

Simultaneous Reduction to Triangular and Companion Forms of Pairs of Matrices: The Case $\text{rank}(I - AZ) = 1$

H. Bart

*Econometric Institute
Erasmus University Rotterdam
P.O. Box 1738
3000 DR Rotterdam, The Netherlands*

and

H. K. Wimmer

*Mathematisches Institut
Universität Würzburg
D-8700 Würzburg, Germany*

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ABSTRACT

This paper is concerned with simultaneous reduction to triangular and companion forms of pairs of matrices A, Z with $\text{rank}(I - AZ) = 1$. Special attention is paid to the case where A is a first and Z is a third companion matrix. Two types of simultaneous triangularization problems are considered: (1) the matrix A is to be transformed to upper triangular and Z to lower triangular form, (2) both A and Z are to be transformed to the same (upper) triangular form. The results on companions are made coordinate free by characterizing the pairs A, Z for which there exists an invertible matrix S such that $S^{-1}AS$ is of first and $S^{-1}ZS$ is of third companion type. One of the main theorems reads as follows: If $\text{rank}(I - AZ) = 1$ and $\alpha\zeta \neq 1$ for every eigenvalue α of A and every eigenvalue ζ of Z , then A and Z admit simultaneous reduction to complementary triangular forms.

1. INTRODUCTION

Let A and Z be (complex) $m \times m$ matrices. We say that A and Z admit *simultaneous reduction to upper triangular forms* if there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ and $S^{-1}ZS$ are upper triangular. In 1896, Frobenius [8] investigated this property, which has since become well

understood. It can be characterized in different equivalent forms, e.g. as semicommutativity of A and Z or as solvability of the Lie algebra generated by A and Z . For details, see [14], [11], [12], and the references given there (cf. also [16]).

Recently another kind of simultaneous triangularization has received attention. We say that A and Z admit *simultaneous reduction to complementary triangular forms* if there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ is upper triangular and $S^{-1}ZS$ is lower triangular. The motivation for this type of reduction comes from systems theory. Indeed, there is a close connection with the problem of complete factorization of rational matrix functions. For more information, see [1] and the references given there. A particular instance of complementary triangularization arises in the theory of 2D systems [9].

For simultaneous reduction to complementary triangular forms no comprehensive theory is yet available. For special classes of matrices, such as upper triangular Toeplitz matrices, companion matrices, and rank one and Jordan matrices, results have been obtained in [1], [2], [3], [4], [5], [6], and [15]. In several of these publications, attention is paid to cases where $A - Z$ has rank 1, and in this context there is a relation with the problem of simultaneous reduction to first (or second) companion forms (cf. [3] and [5]).

In the present paper, we focus on the multiplicative version of the rank one condition, i.e., we consider matrices A and Z for which $\text{rank}(I - AZ) = 1$. Here simultaneous reduction to first and third (or second and fourth) companion matrices enters the picture. For the convenience of the reader we include the definition of these different types of companions.

An $m \times m$ matrix A is called a *first companion matrix* if it has the form

$$A = \begin{bmatrix} 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{m-2} & a_{m-1} \end{bmatrix},$$

where a_0, \dots, a_{m-1} are complex numbers. An $m \times m$ *second companion matrix* is the transpose of an $m \times m$ first companion matrix. An $m \times m$ matrix Z is called a *third companion matrix* if it has the form

$$Z = \begin{bmatrix} \tilde{z}_{m-1} & \tilde{z}_{m-2} & \cdots & \tilde{z}_1 & \tilde{z}_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix},$$

where z_0, \dots, z_{m-1} are complex numbers. An $m \times m$ *fourth companion matrix* is the transpose of an $m \times m$ *third companion matrix*.

Now let us describe the contents of the present paper. In order to make the connection with the additive rank one case, we begin by recalling some of the main results from [1], [3], and [5].

THEOREM 1.1.A [1, Theorem 0.2]. *Let A and Z be $m \times m$ matrices with $\text{rank}(A - Z) = 1$. Suppose A and Z have no common eigenvalues. Then A and Z admit simultaneous reduction to complementary triangular forms.*

THEOREM 1.2.A [1, Theorem 0.3]. *Let A and Z be $m \times m$ first companion matrices. Then A and Z admit simultaneous reduction to complementary triangular forms if and only if there exist orderings $\alpha_1, \dots, \alpha_m$ of the eigenvalues of A and ζ_1, \dots, ζ_m of the eigenvalues of Z such that*

$$\alpha_j \neq \zeta_k, \quad 1 \leq j < k \leq m.$$

If A and Z are $m \times m$ first companion matrices, then $\text{rank}(A - Z) \leq 1$.

Note that [1, Theorem 0.3] is concerned with *second* companion matrices. The result for *first* companion matrices can be obtained by taking transposes.

THEOREM 1.3.A [5, Theorem 0.1]. *Let A and Z be $m \times m$ first companion matrices. Then A and Z admit simultaneous reduction to upper triangular forms if and only if there exist orderings $\alpha_1, \dots, \alpha_m$ of the eigenvalues of A and ζ_1, \dots, ζ_m of the eigenvalues of Z such that*

$$\alpha_k = \zeta_k, \quad k = 1, \dots, m - 1.$$

The result in [5] referred to above actually deals with simultaneous reduction to upper triangular forms of *sets* (rather than *pairs*) of first companion matrices. See also [5, Theorem 0.4].

