A class of marked invariant subspaces with an application to algebraic Riccati equations

Pudji Astuti
Departemen Matematika
Institut Teknologi Bandung
Bandung 40132
Indonesia

Harald K. Wimmer
Mathematisches Institut
Universität Würzburg
D-97074 Würzburg
Germany

March 1, 2006

Abstract

Mathematical Subject Classifications (2000): 15A18, 47A15 15A24, 15A57.

Keywords: invariant subspaces, Jordan chains, marked subspaces, algebraic Riccati equation, Hamiltonian matrix.

Running title: Marked subspaces

Abstract: Invariant subspaces of a matrix A are considered which are obtained by truncation of a Jordan basis of a generalized eigenspace of A. We characterize those subspaces which are independent of the choice of the Jordan basis. An application to Hamilton matrices and algebraic Riccati equations is given.

Address for Correspondence:
Prof. Dr. H. Wimmer
Mathemat. Institut
Universität Würzburg
Am Hubland
D-97074 Würzburg
Germany

e-mail: `wimmer@mathematik.uni-wue rzburg.de`
Fax: +49 931 8 88 46 11

1 Invariant subspaces

Let λ be an eigenvalue of a complex $n \times n$ matrix A and let

$$E_\lambda(A) = \mathrm{Ker}(A - \lambda I)^n$$

be the corresponding generalized eigenspace. Suppose $\dim E_\lambda(A) = k$. If

$$(s-\lambda)^{t_1}, \ldots, (s-\lambda)^{t_k}, \ \ t_1 \le \cdots \le t_k,$$

are the corresponding elementary divisors then $E_\lambda(A)$ is a direct sum of t_i-dimensional cyclic subspaces, i.e.

$$E_\lambda(A) = K_1 \oplus \cdots \oplus K_k$$

with

$$K_i = \mathrm{span}\big\{u_i, \, (A-\lambda I)u_i, \, \ldots, (A-\lambda I)^{t_i-1}u_i\big\}, \tag{1.1}$$

and $(A-\lambda I)^{t_i}u_i = 0, \ \ i = 1, \ldots, k$. We call

$$U = (u_1, \ldots, u_k) \tag{1.2}$$

a tuple of *generators* of $E_\lambda(A)$. From a given U one can construct A-invariant subspaces in the following way. Let $r = (r_1, \ldots, r_k)$ be such that

$$0 \le r_i < t_i, \ i = 1, \ldots, k. \tag{1.3}$$

We set

$$W_{r_i}(U) = \mathrm{span}\big\{(A-\lambda I)^{r_i}u_i, \, (A-\lambda I)^{r_i+1}u_i, \, \ldots, (A-\lambda I)^{t_i-1}u_i\big\} \tag{1.4}$$

and

$$W(r, U) = W_{r_1}(U) \oplus \cdots \oplus W_{r_k}(U). \tag{1.5}$$

The construction of invariant subspaces of the from $W(r, U)$ is a standard procedure in linear algebra and systems theory (see e.g. [8], [6, p.61], [5], [10], [2, p.28]).

If U and $\tilde{U}$ are two different tuples of generators of $E_\lambda(A)$ then the restrictions of A to $W(r, U)$ and $W(r, \tilde{U})$ have the same elementary divisors, namely $(s-\lambda)^{t_i - r_i}$, $i = 1, \ldots, k$. However, in general, the subspaces $W(r, U)$ and $W(r, \tilde{U})$ will be different. Consider the following example with $k = 2$, $t_1 = 2$, $t_2 = 3$, and

$$A = \mathrm{diag}(N_2, N_3), \ N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ N_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.6}$$

Let e_i be a unit vector of $\mathbb{C}^5$. Then $U = \{e_2, e_5\}$ and $\tilde{U} = \{e_2, e_5 + e_2\}$ are tuples of generators of $E_0(A) = \operatorname{Ker} A^5 = \mathbb{C}^5$. If we choose $r = (1, 0)$, then $W(r, U) = \operatorname{span}\{e_1, e_3, e_4, e_5\}$ and $W(r, \tilde{U}) = \operatorname{span}\{e_1, e_3, e_4 + e_1, e_5 + e_2\}$. Thus

$$W(r, U) \neq W(r, \tilde{U}). \tag{1.7}$$

On the other hand, if we choose $r = (1, 2)$, then

$$W(r, U) = W(r, \tilde{U}). \tag{1.8}$$

It is the purpose of our note to determine those tuples $r = (r_1, \ldots, r_k)$ which have the property that the space $W(r, U)$ given by (1.4) and (1.5) is independent of the generator tuple U. The motivation for our study comes from Kucera's survey article [8], which deals with independence of generator tuples in the case of Hamiltonian matrices. In Section 3 we make the connection with [8, p.60] applying a corollary of our main theorem to Hamiltonian matrices and algebraic Riccati equations.

