On the existence of a least and negative-semidefinite solution of the discrete-time algebraic Riccati equation

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Abstract.

The Riccati equation $X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XF - H^*H = 0$ is studied and the existence of a least (and negative-semidefinite) solution X_- is investigated. If F is singular then the zero eigenvalue of F plays a crucial role for X_-. There is no complete analogy between the results for a greatest (and positive-semidefinite) solution and the results for X_-.

Keywords: discrete-time algebraic Riccati equation, least solution, negative-semidefinite solutions.

AMS Subject Classifications: 15A24, 93C55.

1 Introduction

This paper deals with the discrete-time algebraic Riccati equation (DARE)

$$X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XF - H^*H = 0 \tag{1.1}$$

and its hermitian solutions. Here F, G and H are complex matrices of size $n \times n, n \times p$ and $q \times n$ respectively. The study is motivated by discrepancies between (1.1) and the corresponding continuous-time algebraic Riccati equation (CARE)

$$F^*X + XF - XGG^*X + H^*H = 0. \tag{1.2}$$

Since it is the same type of optimal control problem which – depending on the time scale – gives rise to (1.1) or to (1.2) it is clear that those two equations should have many features in common. For example if (F, G) is stabilizable then both (1.1) and (1.2) have a greatest solution X_+ such that $X_+ \geq 0$. In the case of (1.1) X_+ is the unique solution such that the spectrum $\sigma(F_{X_+})$ of the associated closed loop matrix

$$F_{X_+} = F - G(I + G^*X_+G)^{-1}G^*X_+F = (I + GG^*X_+)^{-1}F$$

lies in the closed unit disc. In the case of (1.2) the feedback matrix $F_{X_+} = F - GG^*X_+$ has all its eigenvalues in the closed left half plane. For the CARE (1.2) it is immediate that the stronger hypothesis of controllability implies in addition to X_+ also the existence of a least solution X_- such that $X_- \leq 0$ and $\sigma(F - GG^*X_-)$ is in the closed right half plane. The following example from [2] shows that an analogous result does not hold for the DARE. Consider (1.1) with

$$F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad H = (1, 2)\,.$$

Then (1.1) has only two solutions, namely

$$X_+ = \begin{pmatrix} 1 & 2 \\ 2 & 2 + \sqrt{5} \end{pmatrix} \quad \text{and} \quad X_\ell = \begin{pmatrix} 1 & 2 \\ 2 & 2 - \sqrt{5} \end{pmatrix}.$$

Clearly X_ℓ is the least solution, but X_ℓ is not negative-semidefinite. Note that

$$F_{X_\ell} = \begin{pmatrix} 0 & 1 \\ 0 & \frac{-2}{3-\sqrt{5}} \end{pmatrix}.$$

Hence X_ℓ is not anti-stabilizing. We have $\sigma(F_{X_\ell}) \subseteq \{\lambda | \lambda = 0 \text{ or } |\lambda| \geq 1\}$. If F is nonsingular and (F, G) is controllable then it is known [3] that a least solution X_- of (1.1) exists such that $X_- \leq 0$ and $|\lambda| \geq 1$ for all $\lambda \in \sigma(F_{X_-})$. Hence the discrepancy between the Riccati equations (1.1) and (1.2) which is displayed by the preceding example can not occur if $0 \notin \sigma(F)$. It is one of the goals of this paper to clarify the role of a zero eigenvalue of F as far as solutions X of (1.1) and their feedback matrices

$$F_X = (I + GG^*X)^{-1}F$$

are concerned. It will be shown that there exists a subspace U_0 of $\mathbb{C}^n$, which can be described in terms of the triple (F, G, H), such that

$$U_0 = \text{Ker } (F_X)^n$$

is a generalized eigenspace which all solutions X have in common. Furthermore $X = Y$ on U_0 for any two solutions X, Y of (1.1). Our main results are the following theorems.