THEOREM 1.4.A [3, Theorem 0.1]. *Let A and Z be $m \times m$ matrices. Then there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ and $S^{-1}ZS$ are first companion matrices if and only if $A - Z$ can be written as*

$$A - Z = bc^T,$$

where $b, c \in \mathbb{C}^m$ and b is a cyclic vector for A .

Recall that $b \in \mathbb{C}^m$ is a *cyclic vector* for A if the vectors $b, Ab, \dots, A^{m-1}b$ are linearly independent (i.e., they form a basis for \mathbb{C}^m). For a generalization

of Theorem 1.4.A to *sets* of matrices and *block* companions, see [5]. Note that $A - Z$ can be written as a dyadic product bc^T if and only if $\text{rank}(A - Z) \leq 1$. The point in Theorem 1.4.A is that b should be cyclic for A . Finally, observe that Theorem 1.4.A can be used to make Theorems 1.2.A and 1.3.A coordinate free (cf. [3] and [5]).

Next, let us turn to the multiplicative analogues of the results cited above.

THEOREM 1.1.M. *Let A and Z be $m \times m$ matrices with $\text{rank}(I - AZ) = 1$. Suppose $\alpha\zeta \neq 1$ for every eigenvalue α of A and every eigenvalue ζ of Z . Then A and Z admit simultaneous reduction to complementary triangular forms.*

If A is an $m \times m$ first companion matrix and Z is an $m \times m$ third companion matrix, then $\text{rank}(I - AZ) \leq 1$.

THEOREM 1.2.M. *Let A be an $m \times m$ first companion matrix and let Z be an $m \times m$ third companion matrix. Then A and Z admit simultaneous reduction to complementary triangular forms if and only if there exist orderings $\alpha_1, \dots, \alpha_m$ of the eigenvalues of A and ζ_1, \dots, ζ_m of the eigenvalues of Z such that*

$$\alpha_j \zeta_k \neq 1, \quad 1 \leq j < k \leq m.$$

THEOREM 1.3.M. *Let A be an $m \times m$ first companion matrix and let Z be an $m \times m$ third companion matrix. Then A and Z admit simultaneous reduction to upper triangular forms if and only if there exist orderings $\alpha_1, \dots, \alpha_m$ of the eigenvalues of A and ζ_1, \dots, ζ_m of the eigenvalues of Z such that*

$$\alpha_k \zeta_k = 1, \quad k = 1, \dots, m - 1.$$

THEOREM 1.4.M. *Let A and Z be $m \times m$ matrices. Then there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ is a first and $S^{-1}ZS$ is a third companion matrix if and only if the matrix $I - AZ$ can be written as*

$$I - AZ = bc^T,$$

where $b, c \in \mathbb{C}^m$ and b is a cyclic vector for A .

Analogues of Theorems 1.2.M–1.4.M for second and fourth companion matrices (instead of first and third) can be obtained by taking transposes.

Theorem 1.4.M may be used to make Theorems 1.2.M and 1.3.M coordinate free. We leave the details to the reader (cf. [3] and [5]). Theorem 1.1.M

is a corollary to Theorems 1.2.M and 1.4.M. Indeed, the rank condition means that $I - AZ$ can be written in the form $I - AZ = bc^T$. The key observation is now that the condition on the eigenvalues implies that b is cyclic for A . For details, see Section 4, which also contains the proof of Theorem 1.4.M.

Theorems 1.2.M and 1.3.M are proved in Sections 2 and 3, respectively. The latter also contains a discussion of the combinatorial condition on the eigenvalues of A and Z appearing in Theorem 1.2.M.

Theorems 1.1.M, 1.2.M, and 1.3.M are concerned with the existence of an invertible matrix S such that $S^{-1}AS$ and $S^{-1}ZS$ are triangular matrices. It is possible to amplify the results by supplying information about the orders in which the eigenvalues of A and Z may appear on the diagonals of $S^{-1}AS$ and $S^{-1}ZS$, respectively. In fact, it is these refined versions that are presented below.

The theorems marked with an M are multiplicative analogues of those marked with an A. A discussion of the relationship between the two sets of results is presented in Section 5. The main observation is that Theorem 1.1.M is a stronger result than Theorem 1.1.A. Section 5 also contains some comments about possible generalizations.

A polynomial approach for simultaneous triangularization of companion matrices is developed in [15].

We finish the introduction with a few remarks about notation and terminology. All matrices to be considered have complex entries. The $n \times n$ identity matrix is denoted simply by I , the order of the matrix always being clear from the context. The determinant and rank of a matrix H are written as $\det H$ and $\text{rank } H$, respectively. The superscript T signals the operation of taking the transpose of a matrix or vector. Whenever convenient, matrices are identified with linear operators. The restriction of a linear operator A to an invariant subspace M of A is denoted by $A|_M$. For the null space and image of A , we use the expressions $\text{Ker } A$ and $\text{Im } A$, respectively. The linear hull of a set of vectors v_1, \dots, v_n in \mathbb{C}^m is denoted by $\text{span}[v_1, \dots, v_n]$. The symbol ■ stands for “end of proof.”

