HYPERINVARIANT, CHARACTERISTIC AND MARKED SUBSPACES

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Abstract. Let $V$ be a finite dimensional vector space over a field $K$ and $f$ a $K$-endomorphism of $V$. In this paper we study three types of $f$-invariant subspaces, namely hyperinvariant subspaces, which are invariant under all endomorphisms of $V$ that commute with $f$, characteristic subspaces, which remain fixed under all automorphisms of $V$ that commute with $f$, and marked subspaces, which have a Jordan basis (with respect to $f|_X$) that can be extended to a Jordan basis of $V$. We show that a subspace is hyperinvariant if and only if it is characteristic and marked. If $K$ has more than two elements then each characteristic subspace is hyperinvariant.

1. Introduction

Let $V$ be an $n$-dimensional vector space over a field $K$ and let $f : V \rightarrow V$ be $K$-linear. We assume that the characteristic polynomial of $f$ splits over $K$ such that all eigenvalues of $f$ are in $K$. In this paper we deal with three types of $f$-invariant subspaces, namely with hyperinvariant, characteristic and marked subspaces. To describe these three concepts we use the following notation. Let $\text{Inv}(V)$ be the lattice of $f$-invariant subspaces of $V$ and let $\text{End}_f(V)$ be the algebra of all endomorphisms of $V$ that commute with $f$. If a subspace $X$ remains invariant for all $g \in \text{End}_f(V)$ then $X$ is called hyperinvariant for $f$ [13, p. 305]. Let $\text{Hinv}(V)$ be the set of hyperinvariant subspaces of $V$. It is obvious that $\text{Hinv}(V)$ is a lattice. Because of $f \in \text{End}_f(V)$ we have $\text{Hinv}(V) \subseteq \text{Inv}(V)$. We refer to [13], [9], [17], [19] for results on hyperinvariant subspaces. The group of automorphisms of $V$ that commute with $f$ will be denoted by $\text{Aut}_f(V)$. A subspace $X$ of $V$ will be called characteristic (with respect to $f$) if $X \in \text{Inv}(V)$ and $\alpha(X) = X$ for all $\alpha \in \text{Aut}_f(V)$. Let $\text{Chinv}(V)$ be set of characteristic subspaces of $V$. Obviously, also $\text{Chinv}(V)$ is a lattice, and $\text{Hinv}(V) \subseteq \text{Chinv}(V)$.

Set $\imath = \text{id}_V$ and $f^0 = \imath$. Let $\langle x \rangle_f = \text{span}\{f^ix, i \geq 0\}$ be the cyclic subspace generated by $x \in V$. If $B \subseteq V$ we define $\langle B \rangle_f = \sum_{b \in B} \langle b \rangle_f$. Let $\lambda$ be an eigenvalue of $f$ such that $V_\lambda = \text{Ker}(f - \lambda \imath)$ is the corresponding generalized eigenspace. Let $\dim \text{Ker}(f - \lambda \imath) = k$, and let $s_1, \ldots, s_k$, be the elementary divisors of $f|_{V_\lambda}$. Then there exist vectors $u_1, \ldots, u_k$, such that

$$V_\lambda = \langle u_1 \rangle_{f - \lambda \imath} \oplus \cdots \oplus \langle u_k \rangle_{f - \lambda \imath},$$