Theorem 1.1. *The following conditions are equivalent:*

(a) There exists a unique solution X_- of (1.1) such that the spectrum of the associated closed loop matrix satisfies

$$\sigma(F_{X_-}) \subseteq \{\lambda \mid \lambda = 0 \ \textit{ or } \ |\lambda| \geq 1\}. \tag{1.3}$$

(b) We have

$$\text{rank } (F - \lambda I,\, G) = n \ \textit{ if } \ 0 < |\lambda| \leq 1. \tag{1.4}$$

Theorem 1.2. *Let X_- be a unique solution of (1.1) satisfying (1.3).*

(i) Then X_- is the least solution of (1.1), i.e. $X_- \leq X$ holds for all solutions of X of (1.1).

(ii) We have $X_- \leq 0$ if and only if $\text{Ker } F^n \subseteq \text{Ker } H$.

The paper is organized as follows. In Section 2 it will be proved that all solutions X of (1.1) satisfy $I + G^*XG > 0$. Section 3 deals with the generalized eigenspace $U_0 = \text{Ker } (F_X)^n$ and Section 4 contains an existence and uniqueness result for unmixed solutions of (1.1). The proof of Theorem 1.1. and 1.2 is given in Section 5.

2 An inertia result

If X and Y are solutions of (1.1) then [1] the matrix $\Delta = X - Y$ satisfies

$$F_Y^* \Delta F_Y - \Delta = F_Y^* \Delta G (I + G^*XG)^{-1} \, G^* \Delta F_Y. \tag{2.1}$$

Also note that (1.1) can be written in an equivalent form as

$$X - F_X^* X F_X = F_X^* X \Gamma X F_X + H^*H. \tag{2.2}$$

Since (2.1) can be regarded as a discrete-time Lyapunov equation we can relate the inertia of Δ to the location of $\sigma(F_X)$ as soon as we know that the righthand side of (2.1) is semidefinite, in particular that $I + G^*XG > 0$. For the proof of the following lemma I am indebted to a referee.

Lemma 2.1. *If X is a solution of (1.1) then we have $I + G^*XG > 0$.*

Proof. Let X be a hermitian $n \times n$ matrix. For $w \in \mathbb{C}^p$ and $y \in \mathbb{C}^n$ define

$$P(w, y) = w^*(I + G^*XG)w - (F_X y + Gw)^* X(F_X y + Gw) + y^*Xy.$$

Then

$$P(w, y) = w^*w - y^*F_X^*XGw - w^*G^*XF_X y + y^*(X - F_X^*XF_X)y.$$

If X is a solution of (1.1) then (2.2) implies

$$\begin{aligned} P(w, y) &= (w^* \quad y^*) \begin{pmatrix} I & -G^*XF_X \\ -F_X^*XG & F_X^*XGG^*XF_X \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} + y^*H^*Hy = \\ &= (w^* \quad y^*) \begin{pmatrix} I \\ -F_X^*XG \end{pmatrix} \begin{pmatrix} I, -G^*XF_X \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} + y^*Hy \geq 0. \end{aligned}$$

Now choose λ such that $|\lambda| = 1$ and $\lambda \notin \sigma(F_X)$. If $\hat{y} = (\lambda I - F_X)^{-1}Gw$ then $F_X\hat{y} + Gw = \lambda\hat{y}$. Therefore

$$(F_X\hat{y} + Gw)^*X(F_X\hat{y} + Gw) = \hat{y}^*X\hat{y}$$

and we obtain $P(w, \hat{y}) = w^*(I + G^*XG)w$. By assumption $I + G^*XG$ is nonsingular. Hence $P(w, y) \geq 0$ yields $I + G^*XG > 0$.

□

3 Generalized eigenspaces related to a zero eigenvalue

If X and Y are solutions of (1.1) then it is known [6] that Ker $(F_X)^n =$ Ker $(F_Y)^n$ and

$$\text{Ker } (F_X)^n \subseteq \text{Ker } (X - Y). \tag{3.1}$$

A proof of those facts, which differs from [6], will be given below. We consider a DARE

$$X - F^*XF + F^*XG(I + G^*XG)^{-1}G^*XG - Q = 0 \tag{3.2}$$

where Q need not be positive-semidefinite. We describe $U_0 =$ Ker $(F_X)^n$ in terms of the triple (F, Γ, Q) where $\Gamma = GG^*$. Since Ker $(F_X)^n$ is a generalized eigenspace we focus on Jordan chains of F_X.