2. SIMULTANEOUS UPPER TRIANGULAR FORMS OF COMPANION MATRICES

We begin with some remarks about the four different types of companion matrices described in the introduction. Let m be a positive integer, and let R denote the $m \times m$ *reversed identity matrix*, i.e.,

$$Re_j = e_{m+1-j}, \quad j = 1, \dots, m,$$

where e_1, \dots, e_m is the standard basis in \mathbb{C}^m . Note that R is its own inverse. An $m \times m$ third (fourth) companion matrix is obtained from an $m \times m$ first (second) companion matrix by pre- and postmultiplying with R . The inverse of an invertible first companion matrix is a third companion matrix. An analogous observation holds, of course, for second, third, and fourth companion matrices.

In the remainder of this section we discuss simultaneous reduction to upper triangular forms of pairs of companion matrices. Necessary and sufficient conditions for pairs of first (or of second) companion matrices were given in [3] and [5]. Here we consider pairs involving a first and a third companion matrix. Similar results hold for pairs consisting of a second and fourth companion matrix.

The following lemma is a counterpart of [5, Lemma 1.1] and [3, Lemma 2.3]. Whenever convenient, matrices will be identified with linear operators.

LEMMA 2.1. *Let A be an $m \times m$ first companion matrix, let Z be an $m \times m$ third companion matrix, and let M be a nontrivial invariant subspace for both A and Z . Then the restriction $A|_M$ of A to M is invertible, and $(A|_M)^{-1} = Z|_M$.*

Proof. Let r be the dimension of M , so $1 \leq r \leq m - 1$. Since A is nonderogatory, we have $M = \text{Ker } p(A)$, where $p(\lambda) = p_0 + p_1\lambda + \dots + p_{r-1}\lambda^{r-1} + \lambda^r$ is a divisor of the characteristic polynomial of A . Put

$$M^\perp = \{x \in \mathbb{C}^m \mid x^T m = 0 \text{ for all } m \in M\}.$$

Then $M^\perp = \text{Im } p(A^T)$, and M^\perp has dimension $s = m - r$.

The matrix A^T is of second companion type. Hence $M^\perp = \text{Im } p(A^T)$ is spanned by the s linearly independent vectors

$$\sum_{j=0}^r p_j e_{j+k}, \quad k = 1, \dots, s, \tag{2.1}$$

where $p_r = 1$ and, as before, e_1, \dots, e_m is the standard basis in \mathbb{C}^m .

Let \hat{M} be the quotient space \mathbb{C}^m / M^\perp . Whenever convenient, the element $x + M^\perp$ of \hat{M} with representative $x \in \mathbb{C}^m$ will be denoted by \hat{x} . Since M^\perp is an invariant subspace for A^T , we can introduce the linear operator \hat{A} on \hat{M} by stipulating that $\hat{A}\hat{x} = A^T x + M^\perp$. From the description of M^\perp as the linear hull of the vectors (2.1), it is clear that \mathbb{C}^m is the direct sum of $\text{span}[e_1, \dots, e_r]$ and M^\perp . Hence $\hat{e}_1, \dots, \hat{e}_r$ is a basis for \hat{M} . With

respect to this basis \hat{A} has the (second companion type) matrix representation

$$\hat{A} = \begin{bmatrix} 0 & 0 & & -p_0 \\ 1 & 0 & & -p_1 \\ 0 & 1 & & -p_2 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 & -p_{r-1} \end{bmatrix}.$$

Note in this context that $p_0e_1 + \dots + p_{r-1}e_r + A^Te_r = p(A^T)e_1 \in M^\perp$.

Next, let us turn to the third companion matrix Z . Write $M = \text{Ker } q(Z)$, where $q(\lambda) = q_0 + q_1\lambda + \dots + q_{r-1}\lambda^{r-1} + \lambda^r$ is a divisor of the characteristic polynomial of Z . Then $M^\perp = \text{Im } q(Z^T)$ is spanned by the s linearly independent vectors

$$\sum_{j=0}^r q_{r-j}e_{j+k}, \quad k = 1, \dots, s, \tag{2.2}$$

where $q_r = 1$. Comparing the systems of vectors (2.1) and (2.2) spanning the same subspace M^\perp , and recalling that $p_r = q_r = 1$, we obtain $p_0q_0 = 1$ and

$$p_j = \frac{1}{q_0}q_{r-j}, \quad j = 0, \dots, r. \tag{2.3}$$

Now introduce the linear operator \hat{Z} on $\hat{M} = \mathbb{C}^m/M^\perp$ by stipulating that $\hat{Z}\hat{x} = Z^Tx + M^\perp$. The matrix representation of \hat{Z} with respect to the basis $\hat{e}_1, \dots, \hat{e}_r$ in \hat{M} has the (fourth companion) form

$$\hat{Z} = \begin{bmatrix} -q_{r-1} & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -q_2 & & & 1 & 0 \\ -q_1 & & & 0 & 1 \\ -q_0 & & & 0 & 0 \end{bmatrix},$$

and (2.3) implies that \hat{A} is the inverse of \hat{Z} .

We finish the argument by taking advantage of the duality between \mathbb{C}^m/M^\perp and M . Indeed, for $m \in M$ and $x \in \mathbb{C}^m$, we have $\hat{Z}\hat{Ax} = \hat{x}$, so

$Z^T A^T x - x \in M^\perp$ and $(Z^T A^T x - x)^T m = 0$. The latter identity can be rewritten as $x^T (AZm - m) = 0$. Since $x \in \mathbb{C}^m$ was taken arbitrarily, it follows that $AZm = m$, $m \in M$. In other words $A|_M$ is the inverse of $Z|_M$. ■

For another proof of Lemma 2.1, in the same vein as that of [5, Lemma 1.1], see [7]. Of course [5, Lemma 1.1] can also be proved using the method employed in the proof of Lemma 2.1.