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and \((f - \lambda t)^{i-1}u_i \neq 0, (f - \lambda t)^i u_i = 0, i = 1, \ldots, k\). We call \(U_\lambda = \{u_1, \ldots, u_k\}\) a set of generators of \(V_\lambda\). Each \(U_\lambda\) gives rise to a Jordan basis of \(V_\lambda\), namely
\[
\{u_1, (f - \lambda t)u_1, \ldots, (f - \lambda t)^{i-1}u_1, \ldots, u_k, (f - \lambda t)u_k, \ldots, (f - \lambda t)^{k-1}u_k\}.
\]
Define \(f_\lambda = f|_{V_\lambda}\). Let \(Y\) be an \(f_\lambda\)-invariant subspace of \(V_\lambda\). Then \(Y\) is said to be marked in \(V_\lambda\) (with respect to \(f_\lambda\)) if there exists a set \(U_\lambda\) of generators of \(V_\lambda\) and corresponding integers \(r_i, 0 \leq r_i \leq t_i\), such that
\[
Y = \langle (f - \lambda t)^{r_1}u_1 \rangle_{f - \lambda_1} + \cdots + \langle (f - \lambda t)^{r_k}u_k \rangle_{f - \lambda_k}.
\]
Thus \(Y\) has a Jordan basis which can be extended to a Jordan basis of \(V_\lambda\). Let \(\sigma(f) = \{\lambda_1, \ldots, \lambda_m\}\) be the spectrum of \(f\). Then
\[
V = V_{\lambda_1} + \cdots + V_{\lambda_m}, \quad (1.1)
\]
If \(X \in \text{Inv}_f V\) then \(X_{\lambda_i} = X \cap V_{\lambda_i}\) is \(f_{\lambda_i}\)-invariant in \(V_{\lambda_i}\), and
\[
X = X_{\lambda_1} + \cdots + X_{\lambda_m} \quad (1.2)
\]
We say that \(X\) is marked in \(V\) if each subspace \(X_{\lambda_i}\) in (1.2) is marked in \(V_{\lambda_i}\). The set of marked subspaces of \(V\) will be denoted by \text{Mark}(V). We assume \(0 \in \text{Mark}(V)\). Marked subspaces can be traced back to [13, p. 83]. They have been studied in [4], [8], [1], and [6]. For marked \((A, C)\)-invariant subspaces we refer to [5] and [7]. We mention applications to algebraic Riccati equations [2] and to stability of invariant subspaces of commuting matrices [15].

The following examples show that to a certain extent the three types of invariant subspaces are independent of each other. Suppose \(f\) is nilpotent. If \(x \in V\) then the smallest nonnegative integer \(\ell\) with \(f^\ell x = 0\) is called the exponent of \(x\). We write \(e(x) = \ell\). A nonzero vector \(x\) is said to have height \(q\) if \(x \in f^q V\) and \(x \notin f^{q+1} V\). In this case we write \(h(x) = q\). We set \(h(0) = -\infty\). For \(j \geq 0\) we define \(V[f^j] = \text{Ker} f^j\).

**Example 1.1.** Let \(K = \mathbb{Z}_2\). Consider \(V = K^4\) and
\[
f = \text{diag}(0, N_3), \quad N_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]
Let \(e_1, \ldots, e_4\), be the unit vectors of \(K^4\). Then \(f^3 = 0\) and \(V = \langle e_1 \rangle_f + \langle e_2 \rangle_f\). Define \(z = e_1 + e_3\) and \(Z = \langle z \rangle_f\). Then
\[
Z = \{0, z, z + e_4, e_4\} = \langle v; e(v) = 2, h(v) = 0, h(fv) = 2 \rangle_f.
\]
If \(\alpha \in \text{Aut}_f(V)\) then \(|\alpha(Z)| = |Z|\). Moreover \(\alpha\) preserves height and exponent. Hence \(\alpha(Z) = Z\), and \(Z\) is characteristic. Let \(g = \text{diag}(1, 0, 0, 0)\) be the orthogonal projection on \(Ke_1\). Then \(g \in \text{End}_f(V)\). We have \(gz = e_1 \in g(Z)\), but \(e_1 \notin Z\). Therefore \(Z\) is not hyperinvariant. The Jordan bases of \(Z\) are \(J_1 = \{z, e_4\}\) and \(J_2 = \{z + e_4, e_4\}\). If \(y \in K^4\) then \(z \neq fy\) and \(z + e_4 \neq fy\). Hence neither \(J_1\) nor \(J_2\) can be extended to a Jordan basis of \(K^4\). Therefore \(Z\) is not marked.
Example 1.2. Let $V = K^2$ and $f = 0$. Then $K^2 = \langle e_1 \rangle_f \oplus \langle e_2 \rangle_f$ and the subspace $X = \langle e_1 \rangle_f$ is marked. From $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Aut}_f(V)$ and $\alpha(e_1) = e_1 + e_2$ follows that $X$ is not characteristic.

In contrast to $\text{Hinv}(V)$ or $\text{Chinv}(V)$ the set $\text{Mark}(V)$ in general is not a lattice.

Example 1.3. $V = K^6$, $f = \text{diag}(0, N_3, N_2)$. The subspaces $Z_1 = \langle e_5 \rangle$ and $Z_2 = \langle e_5 + e_3 + e_1 \rangle$ are marked but $Z_1 + Z_2 = \langle e_5 \rangle \oplus \langle e_3 + e_1 \rangle$ is not marked. Thus the set of marked subspaces is not closed under addition.