Theorem 3.1. *Let X be a solution of (3.2). Put*

$$T = \begin{pmatrix} -F & I+\Gamma Q & \Gamma F^* Q & \Gamma (F^*)^2 Q & \dots \\ 0 & -F & I+\Gamma Q & \Gamma F^* Q & \dots \\ 0 & 0 & -F & I+\Gamma Q & \dots \\ 0 & 0 & 0 & -F & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix}. \tag{3.3}$$

A chain of vectors $v_i \in \mathbb{C}^n$, $i = 0, 1, \dots, m$, satisfies

$$F_X v_0 = 0, \quad F_X v_1 = v_0, \dots, F_X v_m = v_{m-1} \tag{3.4}$$

if and only if the vector

$$\hat{v} = (v_0^T, v_1^T, \dots, v_m^T, 0, 0, \dots)^T \tag{3.5}$$

is a solution of $T\hat{v} = 0$. Assume that (3.4) holds. Then we have

$$X v_i = Q v_i + F^* Q v_{i-1} + \dots + (F^*)^i Q v_0, \ i = 0, 1, \dots, m. \tag{3.6}$$

For $\hat{v}$ given by (3.5) set $(\hat{v})_0 = v_0$ and define

$$U_0 = \text{span } \{(\hat{v})_0, \hat{v} \in \text{ Ker } T\}. \tag{3.7}$$

Then

$$\text{Ker } (F_X)^n = U_0. \tag{3.8}$$

Proof. The DARE (3.2) can be written as a pair of equations

$$F = (I + \Gamma X) F_X \tag{3.9}$$

together with

$$X = Q + F^* X F_X. \tag{3.10}$$

Assume first that (3.4) holds. Then (3.10) implies $Xv_i = Qv_i + F^* X v_{i-1}$, $v_{-1} = 0$, which yields (3.6). From (3.9) we obtain

$$F v_i = (I + \Gamma X) v_{i-1}, \quad i = 0, 1, \dots, m-1, \tag{3.11}$$

and in particular

$$F v_0 = 0. \tag{3.12}$$

From (3.6) and (3.11) we deduce

$$F v_1 = (I + \Gamma Q) v_0 \tag{3.13}$$

and

$$Fv_i = (I + \Gamma Q)v_{i-1} + \Gamma F^* Q v_{i-2} + \ldots + \Gamma (F^*)^{i-1} Q v_0,\ i = 2, \ldots, m. \tag{3.14}$$

Clearly (3.12), (3.13) and (3.14) are equivalent to $T\hat{v} = 0$.

Conversely assume now that (3.12) – (3.14) hold. Then it is easy to see that (3.9) and (3.10) imply $F_X v_0 = 0$, $F_X v_1 = v_0$, $X v_0 = Q v_0$ and $X v_1 = Q v_1 + F^* Q v_0$. An induction argument yields $X v_i = Q v_i + F^* Q v_{i-1}$ and $F_X v_i = v_{i-1}$ for $i = 2, \ldots, m$.

If $v_0 \neq 0$ and (3.4) holds then $(v_0, v_1, \ldots, v_m)$ is a Jordan chain of F_X corresponding to the eigenvalue 0, and Ker $(F_X)^n$ is spanned by such chains. Note that $\hat{v} \in$ Ker T implies

$$S_- \hat{v} = (v_1^T, \ldots, v_m^T, 0, 0, \ldots)^T \in \text{Ker } T$$

which yields (3.8) and (3.7).

