A Galois Correspondence Between Sets of Semidefinite Solutions of Continuous-Time Algebraic Riccati Equations

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ABSTRACT

The sets of negative semidefinite solutions $\mathcal{T}_i$ of two algebraic Riccati equations $\mathcal{R}_i(X) = A_i^* X + XA_i + XB_iB_i^* - C_i^*C_i = (I, X)H_i(I, X)^*$, $i = 1, 2$, are compared under the hypothesis that $H_1 \leqslant H_2$. If X_1^+ and X_2^+ are the greatest solutions in $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, then $X_1^+ \leqslant X_2^+$. A more general result will be proved which allows the comparison of other solutions of $\mathcal{T}_1$ and $\mathcal{T}_2$ besides the extremal ones and which in the case of stabilizability leads to a Galois connection between $\mathcal{T}_1$ and $\mathcal{T}_2$. The comparison results are based on one hand on a decomposition of the equations $\mathcal{R}_i(X) = 0$ into Lyapunov matrix equations and genuine Riccati equations which induce a corresponding decomposition of the solutions in $\mathcal{T}_i$, and the other hand on a parametrization of the Riccati components by A_i-invariant subspaces.

1. INTRODUCTION

Let

$$\mathcal{R}(X) = A^*X + XA + XBB^*X - C^*C = 0 \tag{1.1}$$

be an algebraic Riccati equation (ARE) where A, B, C are complex matrices of sizes $n \times n, n \times p, q \times n$ respectively, and let $\mathcal{T}$ denote the set of negative semidefinite solutions of (1.1),

$$\mathcal{T} = \{X \mid X \leqslant 0, \mathcal{R}(X) = 0\}.$$

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[*] Research supported by the SCIENCE Programme of the European Communities (SC1*/0126-C) and by Deutsche Forschungsgemeinschaft (Kn 164/3-1).

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

If the pair (A, B) is stabilizable, then $\mathscr{T}$ is a lattice [5] with partial ordering $X_1 \leqslant X_2$ defined by $X_1 - X_2 \leqslant 0$, and $\mathscr{T}$ has a greatest element X_+ and a least element X_-. Together with (1.1) we consider another ARE

$$\tilde{\mathscr{R}}(X) = \tilde{A}^* X + X\tilde{A} + X\tilde{b}\tilde{B}\tilde{B}^* X - \tilde{C}^*\tilde{C} = 0, \tag{1.2}$$

where the matrices

$$H = \begin{pmatrix} -C^*C & A^* \\ A & BB^* \end{pmatrix} \tag{1.3}$$

and

$$\tilde{H} = \begin{pmatrix} -\tilde{C}^*\tilde{C} & \tilde{A}^* \\ \tilde{A} & \tilde{B}\tilde{B}^* \end{pmatrix} \tag{1.4}$$

are related by $H \leqslant \tilde{H}$. Under that hypothesis, stabilizability is carried over from (A, B) to the pair $(\tilde{A}, \tilde{B})$, and according to [8] the least element $\tilde{X}_-$ of $\tilde{\mathscr{T}} = \{X \mid X \leqslant 0,\ \tilde{\mathscr{R}}(X) = 0\}$ satisfies $X_- \leqslant \tilde{X}_-$. We shall see that a corresponding inequality is valid also for the greatest solutions of (1.1) and (1.2), i.e. that $H \leqslant \tilde{H}$ implies $X_+ \leqslant \tilde{X}_+$. It is the purpose of this paper to extend the preceding observations to a theorem which allows a comparison of other members of $\mathscr{T}$ and $\tilde{\mathscr{T}}$ besides their least and greatest elements and which establishes a Galois correspondence between the two sets of solutions.

Our investigation is based on results from [9] which will be reviewed in Section 2. We shall associate to $\mathscr{T}$ and $\tilde{\mathscr{T}}$ certain sets $\mathscr{N}$ and $\tilde{\mathscr{N}}$ of A- and $\tilde{A}$-invariant subspaces such that—in the case where (A, B) is stabilizable—order relations between solutions in $\mathscr{T}$ and $\tilde{\mathscr{T}}$ are reflected by inclusion relations between elements of $\mathscr{N}$ and $\tilde{\mathscr{N}}$.

2. DECOMPOSITION AND PARAMETRIZATION OF SOLUTIONS

The subsequent theorems which describe the structure of the solution set $\mathscr{T}$ are taken from [9]. We shall use the following notation. The partitions

$$\mathbb{C} = \mathbb{C}_< \cup \mathbb{C}_= \cup \mathbb{C}_> = \mathbb{C}_\leqslant \cup \mathbb{C}_> \tag{2.1}$$

should be self-explanatory in that the subscripts refer to complex numbers λ with $\operatorname{Re}\lambda < 0$, $\operatorname{Re}\lambda = 0$, etc. Put

$$E_\lambda(A) = \operatorname{Ker}(A - \lambda I)^n,$$

so that $E_\lambda(A)$ is a generalized eigenspace of A if $\lambda \in \sigma(A)$. For $\Lambda \subseteq \mathbb{C}$ define

$$E_\Lambda(A) = \bigoplus_{\lambda \in \Lambda} E_\lambda(A).$$

In the case where $\Lambda \in \{\mathbb{C}_<, \mathbb{C}_=, \mathbb{C}_\leqslant, \mathbb{C}_>\}$ we write $E_<(A)$, $E_=(A)$, etc., so that to (2.1) correspond the decompositions

$$\mathbb{C}^n = E_<(A) \oplus E_=(A) \oplus E_>(A) = E_\leqslant(A) \oplus E_>(A).$$

Let $\operatorname{Inv} A$ denote the lattice of A-invariant subspaces of $\mathbb{C}^n$, and put

$$\operatorname{Inv}(A;C) = \{N \mid N \in \operatorname{Inv} A,\ N \subseteq \operatorname{Ker} C\}.$$

To the triple (A, B, C) we associate the controllable subspace

$$R(A, B) = \operatorname{Im}(B, AB, \dots, A^{n-1}B)$$

and the unobservable subspace

$$V(A,C) = \sup \operatorname{Inv}(A;C) = \operatorname{Ker}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

Define

$$V_\leqslant(A,C) = V(A,C) \cap E_\leqslant(A)$$

and similarly $V_<(A,C)$, $V_=(A,C)$, etc.