In this paper we study the following problems. Under what conditions is a marked subspace characteristic? When is each characteristic subspace hyperinvariant? Because of Lemma 1.4 it suffices to consider an endomorphism $f$ with only one eigenvalue $\lambda$. We shall assume $\sigma(f) = \{0\}$ such that $f^n = 0$. Let 

$$s^1, \ldots, s^k, \ 0 < t_1 \leq \cdots \leq t_k,$$  \hspace{1cm} (2.1)

be the elementary divisors of $f$. We call $U = \langle u_1, \ldots, u_k \rangle$ a generator tuple of $V$ if 

$$V = \langle u_1 \rangle_f \oplus \cdots \oplus \langle u_k \rangle_f$$  \hspace{1cm} (2.2)

2. Auxiliary results

Because of Lemma 1.4 it suffices to consider an endomorphism $f$ with only one eigenvalue $\lambda$. We shall assume $\sigma(f) = \{0\}$ such that $f^n = 0$. Let 

$$s^1, \ldots, s^k, \ 0 < t_1 \leq \cdots \leq t_k,$$  \hspace{1cm} (2.1)

be the elementary divisors of $f$. We call $U = \langle u_1, \ldots, u_k \rangle$ a generator tuple of $V$ if 

$$V = \langle u_1 \rangle_f \oplus \cdots \oplus \langle u_k \rangle_f$$  \hspace{1cm} (2.2)
and if $U$ is ordered according to (2.1) such that
\[ e(u_1) = t_1 \leq \cdots \leq e(u_k) = t_k. \]

Let $\mathcal{U}$ be the set of generator tuples of $V$. In the following we omit the subscript $f$ in (2.2) and we write $\langle u_i \rangle$ instead of $\langle u_i \rangle_f$. We say that a $k$-tuple $r = (r_1, \ldots, r_k)$ of integers is admissible if
\[ 0 \leq r_i \leq t_i, \quad i = 1, \ldots, k. \tag{2.3} \]

Each $U \in \mathcal{U}$ together with an admissible tuple $r$ gives rise to a subspace
\[ W(r, U) = \langle f^{r_1} u_1 \rangle \oplus \cdots \oplus \langle f^{r_k} u_k \rangle, \tag{2.4} \]
which is marked in $V$. Conversely, a subspace $W$ is marked in $V$ only if $W = W(r, U)$ for some $U \in \mathcal{U}$ and some admissible $r$. The following example shows that, in general, $W(r, U) \neq W(r, \tilde{U})$ if $U \neq \tilde{U}$.

**Example 2.1.** Let $V = K^5$ and $f = \text{diag}(N_2, N_3)$. Then $V = \langle e_1 \rangle \oplus \langle e_3 \rangle$, and $U = (e_1, e_3)$ and $\tilde{U} = (e_1, e_3 + e_1)$ are generator tuples. Choose $r = (1, 0)$. Then the corresponding subspaces $W(r, U) = \langle e_2 \rangle \oplus \langle e_3 \rangle$ and $W(r, \tilde{U}) = \langle e_2 \rangle \oplus \langle e_3 + e_1 \rangle$ are different from each other.

The construction of invariant subspaces of the form $W(r, U)$ is a standard procedure in linear algebra and systems theory. It is used in [16], [12, p.61], [3, p.28], [18]. Hence it is important to know whether for a given $r$ different choices of $U$ will always result in the same subspace. Theorem 3.1 will provide a necessary and sufficient condition for $r$ such that $W(r, U)$ is independent of the choice of $U$. Let $r$ be admissible and define
\[ W(r) = f^{r_1} V \cap V[f^{r_1 - r_1}] + \cdots + f^{r_k} V \cap V[f^{r_k - r_k}]. \tag{2.5} \]
Subspaces of the form $f^r V$ and $V[f^u]$ are hyperinvariant, and $\text{Hinv}(V)$ is a lattice. Therefore (see e.g. [9]) we have $W(r) \in \text{Hinv}(V)$.

The following lemma shows that each $\alpha \in \text{Aut}_f(V)$ is uniquely determined by the image of a given generator tuple.

**Lemma 2.2.** Let $U = (u_1, \ldots, u_k) \in \mathcal{U}$ be given. For $\alpha \in \text{Aut}_f(V)$ define $\Theta_U(\alpha) = (\alpha(u_1), \ldots, \alpha(u_k))$. (i) Then
\[ \alpha \mapsto \Theta_U(\alpha), \quad \Theta_U : \text{Aut}_f(V) \to \mathcal{U}, \]
is a bijection. (ii) If $\tilde{U} = \Theta(\alpha)$ then $W(r, \tilde{U}) = \alpha(W(r, U))$.