□

The kernel of T can be related to the rational matrix $F - s^{-1}I - \Gamma(sI - F^*)^{-1}Q$. Let $y \in \mathbb{C}^n(s)$ be a vector of rational functions with a formal expansion $y(s) = \sum_{\nu=-\infty}^{N} y_\nu s^\nu$ and let $\pi_+ y = y_0 + y_1 s + \ldots + y_N s^N$ be its polynomial part. To a matrix of rational functions $R \in \mathbb{C}^{n \times n}(s)$, $R(s) = R_0 + R_1 s^{-1} + \ldots$, which is assumed to be bounded at $s = \infty$, we associate an infinite block Toeplitz matrix

$$T(R) = \begin{pmatrix} R_0 & R_1 & R_2 & \ldots \\ 0 & R_0 & R_1 & \ldots \\ 0 & 0 & R_0 & \ldots \\ \cdot & \cdot & \cdot & \ldots \end{pmatrix}.$$

It is easy to verify that a polynomial vector $v(s) = v_0 + v_1 s + \ldots + v_m s^m \in \mathbb{C}^n[s]$ satisfies $\pi_+ R v = 0$ if and only if $\operatorname{vec} v = (v_0^T, v_1^T, \ldots, v_m^T, 0, 0, \ldots)^T$ is a solution of $T(R)\operatorname{vec} v = 0$. Furthermore $w = (w_0^T, w_1^T, \ldots, w_k^T, 0, 0, \ldots)^T \in$ Ker $T(R)$ implies $S_- w = (w_1^T, \ldots, w_k^T, 0, 0, \ldots)^T \in$ Ker $T(R)$, and correspondingly $v \in$ Ker $\pi_+ R$ implies $\pi_+ s^{-1} v = v_1 + v_2 s + \ldots + v_m s^{m-1} \in$ Ker $\pi_+ R$. In particular if $R(s) = F - s^{-1}I - \Gamma(sI - F^*)^{-1}Q$ then the matrix $T(R)$ is given by (3.3). Hence a chain $(v_0, v_1, \ldots, v_m)$ satisfies (3.4) if and only if $v(s) = v_0 + v_1 s + \ldots + v_m s^m$ is such that $\pi_+[F - s^{-1}I - \Gamma(sI - F^*)^{-1}Q]v(s) = 0$.

Let

$$V(F, Q) = \text{Ker} \begin{pmatrix} Q \\ QF \\ \vdots \\ QF^{n-1} \end{pmatrix}$$

be the unobservable subspace associated to (F, Q). Define

$$V_0 = V(F, Q) \cap \text{Ker } F^n$$

such that V_0 is the largest F-invariant subspace in Ker Q where the restriction of F is nilpotent.

Lemma 3.2. *For each solution X of (3.2) we have $V_0 \subseteq \text{Ker } X$, and $F = F_X$ on V_0, and $V_0 \subseteq \text{Ker } (F_X)^n$.*

Proof. Let a basis of $\mathbb{C}^n$ be chosen such that

$$V_0 = \text{Im}\begin{pmatrix} I_{n-n_1} \\ 0 \end{pmatrix}. \tag{3.15}$$

Then

$$F = \begin{pmatrix} F_0 & F_{01} \\ 0 & F_1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & Q_1 \end{pmatrix}, \tag{3.16}$$

and $\sigma(F_0) = \{0\}$ and

$$\text{rank}\begin{pmatrix} F_1 - \lambda I \\ Q_1 \end{pmatrix} = n_1 \quad \text{for } \lambda = 0.$$

If we partition

$$X = \begin{pmatrix} X_0 & * \\ X_{10} & * \end{pmatrix}$$

according to (3.16) and write (3.2) as $X - F_X^* X F - Q = 0$ then we obtain

$$\begin{pmatrix} X_0 \\ X_{10} \end{pmatrix} - F_X^* \begin{pmatrix} X_0 \\ X_{10} \end{pmatrix} F_0 = 0. \tag{3.17}$$

Since F_0 is nilpotent the equation (3.17) has only a trivial solution $X_0 = 0$, $X_{10} = 0$. Hence $V_0 \subseteq \text{Ker } X$. Let $\Gamma = \left(\begin{smallmatrix} * & * \\ * & \Gamma_1 \end{smallmatrix}\right)$ be partitioned conforming to (3.16). Then $X = \text{diag }(0, X_1)$ implies

$$F_X = \begin{pmatrix} F_0 & * \\ 0 & (I + \Gamma_1 X_1)^{-1} F_1 \end{pmatrix} \tag{3.18}$$

which shows that F and F_X coincide on V_0.