We know from [3, 4] that there exists a solution $X \leqslant 0$ of (1.1) if and only if

$$V(A,C) + R(A,B) + E_<(A) = \mathbb{C}^n. \tag{2.2}$$

Because of $V_{<}(A,C) \subseteq E_{<}(A)$ the condition (2.2) can be written as

$$V_{=}(A,C) + \left[V_{>}(A,C) + R(A,B) + E_{<}(A)\right] = \mathbb{C}^n.$$

Put

$$U_r = V_{>}(A,C) + R(A,B) + E_{<}(A). \tag{2.3}$$

Then (2.2) is equivalent to

$$\mathbb{C}^n = U_0 \oplus U_r \tag{2.4}$$

for some subspace

$$U_0 \subseteq V_{=}(A,C). \tag{2.5}$$

We call (2.4) with U_0 as in (2.5) an *LR decomposition* with *Riccati part* U_r and a *Lyapunov complement* U_0. If $\mathbb{C}^n$ admits a decomposition (2.4) with nontrivial summands, then (1.1) breaks up into a Lyapunov matrix equation and a genuine Riccati equation.

THEOREM 2.1.

(1) *Let* $\mathbb{C}^n = U_0 \oplus U_r$ *be an LR decomposition. If* $S = (S_0, S_r)$ *is nonsingular such that*

$$\operatorname{Im} S_0 = U_0, \qquad \operatorname{Im} S_r = U_r, \qquad \dim U_r = n_r,$$

then

$$S^{-1}AS = \begin{pmatrix} A_0 & 0 \\ A_{r0} & A_r \end{pmatrix}, \qquad S^{-1}B = \begin{pmatrix} 0 \\ B_r \end{pmatrix}, \qquad CS = (0, C_r), \tag{2.6}$$

$$\sigma(A_0) \subseteq \mathbb{C}_{=}, \tag{2.7}$$

and

$$V_{>}(A_r, C_r) + R(A_r, B_r) + E_{<}(A_r) = \mathbb{C}^{n_r}. \tag{2.8}$$

Assume (2.6)–(2.8). *Then we have* $X \in \mathcal{T}$ *if and only if*

$$X = (S^{-1})^* \begin{pmatrix} X_0 & 0 \\ 0 & X_r \end{pmatrix} S^{-1} \tag{2.9}$$

and $X_0 \leqslant 0$ *satisfies the Lyapunov equation*

$$\mathscr{L}_0(X_0) = A_0^* X_0 + X_0 A_0 = 0 \tag{2.10}$$

and $X_r \leqslant 0$ *is a solution of*

$$\mathscr{R}_r(X_r) = A_r^* X_r + X_r A_r + X_r B_r B_r^* X_r - C_r^* C_r = 0.$$

(2) *Let* $\Pi : \mathbb{C}^n \to U_r$ *be the projection on* U_r *along* U_0. *Put* $\rho(X) = X\Pi$. *Then* $\rho(X) \in \mathscr{T}$ *and* $(I - \Pi)^* X \Pi = 0$. *The projection* Π *is independent of the choice of* U_0. *If* X *is given as in* (2.9), *then*

$$\rho(X) = (S^{-1})^* \begin{pmatrix} 0 & 0 \\ 0 & X_r \end{pmatrix} S^{-1}. \tag{2.11}$$

In our investigation solutions with Lyapunov part $X_0 = 0$ will be of special interest. Put

$$\mathscr{S} = \rho(\mathscr{T}),$$

so that $\mathscr{S}$ contains all solutions of the form (2.11).

LEMMA 2.2.

(1) *If* $\mathscr{T} \neq \varnothing$, *then we have* $\mathscr{S} = \mathscr{T}$ *if and only if*

$$\operatorname{rank}(A - \lambda I, B) = n \qquad \textit{for all} \quad \lambda \in \mathbb{C}_= . \tag{2.12}$$

(2) *Assume* $X \in \mathscr{T}$. *Then* $V_=(A, C) \cap U_r = V(A, C) \cap R(A, B) \cap E_=(A) \subseteq \operatorname{Ker} X$. *Furthermore we have* $X \in \mathscr{S}$ *if and only if*

$$V_=(A, C) \subseteq \operatorname{Ker} X. \tag{2.13}$$

There is an order isomorphism between $\mathscr{S}$ and the following system $\mathscr{N}$ of A-invariant subspaces of $\mathbb{C}^n$. Define

$$\mathscr{N} = \{N \mid N \in \operatorname{Inv} A, V_\leqslant(A, C) \subseteq N \subseteq V(A, C),$$

$$N + R(A, B) + E_<(A) = \mathbb{C}^n\}.$$

THEOREM 2.3. *The map* $\gamma : \mathscr{S} \to \mathscr{N}$ *given by* $\gamma(X) = \operatorname{Ker} X$ *is a bijection, and both* γ *and* γ^{-1} *are order-preserving, i.e., for* $X, Y \in \mathscr{S}$ *and* $M, N \in \mathscr{N}$ *the relations* $X \leqslant Y$ *and* $M \subseteq N$ *imply* $\gamma(X) \subseteq \gamma(Y)$ *and* $\gamma^{-1}(M) \leqslant \gamma^{-1}(N)$.

It is obvious that $\mathscr{N} \neq \varnothing$ (or equivalently $\mathscr{T} \neq \varnothing$ or $\mathscr{S} \neq \varnothing$) holds if and only if $V(A, C) \in \mathscr{N}$, which yields the condition (2.2) for the existence of negative semidefinite solutions of (1.1). Furthermore, $V(A, C)$ is the greatest element of $\mathscr{N}$, and $\gamma^{-1}[V(A, C)] = \sup \mathscr{S}$. Because $X \leqslant \rho(X)$, the greatest elements of $\mathscr{T}$ and $\mathscr{S}$ coincide.

