Pairs of Companion Matrices and Their Simultaneous Reduction to Complementary Triangular Forms

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ABSTRACT

A polynomial approach is described to deal with problems of the following type. Given two companion matrices A and Z, when does there exist a nonsingular matrix S such that $S^{-1}AS$ is upper triangular and $S^{-1}ZS$ is lower triangular, and the eigenvalues of A and Z appear in the main diagonals of $S^{-1}AS$ and $S^{-1}ZS$ in a prescribed ordering? Such problems are related to complete minimal factorizations in systems theory.

1. INTRODUCTION

Let

$$W(\lambda) = I_k + C(\lambda I_m - A)^{-1} B \tag{1.1}$$

be a matrix of rational functions where C, A, and B are complex matrices of size $k \times m$, $m \times m$, and $m \times k$ respectively. Such transfer matrices and their factorizations are important in systems theory (see e.g. [2]). The simplest matrices of the type (1.1) which can occur as (nontrivial) factors of $W(\lambda)$ are of the form

$$I + \frac{1}{\lambda - \alpha} R$$

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where R is of rank 1. A decomposition of $W(\lambda)$ into such *elementary matrices*, i.e.

$$W(\lambda) = \left(I + \frac{1}{\lambda - \alpha_1}R_1\right)\left(I + \frac{1}{\lambda - \alpha_2}R_2\right)\cdots\left(I + \frac{1}{\lambda - \alpha_m}R_m\right)$$

where

$$\operatorname{rank} R_i = 1, \qquad i = 1, \dots, m,$$

is called a *complete factorization* [1]. Whether or not a given $W(\lambda)$ admits such a complete factorization depends on the matrices A and $Z = A - BC$. Note that Z has the property that

$$W^{-1}(\lambda) = I - C(\lambda I - Z)^{-1}B.$$

The following theorem has been the motivation to study complementary triangular forms.

THEOREM 1.1 [1, p. 220]. *The matrix* $W(\lambda) = I + C(\lambda I - A)^{-1}B$ *admits a complete factorization if and only if for* A *and* $Z = A - BC$ *there exists a nonsingular matrix* S *such that*

$(*)$ $S^{-1}AS$ *is upper triangular and* $S^{-1}ZS$ *is lower triangular.*

If $(*)$ holds we say that A and Z admit a *simultaneous reduction to complementary triangular forms.*

Let us consider in some detail the case $k = 1$, where (1.1) is a scalar rational function, as it leads to companion matrices. If $k = 1$, then the right hand side of (1.1) can be written as $1 + g(\lambda)/a(\lambda)$ where g and a are polynomials and $\deg g < \deg a = m$. Conversely, let $w(\lambda) = 1 + g(\lambda)/a(\lambda)$ be given such that $a(\lambda) = a_0 + \cdots + a_{m-1}\lambda^{m-1} + \lambda^m$ and $g(\lambda) = \sum_{i=0}^{m-1} g_i\lambda^i$. Put $c = (0, \dots, 0, 1)$, $b = (g_0, \dots, g_{m-1})^T$, and

$$A = \begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & \cdots & 0 & -a_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & -a_{m-1} \end{pmatrix}. \tag{1.2}$$

Then

$$w(\lambda) = 1 + g(\lambda)/a(\lambda) = 1 + c(\lambda I - A)^{-1}b. \tag{1.3}$$

In this case Z is also a companion matrix,

$$Z = \begin{pmatrix} 0 & \cdots & 0 & -z_0 \\ 1 & \cdots & 0 & -z_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & -z_{m-1} \end{pmatrix} \tag{1.4}$$

with $z_i = a_i + g_i$, $i = 0, \dots, m-1$.

A complete factorization of (1.3) is possible if $(a, g) = 1$. In that case we have $(a, a+g) = 1$, and $a(\lambda) = \prod_{i=1}^{m}(\lambda - \alpha_i)$ and $a(\lambda) + g(\lambda) = \prod_{i=1}^{m}(\lambda - \zeta_i)$ imply

$$w(\lambda) = \prod_{i=1}^{m}\left(1 + \frac{\alpha_i - \zeta_i}{\lambda - \alpha_i}\right).$$

In accordance with [1], we call the matrices A and Z in (1.2) and (1.4) *second companion* matrices. A *fourth companion* matrix is of the form

$$B = \begin{pmatrix} -b_{m-1} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -b_1 & 0 & \cdots & 1 \\ -b_0 & 0 & \cdots & 0 \end{pmatrix}. \tag{1.5}$$

Theorem 1.1 tells us that two second companion matrices A and Z with disjoint spectra admit a simultaneous reduction to complementary triangular forms. H. Bart and H. Hoogland [1] studied such reductions of A and Z in detail. Their results will be stated below. Pairs A, B where A is a second and B is a fourth companion matrix are treated in [4].