□

In the sequel we assume again $Q \geq 0$, $Q = H^* H$. In that case we shall see that all solutions X of (1.1) are positive-semidefinite on $U_0 = \text{Ker } (F_X)^n$. Therefore a solution X satisfies $X \leq 0$ only if $U_0 \subseteq \text{Ker } X$ holds.

Lemma 3.3. *Let X be a solution of (1.1). Then we have*

$$v^* X v \geq 0 \quad \textit{for all } v \in U_0 = \text{Ker } (F_X)^n \tag{3.19}$$

and

$$\text{Ker } (F_X)^n \cap \text{Ker } X = V_0. \tag{3.20}$$

Furthermore

$$U_0 = \text{Ker } (F_X)^n \subseteq \text{Ker } X \tag{3.21}$$

holds if and only if

$$\text{Ker } F^n \subseteq \text{Ker } H. \tag{3.22}$$

Proof. Let F_X be given as

$$F_X = \text{ diag } (\Phi_0, \Phi_2) \tag{3.23}$$

where Φ_0 is nilpotent and Φ_2 is nonsingular. Then $U_0 = \text{ Im } \binom{I}{0}$. Let

$$X = \begin{pmatrix} X_0 & X_{20}^* \\ X_{20} & X_2 \end{pmatrix}, \quad H^* H = Q = \begin{pmatrix} Q_0 & Q_{20}^* \\ Q_{20} & Q_2 \end{pmatrix} \tag{3.24}$$

be partitioned conforming to (3.23). Then (2.2) yields

$$X_0 - \Phi_0^* X_0 \Phi_0 = \Phi_0^* (X_0 \;\; X_{20}^*) \Gamma \begin{pmatrix} X_0 \\ X_{20} \end{pmatrix} \Phi_0 + Q_0 \tag{3.25}$$

and

$$X_{20} - \Phi_2^* X_{20} \Phi_0 = \Phi_2^* (X_{20} \;\; X_2) \Gamma \begin{pmatrix} X_0 \\ X_{20} \end{pmatrix} \Phi_0 + Q_{20}. \tag{3.26}$$

For later use we also note

$$X_2 - \Phi_2^* X_2 \Phi_2 = S_2, \;\; S_2 \geq 0. \tag{3.27}$$

Because of $\Gamma \geq 0$, $Q_0 \geq 0$ and since Φ_0 is nilpotent it is obvious from (3.25) that $X_0 \geq 0$ holds, which proves (3.19).

Now put $D_0 = \text{ Ker } (F_X)^n \cap \text{ Ker } X$. Take $y \in D_0$. Then

$$y = \begin{pmatrix} y_0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} X_0 \\ X_{20} \end{pmatrix} y_0 = 0.$$

From (3.25) and $X_0 \geq 0$ we conclude that

$$X_0 \Phi_0 y_0 = 0, \tag{3.28}$$

and

$$\Gamma\begin{pmatrix} X_0 \\ X_{20} \end{pmatrix}\Phi_0 y_0 = 0, \quad Q_0 y_0 = 0.$$

Then $Q \geq 0$ implies $Q_{20}y_0 = 0$ and (3.26) yields $\Phi_2^* X_{20}\Phi_0 y_0 = 0$. Recall that Φ_2 is nonsingular. Hence $X_{20}\Phi_0 y_0 = 0$, which together with (3.28) shows that

$$XF_X y = \begin{pmatrix} X_0 \\ X_{20} \end{pmatrix}\Phi_0 y_0 = 0. \tag{3.29}$$

Hence D_0 is invariant under F_X. From $F = (I + \Gamma X)F_X$ and (3.29) follows $Fy = F_X y$. We already know that $Qy = 0$. Hence D_0 is an F-invariant subspace in Ker $H \cap$ Ker F^n, which implies $D_0 \subseteq V_0$. From Lemma 3.2 follows $V_0 \subseteq$ Ker $(F_X)^n \cap$ Ker X, which completes the proof of (3.20).

