Characteristic invariant subspaces generated by a single vector

Pudji Astutia, Harald K. Wimmerb,*

*aFaculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung 40132, Indonesia
bMathematisches Institut, Universität Würzburg, 97074 Würzburg, Germany

Abstract

If \( f \) is an endomorphism of a finite dimensional vector space \( V \) over a field \( K \) then an invariant subspace \( X \subseteq V \) is called hyperinvariant (respectively, characteristic) if \( X \) is invariant under all endomorphisms (respectively, automorphisms) that commute with \( f \). The characteristic hull of a subset \( W \) of \( V \) is defined to be the smallest characteristic subspace in \( V \) that contains \( W \). It is known that characteristic subspaces that are not hyperinvariant can only exist when \( |K| = 2 \). In this paper we study subspaces \( X \) which are the characteristic hull of a single element. In the case where \( |K| = 2 \) we derive a necessary and sufficient condition such that \( X \) is hyperinvariant.

Keywords: characteristic subspaces, hyperinvariant subspaces, invariant subspaces, elementary divisors, characteristic hull, exponent, height, generators

2000 MSC: 15A18, 47A15, 15A57

Dedicated to Leiba Rodman on the occasion of his 65th birthday

1. Introduction

Let \( V \) be an \( n \)-dimensional vector space over a field \( K \) and let \( f \) be a \( K \)-endomorphism of \( V \). In this paper we consider two types of invariant

*Corresponding author

Email addresses: pudji@math.itb.ac.id (Pudji Astuti),
wimmer@mathematik.uni-wuerzburg.de (Harald K. Wimmer)

1The work of the first author was supported by the program “Riset dan Inovasi KK ITB” of the Institut Teknologi Bandung.
subspaces. A subspace \( X \subseteq V \) is said to be \textit{hyperinvariant} (under \( f \)) if it remains invariant under all endomorphisms of \( V \) that commute with \( f \) (see e.g. [12, p. 305]). If \( X \) is an \( f \)-invariant subspace of \( V \) and if \( X \) is invariant under all automorphisms of \( V \) that commute with \( f \), then [2] we say that \( X \) is \textit{characteristic} (with respect to \( f \)). Let \( \text{Inv}(V, f) \), \( \text{Hinv}(V, f) \), and \( \text{Chinv}(V, f) \) be sets of invariant, hyperinvariant and characteristic subspaces of \( V \), respectively. These sets are lattices (with respect to set inclusion), and

\[
\text{Hinv}(V, f) \subseteq \text{Chinv}(V, f) \subseteq \text{Inv}(V, f).
\]

Thus, if \( W \) is a subset of \( V \) then there is a smallest characteristic subspace that contains \( W \). We denote it by \( \langle W \rangle^c \) and call it the \textit{characteristic hull} of \( W \). The structure of the lattice \( \text{Hinv}(V, f) \) is well understood ([14], [9], [15], [12, p. 306]). We point out that \( \text{Hinv}(V, f) \) is the sublattice of \( \text{Inv}(V, f) \) generated by \( \text{Ker}f^k, \text{Im}f^k, k = 0, 1, \ldots, n. \)

If the characteristic polynomial of \( f \) splits over \( K \) (such that all eigenvalues of \( f \) are in \( K \)) then one can restrict the study of hyperinvariant and of characteristic subspaces to the case where \( f \) has only one eigenvalue, and therefore to the case where \( f \) is nilpotent. Thus, throughout this paper we shall assume \( f^n = 0 \). Let \( \Sigma(\lambda) = \text{diag}(1, \ldots, 1, \lambda^{t_1}, \ldots, \lambda^{t_m}) \in K^{n \times n}[\lambda] \) be the Smith normal form of \( f \) such that \( t_1 + \cdots + t_m = n \). We say that an elementary divisor \( \lambda^r \) is \textit{unrepeated} if it appears exactly once in \( \Sigma(\lambda) \). Otherwise \( \lambda^r \) is said to be \textit{repeated}. Thus \( \lambda^r \) is unrepeated (see e.g. [13, p. 27] ) if and only if \( \dim(V[f] \cap f^{-1}V/V[f] \cap f^rV) = 1 \). We call a vector \( u \in V \) a \textit{generator} if the \( f \)-cyclic subspace generated by \( u \) has an \( f \)-invariant complement in \( V \).

It is known ([16], [13, p. 63/64], [2]) that all characteristic subspaces are hyperinvariant if the field \( K \) has more than two elements. Hence only if \( K = GF(2) \) one may find \( K \)-endomorphisms \( f \) of \( V \) with a characteristic subspace that is not hyperinvariant. According to Shoda [16, p. 619] such subspaces exist if and only if \( f \) has two unrepeated elementary divisors \( \lambda^r \) and \( \lambda^s \) such that \( r \) and \( s \) are non-consecutive natural numbers (see also [5, Theorem 9, p. 510] and [13, p. 63/64]). In this paper we study subspaces \( X = \langle z \rangle^c \), which are the characteristic hull of a single element \( z \), and in the case where \( |K| = 2 \) we derive a necessary and sufficient condition such that \( X \) is hyperinvariant.
We now describe the content of our paper. Having introduced the required definitions we state an essential part of our main result in Section 1.1. Basic facts on height and exponent are contained in Section 1.2. Our study relies on results of generators in Section 2.1, properties of hyperinvariant subspaces in Section 2.2 and on Baer’s decomposition lemma in Section 2.3. Auxiliary results in Section 3 prepare the proof of Theorem 3.5 (the main theorem of our paper). In Section 3.1 we restrict the map $f$ to $X = \langle z \rangle^c$ and determine the elementary divisors of $f|_X$.

We remark that Shoda [16] deals with abelian groups. But it is known (see e.g. [7]) that in many instances methods or concepts of abelian group theory can be applied to linear algebra if they are translated to modules over principal ideal domains and then specialized to $K[\lambda]$-modules. Conversely, results of linear algebra can be interpreted in the framework of abelian group theory. Thus our results prevail in the setting of finite abelian groups. The language would change, and proofs would carry over almost verbatim to finite abelian $p$-groups [11] with subgroups that are fully invariant (in place of hyperinvariant subspaces) and characteristic subgroups (in place of characteristic subspaces).

1.1. Definitions and notation

We first recall the concepts of exponent and height (see e.g. [11], [13]). Let $\iota$ be the identity automorphism of $V$. We set $f^0 = \iota$. If $x \in V$ then the smallest nonnegative integer $\ell$ with $f^\ell x = 0$ is called the exponent of $x$. We write $e(x) = \ell$. A nonzero vector $x$ is said to have height $s$ if $x \in f^s V$ and $x \not\in f^{s+1} V$. In this case we write $h(x) = s$. We set $h(0) = \infty$. The $n$-tuple

$$H(x) = (h(x), h(fx), \ldots, h(f^{n-1}x))$$

is the indicator [11, p.3] or Ulm sequence [13] of $x$. Thus, if $e(x) = k$ then $H(x) = (h(x), \ldots, h(f^{k-1}x), \infty, \ldots, \infty)$. We say that $H(x)$ has a gap at $r$, if $1 \leq r < e(x)$ and

$$h(f^r x) > 1 + h(f^{r-1}x).$$

We set $V[f^\nu] = \text{Ker } f^\nu$, $\nu \geq 0$. Let $\text{End}(V,f)$ be the algebra of all endomorphisms of $V$ that commute with $f$. The group of automorphisms of $V$ that commute with $f$ will be denoted by $\text{Aut}(V,f)$. If $\alpha \in \text{Aut}(V,f)$ and $U = (u_j)_{j=1}^m \in \mathcal{U}$ then $\alpha$ is determined by the images $\alpha u_j$. We write $x \sim y$ if
$y = \alpha x$ for some $\alpha \in \text{Aut}(V, f)$ and we set $[x] = \{\alpha x; \alpha \in \text{Aut}(V, f)\}$. Let

$$\langle x \rangle = \text{span}\{f^i x, i \geq 0\} = \{c_0 x + c_1 f x + \cdots + c_{n-1} f^{n-1} x; c_i \in K, i = 0, 1, \ldots, n-1\}$$

be the $f$-cyclic subspace generated by $x$. If $B \subseteq V$ we define $\langle B \rangle = \sum_{b \in B} \langle b \rangle$ and $B^c = \langle \alpha b; b \in B, \alpha \in \text{Aut}(V, f) \rangle$.

We call $B^c$ the characteristic hull of $B$. Clearly, if $\alpha \in \text{Aut}(V, f)$ then $\alpha(f^i x) = f^i(\alpha x)$ for all $x \in V$. Hence it is obvious that $e(x) = e(y)$ and $h(x) = h(y)$ if $x \sim y$.

Suppose $\dim \ker f = m$, and let $\lambda^{t_1}, \ldots, \lambda^{t_m}$ be the elementary divisors of $f$ such that $t_1 + \cdots + t_m = \dim V$. Then $V$ can be decomposed into a direct sum of $f$-cyclic subspaces $\langle u_j \rangle$ such that

$$V = \langle u_1 \rangle \oplus \cdots \oplus \langle u_m \rangle \quad \text{and} \quad e(u_j) = t_j, \, j = 1, \ldots, m. \quad (1.1)$$

Let the projections $\pi_j : V \to V, \, j = 1, \ldots, m$, be defined by

$$\text{Im } \pi_j = \langle u_j \rangle \quad \text{and} \quad \ker \pi_j = \langle u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_m \rangle.$$

Note that $\pi_j \in \text{End}(V, f)$. We call $U = (u_1, \ldots, u_m)$ a generator tuple of $V$ (with respect to $f$), if (1.1) holds and if the entries $u_j$ of $U$ are ordered by nondecreasing exponents such that

$$t_1 = e(u_1) \leq \cdots \leq t_m = e(u_m).$$

The tuple $(t_m, \ldots, t_1)$ - with exponents in reversed order - is known as Segre characteristic of $f$. The set of generator tuples of $V$ will be denoted by $\mathcal{U}$.