NOTATION. Let A be the second companion matrix given by (1.2) with characteristic polynomial

$$\det(\lambda I - A) = a(\lambda) = \prod_{i=1}^{m}(\lambda - \alpha_i),$$

and let Z be as in (1.4) and

$$\det(\lambda I - Z) = Z(\lambda) = \prod_{i=1}^{m} (\lambda - \zeta_i).$$

Define the polynomials s_j, $j = 1, \dots, m$, and the $m \times m$ matrix S by

$$s_j = (\lambda - \zeta_1) \cdots (\lambda - \zeta_{j-1})(\lambda - \alpha_{j+1}) \dots (\lambda - \alpha_m) \tag{1.6}$$

and

$$(s_1, \dots, s_m) = (1, \lambda, \dots, \lambda^{m-1}) S. \tag{1.7}$$

THEOREM 1.2 [1, pp. 202–203]. *The matrix S satisfies the intertwining relations $AS = S\tilde{A}$ and $ZS = S\tilde{Z}$, where*

$$\tilde{A} = \begin{pmatrix} \alpha_1 & \alpha_1 - \zeta_1 & \alpha_1 - \zeta_1 & \cdots & \alpha_1 - \zeta_1 & \alpha_1 - \zeta_1 \\ 0 & \alpha_2 & \alpha_2 - \zeta_2 & \cdots & \alpha_2 - \zeta_2 & \alpha_2 - \zeta_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_m \end{pmatrix} \tag{1.8}$$

and

$$\tilde{Z} = \begin{pmatrix} \zeta_1 & 0 & 0 & \cdots & 0 \\ \zeta_2 - \alpha_2 & \zeta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \zeta_{m-1} - \alpha_{m-1} & \zeta_{m-1} - \alpha_{m-1} & \zeta_{m-1} - \alpha_{m-1} & \cdots & 0 \\ \zeta_m - \alpha_m & \zeta_m - \alpha_m & \zeta_m - \alpha_m & \cdots & \zeta_m \end{pmatrix} \tag{1.9}$$

THEOREM 1.3 [1, pp. 205–206]. *If there exists a nonsingular matrix Y such that $Y^{-1}AY = \hat{A} = (\hat{a}_{jk})$ is upper triangular with main diagonal $(\hat{a}_{11}, \dots, \hat{a}_{mm}) = (\alpha_1, \dots, \alpha_m)$ and $Y^{-1}ZY = \hat{Z} = (\hat{z}_{jk})$ is lower triangular with main diagonal $(\hat{z}_{11}, \dots, \hat{z}_{mm}) = (\zeta_1, \dots, \zeta_m)$, then*

$$\zeta_j \neq \alpha_k, \qquad 1 \leqslant j < k \leqslant m. \tag{γ}$$

Conversely, if (γ) is satisfied, then S is nonsingular and $S^{-1}A\,S = \tilde{A}$ and $S^{-1}Z\,S = \tilde{Z}$, where $\tilde{A}$ and $\tilde{Z}$ are given by (1.8) *and* (1.9).

In this note I would like to describe a polynomial approach which leads in a straightforward way to results on simultaneous reduction of companion matrices (and which will be used to given new proofs of the preceding two theorems). It is based on the simple observation that a companion matrix can be regarded as a matrix representation of a shift on a polynomial model.

In Section 2 a polynomial framework is set up to study equations $AY = Y\hat{A}$ where $\hat{A}$ is upper triangular. The approach is elaborated in Section 3 to deal with pairs, A, Z and Z, B where B is a fourth and A and Z are second companion matrices.

2. COMPANION MATRICES AND THE SHIFT OPERATOR

Let (a) be the ideal generated by the characteristic polynomial $a(\lambda) = a_0 + \cdots + + a_{m-1}\lambda^{m-1} + \lambda^m$ of A. Put

$$V_a = \mathbb{C}[\lambda]/(a).$$

If the coset $f + (a)$ is identified with the remainder $f(\text{mod } a)$, then

$$V_a = \{f \mid f \in \mathbb{C}[\lambda], \deg f < m\}.$$

We call $(1, \lambda, \ldots, \lambda^{m-1})$ the *standard basis* of the vector space V_a. For $f \in V_a$ and $p \in \mathbb{C}[\lambda]$ define the product

$$p \cdot f = pf(\text{mod } a).$$

Let Λ be the shift operator on V_a,

$$\Lambda f = \lambda \cdot f, \qquad f \in V_a.$$

We have

$$\Lambda(1, \lambda, \ldots, \lambda^{m-1}) = (1, \lambda, \ldots, \lambda^{m-1})A$$

with A as in (1.2). In other words, the second companion matrix A is the matrix representation of Λ with respect to the standard basis of V_a. The fourth companion matrix B in (1.5) has a similar interpretation,

$$\Lambda(\lambda^{m-1}, \ldots, \lambda, 1) = (\lambda^{m-1}, \ldots, \lambda, 1)B.$$

Hence B describes the action of the shift on V_b on the reversed standard basis.