Because of (3.20) the inclusion (3.21) is equivalent to

$$\text{Ker } (F_X)^n = V_0. \tag{3.30}$$

If we assume (3.15) then (3.18) shows that (3.30) holds if and only if F_1 is nonsingular which is equivalent to Ker $F^n \subseteq$ Ker H.

□

4 An existence and uniqueness theorem

Theorem 1.1 is a special case of a more general result on unmixed solutions of (1.1), which will be derived in this section. Let

$$M - sL = \begin{pmatrix} F & 0 \\ -Q & I \end{pmatrix} - s\begin{pmatrix} I & \Gamma \\ 0 & F^* \end{pmatrix}, \quad Q = H^*H, \quad \Gamma = GG^*,$$

be the symplectic pencil associated to (1.1). If X is a solution of (1.1) then it is well known that

$$(M - sL)\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} I + \Gamma X & 0 \\ F^*X & I \end{pmatrix}\begin{pmatrix} F_X - sI & -s(I + \Gamma X)^{-1}\Gamma \\ 0 & I - sF_X^* \end{pmatrix}. \tag{4.1}$$

For a complex polynomial $f(s) = \prod_{\nu=1}^n (s - \lambda_\nu)$ define $\tilde{f}(s) = \prod_{\nu=1}^n (1 - \bar{\lambda}_\nu s)$. Because of (4.1) a solution X gives rise to a factorization $\det(M - sL) = cg(s)\tilde{g}(s)$, $c \in \mathbb{C}$, where $g(s) = \det(sI - F_X)$. With respect to a suitable basis of $\mathbb{C}^n$ we have

$$F = \begin{pmatrix} F_1 & 0 \\ F_{21} & F_2 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix}$$

where the pair (F_2, G_2) is controllable. Put $h(s) = \det(sI - F_1)$. Then we have

$$h(\lambda) \neq 0 \text{ if and only if } \operatorname{rank}\,(F - \lambda I, G) = n. \tag{4.2}$$

Also note that

$$h \mid \det(M - sL). \tag{4.3}$$

If $X = \left(\begin{smallmatrix} * & * \\ * & X_2 \end{smallmatrix}\right)$ is a solution of (1.1), $\Gamma = \left(\begin{smallmatrix} * & * \\ * & \Gamma_2 \end{smallmatrix}\right)$ is partitioned accordingly and if we set $\hat{F}_2 = (I + \Gamma_2 X_2)^{-1} F_2$ then

$$\det(sI - F_X) = h(s) \det(sI - \hat{F}_2). \tag{4.4}$$

Theorem 4.1. [7, p. 868/869] *Assume that*

$$\operatorname{rank}\,(F - \lambda I, G) = n \quad \textit{if } |\lambda| = 1$$

and let

$$\det(M - sL) = c g(s) \tilde{g}(s), \quad c \in \mathbb{C}, \tag{4.5}$$

be a factorization with the property that

$$g(\lambda) = \tilde{g}(\lambda) = 0 \quad \textit{implies } |\lambda| = 1. \tag{4.6}$$

If

$$(h, \tilde{g}) = 1. \tag{4.7}$$

Then there exists a unique solution X of (1.1) with $\det(sI - F_X) = g(s)$.

In the case where $\det(M - \lambda L) \neq 0$ for $|\lambda| = 1$ the predecing theorem can also be recovered from [8].

Theorem 4.2. *Let g be a complex polynomial such that (4.5) and (4.6) hold. If there exists a unique solution X of (1.1) with*

$$\det(sI - F_X) = g(s) \tag{4.8}$$

then g satisfies (4.7).