The *diagonal* of a matrix $K = (k_{ij})_{i,j=1}^m$ is the ordered m -tuple (k_{11}, \dots, k_{mm}) . The next result is a refined version of Theorem 1.3.M in the introduction.

THEOREM 2.2. *Let A be an $m \times m$ first companion matrix and let Z be an $m \times m$ third companion matrix. Let $\alpha_1, \dots, \alpha_m$ be an ordering of the eigenvalues of A and let ζ_1, \dots, ζ_m be an ordering of the eigenvalues of Z . Then there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ and $S^{-1}ZS$ are upper triangular with diagonals $(\alpha_1, \dots, \alpha_m)$ and $(\zeta_1, \dots, \zeta_m)$, respectively, if and only if*

$$\alpha_k \zeta_k = 1, \quad k = 1, \dots, m - 1. \tag{2.4}$$

Since companion matrices are nonderogatory, the matrix S (provided it exists) is unique up to multiplication on the right with an invertible upper triangular matrix (cf. [2, Proposition 2.11] and the remark following Theorem 3.2 below).

The proof that we shall present below gives more information than is needed to establish Theorem 2.2 as it stands. The extra details are printed in italics.

Proof. We start with the “only if” part. Let e_1, \dots, e_m be the standard basis in \mathbb{C}^m , and put $N = \text{span}[e_1, \dots, e_{m-1}]$. Then N is an invariant subspace for both $S^{-1}AS$ and $S^{-1}ZS$. Applying Lemma 2.1 to $M = S[N]$, we obtain the following result. Write A_0 (Z_0) for the matrix obtained from $S^{-1}AS$ ($S^{-1}ZS$) by omitting the last row and column. Then A_0 is invertible with inverse Z_0 . Note that A_0 is upper triangular with diagonal $(\alpha_1, \dots, \alpha_{m-1})$ and Z_0 is upper triangular with diagonal $(\zeta_1, \dots, \zeta_{m-1})$. It follows that (2.4) is satisfied.

Before we turn to the “if” part of Theorem 2.2, we need some preparations. For z in \mathbb{C} and $k = 0, \dots, m - 1$, write

$$v_k(z) = \sum_{j=k+1}^m \binom{j-1}{k} z^{j-1-k} e_j.$$

The vectors $v_k(z)$ can be used to describe the invariant subspaces of the first companion matrix A (see [13, Section 2.11, Exercise 22]; cf. also [3] and [5]). Analogously, the vectors of the form $w_k(z) = Rv_k(z)$, where R is the $m \times m$ reversed identity matrix, can be employed to describe the invariant subspaces of the third companion matrix Z . There is an interesting relationship between spans of the vectors $v_j(z)$ and spans of the vectors $w_j(1/z)$. Indeed, for $z \neq 0$ and $n = 1, \dots, m$, we have

$$\text{span}[v_0(z), \dots, v_{n-1}(z)] = \text{span}\left[w_0\left(\frac{1}{z}\right), \dots, w_{n-1}\left(\frac{1}{z}\right)\right]. \quad (2.5)$$

This can be proved by brute computation, but we prefer the following “geometric” argument.

By $C_{1,m}(z)$ we denote the $m \times m$ first companion matrix associated with the polynomial $(\lambda - z)^m$. Thus $C_{1,m}(z)$ is the unique $m \times m$ first companion whose characteristic polynomial is $(\lambda - z)^m$. Since $z \neq 0$, the matrix $C_{1,m}(z)$ is invertible, and

$$C_{1,m}(z)^{-1} = C_{3,m}\left(\frac{1}{z}\right), \quad (2.6)$$

where $C_{3,m}(1/z)$ is the $m \times m$ third companion matrix associated with the polynomial $(\lambda - 1/z)^m$. Both $C_{1,m}(z)$ and $C_{3,m}(1/z)$ are unicellular. The unique n -dimensional invariant subspace for $C_{1,m}(z)$ is given by the left hand side of (2.5), and the unique n -dimensional invariant subspace for $C_{3,m}(1/z)$ is given by the right hand side of (2.5). From (2.6) it is clear that $C_{1,m}(z)$ and $C_{3,m}(1/z)$ have the same invariant subspaces, and it follows that (2.5) is satisfied.

After these preparations, we are ready to give the proof proper of the “if” part of Theorem 2.2. For $j = 1, \dots, m-1$, let r_j be the number of $t \in \{1, \dots, j-1\}$ such that $\alpha_t = \alpha_j$. Now let the first $m-1$ columns of the $m \times m$ matrix S be given by

$$Se_j = v_{r_j}(\alpha_j), \quad j = 1, \dots, m-1. \quad (2.7)$$

The last column Se_m of S is taken independently of Se_1, \dots, Se_{m-1} , but otherwise arbitrarily. Since the vectors (2.7) are linearly independent, the matrix S is invertible. A straightforward computation shows that $S^{-1}AS$ is upper triangular with diagonal $(\alpha_1, \dots, \alpha_m)$. *The matrix obtained from $S^{-1}AS$ by omitting the last row and column resembles the upper triangular Jordan form.*