The Λ-invariant subspaces of V_a correspond to factorizations of the polynomial a. If $a = hq$, then

$$\operatorname{Ker} h(\Lambda) = \{qf \,|\deg qf < m\} \tag{2.1}$$

is Λ-invariant, and all such invariant subspaces of V_a are of that form. We have

$$\dim \operatorname{Ker} h(\Lambda) = \deg h. \tag{2.2}$$

NOTATION AND A STANDING ASSUMPTION. In the following all polynomials are in $\mathbb{C}[\lambda]$ and of degree less than m; all matrices are in $\mathbb{C}^{m\times m}$. We say that polynomials y_j, $j = 1, \dots, m$, have the generating matrix Y if

$$(y_1, \dots, y_m) = (1, \lambda, \dots, \lambda^{m-1})Y \tag{2.3}$$

holds. Let the *reversed polynomial* $\tilde{y}$ of y be defined by

$$\tilde{y}(\lambda) = \lambda^{m-1} y(\lambda^{-1}).$$

Then (2.3) implies

$$\left(\tilde{\lambda}_1, \dots, \tilde{y}_m\right) = (1, \lambda, \dots, \lambda^{m-1})EY,$$

where

$$E = \begin{pmatrix} 0 & & & 1 \\ & & \vdots & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix}.$$

To a given ordering $\alpha_1, \dots, \alpha_m$ of the eigenvalues of A we associate the polynomials

$$q_j = (\lambda - \alpha_{j+1})(\lambda - \alpha_{j+2}) \cdots (\lambda - \alpha_m), \quad j = 1, \dots, m-1, \qquad q_m = 1. \tag{2.4}$$

In a complementary manner the polynomials

$$p_1 = 1, p_j = (\lambda - \zeta_1)(\lambda - \zeta_2)\dots(\lambda - \zeta_{j-1}), \qquad j = 2,\dots,m, \quad (2.5)$$

are related to the ordering $\zeta_1,\dots,\zeta_m$ of the eigenvalues of Z. Recall s_j from (1.6). Then $s_j = p_j q_j$.

LEMMA 2.1. *Let Y be the generating matrix for the polynomials y_j, $j = 1,\dots,m$.*

(i) *Let $A_U = (a_{jk})$ be an upper triangular matrix with main diagonal*

$$(a_{11},\dots,a_{mm}) = (\alpha_1,\dots,\alpha_m). \quad (2.6)$$

If Y satisfies $AY = YA_U$, then

$$q_j | y_j, \qquad j = 1,\dots,m. \quad (2.7)$$

(ii) *Let $Z_L = (z_{jk})$ be lower triangular with main diagonal*

$$(z_{11},\dots,z_{mm}) = (\zeta_1,\dots,\zeta_m). \quad (2.8)$$

If Y satisfies $ZY = YZ_L$, then

$$p_j | y_j, \qquad j = 1,\dots,m. \quad (2.9)$$

(iii) *Let $B_L = (b_{jk})$ be lower triangular with*

$$(b_{11},\dots,b_{mm}) = (\beta_1,\dots,\beta_m). \quad (2.10)$$

If Y satisfies $BY = YB_L$, then

$$(\lambda - \beta_1)\dots(\lambda - \beta_{j-1}) | \tilde{y}_j, \qquad j = 2,\dots,m. \quad (2.11)$$

Proof. (i): We regard the polynomials y_j as elements of V_a. Then $AY = YA_U$ is equivalent to $\Lambda(y_1, \dots, y_m) = (y_1, \dots, y_m)A_U$, and we have

$$(\Lambda - \alpha_1)y_1 = 0,$$

$$(\Lambda - \alpha_2)y_2 = a_{12}y_1,$$

$$\vdots$$

$$(\Lambda - \alpha_m)y_m = a_{1m}y_1 + \cdots + a_{m-1,m}y_{m-1}.$$

Hence $y_j \in \operatorname{Ker}(\Lambda - \alpha_1)\dots(\Lambda - \alpha_j)$, $j = 1, \dots, m$. Because $a(\lambda) = (\lambda - \alpha_1)\cdots(\lambda - \alpha_j)q_j(\lambda)$, the relations (2.7) follow from (2.1).

(ii): The underlying space is now V_z. Since Z_L is lower triangular, we have $y_j \in \operatorname{Ker}(\Lambda - \zeta_m)\dots(\lambda - \zeta_j)$, and the factorization $z(\lambda) = (\lambda - \zeta_m)\dots(\lambda - \zeta_j)p_j(\lambda)$ yields (2.9).