In the sequel we assume that $\lambda = 0$ is an eigenvalue of A and we focus on $E_0(A) = \operatorname{Ker} A^n$. With each nonzero vector $v \in E_0(A)$ we associate a *height* $\mathrm{h}(v)$ and an *exponent* $\mathrm{e}(v)$ as follows. Suppose

$$v \in \operatorname{Im} A^q,\ v \notin \operatorname{Im} A^{q+1},\ v \in \operatorname{Ker} A^p,\ v \notin \operatorname{Ker} A^{p-1}.$$

Then we set $\mathrm{h}(v) = q$ and $\mathrm{e}(v) = p$. Thus, if $\lambda = 0$ in (1.1) then the elements of U in (1.2) satisfy $\mathrm{e}(u_1) = t_1 \le \cdots \le \mathrm{e}(u_k) = t_k$ and $\mathrm{h}(u_i) = 0$. We define

$$\langle v \rangle = \operatorname{span}\{A^\nu v, \nu \ge 0\}.$$

Then $\langle v \rangle$ is a cyclic subspace generated by v, and $\dim\langle v \rangle = \mathrm{e}(v)$.

2 The main result

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ and let*

$$s^{t_1}, \ldots, s^{t_k},\ t_1 \le \cdots \le t_k, \tag{2.1}$$

be the elementary divisors corresponding to the eigenvalue $\lambda = 0$. Let

$$U = (u_1, \ldots, u_k)$$

be a tuple of generators of $E_0(A) = \operatorname{Ker} A^n$ such that $\mathrm{e}(u_i) = t_i$, $i = 1, \ldots, k$, and

$$E_0(A) = \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle.$$

Let $r = (r_1, \ldots, r_k)$ *be a* k*-tuple of integers with* $0 \le r_i < t_i$, $i = 1, \ldots, k$. *Define*

$$W(r, U) = \langle A^{r_1} u_1 \rangle \oplus \cdots \oplus \langle A^{r_k} u_k \rangle \tag{2.2}$$

and

$$W(r) = \big(\operatorname{Im} A^{r_1} \cap \operatorname{Ker} A^{t_1 - r_1}\big) + \cdots + \big(\operatorname{Im} A^{r_k} \cap \operatorname{Ker} A^{t_k - r_k}\big). \tag{2.3}$$

Then the following statements are equivalent:
(i) *The* k*-tuple* $r = (r_1, \cdots, r_k)$ *satisfies*

$$r_1 \le \cdots \le r_k, \tag{2.4}$$

and

$$t_1 - r_1 \le \cdots \le t_k - r_k. \tag{2.5}$$

(ii) *The space* $W(r, U)$ *is independent of* U.
Moreover, if (2.4) *and* (2.5) *hold then* $W(r, U) = W(r)$.

Proof. (i) $\Rightarrow$ (ii). We show that (2.4) and (2.5) imply $W(r, U) = W(r)$. Define $W_{r_s}(U) = \langle A^{r_s} u_s \rangle$ such that (1.5) holds. From

$$W_{r_s}(U) \subseteq \operatorname{Im} A^{r_s} \cap \operatorname{Ker} A^{t_s - r_s}$$

we immediately obtain $W(r, U) \subseteq W(r)$. Now let x be in $\operatorname{Im} A^{r_s} \cap \operatorname{Ker} A^{t_s - r_s}$. Then $x = A^{r_s} y$ for some $y \in E_0(A)$, and

$$A^{t_s - r_s} x = A^{t_s} y = 0. \tag{2.6}$$

With respect to the basis

$$\mathcal{B}_U = \{A^{\nu_i} u_i;\ 0 \le \nu_i \le t_i - 1,\ i = i, \ldots, k\} \tag{2.7}$$

we have

$$y = \sum_{i=1}^{k} \sum_{\nu_i = 0}^{t_i - 1} \alpha_{i \nu_i} A^{\nu_i} u_i.$$

Let ℓ be the largest integer sucht that $t_\ell \le t_s$. Then $A^{t_s} u_i = 0$ for $i = 1, \ldots, \ell$. Moreover $A^{t_s + \nu_i} u_i = 0$ if $t_s + \nu_i > t_i$. Therefore

$$A^{t_s} y = \sum_{i > \ell} \sum_{\nu_i = 0}^{t_i - t_s - 1} \alpha_{i \nu_i} A^{t_s + \nu_i} u_i = 0.$$