It remains to prove that $S^{-1}ZS$ is upper triangular with diagonal $(\zeta_1, \dots, \zeta_m)$. For $j = 1, \dots, m - 1$, put $N_j = \text{span}[e_1, \dots, e_j]$ and $M_j = S[N_j]$. From (2.4) and (2.5) it is clear that M_j is Z -invariant. But then N_j is invariant for $S^{-1}ZS$; hence $S^{-1}ZS$ is upper triangular. The statement about the diagonal of $S^{-1}ZS$ follows from (2.4) and the “only if” part of the theorem. ■

3. SIMULTANEOUS COMPLEMENTARY TRIANGULAR FORMS OF COMPANION MATRICES

In this section we consider simultaneous reduction to *complementary* triangular forms of companion matrices. Necessary and sufficient conditions for pairs of second (or of first) companion matrices were given in [1]; see also [3] and [5]. Here we consider pairs involving a first and a third companion matrix. Analogous results hold for pairs consisting of a second and a fourth companion matrix.

We begin with an intertwining lemma similar to [1, Lemma 3.1].

LEMMA 3.1. *Let A be an $m \times m$ first companion matrix and let Z be an $m \times m$ third companion matrix. Let $\alpha_1, \dots, \alpha_m$ be the eigenvalues of A , and let ζ_1, \dots, ζ_m be the eigenvalues of Z . Define*

$$\tilde{A} = \begin{bmatrix} \alpha_1 & \alpha_2 \zeta_2 - 1 & \zeta_2(\alpha_3 \zeta_3 - 1) & \zeta_2 \zeta_3(\alpha_4 \zeta_4 - 1) & \cdots & \zeta_2 \cdots \zeta_{m-1}(\alpha_m \zeta_m - 1) \\ 0 & \alpha_2 & \alpha_3 \zeta_3 - 1 & \zeta_3(\alpha_4 \zeta_4 - 1) & \cdots & \zeta_3 \cdots \zeta_{m-1}(\alpha_m \zeta_m - 1) \\ 0 & 0 & \alpha_3 & \alpha_4 \zeta_4 - 1 & \cdots & \zeta_4 \cdots \zeta_{m-1}(\alpha_m \zeta_m - 1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_{m-1} & \alpha_m \zeta_m - 1 \\ 0 & 0 & \cdots & 0 & 0 & \alpha_m \end{bmatrix},$$

$$\tilde{Z} = \begin{bmatrix} \zeta_1 & 0 & \cdots & 0 & 0 \\ \alpha_1 \zeta_1 - 1 & \zeta_2 & & 0 & 0 \\ \alpha_2(\alpha_1 \zeta_1 - 1) & \alpha_2 \zeta_2 - 1 & & & \cdot \\ \alpha_2 \alpha_3(\alpha_1 \zeta_1 - 1) & \alpha_3(\alpha_2 \zeta_2 - 1) & & & \cdot \\ \vdots & \vdots & & & \cdot \\ \alpha_2 \cdots \alpha_{m-2}(\alpha_1 \zeta_1 - 1) & \alpha_3 \cdots \alpha_{m-2}(\alpha_2 \zeta_2 - 1) & \cdots & \zeta_{m-1} & 0 \\ \alpha_2 \cdots \alpha_{m-1}(\alpha_1 \zeta_1 - 1) & \alpha_3 \cdots \alpha_{m-1}(\alpha_2 \zeta_2 - 1) & \cdots & \alpha_{m-1} \zeta_{m-1} - 1 & \zeta_m \end{bmatrix}.$$

$$E_\zeta = \begin{bmatrix} 1 & -\zeta_2 & 0 & & & \\ 0 & 1 & -\zeta_3 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & & \\ & & & & -\zeta_m & \\ & & & & & 1 \end{bmatrix}.$$

Consider $E_\alpha T$. It is easy to check that

$$E_\alpha T = \begin{bmatrix} 1 & t \\ 0 & D\tilde{T} \end{bmatrix},$$

where $(1 \ t)$ is the first row in T , D is an $(m - 1) \times (m - 1)$ diagonal matrix with diagonal $(\alpha_1 \zeta_2 - 1, \alpha_2 \zeta_3 - 1, \dots, \alpha_{m-1} \zeta_m - 1)$, and \tilde{T} is an $(m - 1) \times (m - 1)$ matrix of the same type as T associated with the $(m - 1)$ -tuples $(\alpha_1, \dots, \alpha_{m-1})$ and $(\zeta_2, \dots, \zeta_m)$. The formula for $\det T$ is now obtained by induction.

To prove the intertwining relation $TA = \tilde{A}T$, it suffices to show that $E_\zeta TA = E_\zeta \tilde{A}T$. The latter identity can be checked by direct computation. The work is facilitated by the fact that $E_\zeta \tilde{A}$ is a simple bidiagonal matrix:

$$E_\zeta \tilde{A} = \begin{bmatrix} \alpha_1 & -1 & 0 & & & \\ 0 & \alpha_2 & -1 & & & \\ 0 & 0 & \alpha_3 & \ddots & & \\ & & & \ddots & & \\ & & & & -1 & \\ & & & & & \alpha_m \end{bmatrix}.$$

Note that $E_\alpha \tilde{Z}$ has an analogous form. This can be used to verify the identity $TZ = \tilde{Z}T$. ■

We are now ready to prove Theorem 1.2.M from the introduction. In fact we shall establish the following more detailed result.