(iii): Transform $BY = YB_L$ into $(EBE^{-1})(EY) = (EY)B_L$. Since EBE^{-1} is a second companion matrix, we are back at (i). The matrix EY generates the reversed polynomials $\tilde{y}_j$. ■

3. COMPLEMENTARY REDUCTIONS

Pairs A, Z of second companion matrices will be considered first. According to Lemma 2.1 the conditions

$$p_j | y_j \quad q_j | y_j, \qquad j = 1, \dots, m, \tag{3.1}$$

are necessary if Y is to reduce A and Z simultaneously to complementary diagonal forms.

LEMMA 3.1. *Let y_j, $j = 1, \dots, m$, be nonzero polynomials generated by the matrix Y. Assume that* (3.1) *holds.*

(i) *If*

$$\zeta_j \neq \alpha_k, \qquad 1 \leqslant j < k - 1 \leqslant m - 1, \tag{3.2}$$

then $y_j = s_j\gamma_j$, $j = 1, \dots, m$, for some nonzero $\gamma_j \in \mathbb{C}$.

(ii) *The matrix Y is nonsingular if and only if*

$$\zeta_j \neq \alpha_k, \qquad 1 \leqslant j < k \leqslant m. \tag{3.3}$$

Proof. (i): The condition (3.2) is equivalent to $(p_j, q_j) = 1, j = 1, \dots, m$. Hence $s_j | y_j$. On the other hand $\deg y_j \leqslant m - 1 = \deg s_j$.

(ii): Suppose Y is singular. Then there is an $r \geqslant 1$ such that the polynomials $y_1, \dots, y_r$ are linearly independent whereas y_{r+1} depends on $y_1 \dots y_r$. Let $\langle y_1, \dots, y_r \rangle$ denote the span of y_j, $j = 1, \dots, r$. Then (2.1) and (2.2) imply

$$\operatorname{Ker}(\Lambda - \alpha_1) \dots (\Lambda - \alpha_r) = \langle q_1, \dots, q_r \rangle = \langle y_1, \dots, y_r \rangle.$$

From $y_{r+1} \in \langle q_1, \dots, q_r \rangle$ follows $q_r | y_{r+1}$. By the assumption (3.1) we have $p_{r+1} | y_{r+1}$, and because $\deg y_{r+1} < m$, $\deg q_r = m - r$, and $\deg p_{r+1} = r$ the polynomials q_r and p_{r+1} have a common zero. Thus

$$\zeta_\mu = \alpha_\nu \qquad \text{for some } \mu \leqslant r < \nu. \tag{$**$}$$

Conversely, suppose now that $(**)$ holds. Put $\pi = \zeta_\mu = \alpha_\nu$. Since ζ_μ is a root of $p_{\mu+1}, \dots, p_m$ and α_ν is a root of $q_1, \dots, q_{\nu-1}$, we see that $p_j(\pi) = 0$ or $q_j(\pi) = 0$ for all j. Hence

$$(1, \pi, \dots, \pi^{m-1})Y = 0$$

and Y is singular. ■

COROLLARY 3.2 [1]. *The matrix S given by* (1.7) *is nonsingular if and only if* (3.3) *holds.*

We combine the results of Lemma 2.1 with the preceding lemma.

THEOREM 3.3. *Let Y be a matrix which generates the polynomials y_j, $j = 1, \dots, m$. Suppose that Y satisfies the equations $AY = YA_U$ and $ZY = YZ_L$ where A_U is upper and Z_L is lower triangular with respective main diagonals* *given by* (2.6) *and* (2.8). *Then we have*

$$(\lambda - \zeta_1) \cdots (\lambda - \zeta_{j-1}) | y_j, \qquad j = 2, \dots, m,$$

and

$$(\lambda - \alpha_{j+1}) \cdots (\lambda - \alpha_m) | y_j, \qquad j = 2, \dots, m-1.$$

If Y has no zero column, then Y is nonsingular if and only if

$$\zeta_j \neq \alpha_k, \qquad 1 \leqslant j < k \leqslant m. \tag{3.3}$$

In this case $Y = S\Gamma$ *for some nonsingular diagonal matrix* Γ.

The similarity S in Theorem 1.2 transforms A into an upper triangular matrix $\tilde{A}$ of a special row structure. The role of $\tilde{A}$ in (1.8) is clarified by the following observation.

