Contour integral solutions of Sylvester-type matrix equations

Harald K. Wimmer

Mathematisches Institut, Universität Würzburg, 97074 Würzburg, Germany

Abstract

The linear matrix equations $AXB - CXD = E$, $AX - X^*D = E$, and $AXB - X^* = E$ are studied. In the case of uniqueness the solutions are expressed in terms of contour integrals.

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1. Introduction

In this note we use contour integrals to study the matrix equations

$$AXB - CXD = E$$

(1)

and

$$AX - X^*D = E$$

(2)

and

$$AXB - X^* = E.$$  

(3)

We assume $A, C \in \mathbb{C}^{m \times m}$ and $B, D \in \mathbb{C}^{n \times n}$ and $E \in \mathbb{C}^{m \times n}$, and $m = n$ in (2) and (3). The generalized Sylvester equation (1) appears in sensitivity problems of eigenvalues [12], in numerical solutions of implicit ordinary differential equations [6], and in a Lyapunov theory of descriptor systems [1]. Equation (2) - where $X^*$ denotes the conjugate transpose of $X$ - plays a role in palindromic eigenvalue problems [2, 9]. The main reference to (3)
is [13]. Besides (2) and (3) equations of the form \(AX - X^TD = E\) and \(X - CX^TD = E\) - where \(X^T\) is the transpose of \(X\) - have been studied e.g. in [5], [13] and [3].

Conditions for uniqueness of solutions of (1) - (3) involve matrix pencils and their spectra. Let the pencil \(zC - A\) be regular, that is \(\det(\lambda C - A) \neq 0\) for some \(\lambda \in \mathbb{C}\). We set

\[
\sigma(zC - A) = \{\lambda; \det(\lambda C - A) = 0\} \cup \{\lambda; \det(C - \lambda^{-1}A) = 0\}.
\]

Then \(0 \in \sigma(zC - A)\) is equivalent to \(\det A = 0\). By convention \(1/\infty = 0\) and \(1/0 = \infty\). Thus \(\infty \in \sigma(zC - A)\) if and only if \(\det C = 0\). A contour in the complex plane will always mean a positively oriented simple closed curve.

2. Explicit solutions by contour integrals

2.1. The generalized Sylvester equation

It is known (see [4]) that equation (1) has a unique solution if and only if the matrix pencils

\[
zC - A \text{ and } zB - D \text{ are regular,}
\]

and

\[
\sigma(zC - A) \cap \sigma(zB - D) = \emptyset.
\]

In the following we give an explicit description of the unique solution of (1).

**Theorem 2.1.** If the conditions (4) and (5) are satisfied then there exists a contour \(\gamma\) such that \(\sigma(zC - A)\) is in the interior of \(\gamma\) and \(\sigma(zB - D)\) is outside of \(\gamma\). The unique solution of (1) is given by

\[
X = \frac{1}{2\pi i} \oint_{\gamma} (zC - A)^{-1}E(zB - D)^{-1}dz.
\]

**Proof.** Let \(T\) and \(W\) be nonsingular complex \(n \times n\) matrices that transform the regular pencil \(zB - D\) into Weierstrass canonical form (see [7, Section XII]). Then

\[
T(zB - D)W = z\tilde{B} - \tilde{D} = z\begin{pmatrix} I_r & 0 \\ 0 & N \end{pmatrix} - \begin{pmatrix} D_0 & 0 \\ 0 & I_{n-r} \end{pmatrix},
\]
where $N$ is nilpotent. We have $\sigma(zB - D) = \sigma(z\tilde{B} - \tilde{D})$. Set $\tilde{X} = XT^{-1}$ and $\tilde{E} = EW$. Then $X$ is a solution of (1) if and only if $\tilde{X}$ is a solution of $A\tilde{X}\tilde{B} - C\tilde{X}\tilde{D} = \tilde{E}$, and it is not difficult to see that we can assume that $B$ and $D$ have the form

$$B = \begin{pmatrix} I_r & 0 \\ 0 & N \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D_0 & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

where $\sigma(N) = \{0\}$. If the pencils $zC - A$ and $zB - D$ are regular then (see [10], [8]) the identity (1) is equivalent to