In accordance with [10, p.4] we call $u \in V$ a generator of $V$ if $u \in U$ for some $U \in \mathcal{U}$. In other words, $u \in V$ is a generator if and only if $u \neq 0$ and

$$V = \langle u \rangle \oplus V_2 \quad \text{for some} \quad V_2 \in \text{Inv}(V, f). \quad (1.2)$$

By definition an elementary divisor $\lambda^{t_r}$ is unrepeated if $t_j \neq t_r$ for $j \neq r$. Similarly, a generator $u_r \in U = (u_1, \ldots, u_m)$ will be called unrepeated if $e(u_j) \neq e(u_r)$ for $j \neq r$. We now give a preview of our main result.
Theorem 1.1. For a given nonzero $z \in V$ there exists a generator tuple $U = (u_1, \ldots, u_m)$ such that $z$ can be represented in the form

$$z = f^{\mu_{p_1}}u_{p_1} + \cdots + f^{\mu_{p_k}}u_{p_k},$$

with $0 \leq \mu_{p_1} < \cdots < \mu_{p_k}$ and $0 < t_{p_1} - \mu_{p_1} < \cdots < t_{p_k} - \mu_{p_k}$. The following statements are equivalent.

(i) The subspace $X = \langle z \rangle^c$ is not hyperinvariant.

(ii) At least two of the generators $u_{p_i}$ in (1.3) are unrepeated.

The extended version of the preceding theorem will be proved in Section 3.

1.2. Height and exponent, an example

We illustrate previously defined concepts in a simple example.

Example 1.2. [13, p. 63/64] Let $|K| = 2$ and $\dim V = 4$, and let $f$ be a nilpotent endomorphism of $V$ with $\dim \ker f = 2$, $\dim \ker f^2 = 3$. Then

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is the Jordan form of $f$. Let $U = (u_1, u_2) \in U$ be such that $V = \langle u_1 \rangle \oplus \langle u_2 \rangle$ and $e(u_1) = 1$, $e(u_2) = 3$. Set $z = u_1 + fu_2$. Then $h(z) = 0$, and $h(fz) = 2 > 1 + h(z)$. Hence $H(z) = (0, 2, \infty, \infty)$ and $z$ has a gap at $r = 1$. With regard to the inequality (1.4) below we note

$$1 = h(z + u_1) > \min\{h(z), h(u_1)\} = 0.$$

Set $x = fu_2$ and $y = fu_2 + f^2u_2$. Then $y \in \langle x \rangle$ and $h(x) = h(y) = 1$, and $\langle x \rangle = \langle y \rangle = \{0, fu_2, f^2u_2, fu_2 + f^2u_2\}$. □

Suppose $x \in V$, $x \neq 0$, and $e(x) = k$. Then

$$e(f^\nu x) = k - \nu \text{ and } h(f^\nu x) \geq \nu + h(x), \nu = 0, \ldots, k - 1.$$

If $x_1, \ldots, x_r \in V$ then

$$h(x_1 + \cdots + x_r) \geq \min\{h(x_j); 1 \leq j \leq r\}.$$  \hspace{1cm} (1.4)
Lemma 1.3. ([8], [4]) Let $U = (u_1, \ldots, u_m) \in \mathcal{U}$. Suppose $x \neq 0$ and
\[ x = \sum_{j=1}^{m} x_j, \quad x_j \in \langle u_j \rangle, \quad j = 1, \ldots, m. \]

(i) Then $e(x) = \max\{e(x_j); 1 \leq j \leq m\}$ and
\[ h(x) = \min\{h(x_j); 1 \leq j \leq m\}. \]

(ii) If $U$ is such that $e(u_j) = t$, $j = 1, \ldots, m$, then $h(x) + e(x) = t$. Moreover $h(x) = 0$ is equivalent to $h(f^{t-1}x) = t - 1$.

(iii) Suppose $y \in \langle x \rangle$. Then $\langle x \rangle = \langle y \rangle$ if and only if $h(y) = h(x)$.

Vectors $u$ that satisfy
\[ h(f^{\nu}u) = \nu + h(u), \quad \nu = 0, \ldots, e(u) - h(u) - 1. \quad (1.5) \]
have no gaps in their indicator sequence. In the terminology of abelian $p$-groups or $p$-primary modules ([5], [1]) such elements are called regular. They play a role in the theory of marked subspaces [8]. The following observation concerns the special case where $h(u) = 0$.

Lemma 1.4. Suppose $u \in V$, $u \neq 0$, and $e(u) = t$. The following conditions are equivalent. (i) $h(f^{t-1}u) = t - 1$. (ii) $h(f^{\nu}u) = \nu$, $\nu = 0, \ldots, t - 1$. (iii) $\mathcal{H}(u) = (0, 1, \ldots, t - 1, \infty, \ldots, \infty)$.

2. Generators, hyperinvariant subspaces and Baer’s lemma

2.1. Generators

In this section we deal with generators $u \in V$ and we determine corresponding equivalence classes $[u] = \{\alpha u; \alpha \in \text{Aut}(V,f)\}$. Suppose $u$ is a generator of $V$ and $e(u) = t$. Then (1.2) implies $f^{t-1}V = \langle f^{t-1}u \rangle \oplus f^{t-1}V_2$ and $f^tV = f^tV_2$. Hence $f^{t-1}u \notin f^tV$, and therefore $h(f^{t-1}u) = t - 1$. We shall see in Lemma 2.2 that the condition $h(f^{e(u)-1}u) = e(u) - 1$ is not only a necessary but also a sufficient condition for $u$ to be a generator.

Let $t \in \{1, \ldots, n\}$ be fixed. To a generator tuple $U = (u_1, \ldots, u_m)$ we associate the subspaces
\[ V_\kappa = \sum_{e(u_j) < t} \langle u_j \rangle, \quad V_\epsilon = \sum_{e(u_j) = t} \langle u_j \rangle \quad \text{and} \quad V_\gamma = \sum_{e(u_j) > t} \langle u_j \rangle. \quad (2.1) \]
Then
\[ V = V_\kappa \oplus V_\epsilon \oplus V_\gamma, \]  
and
\[ f^{t-1}V = f^{t-1}V_\epsilon \oplus f^{t-1}V_\gamma, \quad f^tV = f^tV_\gamma \text{ and } V_\gamma[f^t] \subseteq fV_\gamma. \]

We define
\[ M = \{ v; \ v = v_\kappa + v_\epsilon + v_\gamma, \text{ with } v_\kappa \in V_\kappa, \ v_\epsilon \in V_\epsilon, \ h(v_\epsilon) = 0 \text{ and } v_\gamma \in V_\gamma[f^t] \}. \]  

We note an intermediate result.

**Lemma 2.1.** Let \( u \in V, \ u \neq 0 \). Let \( U = (u_1, \ldots, u_m) \in U \), and let \( V_\kappa, V_\epsilon, V_\gamma \) and \( M \) be defined by (2.1) and (2.3), respectively.

(i) If
\[ e(u) = t \quad \text{and} \quad h(f^{t-1}u) = t - 1 \]  
then \( \lambda' \) is an elementary divisor of \( f \).

(ii) We have (2.4) if and only if \( u \in M \).

(iii) If (2.4) holds and then \([u] \subseteq M\).

**Proof.** (i+ii) Let \( u \) be decomposed according to (2.2) such that
\[ u = v_\kappa + v_\epsilon + v_\gamma, \quad \text{with} \quad (v_\kappa, v_\epsilon, v_\gamma) \in (V_\kappa, V_\epsilon, V_\gamma). \]  
Then (2.4) implies \( 0 = f^t u = f^t v_\gamma \). Hence, if \( v_\gamma \neq 0 \) then \( h(v_\gamma) \geq 1 \), and therefore \( h(f^{t-1}v_\gamma) \geq t \). If \( v_\gamma = 0 \) then \( h(f^{t-1}v_\gamma) = h(0) = \infty > t \). We have
\[ f^{t-1}u = f^{t-1}v_\epsilon + f^{t-1}v_\gamma \]  
and therefore
\[ t - 1 = h(f^{t-1}u) = \min\{h(f^{t-1}v_\epsilon), h(f^{t-1}v_\gamma)\}. \]

Hence \( h(f^{t-1}v_\gamma) \geq t \) yields \( h(f^{t-1}v_\epsilon) = t - 1 \), which is equivalent to \( h(v_\epsilon) = 0 \) (by Lemma 1.3(ii)). Thus \( v_\epsilon \neq 0 \) and therefore \( V_\epsilon \neq 0 \), and \( \lambda' \) is an elementary divisor of \( f \). The preceding arguments also show that \( u \in M \). Conversely, if \( u \in M \) then it is easy to verify that \( u \) satisfies (2.4).

(iii) The condition (2.4) remains valid if \( u \) is replaced by \( \alpha u \). Therefore part (ii) implies \( \alpha u \in M \). \( \square \)
The following result is an exchange lemma. It shows that a given generator can replace in any generator tuple a suitably chosen element.

**Lemma 2.2.** Let $U = (u_1, \ldots, u_m) \in \mathcal{U}$. Suppose $u \in V$, $u \neq 0$, satisfies $e(u) = t$ and $h(f^{t-1}u) = t - 1$. Then there exists an $r \in \{1, \ldots, m\}$ such that

$$e(u_r) = t \quad \text{and} \quad f^{t-1}u \notin f^{t-1}\langle u_1, \ldots, u_{r-1}, u_{r+1}, \ldots, u_m \rangle.$$  \hfill (2.7)

If (2.7) holds then

$$(u_1, \ldots, u_{r-1}, u, u_{r+1}, \ldots, u_m) \in \mathcal{U}, \hfill (2.8)$$

and the vector $u$ is a generator.

