal{R}(X) = A^*X + XA + XBB^*X - C^*C = 0, \tag{1.1}$$

where A, B and C are complex matrices of dimensions $n \times n$, $n \times p$ and $q \times n$ respectively. We focus on the set

$$\mathcal{T} = \{X \mid \mathcal{R}(X) = 0,\ X \leq 0\}$$

of negative-semidefinite solutions and we shall characterize those elements X of $\mathcal{T}$ which are isolated (in the topology which $\mathcal{T}$ inherits as a subset of the normed space $\mathbb{C}^{n\times n}$). Such an isolated X has the property that for a sufficiently small ϵ there is no solution $Y \in \mathcal{T}$, $Y \neq X$, with $\|X - Y\| < \epsilon$.

For $\lambda \in \sigma(A)$ let

$$E_\lambda(A) = \operatorname{Ker}(A - \lambda I)^n$$

denote the corresponding generalized eigenspace and let

$$V(A, C) = \operatorname{Ker} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

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be the unobservable subspace. It is the purpose of this paper to prove the following result.

THEOREM 1.1. *Assume that (1.1) has a solution* $X \leq 0$. *Then* $\mathcal{T}$ *has an isolated element if and only if all purely imaginary eigenvalues of* A *are* B*-controllable. A solution* $X \in \mathcal{T}$ *is isolated in* $\mathcal{T}$ *if and only if its kernel has the following property: If* $\operatorname{Re} \lambda > 0$ *and*

$$\operatorname{rank} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} \leq n - 2$$

then either

$$\operatorname{Ker} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} \cap \operatorname{Ker} X = 0$$

or

$$V(A, C) \cap E_\lambda(A) \subseteq \operatorname{Ker} X.$$

Isolated solutions of quadratic matrix equations were studied for the first time by J. Daughtry [Da]. We note the corresponding result for the ARE

$$A^*X + XA + XBB^*X - Q = 0, \tag{1.2}$$

where $Q = Q^*$.

THEOREM 1.2 [RR]. *Assume that* (A, B) *is controllable. Then* X *is isolated in the set of hermitian solutions of (1.2) if and only if each common eigenvalue of* $A + BB^*X$ *and* $-(A + BB^*X)^*$ *which is not purely imaginary is an eigenvalue of*

$$H = \begin{pmatrix} A & BB^* \\ Q & -A^* \end{pmatrix}$$

of geometric multiplicity one.

Our study is based on results of [W1] which will be reviewed in Section 2. If A has no uncontrollable modes on the imaginary axis then – according to [W1] – there is a bijection between $\mathcal{T}$ and a well-defined set $\mathcal{N}$ of A-invariant subspaces. Some facts on the gap metric and on isolated invariant subspaces are put together in Section 3. In Section 4 we show that the bijection between $\mathcal{T}$ and $\mathcal{N}$ mentioned above is a homeomorphism. We give a proof of Theorem 1.1 in Section 5.

The following notation will be used. In the partitions

$$\mathbb{C} = \mathbb{C}_{\leq} \cup \mathbb{C}_{>} = \mathbb{C}_{<} \cup \mathbb{C}_{=} \cup \mathbb{C}_{>} \tag{1.3}$$

the subscripts refer to real parts such that $\mathbb{C}_{\leq} = \{\lambda \mid \lambda \in \mathbb{C}, \mathrm{Re}\ \lambda \leq 0\}$, etc. To (1.3) correspond the decompositions

$$\mathbb{C}^n = E_{\leq}(A) \oplus E_{>}(A) = E_{<}(A) \oplus E_{=}(A) \oplus E_{>}(A)$$

where $E_{\leq}(A) = \oplus\{E_\lambda(A), \lambda \in \mathbb{C}_{\leq}\}$, etc. Let $\mathrm{Inv}A$ denote the lattice of A-invariant subspaces of $\mathbb{C}^n$. To the triple (A, B, C) we associate the controllable subspace

$$R(A, B) = \mathrm{Im}(B, AB, \ldots, A^{n-1}B)$$

and the unobservable subspace $V(A, C)$. It will be convenient to define

$$V_{\leq}(A, C) = V(A, C) \cap E_{\leq}(A)$$

and similarly $V_{=}(A, C)$, $V_{\geq}(A, C)$, etc. With this notation $V_{\geq}(A, C)$ is the undedectable and $R(A, B) + E_{<}(A)$ is the stabilizable subspace.