Proof. If there exists a solution X with the property (4.8) then (4.4) implies $h|g$, and because of (4.6) all common zeros of h and $\tilde{g}$ (if any) have modulus 1. We shall see that $h(\lambda) \neq 0$ if $|\lambda| = 1$ provided that the solution X with (4.8) is unique. Suppose $h(\alpha) = 0$, i.e. $\operatorname{rank}\,(F - \alpha I, G) < n$ for some α with $|\alpha| = 1$. Let $R(F, G) = \operatorname{Im}\,(G, FG, \ldots, F^{n-1}G)$ be the controllable subspace of (F, G). Put

$$K = \oplus\{\operatorname{Ker}\,(\lambda I - F_X)^n, \lambda \neq \alpha\} + R(F, G).$$

Because of $R(F,G) = R(F_X,G)$ the space K is invariant under F_X. Choose a basis of $\mathbb{C}$ such that $K = \operatorname{Im} \binom{I}{0}$. Then

$$F_X = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ 0 \end{pmatrix}$$

and $\sigma(A_2) = \{\alpha\}$. Let $\Delta_2 \neq 0$ be a hermitian solution of $A_2^* \Delta_2 A_2 - \Delta_2 = 0$. For example, if A_2 is a single Jordan block

$$A_2 = \begin{pmatrix} \alpha & 1 & & \\ & \alpha & \ddots & \\ & & \ddots & 1 \\ & & & \alpha \end{pmatrix}$$

one may choose $\Delta_2 = \operatorname{diag}(0,\dots,0,1)$. Put $\Delta = \operatorname{diag}(0,\Delta_2)$. Then $\Delta \neq 0$ satisfies

$$F_X^* \Delta F_X - \Delta = F_X^* \Delta G (I + G^* X G + G^* \Delta G)^{-1} G^* \Delta F_X.$$

Hence $Y = \Delta + X$ is a solution of the DARE (1.1) and $Y \neq X$. From $\Gamma\Delta = 0$ and $F_Y = F_X + (I + \Gamma X)^{-1} \Gamma \Delta F_Y$ follows $F_Y = F_X$, which shows that $h(\alpha) = 0$ for some α, $|\alpha| = 1$, yields a solution Y, $Y \neq X$, with $\det(sI - F_Y) = g(s)$.

□

5 A least solution

In this section Theorem 1.1 and 1.2 will be proved. We recall some facts on unimodular eigenvalues of F_X and their generalized eigenspaces. Define

$$E_=(F) = \oplus\{\text{ Ker } (F - \lambda I)^n,\ |\lambda| = 1\},$$

let $V(F,H)$ be the unobservable subspace of (F,H) and put

$$V_= = V(F,H) \cap E_=(F).$$

Lemma 5.1 [7]. *Assume* rank $(F - \lambda I, G) = n$ *if* $|\lambda| = 1$*. Let* X *be a solution of (1.1). Then we have* $E_=(F_X) = V_=$*, and* $V_= \subseteq$ Ker X*, and* $F_X = F$ *on* $V_=$*.*

Proof of Theorem 1.1. If $\lambda \neq 0$ is a characteristic root of the pencil $M - sL$, i.e. if $\det(M - \lambda L) = 0$, then also $1/\bar{\lambda}$ is a characteristic root, and if $|\lambda| \neq 1$ then (see e.g. [4]) the corresponding elementary divisors appear in pairs $(s - \lambda)^k$, $(s - 1/\bar{\lambda})^k$. Furthermore (see e.g. [5]) if $|\lambda| = 1$ and

$$\text{rank } (F - \lambda I,\ G) = n, \tag{5.1}$$

then the multiplicity of λ as a root of $\det(M - sL)$ is even (or 0). Therefore if (5.1) holds for all λ with $|\lambda| = 1$ then there exist a unique factorization

$$\det(M - sL) = cg(s)\tilde{g}(s) \tag{5.2}$$

such that

$$g(\lambda) = 0 \text{ only if } \lambda = 0 \text{ or } |\lambda| \geq 1. \tag{5.3}$$

In that case (4.6) is valid and we have $\tilde{g}(\lambda) = 0$ only if $0 < |\lambda| \leq 1$. Now let (5.2) be a factorization with the property (5.3). Then (4.3), i.e. $h|g\tilde{g}$, implies that $(h, \tilde{g}) = 1$ holds if and only if $h(\lambda) \neq 0$ for all λ with $0 < |\lambda| \leq 1$. From (4.2) follows that (4.7) is equivalent to (1.4). Recall that (4.1) yields $\det(M - sL) = c \det(sI - F_X)\det(I - sF_X^*)$ for a solution X. Therefore $\det(sI - F_X) = g(s)$ together with (5.3) is equivalent to (1.3). At this point it is obvious that Theorem 1.1 is a special case of Theorem 4.1 combined with Theorem 4.2.