**Proof.** Let $u = v_\kappa + v_\epsilon + v_\gamma$ be given as in (2.5). We know from Lemma 2.1 that $f^{t-1}v_\epsilon \neq 0$. Set

$$D_j = \langle u_1, \ldots, u_j-1, u_j+1, \ldots, u_m \rangle, \quad j = 1, \ldots, m.$$  

Suppose $f^{t-1}u \in f^{t-1}D_j$ for all $j$ with $e(u_j) = t$. In that case

$$f^{t-1}u \in f^{t-1}\left( \bigcap_{1 \leq j \leq m; e(u_j) = t} D_j \right) = f^{t-1}(V_\kappa + V_\gamma),$$

and therefore $f^{t-1}u = f^{t-1}(v_\kappa + v_\gamma)$. But this is a contradiction to $f^{t-1}v_\epsilon \neq 0$. Hence there exists an $r$ such that $e(u_r) = t$ and

$$f^{t-1}u \notin f^{t-1}D_r. \hfill (2.9)$$

Let us now prove $V = \langle u \rangle \oplus D_r$. Because of $U \in \mathcal{U}$ and $u \notin D_r$ we have

$$u = w_r + d_r \quad \text{with} \quad w_r \in \langle u_r \rangle \subseteq V_\epsilon, \quad w_r \neq 0, \quad d_r \in D_r.$$  

Moreover, (2.9) implies $f^{t-1}w_r \neq 0$. Then $w_r \in V_\epsilon$ yields $h(w_r) = 0$. Therefore the vectors $\{w_r, fw_r, \ldots, f^{t-1}w_r\}$ are linearly independent. If $x \in \langle u \rangle$ then

$$x = \sum_{\nu=0}^{t-1} c_\nu f^\nu u = \sum_{\nu=0}^{t-1} c_\nu f^\nu w_r + \sum_{\nu=0}^{t-1} c_\nu f^\nu d_r$$

with $c_0, \ldots, c_{t-1} \in K$. Hence, if $x \in \langle u \rangle \cap D_r$ then $\sum_{\nu=0}^{t-1} c_\nu f^\nu w_r = 0$, and we obtain $c_\nu = 0$, $\nu = 0, \ldots, t - 1$. Therefore $x = 0$, and $\langle u \rangle \cap D_r = 0$. From $\dim \langle u \rangle = \dim \langle u_r \rangle = t$ follows $\dim \langle u \rangle + \dim D_r = \dim V$. Hence $V = \langle u \rangle \oplus D_r$. From (2.8) it is obvious that $u$ is a generator. \hfill $\square$
The next lemma shows that the equivalence class of a generator \( u \) consists of generators with the same exponent as \( u \).

**Lemma 2.3.** Let \( u \in V \) be a generator. Then \( v \sim u \) if and only if \( v \) is a generator and \( e(v) = e(u) \).

**Proof.** Recall that \( u \) is a generator if and only if \( V = \langle u \rangle \oplus V_2 \) and \( V_2 \in \text{Inv}(V,f) \). Suppose \( v = \alpha u \) for some \( \alpha \in \text{Aut}(V,f) \). Then \( e(v) = e(u) \), and \( V = \langle \alpha u \rangle \oplus \alpha V_2 \) and \( \alpha V_2 \in \text{Inv}(V,f) \).

Suppose \( e(u) = e(v) = t \). Then both \( u \) and \( v \) are generators if \( h(f^{t-1}u) = h(f^{t-1}v) = t-1 \). Let \( U = (u_1, \ldots, u_m) \in U \). Then \( u \sim u_r, v \sim u_\ell \) for some \( r, \ell \) such that \( e(u_r) = e(u_\ell) = t \). Because of \( u_r \sim u_\ell \) we obtain \( u \sim v \). \( \square \)

We now describe the equivalence class of a generator.

**Proposition 2.4.** Suppose \( u \in V \) is a generator with \( e(u) = t \). Let \( U = (u_j)_{j=1}^m \in U \) and let \( M \) be defined by (2.3).

(i) Then \( [u] = M \).

(ii) Suppose \( |K| = 2 \). If the elementary divisor \( \lambda^t \) is unrepeated and \( t = t_s \) such that \( V_\epsilon = \langle u_s \rangle \) then

\[
[u_s] = \left\{ v_\kappa + (u_s + fy_\kappa) + v_\gamma; \right. \\
v_\kappa \in \langle u_1, \ldots, u_{s-1} \rangle, y_\kappa \in \langle u_s \rangle, v_\gamma \in \langle u_{s+1}, \ldots, u_m \rangle [f^t]\right\}. \tag{2.10}
\]

**Proof.** (i) If \( u \) is generator with \( e(u) = t \) then \( u \) satisfies (2.4). Hence Lemma 2.1(iii) implies \( [u] \subseteq M \). To show the converse inclusion suppose \( w \in M \). Then it follows from Lemma 2.1 that \( e(w) = t \) and \( h(f^{t-1}w) = t-1 \). Hence \( w \) is a generator (by Lemma 2.2). Therefore \( w \sim u \) (by Lemma 2.3). Thus we conclude that \( M \subseteq \{ v; v \sim u \} = [u] \).

(ii) The summand \( v_\epsilon \in V_\epsilon \) in (2.3) satisfies \( h(v_\epsilon) = 0 \). If \( V_\epsilon = \langle u_s \rangle \) then \( v_\epsilon = cu_s + w_\gamma \) with \( c \neq 0 \) and \( w_\gamma \in f \langle u_s \rangle \). If \( |K| = 2 \) then \( c = 1 \). \( \square \)

**Corollary 2.5.** If \( u \) is a generator with \( e(u) = t \) then

\[
(f^\nu u)^c = \text{Im} f^\nu \cap \text{Ker} f^{t-\nu}, \nu = 0, 1, \ldots, t.
\]

In particular, \( \langle u \rangle^c = \text{Ker} f^t \).
Proof. From \( [u] = M \) follows \( \langle u \rangle^c = \langle M \rangle \). Hence \( \langle f^\nu u \rangle^c = f^\nu \langle M \rangle \). The definition of \( M \) implies \( \langle M \rangle = V_\kappa + V_\epsilon + V_\gamma [f] \). Then \( V = V_\kappa + V_\epsilon + V_\gamma \) yields

\[
\text{Im} f^\nu \cap \text{Ker} f^{t-\nu} = (f^\nu V_\kappa \cap V_\kappa[f^{t-\nu}]) + (f^\nu V_\epsilon \cap V_\epsilon[f^{t-\nu}]) + (f^\nu V_\gamma \cap V_\gamma[f^{t-\nu}]) = f^\nu V_\kappa + f^\nu V_\epsilon + (f^\nu V_\gamma)[f^{t-\nu}] = f^\nu \langle M \rangle.
\]

\[\square\]

Example 2.6. Let \( V, f \) and \( z \) be given as in Example 1.2. Then \( (t_1, t_2) = (1, 3) \). Set \( X = \langle z \rangle^c = \langle u_1 + fu_2 \rangle^c \), and \( M_1 = [u_1], M_3 = [u_2] \). Then

\[
\langle u_2 \rangle^c = \langle M_3 \rangle = \text{Ker} f^3 = V,
\]

and

\[
M_1 = \{u_1, u_1 + f^2u_2\}, \quad fM_3 = \{fu_2, fu_2 + f^2u_2\}, \quad \text{and}
\]

\[
[z] = \{u_1 + \eta_1 f^2u_2 + fu_2 + \eta_2 f^2u_2; \eta_i = 0, 1, i = 1, 2\} = \{u_1 + fu_2, u_1 + fu_2 + f^2u_2\}.
\]

Hence

\[
X = \{0, u_1 + fu_2, u_1 + fu_2 + f^2u_2, f^2u_2\} = \langle z, f^2u_2 \rangle.
\]

The projection \( \pi_1 : V \rightarrow \langle u_1 \rangle \) is an endomorphism of \( V \) that commutes with \( f \). We have \( z \in X \) and \( \pi_1 z = u_1 \notin X \). Hence \( \pi_1 X \notin X \), and therefore \( X \) is a characteristic subspace that is not hyperinvariant. The subspace \( X_H = \langle f^2u_2 \rangle = \text{Ker} f^2 \) is the largest hyperinvariant subspace contained in \( X \).

2.2. Hyperinvariant subspaces

Suppose \( \dim \text{Ker} f = m \). Let

\[
\lambda_{ij}, \quad j = 1, \ldots, m, \quad 0 < t_1 \leq \cdots \leq t_m,
\]

be the \( m \)-tuple of elementary divisors of \( f \). Set \( t = (t_1, \ldots, t_m) \). Let \( L(t) \) denote the set of \( m \)-tuples \( \vec{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m \) satisfying

\[
0 \leq p_1 \leq \cdots \leq p_m \quad \text{and} \quad 0 \leq t_1 - p_1 \leq \cdots \leq t_m - p_m.
\]

We write \( \vec{p} \preceq \vec{q} \) if \( \vec{p} = (p_i)_{i=1}^m, \vec{q} = (q_i)_{i=1}^m \in L(t) \) and \( p_i \leq q_i, \quad i = 1, \ldots, m \). Then \( (L(t), \preceq) \) is a lattice. The following theorem is due to Fillmore, Herrero and Longstaff [9]. We refer to [12] for a proof.
Theorem 2.7. Let \( f : V \to V \) be nilpotent with elementary divisors \((2.11)\).