2. DECOMPOSITION AND PARAMETRIZATION OF SOLUTIONS

The subsequent theorems which describe the structure of the solution set $\mathcal{T}$ are taken from [W1]. According to [G], [GH] there exists a solution $X \leq 0$ of (1.1) if and only if

$$V(A, C) + R(A, B) + E_{<}(A) = \mathbb{C}^n. \tag{2.1}$$

Because of $V_{<}(A, C) \subseteq E_{<}(A)$ the preceding condition (2.1) can be written as

$$V_{=}(A, C) + [V_{>}(A, C) + R(A, B) + E_{<}(A)] = \mathbb{C}^n. \tag{2.2}$$

Put

$$U_r = V_{>}(A, C) + R(A, B) + E_{<}(A). \tag{2.3}$$

Then (2.2) is equivalent to

$$\mathbb{C}^n = U_0 \oplus U_r \tag{2.4}$$

for some subspace $U_0 \subseteq V_{=}(A, C)$. We call (2.4) an LR-decomposition with Riccati part U_r and a Lyapunov complement U_0. If $\mathbb{C}^n$ admits a decomposition (2.4) with nontrivial summands then (1.1) breaks up into a Lyapunov matrix equation and an irreducible Riccati equation.

THEOREM 2.1 [W1]. *(1) Let $\mathbb{C}^n = U_0 \oplus U_r$ be an LR-decomposition. If $S = (S_0, S_r)$ is nonsingular such that* $\mathrm{Im}\ S_0 = U_0$, $\mathrm{Im}\ S_r = U_r$, $\dim U_r = n_r$, *then*

$$S^{-1}AS \begin{pmatrix} A_0 & 0 \\ A_{r0} & A_r \end{pmatrix}, \quad S^{-1}B = \begin{pmatrix} 0 \\ B_r \end{pmatrix}, \quad CS = (0, C_r) \tag{2.5}$$

and

$$\sigma(A_0) \subseteq \mathbb{C}_= \tag{2.6}$$

and

$$V_>(A_r, C_r) + R(A_r, B_r) + E_<(A_r) = \mathbb{C}^{n_r}. \tag{2.7}$$

Assume (2.5) – (2.7). Then we have $X \in \mathcal{T}$ if and only if

$$X = (S^{-1})^* \begin{pmatrix} X_0 & 0 \\ 0 & X_r \end{pmatrix} S^{-1} \tag{2.8}$$

and $X_0 \leq 0$ satisfies the Lyapunov equation

$$\mathcal{L}_0(X_0) = A_0^* X_0 + X_0 A_0 = 0$$

and X_r is a solution of

$$\mathcal{R}_r(X_r) = A_r^* X_r + X_r A_r + X_r B_r B_r^* X_r - C_r^* C_r = 0.$$

(2) Let $\Pi : \mathbb{C}^n \to U_r$ be the projection on U_r along U_0. Put $\rho(X) = X\Pi$. Then $\rho(X) \in \mathcal{T}$ and $(I - \Pi)^* X \Pi = 0$. If X is given as in (2.8) then

$$\rho(X) = (S^{-1})^* \begin{pmatrix} 0 & 0 \\ 0 & X_r \end{pmatrix} S^{-1}.$$

Put $\mathcal{S} = \rho(\mathcal{T})$ such that $\mathcal{S}$ contains all solutions of the form (2.8) with Lyapunov part $X_0 = 0$. Whether a Lyapunov complement U_0 does appear or not depends on purely imaginary eigenvalues of A.

LEMMA 2.2 [W1]. *Assume $\mathcal{T} \neq \emptyset$. Then we have $\mathcal{S} = \mathcal{T}$ if and only if*

$$\operatorname{rank}(A - \lambda I, B) = n \quad \text{for all } \lambda \in \mathbb{C}_=. \tag{2.9}$$

In the case where (A, B) is stabilizable the conditions (2.1) and (2.9) are satisfied and we have $\mathcal{T} \neq \emptyset$ and $\mathcal{T} = \mathcal{S}$. The following observation will be useful.

