Systems & Control Letters 19 (1992) 233–239
North-Holland

A comparison theorem for matrix Riccati difference equations

Harald K. Wimmer *

Mathematisches Insitut, Universität Würzburg, D-8700 Würzburg, Germany

Michele Pavon

Dipartimento di Elettronica e Informatica, Università di Padova, and LABSEB-CNR, I-35100 Padova, Italy

Received 10 January 1992
Revised 25 April 1992

Abstract: Difference equations of the form $X(t) = F^*(t)X(t-1)F(t) - F^*(t)X(t-1)G(t)[I + G^*(t)X(t-1)G(t)]^{-1}G^*(t)X(t-1)F(t) + Q(t)$ and their associated Hermitian matrices $H(t) = \left(\begin{smallmatrix} Q & F^* \\ F & -GG^* \end{smallmatrix}\right)(t)$ are studied. Solution of different Riccati equations can be compared if the difference of their corresponding Hermitian matrices is semidefinite for all t. An application to the discrete-time LQ optimal control problem is given.

Keywords: Matrix Riccati difference equations; monotonicity of solutions; discrete-time LQ optimal control.

1. Introduction

Discrete-time linear-quadratic optimal control problems (see e.g. [1,6]) and linear least-squares estimation (see e.g. [5]) are two areas where matrix Riccati difference equations are important. To provide a motivation for our study and an application of our main result let us first review the LQ optimal control problem.

Consider the time-variant control system

$$x(t+1) = F(t)x(t) + G(t)u(t), \quad x(0) = c, \tag{1.1}$$

together with a quadratic cost functional given by

$$J = x(N)^* D_N x(N) + \sum_{i=0}^{N-1} [x(i)^* Q(i) x(i) + u(i)^* R(i) u(i)] \tag{1.2}$$

where $Q(i) \geq 0$, $R(i) > 0$ for $i = 0, 1, \ldots, N-1$, and $D_N \geq 0$. The optimal control which minimizes J is of the form

$$u_{\text{opt}}(t) = -[R(t) + G(t)^* D(t+1) G(t)]^{-1} G(t)^* D(t+1) F(t) x(t) \tag{1.3}$$

Correspondence to: Dr. H.K. Wimmer, Mathematisches Institut, Universität Würzburg, D-8700 Würzburg, Germany.

* Research supported by the SCIENCE-Programme of the European Communities (SC1*/0126-C) and Deutsche Forschungsgemeinschaft (Kn 164/3-1).

where $D(t)$ is a solution of the Riccati difference equation

$$\begin{aligned} D(t) &= F(t)^* D(t+1) F(t) \\ &\quad - F(t)^* D(t+1) G(t) [R(t) + G(t)^* D(t+1) G(t)]^{-1} G(t)^* D(t+1) F(t) \\ &\quad + Q(t), \quad D(N) = D_N. \end{aligned} \tag{1.4}$$

Thus (1.3) yields the closed loop system $x(t+1) = F_D(t)x(t)$ with $F_D(t) = F(t) - G(t)[R(t) + G(t)^*D(t+1)G(t)]^{-1}G(t)^*D(t+1)F(t)$. The minimum of the cost functional (see e.g. [1,6]) equals

$$J_{\text{opt}} = c^* D(0) c \tag{1.5}$$

where $D(0)$ is obtained from (1.4).

Now suppose (1.2) is replaced by a different cost functional

$$\tilde{J} = x(N)^* \tilde{D}_N x(N) + \sum_{i=0}^{N-1} \left[x(i)^* \tilde{Q}(i) x(i) + u(i)^* \tilde{R}(i) u(i) \right] \tag{1.6}$$

such that (1.2) and (1.6) are related by

$$\tilde{D}_N \geq D_N, \qquad \tilde{Q}(i) \geq Q(i), \quad \tilde{R}(i) \geq R(i) \quad \text{for } i = 0, \ldots, N-1. \tag{1.7}$$

It is obvious that the assumptions (1.7) induce a monotonicity relation $\tilde{J}_{\text{opt}} \geq J_{\text{opt}}$ of the respective minimal costs. A more general situation arises when we consider a linear system

$$x(t+1) = \tilde{F}(t) x(t) + \tilde{G}(t) u(t), \quad x(0) = c, \tag{1.8}$$

which is different form (1.1), together with the new cost functional (1.6). In this note we shall prove the following comparison result.