THEOREM 3.2. *Let A be an $m \times m$ first companion matrix and let Z be an $m \times m$ third companion matrix. Let $\alpha_1, \dots, \alpha_m$ be an ordering of the eigenvalues of A , and let ζ_1, \dots, ζ_m be an ordering of the eigenvalues of Z . Then there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ is upper triangular with diagonal $(\alpha_1, \dots, \alpha_m)$ and $S^{-1}ZS$ is lower triangular with diagonal $(\zeta_1, \dots, \zeta_m)$ if and only if*

$$\alpha_j \zeta_k \neq 1, \quad 1 \leq j < k \leq m.$$

Since companion matrices are nonderogatory, the matrix S (provided it exists) is unique up to multiplication on the right with an invertible diagonal matrix. Here, as in the remark following Theorem 2.2, one might refer to [2, Proposition 2.1]. What we use of this proposition is this: If P is a nonderogatory upper (lower) triangular matrix and U is an invertible matrix such that $U^{-1}PU$ is again upper (lower) triangular and has the same diagonal as P , then U is upper (lower) triangular too.

Proof. For the “only if” part, use a straightforward modification of the corresponding argument in the proof of [1, Theorem 3.2]. For the “if” part, employ Lemma 3.1. Thus S (or rather its inverse T), $S^{-1}AS$, and $S^{-1}ZS$ are obtained in an explicit way. ■

Let A be an $m \times m$ first companion matrix, and let Z be an $m \times m$ second companion matrix. There are several cases in which one can verify that there do exist orderings $\alpha_1, \dots, \alpha_m$ of the eigenvalues of A and ζ_1, \dots, ζ_m of the eigenvalues of Z such that

$$\alpha_j \zeta_k \neq 1, \quad 1 \leq j < k \leq m. \tag{3.1}$$

In fact, the situation is rather similar to that in [1, Section 3].

Let us give an example. Suppose A is an arbitrary $m \times m$ first companion matrix and Z is the $m \times m$ nilpotent lower triangular Jordan block (i.e., Z is the nilpotent $m \times m$ third companion matrix). Then, given an ordering $\alpha_1, \dots, \alpha_m$ of the eigenvalues of A , there exists an invertible (lower triangular) $m \times m$ matrix S such that $S^{-1}AS$ is upper triangular with diagonal $(\alpha_1, \dots, \alpha_m)$ and $S^{-1}ZS$ is lower triangular (with zeros on the diagonal). Additional details can be obtained from Lemma 3.1. These fit nicely with [2, Theorem 3.3].

We conclude this section with a remark concerning the (combinatorial) ordering condition (3.1). Let A and Z be $m \times m$ matrices (companions or not). By $\mathcal{S}(A, Z)$, we mean (the possibly empty) set of all nonzero eigenvalues τ of A such that τ^{-1} is an eigenvalue of Z . There exist orderings $\alpha_1, \dots, \alpha_m$ of the eigenvalues of A and ζ_1, \dots, ζ_m of the eigenvalues of Z such that (3.1) is satisfied if and only if there exists an ordering τ_1, \dots, τ_s of the (different) elements of $\mathcal{S}(A, Z)$ for which

$$\sum_{p=1}^q m_A(\tau_p) + \sum_{p=q}^s m_Z\left(\frac{1}{\tau_p}\right) \leq m + 1, \quad q = 1, \dots, s.$$

Here $m_A(\tau_p)$ denotes the algebraic multiplicity of τ_p as an eigenvalue of A , and $m_Z(1/\tau_p)$ denotes the algebraic multiplicity of $1/\tau_p$ as an eigenvalue of Z . The proof is similar to that of [5, Proposition 2.2].

4. SIMULTANEOUS COMPANION FORMS OF PAIRS OF MATRICES

The aim of this section is to prove Theorem 1.1.M and to show that our previous results on pairs of companion matrices can be made coordinate free. The latter is accomplished by proving Theorem 1.4.M.

Proof of Theorem 1.4.M. The “only if” part of the theorem is obvious. So we can concentrate on the “if” part.

Assume that $I - AZ$ can be written as $I - AZ = bc^T$, where $b, c \in \mathbb{C}^m$ and b is a cyclic vector for A . Then $[b \ Ab \ \cdots \ A^{m-1}b]$ is invertible with inverse V , say. For $j = 0, \dots, m - 1$, let v_j be the $(j + 1)$ th row of V , and put $a_j = -v_j A^m b$. Now introduce the invertible $m \times m$ Hankel matrix

$$H = \begin{bmatrix} a_1 & a_2 & \cdots & a_{m-1} & 1 \\ a_2 & \cdots & a_{m-1} & 1 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ a_{m-1} & 1 & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and write $S = V^{-1}H$. Then $S^{-1}AS$ is a first companion matrix (with $-a_0, \dots, -a_{m-1}$ in the last row). It remains to prove that $S^{-1}ZS$ is of third companion type.