LEMMA 2.3. *For $X \in \mathcal{T}$ we have*

$$E_=(A + BB^*X) = V_=(A, C).$$

PROOF. Put

$$A_X = A + BB^*X.$$

Then $\mathcal{R}(X) = 0$ is equivalent to

$$A_X^*X + XA_X = XB^*BX + C^*C. \tag{2.10}$$

We recall the well known fact that a Lyapunov equation

$$F^*Y + YF = D^*D$$

where $\sigma(F) \subseteq \mathbb{C}_=$ and Y is semidefinite implies $D = 0$. Assume

$$E_=(A_X) = \operatorname{Im} \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Then

$$A_X = \begin{pmatrix} \hat{A}_1 & \hat{A}_{12} \\ 0 & \hat{A}_2 \end{pmatrix} \tag{2.11}$$

and $\sigma(\hat{A}_1) \subseteq \mathbb{C}_=$. Let

$$X = \begin{pmatrix} X_1 & X_{21}^* \\ X_{21} & X_2 \end{pmatrix} \tag{2.12}$$

and $C = (C_1 \;\; C_2)$ be partitioned accordingly. Then (2.10) yields

$$\hat{A}_1^*X_1 + X_1\hat{A}_1 = (X_1 \;\; X_{21}^*)BB^* \begin{pmatrix} X_1 \\ X_{21} \end{pmatrix} + C_1^*C_1.$$

Hence we obtain

$$BB^* \begin{pmatrix} X_1 \\ X_{21} \end{pmatrix} = 0 \tag{2.13}$$

and $C_1 = 0$ and

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \tag{2.14}$$

with $A_1 = \hat{A}_1$. Therefore we have $E_=(A_X) \subseteq V_=(A, C)$. To prove the converse inclusion

$$V_=(A, C) \subseteq E_=(A_X) \tag{2.15}$$

assume now that a basis has been chosen such that A and C are in the form (2.14) and $C = (0, C_2)$. By a slight abuse of notation take X as in (2.12). Then

$$A_1^* X_1 + X_1 A_1 = -(X_1 \;\; X_{21}^*) BB^* \begin{pmatrix} X_1 \\ X_{21} \end{pmatrix}. \tag{2.16}$$

As before we can regard (2.16) as a Lyapunov equation and conclude that (2.13) holds. From (2.14) we obtain A_X in the form (2.11) with $\hat{A}_1 = A_1$ which yields (2.15).

□

There is an order isomorphism between $\mathcal{S}$ and the following system $\mathcal{N}$ of A-invariant subspaces of $\mathbb{C}^n$. Define

$$\begin{aligned} \mathcal{N} \;\; = \;\; & \{N \mid N \in \mathrm{Inv} A, V_{\leq}(A, C) \subseteq N \subseteq V(A, C), N + \\ & \qquad + R(A, B) + E_{<}(A) = \mathbb{C}^n\}. \end{aligned} \tag{2.17}$$

THEOREM 2.4 [W1]. *(1) The map* $\gamma : \mathcal{S} \to \mathcal{N}$ *given by* $\gamma(X) = \mathrm{Ker}\, X$ *is a bijection, and both* γ *and* γ^{-1} *are order preserving, i.e. for* $X, Y \in \mathcal{S}$ *and* $M, N \in \mathcal{N}$ *the relations* $X \leq Y$ *and* $M \subseteq N$ *imply* $\gamma(X) \subseteq \gamma(Y)$ *and* $\gamma^{-1}(M) \leq \gamma^{-1}(N)$.
(2) For $X \in \mathcal{S}$ *we have*