$$(zC - A)^{-1}CX - XB(zB - D)^{-1} = (zC - A)^{-1}E(zB - D)^{-1}. \quad (7)$$

Because of (5) we have $\infty \notin \sigma(zC - A) \cap \sigma(zB - D)$. Suppose $\infty \notin \sigma(zC - A)$, that is $\det C \neq 0$. Then (7) can be written as

$$(zI - C^{-1}A)^{-1}X - X \begin{pmatrix} (zI_r - D_0)^{-1} & 0 \\ 0 & N(zN - I_{n-r})^{-1} \end{pmatrix} = (zC - A)^{-1}E(zB - D)^{-1}. \quad (8)$$

The assumption (5) implies that there exists a contour $\gamma$ surrounding the set $\sigma(C^{-1}A) = \sigma(zC - A)$ and leaving $\sigma(zB - D)$ in the exterior. In particular, there is no eigenvalue of $D_0$ in the interior of $\gamma$. Note that $(zN - I_{n-r})^{-1}$ is a polynomial matrix. Hence the spectral calculus for matrices yields

$$\frac{1}{2\pi i} \oint_{\gamma} (zI - C^{-1}A)^{-1}dz = I_n,$$

and

$$\frac{1}{2\pi i} \oint_{\gamma} (zI_r - D_0)^{-1}dz = 0 \quad \text{and} \quad \frac{1}{2\pi i} \oint_{\gamma} (zN - I_{n-r})^{-1}dz = 0.$$

Then (6) follows immediately from (8). Now suppose $\infty \notin \sigma(zB - D)$. Then $\det B \neq 0$. In that case we consider the equation $D^TX^TC^T - B^TX^TA^T = -E^T$ and represent $X^T$ in accordance with (6). We obtain

$$X = \frac{-1}{2\pi i} \oint_{\beta} (zC - A)^{-1}E(zB - D)^{-1}dz,$$

where $\beta$ is a contour such that $\sigma(zB - D)$ is in the interior of $\beta$ and $\sigma(zC - A)$ is in the exterior of $\beta$. Let $-\beta$ denote the simple closed curve orientated in the opposite direction of $\beta$. Then we have (6) with $\gamma = -\beta$. □
The Sylvester equation

\[ AX - XD = E \] \hspace{1cm} (9)

is a special case of (1). Suppose \( \sigma(A) \cap \sigma(C) = \emptyset \) and let \( \gamma \) be a contour with \( \sigma(A) \) in the interior and \( \sigma(D) \) in the outside. Then (6) implies (see e.g. [11]) that

\[ X = \frac{1}{2\pi i} \oint_{\gamma} (zI - A)^{-1}E(zI - D)^{-1}dz \]

is the solution of (9). The following result on the discrete-time Sylvester equation (= Stein equation)

\[ AXB - X = E \] \hspace{1cm} (10)

is also a consequence of Theorem 2.1.

**Corollary 2.2.** Suppose \( 1 \notin \sigma(A)\sigma(B) \). Let \( \gamma \) be a contour with \( \sigma(A) \) in the interior and \( \{1/\lambda; \lambda \in \sigma(B), \lambda \neq 0\} \) in the exterior. Then the unique solution of (10) is given by

\[ X = \frac{1}{2\pi i} \oint_{\gamma} (zI - A)^{-1}E(zB - I)^{-1}dz. \]

2.2. The \((X, X^*)\)-Sylvester equation

In [2] and [9, p. 223] it was shown that the \((X, X^*)\)-Sylvester equation

\[ AX - X^*D = E \] \hspace{1cm} (11)

has a unique solution if and only if the pencil

\[ A - zD^* \] is regular, \hspace{1cm} (12)

and

\[ \sigma(A - zD^*) \cap \sigma(D - zA^*) = \emptyset. \] \hspace{1cm} (13)

Condition (13) means that \( \lambda \in \sigma(A - zD^*) \) implies \( 1/\lambda \notin \sigma(A - zD^*) \). The following result gives a contour integral solution of (11).