LEMMA 3.4. *Let* $y_1, \dots, y_m$ *be monic polynomials of degree* $m-1$ *generated by the matrix* Y, *and let* $c_1, \dots, c_{m-1}$ *be complex numbers. The following statements are equivalent*:

(i) *The polynomials* y_j *satisfy the relations*

$$(\Lambda - \alpha_1) y_1 = 0, \tag{3.4a}$$

$$(\Lambda - \alpha_j) y_j = c_1 y_1 + \cdots + c_{j-1} y_{j-1}, \qquad j = 2, \dots, m. \tag{3.4b}$$

(ii) *The matrix* Y *satisfies an equation* $AY = Y\hat{A}$ *where* $\hat{A}$ *has the form*

$$\hat{A} = \begin{pmatrix} \alpha_1 & c_1 & c_1 & c_1 & \cdots & c_1 \\ & \alpha_2 & c_2 & c_2 & \cdots & c_2 \\ & & \cdot & & & \vdots \\ & & & \cdot & & \vdots \\ & & & & \cdot & \vdots \\ 0 & & & & & \alpha_m \end{pmatrix}.$$

(iii) *The polynomials* y_j *are of the form*

$$y_1 = q_1, \qquad y_2 = (\lambda - \tau_1)q_2, \dots, \qquad y_m = (\lambda - \tau_1)\dots(\lambda - \tau_{m-1})q_m, \tag{3.5}$$

and

$$c_j = \alpha_j - \tau_j, \qquad j = 1, \dots, m-1. \tag{3.6}$$

Proof. (iii) $\Rightarrow$ (i): Let the polynomials y_j be given as in (3.5), and c_j be defined by (3.6). Then

$$(\Lambda - \alpha_{k+1})y_{k+1} = (\lambda - \alpha_{k+1})\cdot(\lambda - \tau_1)\dots(\lambda - \tau_k)$$
$$\times(\lambda - \alpha_{k+2})\cdots(\lambda - \alpha_m)$$

We split the right hand side using $\lambda - \tau_k = (\lambda - \alpha_k) + (\alpha_k - \tau_k)$, so that

$$(\Lambda - \alpha_{k+1})y_{k+1} = (\lambda - \alpha_k)\cdot y_k + (\alpha_k - \tau_k)y_k. \tag{3.7}$$

If we assume as induction hypothesis

$$(\Lambda - \alpha_k)y_k = (\alpha_1 - \tau_1)y_1 + \cdots + (\alpha_{k-1} - \tau_{k-1})y_{k-1}, \tag{3.8}$$

then (3.7) implies that the relation (3.8) is also true for $k+1$.

(i) $\Rightarrow$ (iii): Assume now that (3.4) holds. Then $y_j \in \operatorname{Ker}(\Lambda - \alpha_1)\cdots(\Lambda - \alpha_j)$. Hence $y_j = q_j h_j$ where h_j is monic of degree $j-1$. In particular $h_1(\lambda) = 1$. We want to show that

$$h_{j+1} = (\lambda - \tau_j)h_j, \quad \tau_j = \alpha_j - c_j, \qquad j = 1, \dots, m-1.$$

From (3.4b) follows

$$(\Lambda - \alpha_{j+1}) y_{j+1} = (c_1 y_1 + \cdots + c_{j-1} y_{j-1}) + c_j y_j$$
$$= (\Lambda - \alpha_j) y_j + c_j y_j = (\Lambda - \alpha_j + c_j) y_j.$$

Therefore

$$(\lambda - \alpha_{j+1}) \cdot q_{j+1}(\lambda) h_{j+1} = (\lambda - \alpha_{j+1}) \cdot (\lambda - \alpha_{j+2}) \ldots (\lambda - \alpha_m) h_{j+1}$$
$$= (\lambda - \alpha_j + c_j) \cdot (\lambda - \alpha_{j+1}) \ldots (\lambda - \alpha_m) h_j.$$

Since h_j and h_{j+1} are both monic, we have $h_{j+1} = (\lambda - \alpha_j + c_j) h_j$.
It is obvious that (ii) is the matrix version of (i). ∎

In the corresponding result for the matrix Z, statement (i) will be omitted.

LEMMA 3.5. *Let $y_1, \ldots, y_m$ be monic polynomials of degree $m-1$ generated by Y, and let $d_1, \ldots, d_{m-1}$ be complex numbers. The following statements are equivalent:*

(ii) *The matrix Y satisfies an equation $ZY = Y\hat{Z}$ where $\hat{Z}$ has the form*

$$\hat{Z} = \begin{pmatrix} \zeta_1 & & & & 0 \\ d_2 & \zeta_2 & & & \\ \vdots & \ddots & \ddots & & \\ d_{m-1} & d_{m-1} & \cdots & \zeta_{m-1} & \\ d_m & d_m & \cdots & d_m & \zeta_m \end{pmatrix}.$$

(iii) *The polynomials y_j are of the form $y_m = p_m$,*

$$y_j = (\lambda - \zeta_1) \cdots (\lambda - \zeta_{j-1})(\lambda - \rho_{j+1}) \ldots (\lambda - \rho_m),$$

and

$$d_j = \zeta_j - \rho_j, \qquad j = 1, \ldots, m-1.$$

The preceding two lemmas contain Theorem 1.2. Take $y_j = s_j$ in (iii).