**Proof.** (i) It is easy to see that $\Theta_U(\alpha) \in \mathcal{U}$. Hence $\Theta_U$ maps $\text{Aut}_f(V)$ into $\mathcal{U}$. Let $x \in V$ and
\[ x = \sum_{i=1}^k \sum_{j=0}^{e(u_i)-1} c_{ij} f^j u_i. \tag{2.6} \]
Suppose $\alpha, \beta \in \text{Aut}_f(V)$ and $\Theta_U(\alpha) = \Theta_U(\beta) = (\tilde{u}_1, \ldots, \tilde{u}_k)$. Then
\[ \alpha(x) = \sum \sum c_{ij} f^j \tilde{u}_i = \beta(x). \]
Hence $\alpha = \beta$, and $\Theta_U$ is injective. Now consider $U = (\tilde{u}_1, \ldots, \tilde{u}_k) \in \mathcal{U}$. Let $x \in V$ be the vector in (2.6). Define $\gamma : x \mapsto \sum \sum c_{ij} f^j \tilde{u}_i$. Then $\gamma \in \text{Aut}_f(V)$ and $\tilde{U} = \Theta_U(\gamma)$. Hence $\Theta_U$ is surjective.

(ii) It is obvious that $\alpha(W(r, U)) = \langle f^{r_1} \alpha(u_1) \rangle_f \oplus \cdots \oplus \langle f^{r_k} \alpha(u_k) \rangle_f = W(r, \tilde{U})$. $\square$

In group theory fully invariant subgroups play the role of hyperinvariant subspaces. Hence the decomposition (2.8) below is an analog to a distributive law in Lemma 9.3 in [10, p. 47].

**Lemma 2.3.** Suppose

$$V = V_1 \oplus \cdots \oplus V_q, \quad V_i \in \text{Inv}(V), \quad i = 1, \ldots, q. \quad (2.7)$$

(i) If $X$ is a hyperinvariant subspace of $V$, or

(ii) if $X$ characteristic and $|K| > 2$, then

$$X = (X \cap V_1) \oplus \cdots \oplus (X \cap V_q). \quad (2.8)$$

**Proof.** If $x \in V$ then $x = \sum_{i=1}^q x_i$, $x_i \in V_i$. Set $X_i = X \cap V_i$, and $S = \oplus_{i=1}^q X_i$. Then $S \subseteq X$. To prove the converse inclusion we note that

$$fx = \sum_{i=1}^q f_{|V_i}(x_i). \quad (2.9)$$

(i) Let $\pi_i$ be the projection on $V_i$ induced by (2.7). Then (2.9) implies $\pi_i \in \text{End}_f(V)$. Hence, if $x \in X$ then and $\pi_i(x) = x_i \in X$. Thus $x_i \in X_i$, and therefore $X \subseteq S$.

(ii) Let $a \in K$ be different from 0 and 1, and define $\gamma_i = 1 - a\pi_i$. Then $\gamma_i \in \text{Aut}_f(V)$. Hence $\gamma_i(x) = x - ax_i \in X$ if $x \in X$. Thus we obtain $x_i \in X_i$. $\square$

**Example 2.4.** In Lemma 2.3 (ii) one can not drop the assumption $|K| > 2$. Suppose $|K| = 2$, and let $V$ and $f$ be as in Example 1.1. The subspace $Z = \langle e_1 + e_3 \rangle$ is characteristic. Both $V_1 = \langle e_1 \rangle$ and $V_2 = \langle e_2 \rangle$ are in Inv$(V)$, and we have $V = V_1 \oplus V_2$. But $Z \cap V_1 = 0$ and $Z \cap V_2 = \langle e_4 \rangle$ imply $Z \not\subseteq (Z \cap V_1) \oplus (Z \cap V_2)$.

The next lemma is an intermediate result.