$$\mathrm{Ker}\, X = V_{\leq}(A, C) \oplus E_{>}(A_X). \tag{2.18}$$

3. THE GAP METRIC AND ISOLATED INVARIANT SUBSPACES

For $k \in \mathbb{N}$ let $\mathbb{C}^k$ be endowed with the usual scalar product $(x, y) = x^* y$ and with the corresponding norm $\|x\| = (x^* x)^{1/2}$. The norm of a matrix $G \in \mathbb{C}^{n \times m}$ (or of a linear map $G : \mathbb{C}^m \to \mathbb{C}^n$) is defined accordingly as operator norm, $\|G\| = \max\{\|Gx\|, x \in \mathbb{C}^m, \|x\| = 1\}$. Let P_M denote the orthogonal projection of $\mathbb{C}^n$ onto a subspace M of $\mathbb{C}^n$. The gap $\Theta(L, M)$ between two subspaces L and M of $\mathbb{C}^n$ is defined as

$$\Theta(L, M) = \|P_L - P_M\|.$$

It is a metric in the set of all subspaces of $\mathbb{C}^n$. The following lemma can be used to compute the gap.

LEMMA 3.1. *Let* L *and* M *be subspaces of* $\mathbb{C}^n$ *such that* $1 \leq \dim L = \dim M = r < n$. *Assume*

$$L = \mathrm{Im} \begin{pmatrix} I_r \\ 0 \end{pmatrix}$$

and let

$$U = \begin{pmatrix} U_1 & U_{12} \\ U_{21} & U_2 \end{pmatrix}$$

be unitary such that

$$M = \operatorname{Im} \begin{pmatrix} U_1 \\ U_{21} \end{pmatrix} \quad \textit{and} \quad M^{\perp} = \operatorname{Im} \begin{pmatrix} U_{12} \\ U_2 \end{pmatrix}.$$

Then

$$\Theta(L, M) = \Theta(L^{\perp}, M^{\perp}) = \|U_{21}\| = \|U_{12}\|.$$

PROOF. We have

$$P_L = \begin{pmatrix} I \\ 0 \end{pmatrix} (I, 0) \quad \text{and} \quad P_M = \begin{pmatrix} U_1 \\ U_{21} \end{pmatrix} (U_1^*, U_{21}^*).$$

From

$$\begin{aligned} (P_L - P_M)^2 &= P_L - P_L P_M - P_M P_L + P_M = \\ &= \begin{pmatrix} I - U_1 U_1^* & 0 \\ 0 & U_{21} U_{21}^* \end{pmatrix} \end{aligned}$$

and $\|I - U_1 U_1^*\| = \|I - U_1^* U_1\| = \|U_{21}^* U_{21}\|$ follows $\Theta(L, M) = \|U_{21}\|$. Similarly we have $\Theta(L^{\perp}, M^{\perp}) = \|U_{12}\|$, and $P_L - P_M = (I - P_M) - (I - P_L) = P_{M^{\perp}} - P_{L^T}$ yields $\Theta(L, M) = \Theta(L^{\perp}, M^{\perp})$.

□

For the proofs of the subsequent results we refer to [GLR].

LEMMA 3.2. *Let L, M and V be subspaces of $\mathbb{C}^n$.*

(i) If $\Theta(L, M) < 1$ then $\dim L = \dim M$.

(ii) Assume $L + V = \mathbb{C}^n$. If $\Theta(L, M)$ is sufficiently small, then $M + V = \mathbb{C}^n$.

LEMMA 3.3. *Let $X \in \mathbb{C}^{n \times n}$ be given. Then there exists a constant $\alpha > 0$ such that for all $Y \in \mathbb{C}^{n \times n}$ with*

$$\dim \operatorname{Ker} X = \dim \operatorname{Ker} Y \tag{3.1}$$

we have

$$\Theta(\operatorname{Ker} X, \operatorname{Ker} Y) \le \alpha \|X - Y\|. \tag{3.2}$$

Let $A \in \mathbb{C}^{n\times n}$ be given. A subspace $M \in \mathrm{Inv}A$ is called isolated if there is an $\epsilon > 0$ such that the only subspace $N \in \mathrm{Inv}A$ satisfying $\Theta(M,N) < \epsilon$ is M itself.