(i) If \( \vec{p} \in \mathcal{L}(\vec{t}) \), then
\[
W(\vec{p}) = f^{p_1}V \cap V[f^{t_1-p_1}] + \cdots + f^{p_m}V \cap V[f^{t_m-p_m}]
\]
is a hyperinvariant subspace. Conversely, each \( W \in \text{Hinv}(V,f) \) is of the form \( W = W(\vec{p}) \) for some \( \vec{p} \in \mathcal{L}(\vec{t}) \).

(ii) If \( \vec{p} \in \mathcal{L}(\vec{t}) \) and \( U = (u_i)_{i=1}^{m} \in \mathcal{U} \) then
\[
W(\vec{p}) = f^{p_1}\langle u_1 \rangle \oplus \cdots \oplus f^{p_m}\langle u_m \rangle.
\]

(iii) The mapping \( \vec{p} \mapsto W(\vec{p}) \) is a lattice isomorphism from \( (\mathcal{L}(\vec{t}), \preceq) \) onto \( (\text{Hinv}(V,f), \supseteq) \).

Recall Example 2.6 with the characteristic subspace and note that the subspace \( X = \langle z \rangle^c \) is not hyperinvariant because \( X \) does not satisfy the condition \( \pi_1X \subseteq X \), or equivalently \( \pi_1X = X \cap \langle u_1 \rangle \). This confirms part (iii) of the following theorem. Let \( X_H \) denote the largest hyperinvariant subspace contained in a characteristic subspace \( X \). We give an explicit description of \( X_H \) in \((2.12)\) below.

Theorem 2.8. [3, Lemma 4.5, Lemma 4.2, Theorem 4.3] Suppose \( X \) is a characteristic subspace of \( V \). Let \( U = (u_1, \ldots, u_m) \in \mathcal{U} \).

(i) If \( u_j \) is a repeated generator then \( \pi_jX = X \cap \langle u_j \rangle \).

(ii) The subspace
\[
X_H = \bigoplus_{i=1}^{m} \left(X \cap \langle u_i \rangle\right)
\]
is the largest hyperinvariant subspace contained in \( X \).

(iii) The subspace \( X \) is hyperinvariant if and only if
\[
\pi_jX = X \cap \langle u_j \rangle, \; j = 1, \ldots, m,
\]
or equivalently \( X = \bigoplus_{i=1}^{m} \left(X \cap \langle u_i \rangle\right) \).

In the following consider subspaces that are the characteristic hull of a single element.

Lemma 2.9. Let \( z = \sum_{i=1}^{k} z_k \) and \( X = \langle z \rangle^c \).
(i) If \( z_i \in X, \ i = 1, \ldots, k \), then
\[
X = \sum_{i=1}^{k} \langle z_i \rangle^c. \tag{2.13}
\]

(ii) Suppose (2.13) holds and the subspaces \( \langle z_i \rangle^c, \ i = 1, \ldots, k \), are hyperinvariant. Then \( X \) is hyperinvariant.

**Proof.** (i) The inclusion \( \langle \sum_{i=1}^{k} z_i \rangle^c \subseteq \sum_{i=1}^{k} \langle z_i \rangle^c \) is obvious. On the other hand, if \( z_i \in X \) then \( \langle z_i \rangle^c \subseteq X \), \( i = 1, \ldots, k \). (ii) We note that \( Hinv(V, f) \) is closed under subspace addition. \( \square \)

The next result will be incorporated in our main theorem.

**Lemma 2.10.** Let \( U = (u_1, \ldots, u_m) \in \mathcal{U} \). Suppose
\[
z = f^{\mu_1} u_{\rho_1} + \cdots + f^{\mu_k} u_{\rho_k}, \tag{2.14}
\]
and \( 1 \leq \rho_1 < \cdots < \rho_k \leq m \) and \( 0 \leq \mu_i < t_{\rho_i}, \ i = 1, \ldots, k \). If at most one of the generators \( u_{\rho_i}, \ i = 1, \ldots, k \), is unremarked then \( X = \langle z \rangle^c \) is hyperinvariant and
\[
X = \left( \text{Im} f^{\mu_1} \cap \ker f^{t_{\rho_1} - \mu_1} \right) + \cdots + \left( \text{Im} f^{\mu_k} \cap \ker f^{t_{\rho_k} - \mu_k} \right).
\]

**Proof.** If \( k = 1 \) then Corollary 2.5 immediately yields
\[
X = \langle f^{\mu_1} u_{\rho_1} \rangle^c = \text{Im} f^{\mu_1} \cap \ker f^{t_{\rho_1} - \mu_1} \in Hinv(V, f).
\]
Now let \( k \geq 2 \), and suppose that the generators \( u_{\rho_i} \) in (2.14) are repeating ones - with the possible exception of \( u_{\rho_\kappa} \). Then it follows from Lemma 3.4(ii) that \( \pi_{\rho_i} z = f^{\mu_{\rho_i}} u_{\rho_i} \in X \) for all \( i \neq \kappa \). Hence (2.14) and \( z \in X \) imply \( \pi_{\rho_\kappa} z = f^{\mu_{\rho_\kappa}} u_{\rho_\kappa} \in X \). Therefore \( f^{\mu_\rho_i} u_{\rho_i} \in X \) for all \( i \in \{1, \ldots, k\} \). Then Lemma 2.9 and Corollary 2.5 yield
\[
X = \sum_{i=0}^{k} \langle f^{\mu_\rho_i} u_{\rho_i} \rangle^c = \sum_{i=0}^{k} \left( \text{Im} f^{\mu_\rho_i} \cap \ker f^{t_{\rho_i} - \mu_\rho_i} \right).
\]
Hence \( X \) is hyperinvariant. \( \square \)
2.3. Baer’s lemma

If \( U = (u_1, \ldots, u_m) \in \mathcal{U} \) then each \( z \in V \) can be represented as
\[
z = z_1 + \cdots + z_m \quad \text{with} \quad z_j \in \langle u_j \rangle, \ j = 1, \ldots, m. \tag{2.15}
\]

For a single element \( z \) one can find a generator tuple that is adapted to it in the sense of the following lemma.

**Lemma 2.11.** Let \( z \in V \). There exists a generator tuple \( U' = (u'_1, \ldots, u'_m) \) such that
\[
z = f^\mu_1 u'_1 + \cdots + f^\mu_m u'_m \quad \text{with} \quad 0 \leq \mu_j \leq t_j, \ j = 1, \ldots, m. \tag{2.16}
\]

**Proof.** Let \( U = (u_1, \ldots, u_m) \in \mathcal{U} \) and \( z \) be given by (2.15). Suppose \( z_j \neq 0 \) and \( h(z_j) = \mu_j \). Then \( z_j = f^{\mu_j} y_j \) for some \( y_j \in \langle u_j \rangle \) with \( h(y_j) = 0 \). Note that \( \langle y_j \rangle = \langle u_j \rangle \). If \( z_j = 0 \) then \( z_j = f^{t_j} u_j \). In that case we set \( \mu_j = t_j \). Now define \( u'_j = \begin{cases} u_j & \text{if } z_j = 0 \\ y_j & \text{if } z_j \neq 0. \end{cases} \)

Then (2.16) holds, and \( V = \oplus_{1 \leq j \leq m} \langle u'_j \rangle \) yields \( U' = (u'_j)_{j=1}^m \in \mathcal{U} \). \( \square \)

The numbers \( \mu_j \) in (2.16) are not uniquely determined by \( z \).

**Example 2.12.** Let \( \dim V = 7 \), \( \ker f = 3 \), and \( (t_1, t_2, t_3) = (1, 3, 3) \), and \( V = \langle u_1 \rangle \oplus \langle u_2 \rangle \oplus \langle u_3 \rangle \) with \( e(u_1) = 1 \), \( e(u_2) = e(u_3) = 3 \). Consider
\[
z = z_1 + z_2 + z_3 = u_1 + (fu_2 + f^2 u_2) + (u_3 + fu_3).
\]

Set \( u'_1 = u_1, u'_2 = u_2 + fu_2, u'_3 = u_3 + fu_3 \). Then \( U' = (u'_1, u'_2, u'_3) \in \mathcal{U} \) and
\[
z = u'_1 + fu'_2 + u'_3.
\]

Thus \( (\mu_1, \mu_2, \mu_3) = (1, 1, 0) \). Because of \( e(z) = 3 \) and \( h(f^2 z) = h(f^2 u_3) = 2 \) the vector \( z \) is a generator. If we define
\[
U'' = (u''_1, u''_2, u''_3) = (u_1, u_2, z)
\]
then \( U'' \in \mathcal{U} \) and \( z = u'' \). Thus \( z = \sum f^{\mu_j} u''_j \) with \( (\mu_1, \mu_2, \mu_3) = (1, 3, 0) \). \( \square \)
Under additional conditions one can obtain a representation (2.16) where the numbers $\mu_j$ are uniquely determined. In the case of finite abelian $p$-groups that representation is due to Baer [6] (see [11, p. 4]). To make our note self-contained we include a proof. In contrast to [6] our proof is constructive.