Since the vectors of $\mathcal{B}_U$ are linearly independent we obtain $\alpha_{i\nu_i} = 0$ for $i > \ell$ and $\nu_i = 0, \ldots, t_i - t_s - 1$. Hence

$$y = \sum_{i=1}^{\ell} \sum_{\nu_i=0}^{t_i-1} \alpha_{i\nu_i} A^{\nu_i} u_i + \sum_{i>\ell} \sum_{\nu_i=t_i-t_s}^{t_i-1} \alpha_{i\nu_i} A^{\nu_i} u_i$$

and

$$x = \sum_{i=1}^{\ell} \sum_{\nu_i=0}^{t_i-1} \alpha_{i\nu_i} A^{r_s+\nu_i} u_i + \sum_{i>\ell} \sum_{\nu_i=t_i-t_s}^{t_i-1} \alpha_{i\nu_i} A^{r_s+\nu_i} u_i.$$

Note that $t_s = \cdots = t_\ell$ implies $r_s = \cdots = r_\ell$. Hence, if $1 \le i \le \ell$ then $r_i \le r_s$, and therefore

$$A^{r_s+\nu_i} u_i \in W_{r_i}(U). \tag{2.8}$$

On the other hand, if $i > \ell$ then $t_s - r_s \le t_i - r_i$. In that case $\nu_i \in \{t_i - t_s, \ldots, t_i - 1\}$ implies

$$r_s + \nu_i \ge r_s + (t_i - t_s) \ge r_i.$$

Thus, we again have (2.8). Hence $x \in W(r, U)$ and therefore $W(r) \subseteq W(r, U)$.

(ii) $\Rightarrow$ (i). We assume that $W(r, U)$ is independent of U. Let us show first that

$$r_i = r_j \;\text{ if }\; t_i = t_j. \tag{2.9}$$

Suppose $r = (r_1, \ldots, r_k)$ is such that $t_s = t_{s+1}$ and $r_s \ne r_{s+1}$, e.g.

$$r_{s+1} < r_s \;\text{ for some }\; s \in \{1, \ldots, k-1\}. \tag{2.10}$$

Let $V = (v_1, \ldots, v_k)$ be such that $(v_s, v_{s+1}) = (u_{s+1}, u_s)$, and $v_i = u_i$ if $i \notin \{s, s+1\}$. Then $A^{r_{s+1}} u_{s+1} \in W(r, U)$ but $A^{r_{s+1}} u_{s+1} = A^{r_{s+1}} v_s \notin W(r, V)$. Therefore the tuples U and V contain the same elements, but $W(r, U) \ne W(r, V)$.

Now suppose that (2.4) is not satisfied. Then we have (2.10), and

$$A^{r_{s+1}} u_s \notin W(r, U). \tag{2.11}$$

Let $V = (v_1, \ldots, v_k)$ be given by $v_{s+1} = u_{s+1} + u_s$, and $v_i = u_i$, if $i \ne s+1$. Thus V is a tuple of generators of $E_0(A)$ with $\mathrm{e}(v_i) = \mathrm{e}(u_i)$. Consider

$$A^{r_{s+1}} u_{s+1} + A^{r_{s+1}} u_s = A^{r_{s+1}} v_{s+1} \in W(r, V).$$

Then $A^{r_{s+1}} v_{s+1} \notin W(r, U)$. Otherwise $A^{r_{s+1}} u_{s+1} \in W(r, U)$ would imply $A^{r_{s+1}} u_s \in W(r, U)$, which is a contradiction to (2.11).

Suppose $r = (r_1, \ldots, r_k)$ does not satisfy (2.5). Then $t_s - r_s > t_{s+1} - r_{s+1}$ for some $s \in \{1, \ldots, k-1\}$. Because of (2.9) we have $t_{s+1} \neq t_s$. Hence $r_{s+1} - r_s > t_{s+1} - t_s > 0$, and $r_s < r_{s+1}$, and

$$r_s + (t_{s+1} - t_s) < r_{s+1}. \tag{2.12}$$

Because (2.2) it is obvious that (2.12) implies

$$A^{r_s + (t_{s+1} - t_s)} u_{s+1} \notin W(r, U). \tag{2.13}$$

Define $v_s = u_s + A^{t_{s+1} - t_s} u_{s+1}$. Then $\mathrm{e}(v_s) = \mathrm{e}(u_s) = t_s$. Therefore

$$V = \{u_1, \ldots, u_{s-1}, v_s, u_{s+1}, \ldots, u_k\} \tag{2.14}$$

is another tuple of generators of $E_0(A)$. Let us show that $W(r, V) \neq W(r, U)$. Clearly, the vector $A^{r_s} v_s$ belongs to $W(r, V)$. Suppose

$$A^{r_s} u_s + A^{r_s + (t_{s+1} - t_s)} u_{s+1} \; = \; A^{r_s} v_s \; \in \; W(r, U).$$