THEOREM 3.4. *(1) A subspace $M \in \mathrm{Inv}A$ is isolated if and only if for each $\lambda \in \sigma(A)$ the primary component $M \cap E_\lambda(A)$ is isolated as an $A|_{E_\lambda(A)}$-invariant subspace.*
(2) A subspace $M \in \mathrm{Inv}(A|_{E_\lambda(A)})$ is isolated if and only either $\dim\mathrm{Ker}(A-\lambda I) = 1$ *or* $\dim\mathrm{Ker}(A-\lambda I) \geq 2$ *and $M = 0$ or $M = E_\lambda(A)$.*

4. A HOMEOMORPHISM

In this section it will be shown that the map $\gamma : \mathcal{S} \to \mathcal{N}$ of Theorem 2.4 and its inverse are continuous. A technical lemma and a theorem on parameter dependence of least solutions will be needed.

LEMMA 4.1. *Let*

$$U = \begin{pmatrix} U_1 & U_{12} \\ U_{21} & U_2 \end{pmatrix}$$

be a unitary $n \times n$ matrix and let

$$X = \begin{pmatrix} 0 & 0 \\ 0 & X_2 \end{pmatrix}, \quad X_2 < 0,$$

and

$$Y = \begin{pmatrix} Y_1 & Y_{12} \\ Y_{12}^* & Y_2 \end{pmatrix} \leq 0, \tag{4.1}$$

be hermitian $n \times n$ matrices which are partitioned conformingly. Assume

$$\mathrm{Ker}\, Y = \mathrm{Im} \begin{pmatrix} U_1 \\ U_{21} \end{pmatrix}. \tag{4.2}$$

Then

$$\Theta(\mathrm{Ker}\, X, \mathrm{Ker}\, Y) = \|U_{21}\| = \|U_{12}\|. \tag{4.3}$$

Assume furthermore $\|U_{12}\| < \frac{1}{2}$. Then U_2 is nonsingular. Put

$$T = (U_{12}U_2^{-1})^*. \tag{4.4}$$

Then

$$Y = \begin{pmatrix} T^* \\ I \end{pmatrix} Y_2 (T \ I), \quad Y_2 < 0, \tag{4.5}$$

and we have

$$\|U_{12}\| \le \|T\| \le \sqrt{2}\|U_{12}\| \tag{4.6}$$

and

$$\|Y - X\| \le (\|T\|^2 + \|T\|)\|Y_2\| + \|Y_2 - X_2\|. \tag{4.7}$$

PROOF. The identity (4.3) is obvious from Lemma 3.1. Now let μ denote the smallest eigenvalue of $U_2^* U_2$. Then

$$U_{12}^* U_{12} + U_2^* U_2 = I \tag{4.8}$$

implies $\mu = 1 - \|U_{12}\|^2 > \frac{3}{4}$. Hence U_2 is nonsingular and

$$\|U_2^{-1}\|^2 = \frac{1}{\mu} < \frac{4}{3} < 2,$$

which implies $\|T\| \le \|U_{12}\|\|U_2^{-1}\| \le \sqrt{2}\|U_{12}\|$. On the other hand we see from (4.8) that $\|U_2\| \le 1$ and conclude that $\|U_{12}\| \le \|T\|\|U_2\| \le \|T\|$. From (4.2) follows $U^*YU = \mathrm{diag}(0, \Lambda)$ for some nonsingular Λ. Hence

$$Y = \begin{pmatrix} U_{12} \\ U_2 \end{pmatrix} \Lambda (U_{12}^* \;\; U_2^*),$$

and in particular $Y_2 = U_2 \Lambda U_2^*$, which yields (4.5). The estimate (4.7) is obtained from

$$Y - X = \mathrm{diag}(T^* Y_2 T, 0) + \begin{pmatrix} 0 & T^* Y_2 \\ Y_2 T & 0 \end{pmatrix} + \mathrm{diag}(0, Y_2 - X_2).$$

□

THEOREM 4.2 [De],[R]. *Let W be the set of all ordered triples (A, D, Q) of complex $n \times n$ matrices with the following properties:*

(i) $D \ge 0$, $Q = Q^*$, (A, D) is stabilizable.

(ii) There exists a solution $X = X^*$ of

$$A^* X + XA + XDX - Q = 0. \tag{4.9}$$

Then (4.9) has a least solution X_-, and X_- is a continuous function of $(A, D, Q) \in W$.