COROLLARY 2.4 [3, 4]. *If* (2.2) *holds, then* $\mathscr{T}$ *has a greatest element* X_+ *which is characterized by* $\operatorname{Ker} X_+ = V(A, C)$.

Since the Lyapunov equation (2.10) with (2.7) has solutions $X_0 \leqslant 0$ with arbitrary large spectral norm, a least element of $\mathscr{T}$ can only exist if $\mathscr{T} = \mathscr{S} = \gamma^{-1}(\mathscr{N})$. In $\mathscr{N}$ the lower bound $V_{\leqslant}(A, C)$ is the most noticeable candidate for a least element. If $V_{\leqslant}(A, C)$ is in $\mathscr{N}$, then we shall see that the hypothesis $H \leqslant \tilde{H}$ ensures the existence of a least element in $\tilde{\mathscr{N}}$ also. We then have $\inf \tilde{\mathscr{N}} = V_{\leqslant}(\tilde{A}, \tilde{C})$, and $V_{\leqslant}(A, C) \subseteq V_{\leqslant}(\tilde{A}, \tilde{C})$. In general, i.e. if $V_{\leqslant}(A, C)$ is not in $\mathscr{N}$, it may happen that $\inf \mathscr{N}$ exists but $\inf \tilde{\mathscr{N}}$ does not. Consider the following example (in the order-isomorphic setting of $\mathscr{S}$ and $\tilde{\mathscr{S}}$). Let

$$A = \tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \tilde{C} = (0, 0), \qquad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \tilde{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be given. Then $H - \tilde{H} = \operatorname{diag}(0, 0, 0, 0, -1) \leqslant 0$. From $\mathscr{R}(X) = 2X = 0$ it follows that $X = 0$ and $\mathscr{T} = \mathscr{S} = \{0\}$. Put

$$X = \begin{pmatrix} x_1 & x_{12} \\ \bar{x}_{12} & x_2 \end{pmatrix}.$$

Then $\tilde{\mathscr{R}}(X) = 2X + X \operatorname{diag}(0, 1)\, X = 0$ implies

$$X = 0 \quad \text{or} \quad X = \begin{pmatrix} -\frac{1}{2}|x_{12}|^2 & x_{12} \\ \bar{x}_{12} & -2 \end{pmatrix}.$$

Therefore—in contrast to $\mathscr{S}$—there is no least solution in $\tilde{\mathscr{S}}$.

LEMMA 2.5.

(1) *The following statements are equivalent*:

(i) *We have*

$$V_{\leqslant}(A,C) \in \mathscr{N}. \tag{2.14}$$

(ii) *We have*

$$V_{\leqslant}(A,C) + R(A,B) + E_{<}(A) = \mathbb{C}^n. \tag{2.15}$$

(iii) *We have* (2.2) *and*

$$\operatorname{rank}(A - \lambda I, B) = n \quad or \quad \operatorname{rank}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n \qquad for\ all \quad \lambda \in \mathbb{C}_{>}. \tag{2.16}$$

(2) *If* (A, B) *is stabilizable, then we have* $\mathscr{T} \neq \varnothing$, $\mathscr{T} = \mathscr{S}$, *and* (2.14).

(3) *If in addition to* (2.15) *also* (2.12), *i.e.* $\mathscr{T} = \mathscr{S}$, *holds, then* (A, B) *is stabilizable*.

Proof. (1): It is clear that (2.14) and (2.15) are equivalent. Now write (2.15) as

$$\begin{aligned} V_{=}(A,C) &+ V_{<}(A,C) + R(A,B) + E_{<}(A) \\ &= \{[V(A,C) + R(A,B)] \cap E_{=}(A)\} \\ &\quad \oplus [R(A,B) \cap E_{>}(A)] \oplus E_{<}(A) \\ &= E_{=}(A) \oplus E_{>}(A) \oplus E_{<}(A) = \mathbb{C}^n. \end{aligned}$$

Then (2.15) holds if and only if both $[V(A,C) + R(A,B)] \cap E_{=}(A) = E_{=}(A)$ and $[R(A,B) \cap E_{>}(A)] = E_{>}(A)$—or equivalently, both

$$E_{=}(A) \subseteq V(A,C) + R(A,B) \tag{2.17}$$

and

$$E_{>}(A) \subseteq R(A,B) \tag{2.18}$$

—are satisfied. The preceding two inclusions can be expressed in terms of Kalman's canonical form of (A, B, C). Let $\mathbb{C}^n = K_1 \oplus K_2 \oplus K_3 \oplus K_4$ be a state space decomposition such that $K_1 = V(A, C) \cap R(A, B)$, $K_1 \oplus K_2 = V(A, C)$, $K_1 \oplus K_3 = R(A, B)$. With respect to a suitable basis we have

$$A = \begin{pmatrix} A_1 & A_{12} & A_{13} & A_{14} \\ 0 & A_2 & 0 & A_{24} \\ 0 & 0 & A_3 & A_{34} \\ 0 & 0 & 0 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ 0 \\ B_3 \\ 0 \end{pmatrix}, \qquad C = (0, 0, C_3, C_4).$$

If we translate (2.17), i.e. $E_=(A) \subseteq K_1 \oplus K_2 \oplus K_3$, into

$$\sigma(A_4) \subseteq \mathbb{C}_< \cap \mathbb{C}_> ,$$

and (2.18), i.e. $E_>(A) \subseteq K_1 \oplus K_3$, into

$$\sigma(A_2) \cup \sigma(A_4) \subseteq \mathbb{C}_\leqslant ,$$

then we find that (2.17) together with (2.18) is equivalent to

$$\sigma(A_4) \subseteq \mathbb{C}_< \tag{2.19}$$

combined with

$$\sigma(A_2) \subseteq \mathbb{C}_\leqslant . \tag{2.20}$$

It is known [3] that (2.19) is equivalent to (2.2), and it is obvious that (2.20) is the condition (2.16).

(2): Note that stabilizability of (A, B) is equivalent to

$$R(A, B) + E_<(A) = \mathbb{C}^n \tag{2.21}$$

or to

$$E_\geqslant(A) \subseteq R(A, B). \tag{2.22}$$

Hence in that case (2.2), (2.12), and (2.14) are satisfied.