Theorem 1.1. *Let* (1.2), (1.1) *and* (1.6), (1.8) *be two LQ optimal control problems given by cost functional and constraint and let* J_{opt} *and* $\tilde{J}_{\text{opt}}$ *be the minimal values of* J *and* $\tilde{J}$. *If* $\tilde{D}_N \geq D_N$ *and*

$$\begin{pmatrix} \tilde{Q}(t) & \tilde{F}(t)^* \\ \tilde{F}(t) & -\tilde{G}(t)\tilde{R}(t)^{-1}\tilde{G}(t)^* \end{pmatrix} \geq \begin{pmatrix} Q(t) & F(t)^* \\ F(t) & -G(t)R(t)^{-1}G^*(t) \end{pmatrix},$$

$i = 0, 1, \ldots, N-1$, *then we have* $\tilde{J}_{\text{opt}} \geq J_{\text{opt}}$.

For the continuous-time LQ problem with a system

$$\dot{x}(t) = F(t)x(t) + G(t)u(t), \quad x(0) = c, \tag{1.9}$$

and a cost functional

$$J = x(T)^* D_T x(T) + \int_0^{*} [x(t)^* Q(t) x(t) + u(t)^* R(t) u(t)] \, \mathrm{d}t, \tag{1.10}$$

the analogue of Theorem 1.1 has been known for a long time (see [4, pp. 52, 53]). The fact that our corresponding discrete-time result has not yet been available may be related to a matrix inequality (2.3) in Lemma 2.1. The counterpart of (2.3) which is needed for the problem (1.9), (1.10) can be proved without effort, whereas we do not know of a short proof of the inequality (2.3).

2. Auxiliary results

Let F, G and Q be complex matrices of sizes $n \times n$, $n \times m$ and $n \times n$ respectively and assume $Q = Q^*$. If X is a Hermitian $n \times n$ matrix and if $I + G^*XG$ is nonsingular we define

$$\mathscr{A}(X) = F^*XF - F^*XG(I + G^*XG)^{-1}G^*XF + Q. \tag{2.1}$$

The matrix

$$F_X = (I + GG^*X)^{-1}F = F - G(I + G^*XG)^{-1}G^*XF$$

can be considered as a closed-loop matrix of a time-invariant system $x(t+1) = Fx + Gu$, $u = -(I + G^*XG)^{-1}G^{\mathrm{T}}XF$. To the Riccati operator $\mathscr{A}$ we associate a Hermitian matrix

$$H = \begin{pmatrix} Q & F^* \\ F & -GG^* \end{pmatrix}. \tag{2.2}$$

Assume that the blocks in a second Hermitian matrix

$$\tilde{H} = \begin{pmatrix} \tilde{Q} & \tilde{F}^* \\ \tilde{F} & -\tilde{G}\tilde{G}^* \end{pmatrix}$$

are all of size $n \times n$ and let

$$\tilde{\mathscr{A}}(X) = \tilde{F}^*X\tilde{F}^* - \tilde{F}^*X\tilde{G}(I + \tilde{G}^*X\tilde{G})^{-1}\tilde{G}^*X\tilde{F} + \tilde{Q}$$

be its corresponding Riccati operator. The matrix inequality mentioned in Section 1 relates two Riccati operators $\mathscr{A}$ and $\tilde{\mathscr{A}}$.

Lemma 2.1 [7]. *Let Y be a Hermitian $n \times n$ matrix. Suppose $H \leq \tilde{H}$. Then*

(1) $I + G^*YG > 0$ *implies* $I + \tilde{G}^*Y\tilde{G} > 0$.

(2) $I + \tilde{G}^*Y\tilde{G} > 0$ *implies*

$$\mathscr{A}(Y) \leq \tilde{\mathscr{A}}(Y). \tag{2.3}$$

The next lemma shows how a Riccati operator varies with its argument.