We now consider a pair A, B, where A is a second and B is a fourth companion matrix. Let $(\beta_1, \dots, \beta_m)$ be a given ordering of the eigenvalues of B. Denote by ν_j the number of zeros in the sequence $(\beta_1, \dots, \beta_{j-1})$, and put $\nu_0 = 0$. Then

$$(\lambda - \beta_1) \dots (\lambda - \beta_{j-1}) | \tilde{y}_j, \qquad j = 2, \dots, m,$$

is equivalent to

$$(1 - \lambda\beta_1) \dots (1 - \lambda\beta_{j-1}) | y_j \tag{3.9}$$

and

$$\deg y_j \leqslant m - 1 - \nu_j, \tag{3.10}$$

$j = 2, \dots, m$. We define polynomials r_j and t_j by

$$r_1 = 1, \qquad r_j = (1 - \lambda\beta_1) \cdots (1 - \lambda\beta_{j-1}), \quad j = 2, \dots, m,$$

and

$$t_j = r_j q_j = (1 - \lambda\beta_1) \dots (1 - \lambda\beta_{j-1})(\lambda - \alpha_{j+1}) \dots (\lambda - \alpha_m). \tag{3.11}$$

The matrix T which generates the polynomials t_j such that

$$(t_1, \dots, t_m) = (1, \lambda, \dots, \lambda^{m-1})T \tag{3.12}$$

will play a role similar to the matrix S before.

LEMMA 3.6. *Let $y_1, \dots, y_m$ be nonzero polynomials generated by the matrix Y. Assume that* (3.9), (3.10), *and $q_j | y_j$, $j = 1, \dots, m$, hold.*

(i) *If*

$$1 - \beta_j \alpha_k \neq 0, \qquad 1 \leqslant j < k - 1 \leqslant m - 1, \tag{3.13}$$

then $y_j = t_j \gamma_j$, $j = 1, \dots, m$, for some nonzero $\gamma_j \in \mathbb{C}$.

(ii) *The matrix Y is nonsingular if and only if*

$$1-\beta_j\alpha_k \neq 0, \qquad 1 \leqslant j < k \leqslant m. \tag{3.14}$$

Proof. (i): The condition (3.13) is equivalent to $(r_j, q_j) = 1$ or to $\mathrm{lcm}(r_j, q_j) = t_j$, $j = 1, \dots, m$. On the other hand, the assumptions on y_j imply $\mathrm{lcm}(r_j, q_j) | y_j$ for $1 \leqslant j \leqslant m$. We have $\deg t_j = m - 1 - \nu_j$, and it follows from (3.10) that $y_j = t_j\gamma_j$. with some scalar factor γ_j.

(ii): Suppose Y is singular. Then there is an s such that the polynomials $y_1, \dots, y_s$ are linearly independent and $y_{s+1} \in \langle y_1, \dots, y_s \rangle$. As in the proof of Lemma 3.1, we conclude that $q_s | y_{s+1}$. By assumption we have $r_{s+1} | y_{s+1}$ and $\deg y_{s+1} \leqslant m - \nu_{s+1} - 1$. Furthermore $\deg r_{s+1} = s - \nu_{s+1}$ and $\deg q_s = m - s$. Hence $\deg r_{s+1}q_s = m - \nu_{s+1}$. Since $\mathrm{lcm}(r_{s+1}, q_s)$ is a factor of y_{s+1}, the polynomials r_{s+1} and q_s have a common zero ρ such that for some μ and $\varkappa$

$$\rho = \beta_\mu^{-1} = \alpha_\varkappa, \qquad \mu \leqslant s < \varkappa. \tag{$*$}$$

Conversely, if $(*)$ holds then $q_1(\rho) = \dots = q_s(\rho) = r_{s+1}(\rho) = \dots = r_m(\rho) = 0$. Hence $y_j(\rho) = 0$ for $1 \leqslant j \leqslant m$ and $(1, \rho, \dots, \rho^{m-1})Y = 0$. ■

COROLLARY 3.7. *The matrix T is nonsingular if and only if* (3.14) *holds.*

Omitting the result on A, B which corresponds to Lemma 3.4, we pass directly to the analogue of Theorem 1.3. For the matrices $\hat{A}$ and $\hat{B}$ below put $\pi_i = 1 - \beta_i\alpha_i$.