**Lemma 2.13.** (Baer’s lemma) Let $z \in V$, $z \neq 0$. Then there exists a generator tuple $U = (u_1, \ldots, u_m)$ such that

$$z = f^{\mu_1} u_{\rho_1} + f^{\mu_2} u_{\rho_2} + \cdots + f^{\mu_k} u_{\rho_k}$$

with

$$0 \leq \mu_1 < \cdots < \mu_k \quad \text{and} \quad 0 < t_{\rho_1} - \mu_1 < \cdots < t_{\rho_k} - \mu_k. \quad (2.18)$$

**Proof.** Let $U' = (u'_1, \ldots, u'_m) \in \mathcal{U}$ be such that (2.16) holds. Lemma 1.3(i) implies

$$h(z) = h(f^{\mu_1} u'_1 + \cdots + f^{\mu_m} u'_m) = \min \{h(f^{\mu_j} u'_j); j = 1, \ldots, m\},$$

with $h(f^{\mu_j} u'_j) = \infty$ if $f^{\mu_j} u'_j = 0$. Let

$$\rho_1 = \max \{j; 1 \leq j \leq m, \text{ such that } h(f^{\mu_j} u'_j) = h(z)\}.$$

Thus, $\mu_{\rho_1} = h(z)$, and if $f^{\mu_j} u'_j \neq 0$ then $\mu_j = h(f^{\mu_j} u'_j)$ satisfies

$$\mu_j \geq \mu_{\rho_1} \quad \text{if} \quad j < \rho_1,$$

$$\mu_j > \mu_{\rho_1} \quad \text{if} \quad j > \rho_1.$$

We decompose $z$ as follows.

$$z = \sum_{1 \leq j < \rho_1} f^{\mu_j} u'_j + f^{\mu_{\rho_1}} u'_{\rho_1} + \\
\sum_{\rho_1 < j \leq m, t_{\rho_1} - \mu_{\rho_1} \geq t_j - \mu_j} f^{\mu_j} u'_j + \sum_{\rho_1 < j \leq m, t_{\rho_1} - \mu_{\rho_1} < t_j - \mu_j} f^{\mu_j} u'_j = v + f^{\mu_{\rho_1}} u'_{\rho_1} + w + z_2.$$

Set

$$\tilde{v} = \sum_{1 \leq j < \rho_1} f^{\mu_j - \mu_{\rho_1}} u'_j.$$
Then \( v = f^{\mu_1} \tilde{v} \). If \( j < \rho_1 \) then \( e(u'_{j}) \leq e(u'_{\rho_1}) \). Hence \( e(\tilde{v}) \leq e(u'_{\rho_1}) = t_{\rho_1} \).

Set
\[
\tilde{w} = \sum_{\rho_1 < j \leq m, t_{\rho_1} - \mu_{\rho_1} \geq t_j - \mu_j} f^{\mu_j - \mu_{\rho_1}} u'_j.
\]

Then \( w = f^{\mu_1} \tilde{w} \). We have
\[
f^{\tau_{\rho_1}} \tilde{w} = \sum_{\rho_1 < j \leq m, t_{\rho_1} - \mu_{\rho_1} \geq t_j - \mu_j} f^{\mu_j + (t_{\rho_1} - \mu_{\rho_1})} u'_j = 0.
\]

Hence \( e(\tilde{w}) \leq t_{\rho_1} = e(u'_{\rho_1}) \). Set
\[
u_{\rho_1} = u'_{\rho_1} + \tilde{v} + \tilde{w}.
\]

Then \( h(u_{\rho_1}) = h(u'_{\rho_1}) = 0 \), \( e(u_{\rho_1}) = e(u'_{\rho_1}) = t_{\rho_1} \), and \( h(f^{\tau_{\rho_1}} u_{\rho_1}) = h(f^{\tau_{\rho_1}} u'_{\rho_1}) = t_{\rho_1} - 1 \). Therefore \( u_{\rho_1} \) is a generator. Moreover
\[
u_{\rho_1} \notin D'_{\rho_1} = \langle u'_1, \ldots, u'_{\rho_1 - 1}, u'_{\rho_1 + 1}, \ldots, u'_m \rangle.
\]

By Lemma 2.2 we can replace in \( U' \) the generator \( u'_{\rho_1} \) by \( u_{\rho_1} \) such that
\[
\tilde{U}_{\rho_1} = (u'_1, \ldots, u'_{\rho_1 - 1}, u_{\rho_1}, u'_{\rho_1 + 1}, \ldots, u'_m) \in U.
\]

Then
\[
z = f^{\mu_1} u_{\rho_1} + z_2 \quad (2.19)
\]

with
\[
z_2 = \sum_{\rho_1 < j \leq m, t_{\rho_1} - \mu_{\rho_1} \leq t_j - \mu_j} f^{\mu_j} u'_j, \quad (2.20)
\]

If \( z_2 = 0 \) then the proof is complete. If \( z_2 \neq 0 \) then the preceding arguments yield a decomposition \( z_2 = f^{\mu_2} u_{\rho_2} + z_3 \). We note that the numbers \( \mu_j \) in (2.19) and (2.20) satisfy \( \mu_{\rho_1} < \mu_i \) and \( t_{\rho_1} - \mu_{\rho_1} < t_j - \mu_j \). Hence \( \mu_{\rho_1} < \mu_{\rho_2} \) and \( t_{\rho_1} - \mu_{\rho_1} < t_{\rho_2} - \mu_{\rho_2} \). When this decomposition process comes to an end the outcome is of the form (2.17) and (2.18).

Let \( z \) be decomposed as in (2.17) such that (2.18) holds. Then we have
\[
1 \leq \rho_1 < \cdots < \rho_k \leq m.
\]

If we set \( z_{\rho_i} = f^{\mu_{\rho_i}} u_{\rho_i}, \ i = 1, \ldots, k \), then
\[
z = \sum_{i=1}^{k} z_{\rho_i}.
\]

The vectors \( z_{\rho_i} \) are regular in the sense of (1.5). Moreover, \( h(z_{\rho_i}) = \mu_{\rho_i} \) and \( e(z_{\rho_i}) = t_{\rho_i} - \mu_{\rho_i} \) such that
\[
h(z_{\rho_1}) < \cdots < h(z_{\rho_k}) < \infty \quad \text{and} \quad 0 < e(z_{\rho_1}) < \cdots < e(z_{\rho_k}). \quad (2.21)
\]
We remark that an invariant subspace $W \subseteq V$ is marked if and only if each element $z \in W$ can be written as $z = \sum_{i=1}^{k} z_{\rho_i}$ such that (2.21) holds [1].

The following result completes Lemma 2.13. It shows that the numbers $e(u_{\rho_i})$ and $\mu_{\rho_i}$ can be recovered from the indicator of $z$.

**Lemma 2.14.** If $z$ is given as in (2.17) such that (2.18) holds. Then

$$H(z) = H(f^{\mu_{\rho_1}}u_{\rho_1} + f^{\mu_{\rho_2}}u_{\rho_2} + \cdots + f^{\mu_{\rho_k}}u_{\rho_k}) =$$

$$(\mu_{\rho_1}, \mu_{\rho_1} + 1, \ldots, t_{\rho_1} - 1,$$

$$(\mu_{\rho_2} + (t_{\rho_1} - \mu_{\rho_1}), \mu_{\rho_2} + (t_{\rho_1} - \mu_{\rho_1}) + 1, \ldots, t_{\rho_2} - 1,$$

$$(\mu_{\rho_3} + (t_{\rho_2} - \mu_{\rho_2}), \mu_{\rho_3} + (t_{\rho_2} - \mu_{\rho_2}) + 1, \ldots, t_{\rho_3} - 1,$$

$$(\mu_{\rho_4} + (t_{\rho_3} - \mu_{\rho_3}), \ldots, t_{\rho_{k-1}} - 1,$$

$$(\mu_{\rho_{k-1}} + (t_{\rho_{k-1}} - \mu_{\rho_{k-1}}), \mu_{\rho_{k-1}} + (t_{\rho_{k-1}} - \mu_{\rho_{k-1}}) + 1, \ldots, t_{\rho_k} - 1, \infty, \ldots, \infty).$$

The indicator $H(z)$ has $k - 1$ gaps, namely

$$h(f^{t_{\rho_i}-\mu_{\rho_i}}z) - h(f^{t_{\rho_i}-\mu_{\rho_i}-1}z) = 1 + (\mu_{\rho_{i+1}} - \mu_{\rho_i}), i = 1, \ldots, k - 1.$$

3. The main theorem

The following technical lemma prepares the ground for Lemma 3.2 and subsequently for the construction of hyperinvariant subspaces $X_H$.

**Lemma 3.1.** Let $U = (u_1, \ldots, u_m) \in U$. Suppose $x \in V$, $x \neq 0$, and

$$x = f^{\mu_{\rho_1}}u_{\rho_1} + \cdots + f^{\mu_{\rho_k}}u_{\rho_k} \quad \text{with} \quad t_{\rho_1} < \cdots < t_{\rho_k}, \quad \text{and}$$

$$0 \leq \mu_{\rho_i} < t_{\rho_i}, \quad i = 1, \ldots, k. \quad (3.1)$$

Set $X = \langle x \rangle^c$. Let $\nu \in \{1, \ldots, k\}$. Then

(i) $f^{\mu_{\rho_\nu}}u_j \in X$ if $t_j < t_{\rho_\nu}$.

(ii) $f^{\mu_{\rho_\nu}+1}u_j \in X$ if $t_j = t_{\rho_\nu}$. If $u_{\rho_\nu}$ is a repeated generator such that $t_j = t_{\rho_\nu}$ then $f^{\mu_{\rho_\nu}}u_j \in X$.

(iii) $f^{\mu_{\rho_\nu}+t_j-t_{\rho_\nu}}u_j \in X$ if $t_j > t_{\rho_\nu}$.
Proof. In the following we choose vectors \( w_j \in \langle u_j \rangle \) in an appropriate way such that
\[
h(u_{\rho \nu} + w_j) = 0 \quad \text{and} \quad e(u_{\rho \nu} + w_j) = e(u_{\rho \nu}),
\]
and such that \( u_{\rho \nu} \in U \) can be replaced by \( u_{\rho \nu} + w_j \). Let \( \tilde{\alpha}_j \in \text{Aut}(f, V) \), \( 1 \leq j \leq m \), be the corresponding automorphisms defined by
\[
\tilde{\alpha}_j : u_{\rho \nu} \mapsto u_{\rho \nu} + w_j \quad \text{and} \quad \tilde{\alpha}_j : u_\tau \mapsto u_\tau \quad \text{if} \quad \tau \neq \rho \nu, \quad \tau = 1, \ldots, m.
\]
We consider three cases.