We adapt the preceding theorem for our purposes. Consider the assumption *(iii)* $Q \geq 0$, $(-A, Q)$ *is detectable.*

It is well known (see e.g. [K]) that (i) together with (iii) implies (ii). In that case the least solution X_- is the unique negative-definite solution of (4.9). Theorem 4.2 remains valid if we replace (ii) by (iii). A proof that X_- depends continuously on the parameters of (4.9) could be given along the lines described in [GL, p. 1465] using the implicit function theorem.

THEOREM 4.3. *The map* $\gamma : \mathcal{S} \to \mathcal{N}$ *given by* $\gamma(X) = \operatorname{Ker} X$ *is a homeomorphism.*

PROOF. It is not difficult to show that the map $X \mapsto \gamma(X) = \operatorname{Ker} X$ is continuous. We fix a solution $X \in \mathcal{S}$. According to Lemma 2.3 we have $E_=(A_X) = E_=(A_Y)$ for all $Y \in \mathcal{S}$. Hence if Y is sufficiently close to X then $\dim E_>(A_X) = \dim E_>(A_Y)$ and (2.18) yields $\dim \operatorname{Ker} X = \dim \operatorname{Ker} Y$. Condition (3.1) of Lemma 3.3 is satisfied and (3.2) implies continuity of γ. In order to prove that the map $\gamma^{-1} : \mathcal{N} \to \mathcal{S}$ is continuous we fix a subspace $N \in \mathcal{N}$ and choose an orthonormal basis of $\mathbb{C}^n$ such that

$$N = \operatorname{Im} \begin{pmatrix} I \\ 0 \end{pmatrix}. \tag{4.10}$$

Then the solution $X \in \mathcal{S}$ with $\gamma(X) = \operatorname{Ker} X = N$ is of the form $X = \operatorname{diag}(0, X_2)$. Note that $N = \operatorname{Ker} X \in \operatorname{Inv} A$ and $N \subseteq V(A, C)$ imply

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = (0 \;\; C_2). \tag{4.11}$$

Furthermore $N + R(A, B) + E_<(A) = \mathbb{C}^n$ is equivalent to stabilizability of (A_2, B_2), whereas $V_\leq(A, C) \subseteq N$ means that $(-A_2, C_2)$ is detectable. The matrix X_2 is the unique negative-definite (and thus the least) solution of

$$\mathcal{R}_2(X_2) = A_2^* X_2 + X_2 A_2 + X_2 B_2 B_2^* X_2 - C_2^* C_2 = 0.$$

Now let $P \in \mathcal{N}$ and $Y \in \mathcal{S}$ be such that $\Theta = \Theta(P, N) < \frac{1}{2}$ and $\operatorname{Ker} Y = P$. Then Lemma 3.2 implies $\dim \operatorname{Ker} X = \dim \operatorname{Ker} Y$. Hence we can apply Lemma 4.1 and assume that Y and $\operatorname{Ker} Y$ are given by (4.1) and (4.2). In particular we have $\Theta = \|U_{21}\|$. From $\mathcal{R}(Y) = 0$ and (4.11) we obtain

$$\begin{aligned} &(A_{12}^* \;\; A_2^*) \begin{pmatrix} Y_{21}^* \\ Y_2 \end{pmatrix} + (Y_{21} \;\; Y_2) \begin{pmatrix} A_{12} \\ A_2 \end{pmatrix} + \\ &\quad + (Y_{21} \;\; Y_2) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} (B_1^* \;\; B_2^*) \begin{pmatrix} Y_{21}^* \\ Y_2 \end{pmatrix} - C_2^* C_2 = 0. \end{aligned} \tag{4.12}$$

If $T = (U_{12}U_2^{-1})^*$ is defined as in (4.4) then (4.5) yields

$$(Y_{21} \;\; Y_2) = Y_2(T \;\; I)$$

such that (4.12) can be written as

$$(A_2 + TA_{12})^*Y_2 + Y_2(A_2 + TA_{12}) +$$
$$+Y_2(B_2B_2^* + TB_2B_1^* + B_1B_2^*T^* + TB_1B_1^*T^*)Y_2 - C_2^*C_2 = 0.$$

