Roth's Theorems for Matrix Equations With Symmetry Constraints

H. K. Wimmer*
Mathematisches Institut
Universität Würzburg
D-87074 Würzburg, Germany

Submitted by Ion Zaballa

Dedicated to Ingram Olkin

ABSTRACT

This note deals with the consistency of complex matrix equations $AX - YB = C$ and $AX - XB = C$ under the constraints $Y = X^*$ and $X = X^*$.

Let F be a field, and let A, B, C be matrices over F of respective sizes $m \times n$, $s \times k$, and $m \times k$. Put

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

and

$$M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

The following theorem was given by W. E. Roth.

THEOREM 1 [2].

(1) *The matrix equation*

$$AX - YB = C \tag{1}$$

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has a solution $X \in F^{n\times k}$, $Y \in F^{m\times s}$ *if and only if* M_C *and* M_0 *are equivalent (i.e., have the same rank).*

(2) *Assume* $n = m$, $k = s$. *There exists a solution* $X \in F^{n\times k}$ *of*

$$AX - XB = C \tag{2}$$

if and only if M_C *and* M_0 *are similar.*

In this note we assume $F = \mathbb{C}$ and consider (1) and (2) together with the respective constraints $Y = X^*$ and $X = X^*$. The approach of [1] will be used to prove the following results.

THEOREM 2. *Assume* $n = s$, $k = m$. *The following statements are equivalent*:

(a) *The equation*

$$AX - X^*B = C \tag{3}$$

has a solution $X \in \mathbb{C}^{n\times k}$.

(b) *There exists a nonsingular matrix* $S \in \mathbb{C}^{(n+k)\times(n+k)}$ *such that*

$$\begin{pmatrix} 0 & -A \\ B & 0 \end{pmatrix} = S\begin{pmatrix} C & -A \\ B & 0 \end{pmatrix}S^*. \tag{4}$$

(c) *There exist nonsingular matrices* R, $S \in \mathbb{C}^{(n+k)\times(n+k)}$ *such that*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}R = S\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{5}$$

and

$$S^*\begin{pmatrix} 0 & I_k \\ -I_n & 0 \end{pmatrix}R = \begin{pmatrix} 0 & I_k \\ -I_n & 0 \end{pmatrix}. \tag{6}$$

THEOREM 3. *Assume* $n = m = k = s$. *The following statements are equivalent*:

(a) *The equation*

$$AX - XB = C$$

has a hermitian solution.

(b) *There exists a nonsingular matrix* $R \in \mathbb{C}^{2n\times 2n}$ *which satisfies*

$$R^{-1}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} R = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

and

$$R^{*}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} R = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Proof of Theorem 2. (a) ⇒ (c): For $X \in \mathbb{C}^{n\times k}$ and $Y \in \mathbb{C}^{m\times s}$ define

$$G_X = \begin{pmatrix} I_n & X \\ 0 & I_k \end{pmatrix}$$

and

$$G_Y = \begin{pmatrix} I_m & Y \\ 0 & I_s \end{pmatrix}.$$

As was observed in [2], Equation (1) can be written in an equivalent form

$$M_0 G_X - G_Y M_C = 0. \tag{7}$$

Put

$$J = \begin{pmatrix} 0 & I_k \\ -I_n & 0 \end{pmatrix}.$$

For a nonsingular matrix G define $G^{-*} = (G^*)^{-1}$. Obviously $Y = X^*$ is equivalent to

$$G_Y^* J G_X = J.$$

Hence (7) yields (5) with $R = G_X$ and $S = G_Y$.

(b) ⇔ (c): Put

$$P_C = \begin{pmatrix} C & -A \\ B & 0 \end{pmatrix}$$

such that $P_C J = M_C$. Hence (4) is equivalent to

$$M_0(J^{-1}S^{-*}J) = SM_C.$$

Note that (6) can be written as $R = J^{-1}S^{-*}J$.