(i) Case \( t_j < t_{\rho \nu} \). Set \( w_j = u_j \). Then
\[
\tilde{\alpha}_j x - x = f^{\mu_{\rho \nu}} u_j \in X.
\]

(ii) Case \( t_j = t_{\rho \nu} \). Set \( w_j = fu_j \). Then
\[
\tilde{\alpha}_j x - x = f^{\mu_{\rho \nu} + 1} u_j \in X.
\]
Now suppose \( \lambda_{\rho \nu} x = X \cap \langle u_{\rho \nu} \rangle \). Hence \( \pi_{\rho \nu} x = f^{\mu_{\rho \nu}} u_{\rho \nu} \in X \). From \( t_j = t_{\rho \nu} \) follows
\[
u_j \sim u_{\rho \nu}. \quad \text{Hence} \quad X = X^c \text{ yields} \quad f^{\mu_{\rho \nu}} u_j \in X.
\]

(iii) Case \( t_j > t_{\rho \nu} \). Set \( w_j = f^{t_j - t_{\rho \nu}} u_j \). Then (3.2) holds and
\[
\tilde{\alpha}_j x - x = f^{\mu_{\rho \nu} + t_j - t_{\rho \nu}} u_j \in X.
\]

\[\square\]

In the next lemma we consider an element \( z \) and the subspace \( X = \langle z \rangle^c \).
We associate to \( z \) a tuple \( \bar{r} = (r_j)_{j=1}^m \) of nonnegative integers. It will be used to construct the corresponding hyperinvariant subspace \( X_H \).

**Lemma 3.2.** Let \( z \in V, \ z \neq 0 \), and let \( U = (u_1, \ldots, u_m) \in \mathcal{U} \) be such that
\[
z = f^{\mu_{r_1}} u_{r_1} + \cdots + f^{\mu_{r_k}} u_{r_k},
\]
and
\[
\mu_{r_1} < \cdots < \mu_{r_k} \quad \text{and} \quad 0 < t_{r_1} - \mu_{r_1} < \cdots < t_{r_k} - \mu_{r_k}.
\]
Define \( R_u = \{ \rho_i; 1 \leq i \leq k, \lambda^{r_i} \text{ is unrepeated} \} \) and \( R_r = \{ \rho_i; 1 \leq i \leq k, \lambda^{r_i} \text{ is repeated} \} \), and
\[
r_j = \begin{cases} 
\mu_{r_1} & \text{if } t_j < t_{r_1}, \\
\mu_{r_s} + 1 & \text{if } t_j = t_{r_s}, \rho_s \in R_u, \\
\mu_{r_s} & \text{if } t_j = t_{r_s}, \rho_s \in R_r, \\
\min\{t_j - (t_{r_1} - \mu_{r_1}), \mu_{r_{(s+1)}}\} & \text{if } t_{r_1} < t_j < t_{r_s+1}, \ell = 1, \ldots, k - 1, \\
t_j - (t_{r_k} - \mu_{r_k}) & \text{if } t_j > t_{r_k}.
\end{cases}
\]
(3.3)
Set $X = \langle z \rangle^c$. Then
\[ f^{r_j}u_j \in X, \quad j = 1, \ldots, m, \quad \text{(3.4)} \]
and $\langle f^{r_1}u_1, f^{r_2}u_2, \ldots, f^{r_m}u_m \rangle \subseteq X_H$.

Proof. Define
\[ \hat{X} = \langle f^{r_1}u_1, f^{r_2}u_2, \ldots, f^{r_m}u_m \rangle. \]
Because of $X_H = \bigoplus_{j=1}^m \left( X \cap \langle u_j \rangle \right)$ we have $\hat{X} \subseteq X_H$ if and only if (3.4) holds.

Let $z$ be written in accordance with (3.1) such that
\[ z = \sum_{j=1}^m s_{\rho_i} f^{s_j}u_j \quad \text{with} \quad s_{\rho_i} = \mu_{\rho_i}, \quad 1 \leq i \leq k \quad \text{and} \quad s_j = t_j, \quad j \notin \{\rho_1, \ldots, \rho_k\}. \]

Let $z = z_{t\kappa} + z_{t\epsilon} + z_{t\gamma}$ such that
\[ z_{t\kappa} = \sum_{e(u_j) < t} f^{s_j}u_j, \quad z_{t\epsilon} = \sum_{e(u_j) = t} f^{s_j}u_j, \quad z_{t\gamma} = \sum_{e(u_j) > t} f^{s_j}u_j. \]

Hence, if $t = t_{\rho_i}$ then $z_{t\epsilon} = f^{\mu_{\rho_i}}u_{\rho_i}$ and $h(z_{t\epsilon}) = \mu_{\rho_i}$. In the following we apply Lemma 3.1.

(I) Case $t = t_{\rho(\ell+1)}$, $\ell = 0, \ldots, k - 1$. Then Lemma 3.1(i) implies
\[ f^{\mu_{\rho(\ell+1)}}u_j \in X \quad \text{if} \quad t_j < t_{\rho(\ell+1)}. \]

In particular, $\ell = 0$ yields $f^{\mu_{r1}}u_j \in X$ if $t_j < t_{\rho_1}$.

(II) Case $t = t_{\rho_i}$. If $\rho_i \in R_u$ then Lemma 3.1(ii) implies $f^{\mu_{\rho_i+1}}u_{\rho_i} \in X$, and if $\rho_i \in R_s$ then $f^{\mu_{\rho_i}}u_{\rho_i} \in X$.

(III) Case $t = t_{\rho_k}$, $\ell = 1, \ldots, k$. Then Lemma 3.1(iii) implies
\[ f^{\mu_{\rho_k} + t_j-t_{\rho_k}} \in X \quad \text{if} \quad t_j > t_{\rho_k}. \]

In particular, $\ell = k$ yields $f^{\mu_{\rho_k} + t_j-t_{\rho_k}} \in X$ if $t_j > t_{\rho_k}$.

Suppose $\mu_{\rho_k} < t_j = e(u_j) < t_{\rho(\ell+1)}$. Combining (I) and (III) we obtain
\[ \langle f^{\mu_{\rho(\ell+1)}}u_j, f^{t_j-(t_{\rho_k}-\mu_{\rho_k})}u_j \rangle = \langle f^{\mu_{\rho(\ell+1)}}u_j \rangle + \langle f^{t_j-(t_{\rho_k}-\mu_{\rho_k})}u_j \rangle = \langle f^{\min(t_j-(t_{\rho_k}-\mu_{\rho_k}),\mu_{\rho(\ell+1)})}u_j \rangle \subseteq X. \]

\[ \square \]
At this point we make the assumption that $K = GF(2)$.

**Lemma 3.3.** Assume $|K| = 2$. Let $z \in V$, $z \neq 0$, and let $U = (u_1, \ldots, u_m) \in U$ be such that $z = f^{\mu_1} u_{\rho_1} + \cdots + f^{\mu_k} u_{\rho_k}$, $k \geq 2$, and $0 \leq \mu_\rho < \mu_{\rho+1}$ and $0 < t_{\rho_i} - t_{\rho_i+1} = \mu_{\rho_i+1}$, $i = 1, \ldots, k - 1$. (3.5)

Let the numbers $r_j$, $j = 1, \ldots, m$, be defined by (3.3). Define

$$X = \langle z \rangle^c \quad \text{and} \quad \hat{X} = \langle f^{r_1} u_1, f^{r_2} u_2, \ldots, f^{r_m} u_m \rangle.$$

Then

$$X = \langle z \rangle + \hat{X}. \quad (3.6)$$

Moreover, $\hat{X} \subseteq X_H$.

**Proof.** Let $\alpha \in \text{Aut}(f, V)$. We show that $\alpha z = z + w$ for some $w \in \hat{X}$. We focus on $\pi_j(\alpha(f^{\mu_\rho_i} u_{\rho_i}))$. According to Proposition 2.4(i) we have

$$\alpha u_{\rho_i} = v_\kappa + v_\epsilon + v_\gamma$$

where

$$v_\kappa \in \sum_{t_j < t_{\rho_i}} \langle u_j \rangle, \quad v_\gamma \in \sum_{t_{\rho_i} < t_j} \langle u_j \rangle [f^{t_\gamma}], \quad (3.7)$$

and

$$v_\epsilon \in V_\epsilon = \sum_{t_j = t_{\rho_i}} \langle u_j \rangle \quad \text{with} \quad h(v_\epsilon) = 0. \quad (3.8)$$

If $u_{\rho_i}$ is unrepeated then $v_\epsilon = u_{\rho_i} + fy_{\rho_i}$ with $y_{\rho_i} \in \langle u_{\rho_i} \rangle$. In correspondence with (3.7) and (3.8) we consider the following cases.

(i) Case $t_j < t_{\rho_i}$. Then $r_j \leq \mu_{\rho_i}$. Therefore $\pi_j(f^{\mu_\rho_i} v_\kappa) \in \langle f^{\mu_\rho_i} u_j \rangle \subseteq \langle f^{r_j} u_j \rangle$. Hence $v_\kappa \in \hat{X}$.