(3): Recall that (2.15) implies (2.18). Since (2.12) is equivalent to $E_=(A) \subseteq R(A, B)$, we obtain stabilizability in the form (2.22). ■

Part (1) of the preceding lemma can be generalized. For $\Lambda \subseteq \mathbb{C}$ define

$$V_\Lambda(A,C) = V(A,C) \cap E_\Lambda(A).$$

We note without proof that in the case where $\mathbb{C}_\leqslant \subseteq \Lambda$, Lemma 2.5 remains true if $V_\leqslant(A,C)$ is replaced by $V_\Lambda(A,C)$ in (2.14) and (2.15), and $\lambda \in \mathbb{C}_>$ by $\lambda \in \mathbb{C} \setminus \Lambda$ in (2.16).

As an immediate consequence of the order isomorphism of Theorem 2.3 we obtain the following well-known result.

COROLLARY 2.6 [5–7]. *If* (A, B) *is stabilizable, then* (1.1) *has a least solution* $X_- \leqslant 0$, *and* X_- *is characterized by* $\operatorname{Ker} X_- = V_\leqslant(A,C)$.

3. SOLUTION SETS OF DIFFERENT RICCATI EQUATIONS

Let H and $\tilde{H}$ be given by (1.3) and (1.4). As a consequence of the inequality $H \leqslant \tilde{H}$, the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ inherits important properties from (A, B, C). Whenever in the sequel a subspace $N \in \operatorname{Inv}(A; C)$ appears, we can assume without loss of generality that a basis of $\mathbb{C}^n$ is chosen such that

$$N = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix}, x \in \mathbb{C}^{n-n_2} \right\}. \tag{3.1}$$

Then

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}, \qquad C = (0, C_2), \tag{3.2}$$

where $A_2 \in \mathbb{C}^{n_2 \times n_2}$ and $C_2 \in \mathbb{C}^{q \times n_2}$. Let

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{3.3}$$

be partitioned conformingly. If X satisfies $\mathscr{R}(X) = 0$ and is of the form $X = \operatorname{diag}(X_1, X_2)$ then the block X_2 satisfies

$$\mathscr{R}_2(X_2) = A_2^* X_2 + X_2 A_2 + X_2 B_2 B_2^* X_2 - C_2^* C_2 = 0. \tag{3.4}$$

LEMMA 3.1. *Assume* $H \leqslant \tilde{H}$. *Let* $\Lambda \subseteq \mathbb{C}$ *be given.*

(1) *If* $N \in \operatorname{Inv}(A; C)$, *then* $\tilde{A} = A$ *on* N *and* $N \in \operatorname{Inv}(\tilde{A}; \tilde{C})$. *In particular*

$$V_\Lambda(A, C) \subseteq V_\Lambda(\tilde{A}, \tilde{C}). \tag{3.5}$$

(2) *We have*

$$R(A, B) + E_\Lambda(A) \subseteq R(\tilde{A}, \tilde{B}) + E_\Lambda(\tilde{A}). \tag{3.6}$$

Proof. (1): Obviously

$$H - \tilde{H} = \begin{pmatrix} -C^*C + \tilde{C}^*\tilde{C} & (A - \tilde{A})^* \\ A - A & BB^* - \tilde{B}\tilde{B}^* \end{pmatrix} \leqslant 0 \tag{3.7}$$

implies $\tilde{C}^*\tilde{C} \leqslant C^*C$. Hence

$$\operatorname{Ker} C \subseteq \operatorname{Ker} \tilde{C}. \tag{3.8}$$

From $\operatorname{Ker} C \subseteq \operatorname{Ker}(-C^*C + \tilde{C}^*\tilde{C}) \subseteq \operatorname{Ker}(A - \tilde{A})$ and (3.8) it follows that

$$\operatorname{Ker} C \subseteq \operatorname{Ker}(A - \tilde{A}). \tag{3.9}$$

If $N \in \operatorname{Inv}(A; C)$ is given as in (3.1) so that A and C are of the form (3.2), then (3.8) yields $\tilde{C} = (0, \tilde{C}_2)$. From (3.9) follows

$$\tilde{A} = \begin{pmatrix} A_1 & * \\ 0 & * \end{pmatrix},$$

which shows that $A = \tilde{A}$ on N and $N \in \operatorname{Inv}(\tilde{A}; \tilde{C})$. In the special case where $N = V_\Lambda(A, C)$, we obtain (3.5) from the fact that $V_\Lambda(\tilde{A}, \tilde{C})$ is the largest subspace in $\operatorname{Inv}(\tilde{A}; \tilde{C})$ contained in $E_\Lambda(\tilde{A})$.

(2) Put

$$K = \begin{pmatrix} -BB^* & -A \\ -A^* & C^*C \end{pmatrix},$$

and define $\tilde{K}$ accordingly. Then $H \leqslant \tilde{H}$ is equivalent to $\tilde{K} \leqslant K$. Set $M = \mathbb{C} \setminus \Lambda$. A modified version of (3.5) yields

$$V_M(\tilde{A}^*, \tilde{B}^*) \subseteq V_M(A^*, B^*). \tag{3.10}$$

From (3.10) and

$$V_M(A^*, B^*)^{\perp} = V(A^*, B^*)^{\perp} + E_M(A^*)^{\perp} = R(A, B) + E_{\Lambda}(A)$$

we obtain (3.6). ■

To $\tilde{H}$ and $\tilde{\mathscr{R}}(X) = 0$ we associate the set $\tilde{\mathscr{T}} = \{X \mid X \leqslant 0,\ \tilde{\mathscr{R}}(X) = 0\}$, and we define $\tilde{\mathscr{S}}$, $\tilde{U}_r$, $\tilde{\mathscr{N}}$, etc. accordingly.

COROLLARY 3.2. *Assume* $H \leqslant \tilde{H}$.

