The generalized Sylvester matrix equation, rank minimization and Roth's equivalence theorem

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Abstract

Roth's theorem on the consistency of the generalized Sylvester equation $AX - YB = C$ is a special case of a rank minimization theorem.

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Let $A \in K^{m\times m}$, $B \in K^{n\times n}$ and $C \in K^{m\times n}$ be matrices over a field K. Set $\mathrm{Gl}(n) = \{M \in K^{n\times n};\ \det M \neq 0\}$. The following theorem is due to Roth [1]. It gives a necessary and sufficient condition for the consistency of the generalized Sylvester equation (1) in terms of an equivalence of two associated matrices.

Theorem 1. *The matrix equation*

$$AX - YB = C \tag{1}$$

is solvable with $X, Y \in K^{m\times n}$ *if and only if there exist matrices* $P, Q \in \mathrm{Gl}(m+n)$ *such that*

$$P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Q.$$

We shall see that Roth's theorem is a special case of a result on rank minimization. It is the purpose of this note to prove the following.

Theorem 2. *We have*

$$\min\bigl\{\mathrm{rank}(AX - YB - C);\ X, Y \in K^{m\times n}\bigr\} = \min\left\{\mathrm{rank}\left[P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Q\right];\ P, Q \in \mathrm{Gl}(m+n)\right\}.$$

Proof. Let $X, Y \in K^{m\times n}$ and $P, Q \in \mathrm{Gl}(m+n)$. Set

$$\phi(X,Y) = AX - YB - C \quad \text{and} \quad \Phi(P,Q) = P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Q,$$

and

$$\gamma = \min \mathrm{rank}\bigl\{\phi(X,Y);\ X, Y \in K^{m\times n}\bigr\}$$

and

$$\Gamma = \min \mathrm{rank}\{\Phi(P,Q);\ P, Q \in \mathrm{Gl}(m+n)\}.$$

From [2] we know that

$$\gamma = \mathrm{rank}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \mathrm{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

For $P, Q \in \mathrm{Gl}(m+n)\}$ we obtain

$$\mathrm{rank}\,\Phi(P,Q) \geq \mathrm{rank}\,P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \mathrm{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Q = \mathrm{rank}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \mathrm{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \gamma.$$

Hence $\Gamma \geq \gamma$. With $X, Y \in K^{m\times n}$ we associate the matrices

$$P_Y = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \quad \text{and} \quad Q_X = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Then

$$\Phi(P_Y, Q_X) = \begin{pmatrix} 0 & -\phi(X,Y) \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\Gamma \leq \min\{\operatorname{rank}\Phi(P_Y, Q_X);\ X, Y \in K^{m\times n}\} = \min\{\operatorname{rank}\phi(X,Y)\} = \gamma.$$

This completes the proof.

□

The following theorem deals with the Sylvester equation (2). It is known as Roth's similarity theorem.

Theorem 3. [1] *The matrix equation*

$$AX - XB = C \tag{2}$$

is solvable with $X \in K^{m\times n}$ if and only if there exists a matrix $P \in \mathrm{Gl}(m+n)$ such that

$$P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} P.$$

There is evidence that Theorem 3 is also a special case of a result on rank minimization. We conjecture that the identity

$$\min\Big\{\operatorname{rank}(AX - XB - C);\ X \in K^{m\times n}\Big\} = \min\Big\{\operatorname{rank}\Big[P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} P\Big];\ P \in \mathrm{Gl}(m+n)\Big\}$$

holds, which extends Theorem 3.

References

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