(ii) Case $t_{\rho_i} < t_j$. Then $r_j \leq t_j - \mu_{\rho_i} + \mu_{\rho_i}$. Moreover $h(\pi_j v_\gamma) \geq t_j - t_{\rho_i}$. Therefore $\pi_j(f^{\mu_\rho_i} v_\gamma) \in \langle f^{\mu_\rho_i}+\mu_{\rho_i}-(t_j-t_{\rho_i}) u_j \rangle \subseteq \langle f^{r_j} u_j \rangle$. Hence $v_\gamma \in \hat{X}$.

(iii) Case $t_j = t_{\rho_i}$.

(u) Suppose $u_{\rho_i}$ is unrepeated. Then $r_j = \mu_{\rho_i} + 1$. Therefore

$$f^{\mu_\rho_i} v_\epsilon = f^{\mu_\rho_i} u_{\rho_i} + f^{\mu_\rho_i} fy_{\rho_i} = f^{\mu_\rho_i} u_{\rho_i} + f^{r_\rho_i} y_{\rho_i} \in f^{\mu_\rho_i} u_{\rho_i} + f^{r_\rho_i} \langle u_{\rho_i} \rangle.$$

We obtain $\alpha(f^{\mu_\rho_i} u_{\rho_i}) = f^{\mu_\rho_i}(v_\kappa + v_\epsilon + v_\gamma) \in f^{\mu_\rho_i} u_{\rho_i} + \hat{X}$. 

19
(r) Suppose \( u_{\rho_i} \) is repeated. Then \( r_j = \mu_{\rho_i} \). Therefore \( f^{\mu_{\rho_i}} u_j = f^{r_j} u_j \in \hat{X} \), and we conclude that

\[
f^{\mu_{\rho_i}} v_\epsilon \in f^{\mu_{\rho_i}} V_\epsilon = \sum_{t_j = t_{\rho_i}} \langle f^{\mu_{\rho_i}} u_j \rangle \subseteq \hat{X}.
\]

Hence \( \alpha(f^{\mu_{\rho_i}} u_{\rho_i}) = f^{\mu_{\rho_i}} (v_\kappa + v_\epsilon + v_\gamma) \in \hat{X} \). It follows that

\[
\alpha(f^{\mu_{\rho_i}} u_{\rho_i}) - f^{\mu_{\rho_i}} u_{\rho_i} \in \hat{X},
\]

and

\[
\alpha(f^{\mu_{\rho_i}} u_{\rho_i}) = f^{\mu_{\rho_i}} u_{\rho_i} + [\alpha(f^{\mu_{\rho_i}} u_{\rho_i}) - f^{\mu_{\rho_i}} u_{\rho_i}] \in f^{\mu_{\rho_i}} u_{\rho_i} + \hat{X}. \quad (3.9)
\]

Thus in both cases, (u) and (r), we obtain \( \alpha z = \sum_{i=1}^{k} \alpha(f^{\mu_{\rho_i}} u_{\rho_i}) \in z + \hat{X} \).

Hence \( X = (z)^c \subseteq (z) + \hat{X} \). Then \( z \in X \), and \( \hat{X} \subseteq X \) imply \( X = (z) + \hat{X} \).

To prove \( X \subseteq X_H \) we observe that \( f^{r_j} u_j \in X \cap \langle u_j \rangle \). Hence Theorem 2.8 implies \( \hat{X} = \sum_{j=1}^{m} \langle f^{r_j} u_j \rangle \subseteq X_H \).

\[ \square \]

The following auxiliary result is a special case of Theorem 3.5 below. We assume \( \dim \ker f = 2 \).

**Corollary 3.4.** [4] Suppose \( |K| = 2 \). Let \( V = \langle u_1 \rangle \oplus \langle u_2 \rangle \) with

\[
e(u_1) = t_1, \quad e(u_2) = t_2, \quad t_1 + 1 < t_2.
\]

Let \( z = f^{s_1} u_1 + f^{s_2} u_2 \) such that \( 0 \leq s_1 < s_2, \ 0 < t_1 - s_1 < t_2 - s_2 \). Set \( X = (z)^c \). Then \( \pi_1 z = f^{s_1} u_i \notin X, \ i = 1, 2, \) and \( X \) is not hyperinvariant.

**Proof.** Since \( \lambda^{s_1} \) and \( \lambda^{s_2} \) are unrepeated elementary divisors we have \( r_i = s_i + 1, \ i = 1, 2 \). Then (3.6) implies

\[
X = \langle z, f^{s_1 + 1} u_1, f^{s_2 + 1} u_2 \rangle = \langle z, f^{s_1 + 1} u_1, \rangle.
\]

We show that \( \pi_1 z = f^{s_1} u_1 \notin X \). Suppose \( f^{s_1} u_1 \in X \). Then

\[
f^{s_1} u_1 = \sum_{i=0}^{t_2-s_2-1} c_i f^i z + \sum_{j \geq 0} d_j f^j f^{s_1+1} u_1 = \\
\sum_{i \geq 0} c_i f^i f^{s_1} u_1 + \sum_{i=0}^{t_2-s_2-1} d_j f^j f^{s_1+1} u_1.
\]

We have

\[
\sum_{i \geq 0} c_i f^i f^{s_1} u_1 + \sum_{j \geq 0} d_j f^j f^{s_1+1} u_1.
\]

20
Hence
\[ t_2 - s_2 - 1 \sum_{i=0}^{c_2 f^2 u_2 = 0.} \]
Then \( e(f^s u_2) = t_2 - s_2 \) implies \( c_0 = \cdots = c_{t_2 - s_2 - 1} = 0 \). But \( f^s u_1 = j \sum_{j \geq 0} d_j f^j u_1 \) would imply \( u_1 = 0 \).

Our main theorem is the following.

**Theorem 3.5.** Assume \(|K| = 2\). Let \( z \in V, z \neq 0 \). Then there exists a generator tuple \( U = (u_1, \ldots, u_m) \) such that

\[ z = f^{\mu_1} u_1 + \cdots + f^{\mu_k} u_k, \tag{3.10} \]

and \( 0 \leq \mu_1 < \cdots < \mu_k \) and \( 0 < t_1 - \mu_1 < \cdots < t_k - \mu_k \). Let the numbers \( r_j, j = 1, \ldots, m, \) be defined by (3.3).

(i) If all generators \( u_{\rho_i} \) in (3.10) are unrepeated or if only one of them is repeated then the space \( X = \langle z \rangle \) is hyperinvariant, and

\[ X = (\text{Im} f^{\mu_1} \cap \text{Ker} f^{t_1 - \mu_1}) + \cdots + (\text{Im} f^{\mu_k} \cap \text{Ker} f^{t_k - \mu_k}). \]

(ii) If at least two of the generators \( u_{\rho_i} \) in (3.10) are unrepeated then the space \( X = \langle z \rangle \) is not hyperinvariant, and

\[ X_H = \langle f^{r_1} u_1, f^{r_2} u_2, \ldots, f^{r_m} u_m \rangle, \tag{3.11} \]

and \( X_H \neq X \).

**Proof.** The first part of the theorem is Baer’s lemma. Part (i) is a special case of Lemma 2.10. We now prove part (ii). Let \( u_{\rho_r} \) and \( u_{\rho_r} \) be two unrepeated generators. Define \( T = \langle u_{\rho_r}, u_{\rho_r} \rangle, \pi_T = \pi_{\rho_r} + \pi_{\rho_r}, \)

\[ \tilde{z} = \pi_T z = f^{\mu_1} u_{\rho_r} + f^{\mu_1} u_{\rho_r} \quad \text{and} \quad \langle \tilde{z} \rangle = \langle \alpha_T \tilde{z}; \alpha_T \in \text{Aut}(f_T, V) \rangle. \]

By Corollary 3.4 the subspace \( \langle \tilde{z} \rangle \) is not hyperinvariant (in \( T \)), and

\[ f^{\mu_1} u_{\rho_r} \notin \langle \tilde{z} \rangle \quad \text{and} \quad f^{\mu_1} u_{\rho_r} \notin \langle \tilde{z} \rangle. \tag{3.12} \]

Then (3.6) implies

\[ \pi_T X = \pi_T \langle z \rangle + \pi_T \hat{X} = \langle \tilde{z} \rangle + \langle f^{r_1} u_{\rho_r}, f^{r_2} u_{\rho_r} \rangle = \langle f^{\mu_1} u_{\rho_r} + f^{\mu_1} u_{\rho_r} \rangle + \langle f^{\mu_1 + 1} u_{\rho_r}, f^{\mu_1 + 1} u_{\rho_r} \rangle = \langle \tilde{z} \rangle. \]
Suppose \( \pi_{\rho_s} z = f^{\mu_{\rho_s}} u_{\rho_s} \in X \). Then
\[
\pi_T f^{\mu_{\rho_s}} u_{\rho_s} = f^{\mu_{\rho_s}} u_{\rho_s} \in \pi_T X = \langle \tilde{z} \rangle^c_T,
\]
in contradiction to (3.12). Hence \( \pi_{\rho_s} z \notin X \) and also \( \pi_{\rho_t} z \notin X \). Therefore \( X \) is not hyperinvariant.