(1) *If* (A, B) *is stabilizable then* $(\tilde{A}, \tilde{B})$ *is stabilizable.*
(2) *If* $\mathscr{T} \neq \varnothing$ *then* $\tilde{\mathscr{T}} \neq \varnothing$.
(3) *If* $\mathscr{T} \neq \varnothing$ *and* $\mathscr{T} = \mathscr{S}$ *then* $\tilde{\mathscr{T}} = \tilde{\mathscr{S}}$.

Proof. From (3.6) and (3.5) follow

$$R(A, B) + E_<(A) \subseteq R(\tilde{A}, \tilde{B}) + E_<(\tilde{A}),$$

$$V(A, C) \subseteq V(\tilde{A}, \tilde{C}),$$

and

$$U_r = V_>(A, C) + R(A, B) + E_<(A) \subseteq \tilde{U}_r. \tag{3.11}$$

Hence (2.21) implies $R(\tilde{A}, \tilde{B}) + E_<(\tilde{A}) = \mathbb{C}^n$, which proves (1). Furthermore, (2.2) yields $V(\tilde{A}, \tilde{C}) + R(\tilde{A}, \tilde{B}) + E_<(\tilde{A}) = \mathbb{C}^n$, which is equivalent to $\tilde{\mathscr{T}} \neq \varnothing$. Recall that $\mathscr{T} = \mathscr{S}$ if and only if $U_r = \mathbb{C}^n$. Then (3) follows from (3.11). ■

The main result of this section extends Theorem 2.3. The symbol γ will also be used for (1.2) in the sense that $\gamma: \tilde{\mathscr{S}} \to \tilde{\mathscr{N}}$ is the bijection given by $\gamma(\tilde{X}) = \operatorname{Ker} \tilde{X}$. Similarly $\rho: \tilde{\mathscr{T}} \to \tilde{\mathscr{S}}$ is the closure operator on $\tilde{\mathscr{T}}$ induced by an LR decomposition $\mathbb{C}^n = \tilde{U}_o \oplus \tilde{U}_r$.

THEOREM 3.3. *Assume* $H \leqslant \tilde{H}$ *and* $\mathscr{S} \neq \varnothing$. *If* $X \in \mathscr{S}$, $\tilde{X} \in \tilde{\mathscr{S}}$, *and* $X \leqslant \tilde{X}$, *then* $\gamma(X) \subseteq \gamma(\tilde{X})$. *If* $N \in \mathscr{N}$, $\tilde{N} \in \tilde{\mathscr{N}}$, *and* $N \subseteq \tilde{N}$, *then we have*

$$\gamma^{-1}(N) \leqslant \gamma^{-1}(\tilde{N}). \tag{3.12}$$

Proof. It is obvious that $X \leqslant \tilde{X} \leqslant 0$ implies $\gamma(X) = \operatorname{Ker} X \subseteq \gamma(\tilde{X}) = \operatorname{Ker} \tilde{X}$. To prove that also γ^{-1} is order-preserving, let N be given as in (3.1), so that A, B, C are as in (3.2) and (3.3). From $N + R(A, B) + E_{<}(A) = \mathbb{C}^n$ follows

$$R(A_2, B_2) + E_{<}(A_2) = \mathbb{C}^{n_2}, \tag{3.13}$$

i.e., (A_2, B_2) is stabilizable. Now $V_{\leqslant}(A, C) \subseteq N$ implies $V_{\leqslant}(A_2, C_2) = 0$. Put $X = \gamma^{-1}(N)$, so that $X \in \mathcal{T}$ is uniquely defined by $\operatorname{Ker} X = N$. Hence $X = \operatorname{diag}(0, X_2)$, and $X_2 < 0$ is a solution of the ARE $\mathcal{R}_2(X_2) = 0$ in (3.4). Because of (3.13) the subspace system $\mathcal{N}_2$ associated to $\mathcal{R}_2 = 0$ has a least element, which is $V_{\leqslant}(A_2, C_2) = 0 = \operatorname{Ker} X_2$. Therefore X_2 is the least negative semidefinite solution of (3.4). Of course it is well known that the preceding hypotheses for (3.4) and X_2 imply $\sigma(A_2 + B_2 B_2^* X_2) \subseteq \mathbb{C}_{\leqslant}$ and the extremal property of X_2.

From $N \in \operatorname{Inv}(A; C)$ and Lemma 3.1 follows

$$\tilde{A} = \begin{pmatrix} A_1 & \tilde{A}_{12} \\ 0 & \tilde{A}_2 \end{pmatrix}, \qquad \tilde{C} = (0, \tilde{C}),$$

where the block structures correspond to (3.2). Let $\tilde{X}$ be the unique element of $\tilde{\mathcal{S}}$ with $\gamma(\tilde{X}) = \operatorname{Ker} \tilde{X} = \tilde{N}$. Then $N \subseteq \tilde{N}$ implies that $\tilde{X} = \operatorname{diag}(0, \tilde{X}_2)$ and $\tilde{X}_2 \leqslant 0$ satisfies an ARE

$$\tilde{\mathcal{R}}_2(\tilde{X}_2) = \tilde{A}_2^* \tilde{X}_2 + \tilde{X}_2 \tilde{A}_2 + \tilde{X}_2 \tilde{B}_2 \tilde{B}_2^* \tilde{X}_2 - \tilde{C}_2^* \tilde{C}_2 = 0. \tag{3.14}$$

It remains to prove $X_2 \leqslant \tilde{X}_2$. Define

$$H_2 = \begin{pmatrix} -C_2^* C_2 & A_2^* \\ A_2 & B_2 B_2^* \end{pmatrix}$$

and correspondingly $\tilde{H}_2$. Then $H_2 \leqslant \tilde{H}_2$. Since (A_2, B_2) is stabilizable, we know from Theorem 1 in [8] that $X_2 \leqslant \tilde{X}_2$ holds for the least solution X_2 of (3.4) and for an arbitrary solution $\tilde{X}_2$ of (3.14). ■

A correspondence of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ will be described in Section 4. For the moment let us focus on extremal solutions.

THEOREM 3.4. *Assume* $H \leqslant \tilde{H}$.