Let e_1, \dots, e_m be the standard basis in \mathbb{C}^m . Then $S^{-1}b = H^{-1}Vb = H^{-1}e_1 = e_m$. Hence, for $j = 2, \dots, m$,

$$e_{j-1}^T (S^{-1}AS)(S^{-1}ZS) = e_{j-1}^T (I - S^{-1}bc^T S) = e_{j-1}^T.$$

Since $S^{-1}AS$ is a first companion matrix, we have that $e_{j-1}^T S^{-1}AS = e_j^T$. So, for $j = 2, \dots, m$, the j th row of $S^{-1}ZS$ is equal to e_{j-1}^T . In other words, $S^{-1}ZS$ is a third companion matrix. ■

PROPOSITION 4.1. *Let A and Z be $m \times m$ matrices with $\text{rank}(I - AZ) = 1$. Suppose that for each complex λ , the rank condition*

$$\text{rank} \begin{bmatrix} \lambda I - A & \lambda Z - I \end{bmatrix} = m$$

is satisfied. Then there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ is a first and $S^{-1}ZS$ is a third companion matrix.

An analogous result involving second and fourth companion matrices can be obtained by taking transposes. The rank condition amounts to requiring that the (linear) matrix polynomials $\lambda I - A$ and $\lambda Z - I$ be (left) polynomially coprime (cf. [1, Theorem 5.2]). Proposition 4.1 is an amplification of Theorem 1.4.M. There is an analogous amplification of Theorem 1.4.A involving the rank condition $\text{rank} \begin{bmatrix} \lambda I - A & \lambda I - Z \end{bmatrix} = m, \lambda \in \mathbb{C}$.

Proof. Write $I - AZ = bc^T$ with $b, c \in \mathbb{C}^m$. Take $\lambda \in \mathbb{C}, u \in \mathbb{C}^m$, and assume that

$$u^T(\lambda I - A) = 0, \quad u^T b = 0.$$

Then $u^T(\lambda Z - I) = \lambda u^T Z - u^T = u^T(AZ - I) = u^T bc^T = 0$. Since $\text{rank} \begin{bmatrix} \lambda I - A & \lambda Z - I \end{bmatrix} = m$, it follows that $u = 0$. So $\text{rank} \begin{bmatrix} \lambda I - A & b \end{bmatrix} = m, \lambda \in \mathbb{C}$. But then, by the Hautus test from systems theory (see [10]), the vector b is cyclic for A . Apply now Theorem 1.4.M. ■

THEOREM 4.2. *Let A and Z be $m \times m$ matrices with $\text{rank}(I - AZ) = 1$. The following statements are equivalent:*

- (i) *There exist invertible $m \times m$ matrices S and T such that $S^{-1}AS, S^{-1}ZS, T^{-1}AT$, and $T^{-1}ZT$ are first, third, second, and fourth companion matrices, respectively.*
- (ii) *The matrix $I - AZ$ can be written as $I - AZ = bc^T$ with b a cyclic vector for A and c a cyclic vector for Z^T .*
- (iii) *For each complex λ , the rank condition*

$$\text{rank} \begin{bmatrix} \lambda I - A & \lambda Z - I \end{bmatrix} = \text{rank} \begin{bmatrix} \lambda I - A \\ \lambda Z - I \end{bmatrix} = m$$

is satisfied.

- (iv) *For every eigenvalue α of A and every eigenvalue ζ of Z , the product $\alpha\zeta$ is not equal to 1.*

Clearly (i) and (ii) are symmetric in A and Z [for (i) use the reversed identity matrix]. As is well known, the matrices $I - AZ$ and $I - ZA$ always have the same rank. From this we see that (iv) is symmetric in A and Z too. The symmetry in (iii) is obvious from the fact that one can restrict oneself to nonzero λ . This also applies to Proposition 4.1.

Proof. Write $I - AZ = bc^T$ with $b, c \in \mathbb{C}^m$, and note that b and c are essentially unique. The equivalence of (i) and (ii) is now immediate from Theorem 1.4.M. Clearly (iv) implies (iii), and we see from Proposition 4.1 that (iii) implies (i). It remains to prove that (ii) implies (iv).

Let α and ζ be eigenvalues of A and Z , respectively. Choose nonzero vectors x and y in \mathbb{C}^m such that $x^T A = \alpha x^T$ and $Zy = \zeta y$. Since b is cyclic for A , we have that

$$x^T [b \quad Ab \quad \cdots \quad A^{m-1}b] \neq 0.$$

Now $x^T [b \quad Ab \quad \cdots \quad A^{m-1}b] = x^T b(1 \quad \alpha \quad \cdots \quad \alpha^{m-1})$; hence $x^T b \neq 0$. In an analogous way, using the fact that c is cyclic for Z^T , one proves that $c^T y \neq 0$. But then

$$(1 - \alpha\zeta)x^T y = x^T(I - AZ)y = (x^T b)(c^T y) \neq 0,$$

and it follows that $\alpha\zeta \neq 1$. ■

Theorem 1.1.M is now an immediate corollary to Theorems 3.2 and 4.2. In fact we have the following more detailed result.

COROLLARY 4.3. *Let A and Z be $m \times m$ matrices with $\text{rank}(I - AZ) = 1$. Suppose $\alpha\zeta \neq 1$ for every eigenvalue α of A and every eigenvalue ζ of Z . Let $\alpha_1, \dots, \alpha_m$ be the eigenvalues of A (in any order), and let ζ_1, \dots, ζ_m be the eigenvalues of Z (in any order). Then there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ is upper triangular with diagonal $(\alpha_1, \dots, \alpha_m)$ and $S^{-1}ZS$ is lower triangular with diagonal $(\zeta_1, \dots, \zeta_m)$.*

Again the matrix S is unique up to multiplication on the right with an invertible diagonal matrix. Combining the proofs of Theorems 1.4.M and 3.2, (the inverse of) one of the matrices S having the desired properties can be obtained in a rather explicit way. For this S , we have that $S^{-1}AS = \tilde{A}$ and $S^{-1}ZS = \tilde{Z}$, where \tilde{A} and \tilde{Z} are as in Theorem 2.4.