(c) ⇒ (a): Let

$$U = \begin{pmatrix} U_1 & U_{12} \\ U_{21} & U_2 \end{pmatrix}, \qquad W = \begin{pmatrix} W_1 & W_{12} \\ W_{21} & W_2 \end{pmatrix} \tag{8}$$

be complex $(n+k)\times(n+k)$ matrices where $U_{12} \in \mathbb{C}^{n\times k}$ and $W_{12} \in \mathbb{C}^{k\times n}$. Put

$$\Gamma_C = \{(U,W)\,|\,M_0U - WM_C = 0\} \tag{9}$$

and

$$\Delta_C = \{(U,W)\,|\,JU^*J^*M_0 - M_CJ^*W^*J = 0\}. \tag{10}$$

The conditions for $(U,W) \in \Gamma_C$ are

$$\begin{aligned} AU_1 - W_1A &= 0, & AU_{12} - W_1C - W_{12}B &= 0, \\ BU_{21} - W_{21}A &= 0, & BU_2 - W_{21}C - W_2B &= 0, \end{aligned} \tag{11}$$

and those for $(U,W) \in \Delta_C$ are given explicitly by

$$\begin{aligned} U_2^*A - AW_2^* + CW_{21}^* &= 0, & -U_{12}^*B + AW_{12}^* - CW_1^* &= 0, \\ -U_{21}^*A + BW_{21}^* &= 0, & U_1^*B - BW_1^* &= 0. \end{aligned} \tag{12}$$

Put

$$D_C = \Gamma_C \cap \Delta_C. \tag{13}$$

Clearly D_C is a vector space over $\mathbb{C}$. For $C = 0$ let Γ_0, Δ_0, and D_0 be defined by (9), (10), and (13). It is not difficult to verify that $M_C = S^{-1}M_0R$ together with (6) implies that $(U,W) \in D_C$ is equivalent to $(UR^{-1}, WS^{-1}) \in D_0$. Hence

$$\dim D_C = \dim D_0. \tag{14}$$

Suppose there exists a pair $(U, W) \in D_C$ such that $W_1 = I$. Then (11) and (12) yield

$$AU_{12} - W_{12}B = C \tag{15}$$

and

$$AW_{12}^* - U_{12}^*B = C, \tag{16}$$

and

$$X = \tfrac{1}{2}(U_{12} + W_{12}^*) \tag{17}$$

is a solution of (3).

Set $E = \mathbb{C}^{(n+k)\times(n+k)}$. We introduce a linear map $\varphi : E \times E \to \mathbb{C}^{(n+k)\times k}$ and define

$$\varphi(U, W) = \begin{pmatrix} W_1 \\ W_{21} \end{pmatrix}.$$

The aim is to prove that

$$\begin{pmatrix} I \\ 0 \end{pmatrix} \in \varphi(D_C). \tag{18}$$

It is obvious that in the case $C = 0$ we have $(U, W) = (I, I) \in D_0$ and therefore

$$\begin{pmatrix} I \\ 0 \end{pmatrix} \in \varphi(D_0). \tag{19}$$

Put $\varphi_C = \varphi|_{D_C}$ and $\varphi_0 = \varphi|_{D_0}$. From (11) and (12) we see that

$$\operatorname{Ker} \varphi_C = \operatorname{Ker} \varphi_0. \tag{20}$$

Let U and W be as in (8), and put

$$\tilde{U} = \begin{pmatrix} U_1 & 0 \\ U_{21} & 0 \end{pmatrix}, \qquad \tilde{W} = \begin{pmatrix} W_1 & 0 \\ W_{21} & 0 \end{pmatrix}.$$

If $(U, W) \in D_C$ then $(\tilde{U}, \tilde{W}) \in D_0$. Therefore we have

$$\operatorname{Im} \varphi_C \subseteq \operatorname{Im} \varphi_0. \tag{21}$$

Note that

$$\dim \operatorname{Ker} \varphi_C + \dim \operatorname{Im} \varphi_C = \dim D_C$$

and

$$\dim \operatorname{Ker} \varphi_0 + \dim \operatorname{Im} \varphi_0 = \dim D_0.$$

Then (14) and (20) imply $\dim \operatorname{Im} \varphi_C = \dim \operatorname{Im} \varphi_0$, and (21) yields $\varphi(D_C) = \varphi(D_0)$. From (19) we obtain (18), which completes the proof. ■

Proof of Theorem 3. To show that (b) implies (a) let Γ_C and Δ_C be defined as in (9) and (10). Put $\Lambda = \{(U, W) \mid U = W\}$, and replace D_C in (13) by $D_C = \Gamma_C \cap \Delta_C \cap \Lambda$. The arguments of the preceding proof remain unchanged. They lead to (15) and (16), but now with $U_{12} = W_{12}$. Hence in (17) we have $X = X^*$. ■

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