(1) *If* $\mathcal{S} \neq \varnothing$, *then greatest elements* Y *and* $\tilde{Y}$ *of* $\mathcal{S}$ *and* $\tilde{\mathcal{S}}$ *exist, and they are related by* $Y \leqslant \tilde{Y}$.

(2) *If*

$$V_{\leqslant}(A,C) + R(A,B) + E_{<}(A) = \mathbb{C}^n \tag{3.15}$$

holds, then least elements W *and* $\tilde{W}$ *of* $\mathcal{S}$ *and* $\tilde{\mathcal{S}}$ *exist and they satisfy* $W \leqslant \tilde{W}$.

Proof. (1): According to Corollary 2.4 we have $Y = \gamma^{-1}[V(A,C)] = \sup \mathcal{T} = \sup \mathcal{S}$. From $V(A,C) \subseteq V(\tilde{A},\tilde{C})$ and (3.12) follows $Y \leqslant \tilde{Y}$.

(2): Since (3.15) implies the condition $V_{\leqslant}(\tilde{A},\tilde{C}) + R(\tilde{A},\tilde{B}) + E_{<}(\tilde{A}) = \mathbb{C}^n$, both $\mathcal{N}$ and $\tilde{\mathcal{N}}$ have a least element, namely $V_{\leqslant}(A,C)$ and $V_{\leqslant}(\tilde{A},\tilde{C})$ respectively. As before, (3.5) and (3.12) yield

$$W = \gamma^{-1}\big[V_{\leqslant}(A,C)\big] = \inf \mathcal{S} \subseteq \tilde{W} = \gamma^{-1}\Big[V_{\leqslant}(\tilde{A},\tilde{C})\Big] = \inf \tilde{\mathcal{S}}.$$

■

At this point we digress from the main line of our investigation to prove a comparison result which besides the Riccati parts also considers the Lyapunov parts of solutions. Recall the definition $\rho(X) = X\Pi$, $X \in \mathcal{T}$, where $\Pi : \mathbb{C}^n \to U_r$ is a projection along a Lyapunov complement U_0 of U_r. Put $\sigma(X) = X(I - \Pi)$. In the setting (2.6)–(2.9) of Theorem 2.1 we have

$$\sigma(X) = (S^{-1})^* \operatorname{diag}(X_0, 0)\, S^{-1}.$$

Similarly we define $\sigma(\tilde{X}) = \tilde{X} - \rho(\tilde{X})$ for $\tilde{X} \in \tilde{\mathcal{T}}$.

THEOREM 3.5. *Assume* $H \leqslant \tilde{H}$ *and* $\mathcal{T} \neq \varnothing$. *Let* $X \in \mathcal{T}$ *and* $\tilde{X} \in \tilde{\mathcal{T}}$ *be given. Then we have* $X \leqslant \tilde{X}$ *if and only if both* $\sigma(X) \leqslant \sigma(\tilde{X})$ *and* $\rho(X) \leqslant \rho(\tilde{X})$ *hold.*

Proof. Let

$$\mathbb{C}^n = U_0 \oplus U_r = \tilde{U}_0 \oplus \tilde{U}_r \tag{3.16}$$

be two LR decompositions induced by (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ respectively. From $U_r \subseteq \tilde{U}_r$ we obtain $U_0 \oplus U_r \subseteq U_0 + \tilde{U}_r = \mathbb{C}^n$. Hence there is a subspace $\hat{U}_0$ of U_0 such that

$$\mathbb{C}^n = \hat{U}_0 \oplus \tilde{U}_r. \tag{3.17}$$

From

$$U_0 \subseteq V_{=}(A,C) \subseteq V_{=}(\tilde{A},\tilde{C}) \tag{3.18}$$

it follows that (3.17) is an LR decomposition. Hence we can assume in (3.16) that $\tilde{U}_0 \subseteq U_0$. Set $T = U_0 \cap \tilde{U}_r$. Then the modular law yields $\tilde{U}_0 + T = U_0 \cap (\tilde{U}_0 + \tilde{U}_r) = U_0$, and similarly, $T + U_r = \tilde{U}_r$. Furthermore we have $\tilde{U}_0 \cap T = 0$ and $T \cap U_r = 0$, and therefore $\tilde{U}_0 \oplus T \oplus U_r = \mathbb{C}^n$. Now let X and $\tilde{X}$ be partitioned in accordance with that decomposition as

$$X = \begin{pmatrix} X_\alpha & X_{\alpha\beta} & 0 \\ X_{\alpha\beta}^* & X_\beta & 0 \\ 0 & 0 & X_r \end{pmatrix} \quad \text{and} \quad \tilde{X} = \begin{pmatrix} \tilde{X}_0 & 0 & 0 \\ 0 & \tilde{X}_\beta & * \\ 0 & * & * \end{pmatrix}.$$

Then $X \leqslant \tilde{X}$ implies

$$X_0 = \begin{pmatrix} X_\alpha & X_{\alpha\beta} \\ X_{\alpha\beta}^* & X_\beta \end{pmatrix} \leqslant \begin{pmatrix} \tilde{X}_0 & 0 \\ 0 & \tilde{X}_\beta \end{pmatrix}.$$

From $\operatorname{diag}(\tilde{X}_0, \tilde{X}_\beta) \leqslant \operatorname{diag}(\tilde{X}_0, 0)$ we conclude that

$$\sigma(X) = \operatorname{diag}(X_0, 0) \leqslant \sigma(\tilde{X}) = \operatorname{diag}\big(\tilde{X}_0, 0, 0\big).$$

We now turn to the operator ρ. From Lemma 2.2 we know that $V_{=}(\tilde{A}, \tilde{C}) \subseteq \operatorname{Ker} \rho(\tilde{X})$. Hence (3.18) yields $U_0 \subseteq \operatorname{Ker} \rho(\tilde{X})$. If $X \leqslant \tilde{X} \leqslant 0$ holds, then $\operatorname{Ker}\rho(X) = U_0 + \operatorname{Ker} X \subseteq U_0 + \operatorname{Ker} \tilde{X} \subseteq U_0 + \operatorname{Ker} \rho(\tilde{X}) = \operatorname{Ker} \rho(\tilde{X})$. Applying (3.12) to $\operatorname{Ker} \rho(X) \subseteq \operatorname{Ker} \rho(\tilde{X})$ yields $\rho(X) \leqslant \rho(\tilde{X})$.