**Lemma 2.5.** Each hyperinvariant subspace of $V$ is marked, and

$$\text{Hinv}(V) \subseteq \text{Mark}(V) \cap \text{Chin}(V). \quad (2.10)$$

**Proof.** Let $U = (u_1, \ldots, u_k) \in \mathcal{U}$. If $X$ is invariant then $X \cap \langle u_i \rangle = \langle f^{r_i}u_i \rangle$ for some $r_i$. Thus, if $X$ is hyperinvariant then (2.8) in Lemma 2.3 implies $X = \oplus_{i=1}^k \langle f^{r_i}u_i \rangle$. Therefore $X$ is marked, and $\text{Hinv}(V) \subseteq \text{Chin}(V)$ yields the inclusion (2.10). $\square$
3. Hyperinvariant = characteristic + marked

We now characterize those marked subspaces which are characteristic. The theorem below includes results from [2] with new proofs.

**Theorem 3.1.** Let $U \in \mathcal{U}$ and let $r = (r_1, \ldots, r_k)$ be admissible. Then the following statements are equivalent.

(i) The subspace $W(r, U)$ is characteristic.

(ii) The subspace $W(r, U)$ is independent of the generator tuple $U$, i.e.

$$W(r, U) = W(r, \tilde{U}) \text{ for all } \tilde{U} \in \mathcal{U}. \quad (3.1)$$

(iii) The tuples $t = (t_1, \ldots, t_k)$ and $r = (r_1, \ldots, r_k)$ satisfy

$$r_1 \leq \cdots \leq r_k \quad (3.2)$$

and

$$t_1 - r_1 \leq \cdots \leq t_k - r_k. \quad (3.3)$$

(iv) We have $W(r, U) = W(r)$.

(v) $W(r, U)$ is the unique marked subspace $W$ such that the elementary divisors of $W$ and of $V/W$ are

$$s^{1-r_1}, \ldots, s^{k-r_k}, \quad \text{and} \quad s^{r_1}, \ldots, s^{r_k}. \quad (3.4)$$

(vi) The subspace $W(r, U)$ is hyperinvariant.

**Proof.** (i) $\Leftrightarrow$ (ii) It follows from Lemma 2.2 that the two statements are equivalent.

(iv) $\Rightarrow$ (vi) This follows from the fact that $W(r)$ is hyperinvariant.

(v) $\Rightarrow$ (ii) Let $\tilde{U} \in \mathcal{U}$. Then $W(r, U)$ and the quotient space $V/W(r, U)$, and also $W(r, \tilde{U})$ and $V/W(r, \tilde{U})$, have elementary divisors given by (3.4). (Note that in the right-hand side of (2.4) there may be summands of the form $\langle u_i \rangle$ or $\langle ft_i u_i \rangle = 0$. Thus (3.4) may contain trivial entries of the form $s^0 = 1$.)

(vi) $\Rightarrow$ (i) Obvious, because of $\text{Hinv}(V) \subseteq \text{Chinv}(V)$.

(iii) $\Rightarrow$ (iv) From $e(u_i) = t_i$ follows

$$\langle ft_i u_i \rangle = \langle u_i \rangle [f^{t_i - r_i}] \subseteq f^{t_i} V \cap V [f^{t_i - r_i}].$$

Hence $W(r, U) \subseteq W(r)$. We have to show that the conditions (3.2) and (3.3) imply the converse inclusion

$$W(r) = f^{t_1} V \cap V [f^{t_1 - r_1}] + \cdots + f^{t_k} V \cap V [f^{t_k - r_k}] \subseteq W(r, U).$$

With regard to the decomposition $V = \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle$ we define

$$D(\mu, \nu) = f^{r_\nu} \langle u_\mu \rangle \cap \langle u_\mu \rangle [f^{r_\nu - r_\nu}].$$
The subspaces \( f^r v \cap V[f^{s-r}v] \) are hyperinvariant. Therefore Lemma 2.3(i) yields
\[
f^r v \cap V[f^{s-r}v] = \bigoplus_{\mu=1}^{k} (f^r v \cap V[f^{s-r}v] \cap \langle u_\mu \rangle) = \bigoplus_{\mu=1}^{k} D(\mu, v).
\]

Hence
\[
W(r) = \sum_{\mu, v=1}^{k} D(\mu, v). \quad (3.5)
\]

Set \( q(\mu, v) = \max\{r_v, t_\mu - (t_v - r_v)\} \). We have
\[
\langle u_\mu \rangle[f^{s-r}v] = \begin{cases} 
\langle u_\mu \rangle, & \text{if } t_v - r_v \geq t_\mu, \\
\langle f^r u_\mu - (t_v - r_v) \rangle, & \text{if } t_v - r_v \leq t_\mu.
\end{cases}
\]