**Theorem 2.3.** Let (12) and (13) be satisfied. Then there exists a contour \( \gamma \) such that \( \sigma(D - zA^*) \) is in the interior of \( \gamma \) and \( \sigma(A - zD^*) \) is outside of \( \gamma \). The unique solution of (11) is given by

\[ X = \frac{1}{2\pi i} \oint_{\gamma} (zD^* - A)^{-1}(E + zE^*)(D - zA^*)^{-1}A^*dz. \] \hspace{1cm} (14)
Proof. Note that (11) is equivalent to
\[(A - zD^*)X - X^*(D - zA^*) = E + zE^*.\] (15)
Hence, if \(A - zD^*\) is regular then (15) can be written as
\[X(zA^* - D)^{-1} - (zD^* - A)^{-1}X^* = (zD^* - A)^{-1}(E + zE^*)(D - zA^*)^{-1}.\] (16)
Suppose \(\det A \neq 0\). Then (13) implies that there exists a contour \(\gamma\) such that \(\sigma(zA^* - D) = \sigma(zI - D(A^*)^{-1})\) is in the interior of \(\gamma\) and \(\sigma(A - zD^*)\) is outside of \(\gamma\). Then
\[\frac{1}{2\pi i} \oint_{\gamma} X(zA^* - D)^{-1}dz = \frac{1}{2\pi i} \oint_{\gamma} X(A^*)^{-1}(zI - D(A^*)^{-1})^{-1}dz = X(A^*)^{-1}\]
and
\[\frac{1}{2\pi i} \oint_{\gamma} (zD^* - A)^{-1}X^*dz = 0.\]
Hence (14) follows from the identity (16). Suppose \(\det A = 0\). Then we obtain (14) by a perturbation argument. We consider the equation \(\tilde{A}X - X^*D = E\) where \(\tilde{A} = A + \epsilon I, \epsilon > 0\). If \(\epsilon\) is sufficiently small then \(\det \tilde{A} \neq 0, \tilde{A} - zD^*\) is regular, and \(\sigma(\tilde{A} - zD^*) \cap \sigma(D - zA^*) = \emptyset\).

2.3. The \((X, X^*)\)-Stein equation

Zhou, Lam and Duan [13] investigated the equation
\[AXB - X^* = E.\] (17)
Using Kronecker products and the vec-permutation matrix the following result was proved in [13, p.1390, Lemma 13].

Lemma 2.4. The condition \(\sigma(zI - B^*A) \cap \sigma(I - zBA^*) = \emptyset\), or equivalently,
\[1 \notin \sigma(AB^*)\sigma(A^*B),\] (18)
is necessary and sufficient for the existence of a unique solution of (17).

We apply Corollary 2.2 to derive a contour integral solution of (17).
Theorem 2.5. Assume (18). Let $\gamma$ be a contour that has the eigenvalues of $B^*A$ in the interior and the set $\{1/\bar{\lambda}; \lambda \in \sigma(B^*A), \lambda \neq 0\}$ in the exterior. Then
\[ X = \frac{1}{2\pi i} \oint_{\gamma} (zI - B^*A)^{-1}(E^* + B^*EA^*)(zBA^* - I)^{-1}dz \] (19)
is the unique solution of (17).

Proof. Let $W$ denote the right-hand side of (19). Then $W$ satisfies the Stein equation
\[ B^*AWBA^* - W = E^* + B^*EA^*. \] (20)
(by Corollary 2.2). It follows from [13, Theorem 7], which deals with equation (20), that $W$ is a solution of (17). We can show this directly as follows. Define $P = AWB$ and $Q = W^*$. Then
\[ P = \frac{1}{2\pi i} \oint_{\gamma} [(zI - AB^*)^{-1}A(E^* + B^*EA^*)B(zA^*B - I)^{-1}dz \]
is the solution of
\[ (AB^*)P(A^*B) - P = A(E^* + B^*EA^*)B, \]
and (20) yields
\[ (AB^*)Q(A^*B) - Q = E + AE^*B. \]
Set $S = P - Q - E$. Then $(AB^*)S(A^*B) - S = 0$. Therefore (18) implies $S = 0$. Hence $AWB - W^* - E = 0$. $\square$

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