The second part of the proof is obvious from $X = \sigma(X) + \rho(X)$. ■

4. A GALOIS CORRESPONDENCE

The inequality $H \leqslant \tilde{H}$ gives rise to a connection between the solution sets $\mathcal{S}$ and $\tilde{\mathcal{S}}$ which will be described in this section. We shall see that there are sublattice homomorphisms which establish a Galois correspondence between $\mathcal{S}$ and $\tilde{\mathcal{S}}$.

In the following we use the terminology of [2]. Let $\mathcal{P}$ and $\mathcal{Q}$ be two sets with a partial ordering $\leqslant$. A pair (φ, ψ) of maps $\varphi: \mathcal{P} \to \mathcal{Q}$ and $\psi: \mathcal{Q} \to \mathcal{P}$ is a *Galois correspondence* between $\mathcal{P}$ and $\mathcal{Q}$ if

(1) φ and ψ are order-preserving,
(2) $\psi\varphi(P) \geqslant P$ and $\varphi\psi(Q) \leqslant Q$ hold for all $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$.

We regard $\mathcal{N}$ as a substructure of the lattice $\mathcal{U} = \mathcal{U}(\mathbb{C}^n)$ of subspaces of $\mathbb{C}^n$ or of its sublattice Inv A, where the join and the meet operations are given by $M_1 \vee M_2 = M_1 + M_2$ and $M_1 \wedge M_2 = M_1 \cap M_2$, $M_i \in \mathcal{U}$, $i = 1, 2$. Suppose N_1, $N_2 \in \mathcal{N}$. Then the definition of $\mathcal{N}$ implies $N_1 + N_2 \in \mathcal{N}$. Hence $\mathcal{N}$ is a $\vee$-semilattice. If $V_{\leqslant}(A, C) \in \mathcal{N}$, i.e.

$$V_{\leqslant}(A, C) + R(A, B) + E_{<}(A) = \mathbb{C}^n, \tag{4.1}$$

then $V_{\leqslant}(A, C) \subseteq N_1 \cap N_2$ and (4.1) imply $(N_1 \cap N_2) + R(A, B) + E_{<}(A) = \mathbb{C}^n$. Therefore $N_1 \cap N_2 \in \mathcal{N}$. Thus under the hypothesis (4.1) $\mathcal{N}$ is a lattice given by the interval $[V_{\leqslant}(A, C), V(A, C)] = \{N | N \in \text{Inv } A, V_{\leqslant}(A, C) \subseteq N \subseteq V(A, C)\}$.

Lemma 4.1. *Assume $H \leqslant \tilde{H}$. Let $N \in \mathcal{N}$ and $\tilde{N} \in \tilde{\mathcal{N}}$ be given. Then*

(1) $N + V_{\leqslant}(\tilde{A}, \tilde{C}) \in \tilde{\mathcal{N}}$.
(2) $N \in \tilde{\mathcal{N}}$ *is equivalent to* $V_{\leqslant}(\tilde{A}, \tilde{C}) \subseteq N$.

Suppose (4.1) *holds. Then*

(3) $\tilde{N} \cap V(A, C) \in \mathcal{N}$.
(4) $\tilde{N} \in \mathcal{N}$ *is equivalent to* $\tilde{N} \subseteq V(A, C)$.

Proof. (1): We know from Lemma 3.1 that $N \in \mathcal{N}$ satisfies $N \in \text{Inv}(\tilde{A}; \tilde{C})$ and $N + R(\tilde{A}, \tilde{B}) + E_{<}(\tilde{A}) = \mathbb{C}^n$. Put $\tilde{M} = N + V_{\leqslant}(\tilde{A}, \tilde{C})$. Then $V_{\leqslant}(\tilde{A}, \tilde{C}) \subseteq \tilde{M}$, and the properties of N ensure that $\tilde{M} \in \tilde{\mathcal{N}}$.

(2): Clearly, for $N \in \tilde{\mathcal{N}}$ we have

$$V_{\leqslant}(\tilde{A}, \tilde{C}) \subseteq N. \tag{4.2}$$

Conversely, (4.2) is equivalent to $N = N + V_{\leqslant}(\tilde{A}, \tilde{C})$. Hence (1) yields $N \in \tilde{\mathscr{N}}$.

(3): Put $D = \tilde{N} \cap V(A, C)$. According to Lemma 3.1 we have $A = \tilde{A}$ on $V(A, C)$. Hence $D \in \operatorname{Inv} A$. From $V_{\leqslant}(A, C) \subseteq V_{\leqslant}(\tilde{A}, \tilde{C}) \subseteq \tilde{N}$ and the trivial inclusion $V_{\leqslant}(A, C) \subseteq V(A, C)$ it follows that $V_{\leqslant}(A, C) \subseteq D$. Therefore (4.1) implies $D + R(A, B) + E_{<}(A) = \mathbb{C}^n$, and D has all the properties of a member of $\mathscr{N}$.

It is not difficult to see that (4) is a consequence of (3). ■

COROLLARY 4.2. *Assume $H \leqslant \tilde{H}$. Let* (4.1) *be satisfied. Then the greatest and the least elements of $\mathscr{S}$ and $\tilde{\mathscr{S}}$ exist. Let them be denoted by $Y, \tilde{Y}$ and $W, \tilde{W}$ respectively. Then*

$$\mathscr{S} \cap \tilde{\mathscr{S}} = \{X \in \mathscr{S} | \tilde{W} \leqslant X\} = \{\tilde{X} \in \tilde{\mathscr{S}} | \tilde{X} \leqslant Y\}.$$

In particular, if $\tilde{W} \leqslant Y$, then $\tilde{W}$ is a solution of $\mathscr{R}(X) = 0$ and Y is a solution of $\tilde{\mathscr{R}}(X) = 0$.