Hence
\[
D(\mu, v) = f^q(\mu, v) \langle u_\mu \rangle.
\]

Let us show that \( r_\mu \leq q(\mu, v) \) for all \( \mu \). If \( \mu \geq v \), then (3.3) implies
\[
q(\mu, v) = (t_\mu - t_v) + r_v = (t_\mu - r_\mu) - (t_v - r_v) + r_\mu \geq r_\mu.
\]

If \( \mu \leq v \) then \( t_\mu - t_v \leq 0 \), and therefore \( q(\mu, v) = r_v \). Hence (3.2) implies \( q(\mu, v) \geq r_\mu \). It follows that
\[
D(\mu, v) = f^q(\mu, v) \langle u_\mu \rangle \subseteq f^{r_v} \langle u_\mu \rangle \subseteq W(r, U).
\]

for all \( \mu, v \). Thus (3.5) yields \( W(r) \subseteq W(r, U) \).

(ii) \( \Rightarrow \) (iii) We modify the entries of \( U \) and replace \( u_k \) by \( \tilde{u}_k = u_{k-1} + u_k \). Then \( \tilde{U} = (u_1, \ldots, u_{k-1}, \tilde{u}_k) \in \mathcal{U} \). Set \( Y_k = \bigoplus_{i=1}^{k-1} \langle f^{r_i} u_i \rangle \). Then \( W(r, U) = W(r, \tilde{U}) \) implies
\[
Y_k \oplus \langle f^{r_k} u_k \rangle = Y_k \oplus \langle f^{r_k} (u_{k-1} + u_k) \rangle.
\]

From
\[
f^{r_k} u_{k-1} + f^{r_k} u_k \in \langle f^{r_{k-1}} u_{k-1} \rangle \oplus \langle f^{r_k} u_k \rangle
\]
follows \( r_{k-1} \leq r_k \). Proceeding in this manner we obtain the chain of inequalities in (3.2). In order to prove (3.3) we start with the entry of \( U \) of \( \mathcal{U} \) and replace it by \( u_1 + f^{t_2-t_1} u_2 \). Because of \( e(u_1 + f^{t_2-t_1} u_2) = e(u_1) \) we have \( \tilde{U} = (u_1 + f^{t_2-t_1} u_2, u_2, \ldots, u_k) \in \mathcal{U} \). Set \( Y_1 = \bigoplus_{i=2}^{k} \langle f^{r_i} u_i \rangle \). Then \( W(r, U) = W(r, \tilde{U}) \) implies
\[
\langle f^{r_1} u_1 \rangle \oplus Y_1 = \langle f^{r_1} (u_1 + f^{t_2-t_1} u_2) \rangle \oplus Y_1.
\]

From
\[
f^{r_1} u_1 + f^{r_1 + (t_2-t_1)} u_2 \in \langle f^{r_1} u_1 \rangle \oplus \langle f^{r_2} u_2 \rangle
\]
follows \( r_2 \leq r_1 + (t_2 - t_1) \), i.e. \( t_1 - r_1 \leq t_2 - r_2 \), such that we the end up with (3.3). \( \square \)

Let \( [k] \) denote the greatest integer less than or equal to \( k \). If \( c \in \mathbb{R} \) and \( 0 < c < 1 \), then \( r = ([ct_1], \ldots, [ct_m]) \) is admissible, and it is not difficult to verify that \( r \) satisfies (3.2) and (3.3). We remark that admissible tuples of the form \( \tilde{r} = ([\frac{1}{2} t_1], \ldots, [\frac{1}{2} t_k]) \) play a role in the study of maximal invariant neutral subspaces [18]. It follows from Theorem 3.1 that the construction of such subspaces is independent of the choice of the underlying Jordan basis.
THEOREM 3.2. (i) We have
\[
\text{Hinv}(V) = \text{Chinv}(V) \cap \text{Mark}(V). \tag{3.6}
\]
(ii) [9] A subspace \( W \) of \( V \) is hyperinvariant if and only if \( W = W(r) \) for some \( r \) satisfying (3.2) and (3.3).

Proof. (i) From Theorem 3.1 follows \( \text{Mark}(V) \cap \text{Chinv}(V) \subseteq \text{Hinv}(V) \). The reverse inclusion is (2.10) in Lemma 2.5. This yields (3.6). Hence a subspace is hyperinvariant if and only if it is both characteristic and marked.
(ii) If \( W \) is hyperinvariant then \( W \) is marked, that is \( W = W(r, U) \). Therefore we can apply Theorem 3.1(iv). It was noted earlier that \( W(r) \in \text{Hinv}(V) \). □

We note that hyperinvariant subspaces can be characterized completely by the distributive law in Lemma 2.3.