It remains to prove (3.11). From \( fz = \sum_{i=1}^{k} f^{\mu_{\rho_i}} u_{\rho_i} \) and \( f^{\mu_{\rho_i}+1} u_{\rho_i} \in \langle f^{r_i} u_{\rho_i} \rangle \), \( i = 1, \ldots, k \), follows \( fz \in \hat{X} \). Then \( \langle z \rangle = \text{span}\{z, fz, \ldots, f^{n-1}z\} \) implies \( X/\hat{X} = \{\hat{X}, z + \hat{X}\} \). Therefore \( \dim(X/\hat{X}) = 1 \), and subspace \( \hat{X} \) has codimension 1 in \( X \). We have \( \hat{X} \subseteq X_H \subseteq X \). Since \( X \) is not hyperinvariant we obtain \( \hat{X} = X_H \). \( \square \)

3.1. Elementary divisors of \( f|_X \)

In this section we determine the elementary divisors of the restricted map \( f|_X \) where \( X = \langle z \rangle^c \). We assume that \( X \) is not hyperinvariant. Recall
\[
X = \langle z \rangle + X_H \quad \text{with} \quad X_H = \langle f^{r_1} u_1, \ldots, f^{r_m} u_m \rangle.
\]
By definition the numbers \( r_j \) satisfy \( 0 \leq r_j \leq t_j \), \( j = 1, \ldots, m \). Hence, if \( r_j = t_j \) then \( f^{r_j} u_j = 0 \), such that in (3.11) the corresponding summand \( \langle f^{r_j} u_j \rangle \) of \( X_H \) can be discarded. Suppose \( q \in \{1, \ldots, m\} \) is such that \( r_q < t_q \). Then Theorem 2.7 implies \( 0 < t_q - r_q \leq t_{q+1} - r_{q+1} \leq \cdots \leq t_m - r_m \). Therefore \( \langle f^{r_j} u_j \rangle \neq 0 \) if \( j \geq q \). Let \( j_0 \) satisfy
\[
r_{j_0} < t_{j_0} \quad \text{and} \quad r_j = t_j \quad \text{if} \quad j < j_0,
\]
then
\[
\langle f^{r_j} u_j \rangle \neq 0 \quad \text{if and only if} \quad j \geq j_0. \tag{3.13}
\]
Thus \( X_H = \sum_{j_0 \leq j \leq m} \langle f^{r_j} u_j \rangle \).

**Theorem 3.6.** Suppose \( |K| = 2 \). Let
\[
z = f^{\mu_{\rho_1}} u_{\rho_1} + \cdots + f^{\mu_{\rho_k}} u_{\rho_k}, \quad k \geq 2,
\]
be such that (3.5) holds. Assume that among the elementary divisors \( \lambda^{\mu_i} \), \( i = 1, \ldots, k \), there are at least two unrepeated ones, and let \( \ell \in \{i; 1 \leq i \leq k\} \) be the largest integer such that \( \lambda^{\mu_\ell} \) is unrepeated. Define
\[
z(\ell) = f^{\mu_{\rho_1}} u_{\rho_1} + \cdots + f^{\mu_{\rho_\ell}} u_{\rho_{\ell}}.
\]
Let the numbers $r_j, j = 1, \ldots, m$, be given by (3.3). Let $j_0 \in \{ j; 1 \leq j \leq m \}$ be the smallest integer such that $r_j < t_j$. Set $X = \langle z \rangle^c$. Then

$$X = \langle z(\ell) \rangle \oplus \left( \bigoplus_{j_0 \leq j \leq m; j \neq \rho_\ell} \langle f^{r_j}u_j \rangle \right). \tag{3.14}$$

Define

$$\tilde{u}_j = \begin{cases} f^{r_j}u_j & \text{if } j \in \{ j_0, j_0 + 1, \ldots, m \} \setminus \{ \rho_\ell \}, \\ z(\ell) & \text{if } j = \rho_\ell. \end{cases}$$

Then $\tilde{U} = (\tilde{u}_{j_0}, \ldots, \tilde{u}_m)$ is a generator tuple of $X$ (with respect to $f|_X$). The elementary divisors of $f|_X$ are $\tilde{t}_j$, $j = j_0, j_0 + 1, \ldots, m$, where

$$\tilde{t}_j = \begin{cases} t_j - r_j & \text{if } j \in \{ j_0, j_0 + 1, \ldots, m \} \setminus \{ \rho_\ell \}, \\ t_j - r_j + 1 & \text{if } j = \rho_\ell. \end{cases}$$

**Proof.** If $i \in \{ \ell + 1, \ldots, k \}$ then $u_{\rho_i}$ is a repeated generator. Therefore $f^{\mu_{\rho_i}}u_{\rho_i} = f^{r_{\rho_i}}u_{\rho_i} \in X, i = \ell + 1, \ldots, k$. We define

$$X(\rho_\ell) = \sum_{1 \leq j \leq m; j \neq \rho_\ell} \langle f^{r_j}u_j \rangle,$$

and

$$z(\ell - 1) = f^{\mu_{\rho_1}}u_{\rho_1} + \cdots + f^{\mu_{\rho_{\ell - 1}}}u_{\rho_{\ell - 1}} \quad \text{and} \quad z(\ell) = z(\ell - 1) + f^{\mu_{\rho_\ell}}u_{\rho_\ell}.$$

Then

$$X_H = X(\rho_\ell) + \langle f^{r_{\rho_\ell}}u_{\rho_\ell} \rangle \tag{3.15}$$

and

$$X = \langle z \rangle + \langle f^{r_1}u_1, \ldots, f^{r_m}u_m \rangle = \langle z(\ell) \rangle + \langle f^{r_1}u_1, \ldots, f^{r_m}u_m \rangle.$$

Recall that $r_{\rho_i} = \mu_{\rho_i}$ or $r_{\rho_i} = \mu_{\rho_i} + 1$. Therefore, in any case,

$$f^{\mu_{\rho_i} + 1}u_{\rho_i} \in \langle f^{r_{\rho_i}}u_{\rho_i} \rangle \subseteq X,$$

$i = 1, \ldots, m$. Hence $fz(\ell) \in X$ and $fz(\ell - 1) \in X(\rho_\ell)$. Since $u_{\rho_\ell}$ is unrepeated we have $f^{\mu_{\rho_\ell} + 1}u_{\rho_\ell} = f^{r_{\rho_\ell}}u_{\rho_\ell}$. Therefore $fz(\ell) = fz(\ell - 1) + f^{r_{\rho_\ell}}u_{\rho_\ell}$, and we obtain

$$f^{r_{\rho_\ell}}u_{\rho_\ell} = fz(\ell) + fz(\ell - 1) \in \langle z(\ell) \rangle + X(\rho_\ell).$$
Then (3.15) implies

\[ X = \langle z(\ell) \rangle + X_{(\rho\ell)}. \] (3.16)

Define \( \gamma_i = t_{\rho\ell} - \mu_{\rho\ell}, i = 1, \ldots, k \). Then

\[ e(z(\ell)) = \max\{e(f^{\mu_{\rho\ell}}u_{\rho\ell}); 1 \leq i \leq \ell\} = \max\{\gamma_i; 1 \leq i \leq \ell\} = \gamma_\ell. \] (3.17)

We consider \( \langle z(\ell) \rangle \cap X_{(\rho\ell)} \). If \( w \in \langle z(\ell) \rangle \) then

\[ w = \sum_{s=0}^{\gamma_\ell-1} c_s f^s z(\ell) = \sum_{s=0}^{\gamma_\ell-1} c_s f^s z(\ell-1) + \sum_{s=0}^{\gamma_\ell-1} c_s f^{s+\mu_{\rho\ell}} u_{\rho\ell} = w_{(\ell-1)} + w_{\rho\ell} \]

with

\[ w_{(\ell-1)} \in \sum_{1 \leq j \leq \rho\ell-1} \langle f^j u_j \rangle \subseteq X_{(\rho\ell)} \quad \text{and} \quad w_{\rho\ell} \in \langle u_{\rho\ell} \rangle. \]

Suppose \( w \in \langle z(\ell) \rangle \cap X_{(\rho\ell)} \). Then \( X_{(\rho\ell)} \cap \langle u_{\rho\ell} \rangle = 0 \) implies

\[ w_{\rho\ell} = \sum_{s=0}^{\gamma_\ell-1} c_s f^{s+\mu_{\rho\ell}} u_{\rho\ell} = 0. \]

Because of \( e(f^{\mu_{\rho\ell}}u_{\rho\ell}) = \gamma_\ell \) the vectors \( \{f^{\mu_{\rho\ell}}u_{\rho\ell}, \ldots, f^{\gamma_\ell-1}u_{\rho\ell}\} \) are linearly independent. Hence \( c_0 = \cdots = c_{\gamma_\ell-1} = 0 \), and therefore \( w = 0 \). Hence \( \langle z(\ell) \rangle \cap X_{(\rho\ell)} = 0 \). Together with (3.16) and (3.13) this implies (3.14).

Because of \( e(f^{r_j}u_j) = t_j - r_j \) the generators of \( X_{(\rho\ell)} \) yield elementary divisors \( \lambda^{t_j-r_j} \). From (3.17) follows \( e(z(\ell)) = t_{\rho\ell} - \mu_{\rho\ell} = t_{\rho\ell} - (r_\ell - 1) \). Hence the corresponding elementary divisor is \( \lambda^{t_{\rho\ell}-r_\ell+1} \).

**References**


[5] R. Baer, Types of elements and characteristic subgroups of Abelian

Math. 59 (1937), 99–117.

[7] Kh. Benabdallah and B. Charles, Orbits of invariant subspaces of alge-

invariant subspaces of a matrix, Linear Algebra Appl. 150 (1991), 209–
226.

subspace lattice of a linear transformation, Linear Algebra Appl. 17


[12] I. Gohberg, P. Lancaster, and L. Rodman, Invariant Subspaces of Ma-

Ann Arbor, 1954.

[14] W. E. Longstaff, A lattice-theoretic description of the lattice of hyper-
invariant subspaces of a linear transformation, Can. J. Math. 28 (1976),
1062–1066.

[15] W. E. Longstaff, Picturing the lattice of invariant subspaces of a nilpo-

[16] K. Shoda, Über die charakteristischen Untergruppen einer endlichen