Now set

$$\tilde{A}_2 = A_2 + TA_{12}, \quad \tilde{B}_2\tilde{B}_2^* = B_2B_2^* + (TB_1B_2^* + B_2B_1^*T^* + TB_1B_1^*T^*)$$

such that $Y_2 < 0$ is a solution of

$$\tilde{\mathcal{R}}_2(W_2) = \tilde{A}_2^*W_2 + W_2\tilde{A}_2 + W_2\tilde{B}_2\tilde{B}_2^*W_2 - C_2^*C_2 = 0. \tag{4.13}$$

Recall (4.6) which implies that $\|U_{12}\| \to 0$ is equivalent to $\|T\| \to 0$. Hence if $\Theta = \|U_{12}\|$ is sufficiently small then $\tilde{A}_2$ and $\tilde{B}_2\tilde{B}_2^*$ are close to A_2 and $B_2B_2^*$, respectively. Hence Y_2 is the least solution of (4.13) and according to Theorem 4.2 the solution $Y_2 = Y_2(T)$ is a continuous function of T. Therefore

$$\lim_{\|T\|\to 0} Y_2(T) = X_2.$$

We conclude from (4.7) that $\lim \|\gamma^{-1}(N) - \gamma^{-1}(P)\| = 0$ if $\Theta(N,P) \to 0$.

□

5. ISOLATED SOLUTIONS

For $N \in \text{Inv}A$ define $N_\lambda = N \cap E_\lambda(A)$ such that $N_\lambda \in \text{Inv}A$ and $N = \oplus\{N_\lambda \mid \lambda \in \sigma(A)\}$. In particular

$$V_\lambda(A,C) = V(A,C) \cap E_\lambda(A).$$

Furthermore define

$$\mathcal{N}_\lambda = \{N \mid N \in \text{Inv}A,\ N \subseteq V_\lambda(A,C),$$
$$N + [R(A,B) \cap E_\lambda(A)] = E_\lambda(A)\}.$$

Then $\mathcal{N}_\lambda \neq \emptyset$ is equivalent to $V_\lambda(A,C) \in \mathcal{N}_\lambda$. It is not difficult to characterize an element $N \in \mathcal{N}$ by its components N_λ.

LEMMA 5.1 [W2]. *Assume $\mathcal{N} \neq \emptyset$. Then $N \in \text{Inv}A$ is in $\mathcal{N}$ if and only if the subspaces N_λ satisfy*

$$N_\lambda = V_\lambda(A,C) \quad \textit{for all } \lambda \in \mathbb{C}_\leq$$

and

$$N_\lambda \in \mathcal{N}_\lambda \quad \textit{for all } \lambda \in \mathbb{C}_>.$$

For the proof of Theorem 1.1 we have to determine the isolated elements of $\mathcal{N}$.

LEMMA 5.2. *For a subspace $N \in \mathcal{N}$ the following statements are equivalent: (i) N is isolated in $\mathcal{N}$. (ii) N is isolated in* $\mathrm{Inv}(A|_{V(A,C)})$. *(iii) If* $\mathrm{Re}\,\lambda > 0$ *and*

$$\mathrm{rank}\begin{pmatrix} A-\lambda I \\ C \end{pmatrix} \leq n-2 \tag{5.1}$$

then either

$$\mathrm{Ker}\begin{pmatrix} A-\lambda I \\ C \end{pmatrix} \cap N = 0 \tag{5.2}$$

or

$$V(A,C) \cap E_\lambda(A) \subseteq N. \tag{5.3}$$

PROOF. (i) $\Rightarrow$ (ii). Put $\tilde{A} = A|_{V(A,C)}$ and $\tilde{A}_\lambda = A|_{V_\lambda(A,C)}$. Suppose N is not isolated in $\mathrm{Inv}\tilde{A}$. Consider the decomposition $N = \oplus N_\lambda$. According to Theorem 3.4 there exists an $\alpha \in \sigma(A)$ such that N_α is not isolated in $\mathrm{Inv}\tilde{A}_\alpha$. We have $\alpha \in \mathbb{C}_>$ since $N_\lambda = V_\lambda(A,C)$ if $\lambda \in \mathbb{C}_\leq$. Now replace the component N_α in N by a subspace $M_\alpha \in \mathrm{Inv}\tilde{A}_\alpha$, $M_\alpha \neq N_\alpha$, and set