THEOREM 3.3. A subspace \( X \in \text{Inv}(V) \) is hyperinvariant if and only if \( X \) satisfies
\[
X = (X \cap V_1) \oplus \cdots \oplus (X \cap V_q) \tag{3.7}
\]
when
\[
V = V_1 \oplus \cdots \oplus V_q, \quad V_i \in \text{Inv}(V), \quad i = 1, \ldots, q. \tag{3.8}
\]

Proof. Because of Lemma 2.3 it remains to prove sufficiency. Let \( U = (u_1, \ldots, u_k) \in \mathcal{U} \) and \( \hat{U} = (\hat{u}_1, \ldots, \hat{u}_k) \in \mathcal{U} \). Then
\[
V = \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle = \langle \hat{u}_1 \rangle \oplus \cdots \oplus \langle \hat{u}_k \rangle. \tag{3.9}
\]
Define \( X_i = \langle u_i \rangle \cap X \) and \( \hat{X}_i = \langle \hat{u}_i \rangle \cap X \), \( i = 1, \ldots, k \). Then \( X_i = \langle f^{r_i}u_i \rangle \) and \( \hat{X}_i = \langle f^{\tilde{r}_i}\hat{u}_i \rangle \) for some \( r_i, \tilde{r}_i \). Set \( r = (r_1, \ldots, r_k) \) and \( \tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_k) \). In (3.9) we have two direct sums of the form (3.8). Hence the assumption (3.7) implies \( X = W(r, U) = W(\tilde{r}, \hat{U}) \). We can pass from \( U \) to \( \hat{U} \) in at most \( k \) steps, changing a single entry at each step. Suppose we replace \( u_k \) in \( U \) by \( \hat{u}_k \). Then \( \hat{U} = (u_1, \ldots, u_{k-1}, \hat{u}_k) \in \mathcal{U} \), and \( V = \langle u_1 \rangle \oplus \cdots \langle u_{k-1} \rangle \oplus \langle \tilde{u}_k \rangle \). Set \( Y_k = \oplus_{i=1}^{k-1} \langle f^{r_i}u_i \rangle \). Then
\[
X = Y_k \oplus \langle f^{\tilde{r}_k}\tilde{u}_k \rangle = Y_k \oplus \langle f^{r_k}u_k \rangle.
\]
Considering the elementary divisors of \( V/X \) we deduce \( \tilde{r}_k = r_k \), and at the end we obtain \( r = \tilde{r} \), and therefore \( W(r, U) = W(\tilde{r}, \hat{U}) \). We conclude that \( X = W(r, U) \) is independent of the choice of the generator tuple \( U \). Hence \( X \) is hyperinvariant. □

Let us reexamine Example 1.1 and consider a field \( K \) of characteristic different from 2.

EXAMPLE 1.1 (CONTINUED). Let \( \text{char} K \neq 2 \). Then \( \gamma: (e_1, e_2) \mapsto (2e_1, e_2) \) determines an \( f \)-automorphism. For \( Z = \langle e_1 + e_3 \rangle \) we have \( \gamma(Z) = \langle 2e_1 + e_3 \rangle \neq Z \). Hence in this case \( Z \in \text{Inv}(V) \) is not characteristic.
To identify the characteristic subspaces we screen $\text{Inv}(V)$. Note that

$$\text{Aut}_f(V) = \{ \alpha : (e_1,e_2) \mapsto (ae_1 + be_4 + ce_2 + de_3 + ge_4 + he_1), \ a, b, c, d, g, h \in K, a \neq 0, c \neq 0 \}. $$

The nonzero cyclic subspaces are of the form $H_{inv}$ one eigenvalue. We can assume $Ch_{inv}$ variant, i.e.

3.3, the subspace follows from Lemma 2.3(ii) that (2.7) implies (2.8). Therefore, according to Theorem

is a special case of the following general result (see also [14, p. 67]).

Moreover,

$$X \langle ce \rangle \langle 12 \rangle I. \text{GOHBERG, P. LANCASTER, AND L. RODMAN, } \text{The Theory of Matrices, Vol. I, Chelsea, New York, 1959.}$$

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