THEOREM 3.8. *Define*

$$\hat{A} = \begin{pmatrix} \alpha_1 & \pi_1 & -\beta_2\pi_1 & \cdot & \cdot & \cdot & \cdot & (-1)^{m-2}\beta_2\cdots\beta_{m-1}\pi_1 \\ & \alpha_2 & \pi_2 & \cdot & \cdot & \cdot & \cdot & (-1)^{m-3}\beta_3\cdots\beta_{m-1}\pi_2 \\ & & \cdot & \cdot & & & & \vdots \\ & & & \ddots & & & & \\ & & & & \cdot & \cdot & & -\beta_{m-1}\pi_{m-2} \\ & & & & & \cdot & & \pi_{m-1} \\ 0 & & & & & & & \alpha_m \end{pmatrix}$$

and

$$\hat{B} = \begin{pmatrix} \beta_1 & & & & & & 0 \\ \pi_2 & \beta_2 & & & & & \\ \vdots & \vdots & \ddots & & & & \\ \vdots & \vdots & & \ddots & & & \\ (-1)^{m-1}\alpha_2 \cdots \alpha_{m-2}\pi_{m-1} & (-1)^{m-2}\alpha_3 \cdots \alpha_{m-2}\pi_{m-1} & \cdots & & \pi_{m-1} & \beta_{m-1} & \\ (-1)^{m-2}\alpha_2 \cdots \alpha_{m-1}\pi_m & (-1)^{m-1}\alpha_3 \cdots \alpha_{m-1}\pi_m & \cdots & & -\alpha_{m-1}\pi_m & \pi_m & \beta_m \end{pmatrix}.$$

Then the matrix T given by (3.12) *satisfies* $AT = T\hat{A}$ *and* $BT = T\hat{B}$.

Proof. In V_a the equation

$$(1, \lambda, \dots, \lambda^{m-1})\, AT = (1, \lambda, \dots, \lambda^{m-1})T\hat{A}$$

is equivalent to

$$(\Lambda - \alpha_1)t_1 = 0, \tag{3.15}$$

$$(\Lambda - \alpha_2)t_2 = (1 - \beta_1\alpha_1)t_1,$$

$$\begin{aligned}(\Lambda - \alpha_j)t_j = {} & (-1)^{j-2}\beta_2 \cdots \beta_{j-1}(1 - \beta_1\alpha_1)t_1 \\ & +(-1)^{j-3}\beta_3 \cdots \beta_{j-1}(1 - \beta_2\alpha_2)t_2 \\ & + \cdots + (1 - \beta_{j-1}\alpha_{j-1})t_{j-1}, \qquad j = 3, \dots, m.\end{aligned}$$

The preceding relations can be expressed by (3.15) and the recursions

$$(\Lambda - \alpha_j)t_j = -\beta_{j-1}(\Lambda - \alpha_{j-1})t_{j-1} + (1 - \beta_{j-1}\alpha_{j-1})t_{j-1}, \qquad j = 2, \dots, m. \tag{3.16}$$

According to the definition (3.11) of t_j we have for $j \geqslant 2$

$$\begin{aligned}(\Lambda - \alpha_j)t_j &= (\lambda - \alpha_j)\cdot(1 - \beta_1\lambda)\dots(1 - \beta_{j-1}\lambda)(\lambda - \alpha_{j+1})\dots(\lambda - \alpha_m) \\ &= \left[(1 - \beta_{j-1}\alpha_{j-1}) + (\beta_{j-1}\alpha_{j-1} - \beta_{j-1}\lambda)\right]\cdot t_{j-1} \\ &= (1 - \beta_{j-1}\alpha_{j-1})t_{j-1} - \beta_{j-1}(\lambda - \alpha_{j-1})\cdot t_{j-1},\end{aligned}$$

which proves (3.16).

Along similar lines one can prove $\Lambda(\tilde{t}_1, \dots, \tilde{t}_m) = (\tilde{t}_1, \dots, \tilde{t}_m)\tilde{B}$, which is equivalent to $BT = T\tilde{B}$. ■

The picture is completed with a counterpart of Theorem 1.3.

THEOREM 3.9. *If*

$$1 - \beta_j \alpha_k \neq 0, \qquad 1 \leqslant j < k \leqslant m, \tag{3.13}$$

then the matrix T given as in (3.12) *is nonsingular and satisfies $T^{-1}AT = \hat{A}$ and $T^{-1}BT = \hat{B}$, where $\hat{A}$ and $\hat{B}$ are complementary triangular matrices defined as in Theorem* 3.8.