Because of $A^{r_s} u_s \in W(r, U)$ that would imply

$$A^{r_s + (t_{s+1} - t_s)} u_{s+1} \in W(r, U),$$

which is a contradiction to (2.13). □

Let us consider again Example (1.6). We have $(t_1, t_2) = (2, 3)$. In the case of $r = (1, 0)$ condition (2.4) is violated, which accounts for (1.7). In the case of $r = (1, 2)$ both (2.4) and (2.5) hold, which ensures (1.8).

In accordance with a definition in [7, p. 83] and [3] the space $W(r, U)$ is a *marked* A-invariant subspace of $E_0(A)$. That means $W(r, U)$ has a Jordan basis, in our case

$$\{A^{r_i + \mu_i} u_i;\, 0 \leq \mu_i \leq t_i - r_i - 1,\, i = i, \ldots, k\},$$

which can be extended to a Jordan basis of $E_0(A)$, namely to $\mathcal{B}_U$ in (2.7). Let $\mathcal{M}_r$ be the set of marked subspaces M of $E_0(A)$ such that the elementary divisors of the restriction $A_{|M}$ are $s^{t_1 - r_1}, \ldots, s^{t_k - r_k}$. We have noted before that for each tuple of generators U the corresponding space $W(r, U)$ is in $\mathcal{M}_r$. Suppose (2.4) and (2.5) hold. Then all the spaces $W(r, U)$ coincide with $W(r)$ and one might ask whether $W(r)$ is the only subspaces in $\mathcal{M}_r$. In the following we have an example where $\mathcal{M}_r \supsetneqq \{W(r)\}$. Let $n = 10$, $k = 2$, and $t = (t_1, t_2) = (4, 6)$, and $r = (2, 3)$. Then $t - r = (2, 3)$. Hence the conditions (2.4) and (2.5) are satisfied. Let $U = (u_1, u_2)$ be a tuple of generators such that $\mathrm{e}(u_1) = 4$ and $\mathrm{e}(u_2) = 6$. The subspaces

$$M = W(r, U) = W(r) = \langle A^2 u_1 \rangle \oplus \langle A^3 u_2 \rangle$$

and $\tilde{M} = \langle A u_1 \rangle \oplus \langle A^4 u_2 \rangle$ are marked, the elementary divisors of $A_{|M}$ and $A_{|\tilde{M}}$ are s^2, s^3. Hence $\tilde{M} \in \mathcal{M}_r$, but $\tilde{M} \neq W(r)$.

Let $[m]$ denote the greatest integer of m. If we assume $(t_1, \dots, t_k)$ as in (2.1) and take $r = ([\frac{1}{2}t_1], \dots, [\frac{1}{2}t_k])$ then the conditions (2.4) and (2.5) are satisfied and we note the following corollary of Theorem 2.1.

Corollary 2.2. *Let $A \in \mathbb{C}^{n \times n}$ and $0 \in \sigma(A)$. Let $s^{2m_1}, \dots, s^{2m_k}$, be the elementary divisors of A corresponding to $\lambda = 0$. If $U = (u_1, \dots, u_k)$ is a tuple of generators of $\operatorname{Ker} A^n$ such that $\mathrm{e}(u_i) = 2m_i$, $i = 1, \dots, k$, then $\mathrm{e}(A^{m_i} u_i) = m_i$ for all i, and*

$$\langle A^{m_1} u_1 \rangle \oplus \cdots \oplus \langle A^{m_k} u_k \rangle = \big(\operatorname{Im} A^{m_1} \cap \operatorname{Ker} A^{m_1}\big) + \cdots + \big(\operatorname{Im} A^{m_k} \cap \operatorname{Ker} A^{m_k}\big).$$

3 An application

In this section we apply Corollary 2.2 to the algebraic Riccati equation

$$Q + F^* X + X F - X D X = 0 \tag{3.1}$$

and its associated Hamiltonian matrix

$$H = \begin{pmatrix} F & -D \\ -Q & -F^* \end{pmatrix}. \tag{3.2}$$

Here F, D, Q are complex $m \times m$ matrices, D and Q are hermitian, $D \geq 0$, and the pair (F, D) is assumed to be controllable. Then (see [8, p.59] all elementary divisors corresponding to eigenvalues $i\alpha \in i\mathbb{R}$ have even degree. To fix ideas we assume $\sigma(H) = \{0\}$. The subsequent result complements Lemma 3.2.3 of [8, p.60].