Proof. Under the hypothesis (4.1) we deduce from the preceding lemma that

$$\begin{aligned}\mathscr{N} \cap \tilde{\mathscr{N}} &= \{N \in \operatorname{Inv} A, V_{\leqslant}(\tilde{A}, \tilde{C}) \subseteq N \subseteq V(A, C)\} \\ &= \{\tilde{N} \in \operatorname{Inv} \tilde{A}, V_{\leqslant}(\tilde{A}, \tilde{C}) \subseteq \tilde{N} \subseteq V(A, C)\}.\end{aligned}$$ ■

The fact that the maps $\gamma : \mathscr{S} \to \mathscr{N}$ and $\gamma : \tilde{\mathscr{S}} \to \tilde{\mathscr{N}}$ are isomorphisms between partially ordered sets, combined with Theorem 3.3, allows us to formulate results for $\mathscr{S}$ and $\tilde{\mathscr{S}}$ and to pass to $\mathscr{N}$ and $\tilde{\mathscr{N}}$ for their proofs.

THEOREM 4.3. *Assume $H \leqslant \tilde{H}$ and $\mathscr{S} \neq \varnothing$.*

(1) *For each $X \in \mathscr{S}$ there exists*

$$\alpha(X) = \inf\{\tilde{X} \in \tilde{\mathscr{S}}, X \leqslant \tilde{X}\}, \tag{4.3}$$

and $\alpha(X) = \tilde{U}$ if and only if

$$\operatorname{Ker} \tilde{U} = \operatorname{Ker} X + V_{\leqslant}(\tilde{A}, \tilde{C}).$$

The map $\alpha : \mathscr{S} \to \tilde{\mathscr{S}}$ is a $\vee$-semilattice homorphism.

(2) *Assume* (4.1). *Then for each* $\tilde{X} \in \tilde{\mathcal{S}}$ *there exists*

$$\beta(\tilde{X}) = \sup\{X \in \mathcal{S},\ X \leqslant \tilde{X}\}, \tag{4.4}$$

and $\beta(\tilde{X}) = U$ *if and only if*

$$\operatorname{Ker} U = \operatorname{Ker} \tilde{X} \cap V(A, C).$$

The map $\beta: \tilde{\mathcal{S}} \to \mathcal{S}$ *is a* $\wedge$*-semilattice homomorphism.*

(3) *If* (4.1) *holds, then the pair* (α, β) *is a Galois correspondence between* $\mathcal{S}$ *and* $\tilde{\mathcal{S}}$.

Proof. (1): For $N \in \mathcal{N}$ define

$$\varphi(N) = N + V_{\leqslant}(\tilde{A}, \tilde{C}). \tag{4.5}$$

According to Lemma 4.1 we have $\varphi(N) \in \tilde{\mathcal{N}}$. Since $\tilde{N} \in \tilde{\mathcal{N}}$ and $N \subseteq \tilde{N}$ implies $N + V_{\leqslant}(\tilde{A}, \tilde{C}) \subseteq \tilde{N}$, we conclude that

$$\varphi(N) = \inf\left\{\tilde{N} \in \tilde{\mathcal{N}},\ N \subseteq \tilde{N}\right\}. \tag{4.6}$$

It is obvious that $\varphi: \mathcal{N} \to \tilde{\mathcal{N}}$ is a $\vee$-morphism.

(2): For $\tilde{N} \in \tilde{\mathcal{N}}$ define

$$\psi(\tilde{N}) = \tilde{N} \cap V(A, C). \tag{4.7}$$

Under the hypothesis (4.1) we have $\psi(\tilde{N}) \in \mathcal{N}$. From $N \in \mathcal{N}$ and $N \subseteq \tilde{N}$ follows $N \subseteq \tilde{N} \cap V(A, C)$, and we see that

$$\psi(\tilde{N}) = \sup\left\{N \in \mathcal{N},\ N \subseteq \tilde{N}\right\}. \tag{4.8}$$

The definition (4.4) shows that $\psi: \tilde{\mathcal{N}} \to \mathcal{N}$ is a $\wedge$-morphism.

(3): If φ and ψ are given by (4.6) and (4.7), then according to [2, p. 126] or [1, p. 173] we have a Galois correspondence. Since φ and ψ are given explicitely by (4.5) and (4.7), we could also argue as follows. From (4.5) and (4.7) it is obvious that φ and ψ are order-preserving. Take $N \in \mathcal{N}$ and $\tilde{N} \in \tilde{\mathcal{N}}$. Then

$$\psi\varphi(N) = \left[N + V_{\leqslant}(\tilde{A}, \tilde{C})\right] \cap V(A, C) \supseteq N \cap V(A, C) = N,$$

and

$$\begin{aligned} \varphi\psi(\tilde{N}) &= \left[\tilde{N} \cap V(A,C)\right] + V_{\leqslant}(\tilde{A},\tilde{C}) \\ &\subseteq \left[\tilde{N} \cap V(\tilde{A},\tilde{C})\right] + V_{\leqslant}(\tilde{A},\tilde{C}) = \tilde{N}. \end{aligned}$$ ■

Recall that stabilizability of (A, B) implies $\mathscr{T} \neq \varnothing$, $\mathscr{T} = \mathscr{S}$, and (4.1), and that under the additional assumption $H \leqslant \tilde{H}$ also $(\tilde{A}, \tilde{B})$ is stabilizable. Under those assumptions the preceding theorem describes a connection between the solution sets $\mathscr{T}$ and $\tilde{\mathscr{T}}$.

COROLLARY 4.4. *Assume $H \leqslant \tilde{H}$, and let (A, B) be stabilizable. Then the maps $\alpha : \mathscr{T} \rightarrow \tilde{\mathscr{T}}$ and $\beta : \tilde{\mathscr{T}} \rightarrow \mathscr{T}$ given by* (4.3) *and* (4.4) *are well defined, and the pair (α, β) is a Galois correspondence between $\mathscr{T}$ and $\tilde{\mathscr{T}}$.*

I am grateful to a referee for valuable comments.

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Received 25 November 1992; final manuscript accepted 10 May 1993