Conversely, if there exists a nonsingular matrix Y such that $Y^{-1}AY = A_U$ and $Y^{-1}BY = B_L$ is satisfied, where A_U is upper triangular with main diagonal (2.6) *and B_L is lower triangular with main diagonal as in* (2.10), *then* (3.13) *holds and $Y = T\Gamma$ for some nonsingular diagonal matrix Γ.*

4. A DETERMINANT

The matrices S and T are crucial to the problems of complementary triangularization. Conditions for S and T to be nonsingular are (3.3) and (3.13). In [1] it is shown that

$$\det S = \prod_{l \leqslant j < k \leqslant m} (\zeta_j - \alpha_k). \tag{4.1}$$

We shall see that

$$\det T = \prod_{l \leqslant j < k \leqslant m} (\beta_j \alpha_k - 1). \tag{4.2}$$

The determinants (4.1) and (4.2) are special cases of a more general result which will be derived in this section. We shall first evaluate det S by a method which is different from [1].

Proof of (4.1). Let $\alpha_1, \ldots, \alpha_m, \zeta_1, \ldots, \zeta_m$ be indeterminates over $\mathbb{C}$. Putting aside our standing assumption for a moment, we regard the polynomials

$$s_j = p_j q_j = (\lambda - \zeta_1) \cdots (\lambda - \zeta_{j-1})(\lambda - \alpha_{j+1}) \ldots (\lambda - \alpha_m) \tag{1.6}$$

as polynomials over the field $K = \mathbb{C}(\alpha_1, \ldots, \alpha_m, \zeta_1, \ldots, \zeta_m)$, and the matrix S given by

$$(s_1(\lambda), \ldots, s_m(\lambda)) = (1, \lambda, \ldots, \lambda^{m-1})S \tag{1.7}$$

as a matrix in $K^{m\times m}$. Substituting the values $\lambda = \zeta_1, \dots, \zeta_{m-1}, \alpha_m$ in (1.7) yields

$$\begin{pmatrix} s_1(\zeta_1) & 0 & \dots & 0 & 0 \\ s_1(\zeta_2) & s_2(\zeta_2) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ s_1(\zeta_{m-1}) & s_2(\zeta_{m-1}) & \dots & s_{m-1}(\zeta_{m-1}) & 0 \\ 0 & 0 & \dots & 0 & s_m(\alpha_m) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \zeta_{m-1} & \dots & \zeta_{m-1}^{m-1} \\ 1 & \alpha_m & \dots & \alpha_m^{m-1} \end{pmatrix} S. \quad (4.3)$$

Define matrices C and V such that (4.3) can be written as $C = VS$. Then

$$\det C = \prod_{i=1}^{m-1} p_i(\zeta_i) \prod_{i=1}^{m-1} q_i(\zeta_i) p_m(\alpha_m)$$

$$= \prod_{m-1 \geqslant \sigma > \rho \geqslant 1} (\zeta_\sigma - \zeta_\rho) \prod_{1 \leqslant j < k \leqslant m} (\zeta_j - \alpha_k) \prod_{i=1}^{m-1} (\alpha_m - \zeta_i).$$

Since V is a Vandermonde matrix, we have

$$\det V = \prod_{m-1 \geqslant \sigma > \rho \geqslant 1} (\zeta_\sigma - \zeta_\rho) \prod_{i=1}^{m-1} (\alpha_m - \zeta_i),$$

and (4.1) follows from $\det C = \det V \det S$. ∎

THEOREM 4.1. *Let $\gamma_i, \delta_i, \sigma_i, \tau_i$, $i = 1, \dots, m$, be given complex numbers. Define polynomials w_j by*

$$w_j = (\gamma_1\lambda - \delta_1) \dots (\gamma_{j-1}\lambda - \delta_{j-1})(\sigma_{j+1}\lambda - \tau_{j+1}) \dots (\sigma_m\lambda - \tau_m),$$

$$j = 1, \dots, m,$$

and let W be their generating matrix such that

$$(w_1, \dots, w_m) = (1, \lambda, \dots, \lambda^{m-1})W.$$

Then

$$\det W = \prod_{1 \leqslant j < k \leqslant m} \begin{vmatrix} \delta_j & \tau_k \\ \gamma_j & \sigma_k \end{vmatrix}. \tag{4.4}$$

Proof. As before, we work with indeterminates. Put $\zeta_i = \delta_i/\gamma_i$ and $\alpha_i = \tau_i/\sigma_i$, and let s_j be the polynomial in (1.6). Then

$$(w_1, \dots, w_m) = (s_1, \dots, s_m)\,\mathrm{diag}(\dots, \gamma_1 \cdots \gamma_{j-1}\sigma_{j+1} \cdots \sigma_m, \dots),$$

and the computation of det W is reduced to the special case (4.1). ■

To the polynomials t_j in (3.11) corresponds the quadruple

$$\begin{pmatrix} \delta_j & \tau_k \\ \gamma_j & \sigma_k \end{pmatrix} = \begin{pmatrix} -1 & \alpha_k \\ -\beta_j & 1 \end{pmatrix},$$

and their generating matrix T has a determinant given by (4.2).

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