Proposition 3.1. *Let $s^{2m_1}, \dots s^{2m_k}$ be the elementary divisors of H. Set*

$$W = \big(\operatorname{Im} H^{m_1} \cap \operatorname{Ker} H^{m_1}\big) + \cdots + \big(\operatorname{Im} H^{m_k} \cap \operatorname{Ker} H^{m_k}\big). \tag{3.3}$$

Then W is an H-invariant subspace of $\mathbb{C}^{2m}$ and $\dim W = m$. Let $Y, Z \in \mathbb{C}^{m \times m}$ be such that the columns of $\left(\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right)$ are a basis of W. Then Y is nonsingular and $X = ZY^{-1}$ is the unique hermitian solution of (3.1).

Proof. Set $t = (2m_1, \ldots, 2m_k)$. Let $U = (u_1, \ldots, u_k)$ be a tuple of generators of $E_0(H) = \mathbb{C}^{2m}$. According to [8] we have

$$W(\tfrac{1}{2}t, U) = \operatorname{span}\begin{pmatrix} I_m \\ X \end{pmatrix},$$

where $X \in \mathbb{C}^{m\times m}$ is the unique hermitian solution of (3.1). From Corollary 2.2 we know that $W(\frac{1}{2}t, U)$ is independent of the choice of U. Moreover, $W(\frac{1}{2}t, U) = W$ where W is given by (3.3). Hence, if $W = \operatorname{span}\left(\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right)$ then Y is nonsingular, and

$$\operatorname{span}\begin{pmatrix} Y \\ Z \end{pmatrix} = \operatorname{span}\begin{pmatrix} I \\ ZY^{-1} \end{pmatrix}$$

implies that $X = ZY^{-1}$ is the solution of (3.1). □

4 Conclusions

Results of this note can be considered in a module theoretic framework. In a subsequent paper we shall make the connection of Theorem 2.1 with marked subspaces in [4] and with torsion modules over discrete valuation domains in [1].

Acknowledgement. We would like to thank Dr. G. Dirr for a valuable comment.

References

[1] P. Astuti and H.K. Wimmer, Regular submodules of torsion modules over a discrete valuation domain, to appear in Czechoslov. Math. J.

[2] L. Baratchart, P. Enqvist, A. Gombani, and M. Olivi, Surface acoustic wave filters, unitary extensions and Schur analysis, Report No. 44, 2003, Institut Mittag-Leffler, Stockholm, http://www.mittag-leffler.se/preprints/0203s, submitted for publication.

[3] R. Bru, L. Rodman, and H. Schneider, Extensions of Jordan bases for invariant subspaces of a matrix, Linear Algebra Appl. 150(1991), 209–226.

[4] J. Ferrer, F. Puerta, and X. Puerta, Geometric characterization and classification of marked subspaces, Linear Algebra Appl. 235(1996), 15– 34.

[5] G. Freiling, On the existence of hermitian solutions of general algebraic Riccati equations, Schriftenreihe d. FB Mathematik der Universität-GH-Duisburg, Nr. 446 (1999).

[6] I. Gohberg, P. Lancaster, and L. Rodman, Matrices and Indefinite Scalar Products, Operator Theory: Advances and Applications, Vol. 8, Birkhäuser, Basel, 1983.

[7] I. Gohberg, P. Lancaster, and L. Rodman, Invariant Subspaces of Matrices with Applications, Wiley, New York, 1986.

[8] V. Kučera, Algebraic Riccati equations: Hermitian and definite solutions, in S. Bittanti, A.J. Laub and J.C. Willems (Eds.), *The Riccati Equation*, pp.53–88, Springer Verlag, Berlin 1991.

[9] P. Lancaster and L. Rodman, Algebraic Riccati Equations, Clarendon Press, Oxford 1995.

[10] A.C.M. Ran and L. Rodman, A class of robustness problems in matrix analysis, in D. Alpay, I. Gohberg and V. Vinnikov (Eds.), *Interpolation Theory, Systems Theory and Related Topics, The Harry Dym Anniversary Volume*, Operator Theory: Advances and Applications 134, pp. 337 – 389, Birkhäuser, Basel 2002.