$$M = \oplus\{N_\lambda, \lambda \neq \alpha\} \oplus M_\alpha.$$

The proof of Theorem 3.4 (1) in [GLR, p. 429] shows that for M_α sufficiently close to N_α we have an estimate

$$\Theta(N,M) \leq \kappa\Theta(N_\alpha, M_\alpha) \tag{5.4}$$

where κ is independent of M_α. Recall that $N_\alpha \in \mathcal{N}_\alpha$ has the property

$$N_\alpha + [R(A,B) \cap E_\alpha(A)] = E_\alpha(A).$$

Then Lemma 3.2 implies

$$M_\alpha + [R(A,B) \cap E_\alpha(A)] = E_\alpha(A),$$

which yields $M_\alpha \in \mathcal{N}_\alpha$ and $M \in \mathcal{N}$. From (5.4) we conclude that we can find an $M \in \mathcal{N}$, $M \neq N$, arbitrarily close to N. Hence N is not isolated in $\mathcal{N}$. Because of $\mathcal{N} \subseteq \mathrm{Inv}\tilde{A}$ the implication (ii) $\Rightarrow$ (i) is obvious.

(ii) $\Leftrightarrow$ (iii). Lemma 5.1 and Theorem 3.4 imply that (ii) holds if and only if for all $\lambda \in \mathbb{C}_>$ the subspace N_λ is isolated in $\mathrm{Inv}\tilde{A}_\lambda$. Because of

$$\dim \mathrm{Ker}(\tilde{A}_\lambda - \lambda I) = \dim \mathrm{Ker} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

only those eigenvalues λ have to be considered for which (5.1) holds. If $\dim \mathrm{Ker}(\tilde{A}_\lambda - \lambda I) \geq 2$ then according to Theorem 3.4 the subspace N_λ is isolated in $\mathrm{Inv}\tilde{A}_\lambda$ if and only if $N_\lambda = 0$ or $N_\lambda = V_\lambda(A, C)$. The first case is equivalent to (5.2) the second one to (5.3). □

PROOF OF THEOREM 1.1. Assume $\mathcal{T} \neq \emptyset$. If the controllability condition (2.9) is not satisfied then according to Lemma 2.2 the subspace U_r given by (2.3) is not the whole $\mathbb{C}^n$. Let $X \leq 0$ be a solution of (1.1) and let S be as in Theorem 2.1 such that $X = (S^{-1})^* \mathrm{diag}(X_0, X_r) S^{-1}$ and

$$\mathcal{L}_0(X_0) = A_0^* X_0 + X_0 A_0 = 0,$$

and $\sigma(A_0) \subseteq \mathbb{C}_=$. Put $Y_0 = X_0(1 + \epsilon)$. Then $\mathcal{L}_0(Y_0) = 0$. If ϵ is small then $Y = (S^{-1})^* \mathrm{diag}(Y_0, X_r) S^{-1}$ is a solution of (1.1) which is close to X. Hence X is not isolated in $\mathcal{T}$.

Now assume $\mathcal{T} \neq \emptyset$ and (2.9). Then $\mathcal{T} = \mathcal{S}$ and $\mathcal{N}$ are homeomorphic. Hence a solution X is isolated in $\mathcal{T}$ if and only if $N = \mathrm{Ker}\, X$ is isolated in $\mathcal{N}$, and we can apply Lemma 5.2.

Note that (2.1) is equivalent to $V(A, C) \in \mathcal{N}$. If $\mathcal{N} \neq \emptyset$ then $N = V(A, C)$ is the greatest element of $\mathcal{N}$ and we have (5.3) for all $\lambda \in \mathbb{C}$. Then $\hat{X} = \gamma^{-1}(V(A, C))$ is the greatest negative-semidefinite solution of (1.1) and if (2.9) holds $\hat{X}$ is isolated in $\mathcal{T}$, which completes the proof.

□

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