□

Proof of Theorem 1.2. (i) Let X be a solution of (1.1). Put $\Delta = X - X_-$. From (3.1) and Lemma 5.1 we obtain

$$E_=(F_{X_-}) \oplus \text{Ker } (F_{X_-})^n \subseteq \text{ Ker } \Delta. \tag{5.4}$$

Because of (1.3) we can assume that $F_{X_-} = \text{ diag } (A_1, A_2)$ such that $\lambda = 0$ or $|\lambda| = 1$ if $\lambda \in \sigma(A_1)$ and

$$|\lambda| > 1 \text{ if } \lambda \in \sigma(A_2). \tag{5.5}$$

Take $Y = X_-$ in (2.1), i.e.

$$F_{X_-}^* \Delta F_{X_-} - \Delta = F_{X_-}^* \Delta G(I + G^* XG)^{-1} G^* \Delta F_{X_-}.$$

Then (5.4) implies $\Delta = \text{ diag } (0, \Delta_2)$, and Δ_2 satisfies $A_2^* \Delta_2 A_2 - \Delta_2 = T_2$. According to Lemma 2.1 we have $I + G^* XG > 0$. Hence $T_2 \geq 0$ and (5.5) yields $\Delta_2 \geq 0$. Therefore $\Delta \geq 0$, which shows that X_- is indeed a least solution.

(ii) Take $X = X_-$ in Lemma 3.3 and its proof. Let $X = \begin{pmatrix} * & * \\ * & X_2 \end{pmatrix}$ and $F_X = \text{diag } (\Phi_0, \Phi_2)$ be given as in (3.24) and (3.23) such that Φ_0 is nilpotent. Φ_2 is nonsingular and $\text{Ker } (F_X)^n = \text{Im } \begin{pmatrix} I \\ 0 \end{pmatrix}$. The assumption (1.3) implies $|\lambda| \geq 1$ for $\lambda \in \sigma(\Phi_2)$. Condition (1.4), which is necessary for the existence of X_-, ensures that all unimodular eigenvalues of F are controllable. Hence it follows from Lemma 5.1 that $E_=(F_X) \subseteq \text{Ker } X$, which yields

$$E_=(\Phi_2) \subseteq \text{ Ker } X_2. \tag{5.6}$$

Recall the Lyapunov equation (3.27), i.e. $X_2 - \Phi_2^* X_2 \Phi_2 = S_2$, $S_2 \geq 0$. From the location of the spectrum of Φ_2 and because of (5.6) we obtain $X_2 \leq 0$. Taking (3.19) into account we see that $X = X_- \leq 0$ holds if and only if $U_0 = \text{Ker } (F_X)^n \subseteq \text{Ker } X$, which is equivalent to (3.22).

□

6 Concluding remarks

A singular matrix F in a discrete-time control system $x(t+1) = Fx(t)+Gu(t)$ may give rise to phenomena which can not occur in a continuous-time system $\dot{x}(t) = Fx(t) + Gu(t)$. Accordingly the presence of a zero eigenvalue of F is one of the main reasons why there is no complete analogy between results for discrete- and continuous-time algebraic Riccati equations. This paper clarifies the role of $0 \in \sigma(F)$ for the DARE. There exists an (F, G)-invariant subspace which is the generalized eigenspace corresponding to 0 for all closed loop matrices, which are associated to solutions of the DARE. That basic fact makes it possible to investigate solutions with an antistabilizing property or the existence of a least and negative-semidefinite solution of the DARE.

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