We conclude this section with a remark about cyclic vectors. Let A and Z be $m \times m$ matrices, and suppose $\text{rank}(I - AZ) = 1$. Then $\text{rank}(I - ZA) = 1$

too, and we can write

$$I - AZ = bc^T, \quad I - ZA = pq^T,$$

with $b, c, p, q \in \mathbb{C}^m$ essentially unique. Assume b is a cyclic vector for A . Then there exists an invertible $m \times m$ matrix S such that $S^{-1}AS$ is a first and $S^{-1}ZS$ is a third companion. Let R be the $m \times m$ reversed identity matrix, and put $T = SR$. Then $T^{-1}AT$ is a third and $T^{-1}ZT$ is a first companion matrix. Hence p must be a cyclic vector for Z . Is there a simple relationship between b and p ?

To answer this question, we introduce the vector

$$p_0 = (a_1I + a_2A + \dots + a_{m-1}A^{m-2} + A^{m-1})b,$$

where $\det(\lambda I - A) = \lambda^m + a_{m-1}\lambda^{m-1} + \dots + a_1\lambda + a_0$. Thus p_0 is the first column of the matrix S constructed in the proof of Theorem 1.4.M. This matrix has the property that $S^{-1}AS$ is of first and $S^{-1}ZS$ is of third companion type. But then, with the exception of the first, all rows of $I - (S^{-1}ZS)(S^{-1}AS)$ are zero. Since $I - (S^{-1}ZS)(S^{-1}AS) = S^{-1}pq^TS$, it follows that p is a scalar multiple of the first column of S . In other words, p is a scalar multiple of p_0 .

From the above, we may conclude that the vector p_0 is a cyclic vector for Z . In the case when A is invertible, this can also be seen as follows. By the Cayley-Hamilton theorem, $p_0 = -a_0A^{-1}b$ with $a_0 \neq 0$. Now $A^{-1} - Z = A^{-1}bc^T$ with $A^{-1}b$ cyclic for A^{-1} and Z . Hence p_0 is cyclic for Z .

5. REMARKS AND GENERALIZATIONS

First, let us return to the introduction and discuss the relationship between the two sets of theorems appearing there.

We begin by noting that under the additional assumption that A or Z is invertible, the theorems marked with an M can be derived easily from those marked with an A and vice versa. Indeed, if (for example) A is invertible, then $A - Z$ can be written as $A(I - A^{-1}Z)$ and $I - AZ$ as $A(A^{-1} - Z)$.

For Theorems 1.1.A and 1.1.M something extra can be said. The point is that even when neither A nor Z is invertible, Theorem 1.1.A can be deduced from Theorem 1.1.M. To see this, choose τ such that $A - \tau I$ is invertible. If

A and Z have no common eigenvalues, then $(\alpha - \tau)^{-1}(\zeta - \tau) \neq 1$ for every eigenvalue α of A and every eigenvalue ζ of Z . Also

$$I - (A - \tau I)^{-1}(Z - \tau I) = (A - \tau I)^{-1}(A - Z),$$

so $I - (A - \tau I)^{-1}(Z - \tau I)$ has rank one. Applying Theorem 1.1.M to $A - \tau I$ and $Z - \tau I$, we now see that these matrices admit simultaneous reduction to complementary triangular forms. But then the same is true for A and Z .

These considerations justify the conclusion that Theorem 1.1.M is a stronger result than Theorem 1.1.A.

Next, let us discuss some possibilities for generalizing the results obtained in this paper. Theorems 1.2.A and 1.3.A have been extended to sets (rather than pairs) of matrices (cf. [5]). An analogous generalization can be established for Theorems 1.2.M and 1.3.M. The results can be completed by adding information about the order in which eigenvalues may appear on the diagonals. Theorem 1.4.A admits an extension involving sets of matrices and block companions (see [5]). A similar generalization can be obtained for Theorem 1.4.M. The details are left to the reader.

Finally, we note that the equivalence of (ii) and (iii) in Theorem 4.2 can be placed in a wider context. Indeed, using the Hautus test as in the proof of Proposition 4.1, one can establish the following result.

Let A and Z be $m \times m$ matrices with $\text{rank}(I - AZ) = k$. Write $I - AZ = BC^T$, where B and C are $m \times n$ matrices and $\text{rank } C = k$. Then

$$\text{rank} \begin{bmatrix} B & AB & \cdots & A^{m-1}B \end{bmatrix} = m \quad (5.1)$$

if and only if for each complex λ the condition

$$\text{rank} \begin{bmatrix} \lambda I - A & \lambda Z - I \end{bmatrix} = m \quad (5.2)$$

is satisfied.

The proof of the “if” part does not use the assumption that $\text{rank } C = k$. The condition that (5.2) is satisfied for each complex λ amounts to requiring that the matrix polynomials $\lambda I - A$ and $\lambda Z - I$ be (left) polynomially coprime. In systems theoretical terms, (5.1) can be rephrased by saying that the pair (A, B) is controllable.

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