Lemma 2.2 [3]. *Let X and Y be Hermitian $n \times n$ matrices. Put $\Delta = Y - X$. If $I + G^*XG$ is nonsingular then*

$$\mathscr{A}(Y) - \mathscr{A}(X) = F_Y^*\Delta F_Y + F_Y^*\Delta G(I + G^*XG)^{-1}G^*\Delta F_Y. \tag{2.4}$$

3. A comparison theorem

In the following we deal with a Riccati difference equation which appears in linear least-squares estimation and which in contrast to (1.4) evolves from initial time $t = 0$ in the sense of increasing t. Let

$$\begin{aligned} X(0) &= X_0, \\ X(t) &= \mathscr{A}_t(X(t-1)) \\ &= F^*(t)X(t-1)F(t) \\ &\quad - F^*(t)X(t-1)G(t)[I + G^*(t)X(t-1)G(t)]^{-1}G^*(t)X(t-1)F(t) \\ &\quad + Q(t), \quad t \geq 1, \end{aligned} \tag{3.1}$$

be a Riccati difference equation where the assumptions on $F(t)$, $G(t)$, $Q(t)$ are those we had for F, G, Q in the previous section. Only Hermitian matrices $X(t)$ satisfying (3.1) are considered as solutions. As in (2.2) we define

$$H(t)=\begin{pmatrix} Q(t) & F^*(t) \\ F(t) & -G(t)G^*(t) \end{pmatrix}.$$

To the equation

$$\begin{aligned} X(0) &= \tilde{X}_0, \\ X(t) &= \tilde{\mathscr{A}}_t(X(t-1)) \\ &= \tilde{F}^*(t)X(t-1)\tilde{F}(t) \\ &\quad -\tilde{F}^*(t)X(t-1)\tilde{G}(t)\left[I+\tilde{G}^*(t)X(t-1)\tilde{G}(t)\right]^{-1}\tilde{G}^*(t)X(t-1)\tilde{F}(t) \\ &\quad +\tilde{Q}(t), \quad t\geq 1, \end{aligned} \tag{3.2}$$

we associate a corresponding matrix $\tilde{H}(t)$. For $k, T\in\mathbb{Z}$ put $[k, T]=\{i \mid i\in\mathbb{Z},\ k\leq i\leq T\}$.

Theorem 3.1. *Let* $X(t)=\mathscr{A}_t(X(t-1))$ *and* $X(t)=\tilde{\mathscr{A}}_t(X(t-1))$ *be two Riccati difference equations given by* (3.1) *and* (3.2) *and let* $H(t)$ *and* $\tilde{H}(t)$ *be their associated Hermitian matrices. Assume*

$$X_0\leq\tilde{X}_0 \tag{3.3}$$

and

$$H(t)\leq\tilde{H}(t) \tag{3.4}$$

for $t\in[1, T]$. *Suppose there exists a solution of* (3.1) *which satisfies*

$$I+G^*(t)X(t-1)G(t)>0 \tag{3.5}$$

on $[1, T]$. *Then the solution* $\tilde{X}(t)$ *of* (3.2) *is defined and satisfies the inequalities*

$$X(t)\leq\tilde{X}(t) \tag{3.6}$$

and

$$I+\tilde{G}^*(t)\tilde{X}(t-1)\tilde{G}(t)>0 \tag{3.7}$$

throughout $[1, T]$.

Proof. Induction can be started almost trivially by extending (3.1) and (3.2) by one time interval to the left. If we put $X(-1)=0$ and

$$H(0)=\begin{pmatrix} X_0 & 0 \\ 0 & 0 \end{pmatrix}$$

then we have $\mathscr{A}_0(X(-1))=Q(0)=X_0$ and (3.1) becomes

$$X(-1)=X_{-1}=0, \qquad X(t)=\mathscr{A}_t(X(t-1)), \quad t\geq 0. \tag{3.8}$$

Correspondingly we put

$$\tilde{H}(0)=\begin{pmatrix} \tilde{X}_0 & 0 \\ 0 & 0 \end{pmatrix}$$

and replace (3.2) by

$$X(-1)=\tilde{X}_{-1}=0, \qquad X(t)=\tilde{\mathscr{A}}_t(X(t-1)), \quad t\geq 0. \tag{3.9}$$

A solution of (3.8) on the interval $[-1, 0]$ is given by $X(-1)=0$, $X(0)=X_0$. Note that (3.3) is equivalent to $H(0)\leq\tilde{H}(0)$. In the proof we shall consider the equations (3.8) and (3.9) extended over the interval $[-1, T]$.

Let $i\in[0, T-1]$ be such that the solution $\tilde{X}(t)$ of (3.9) is defined and has the properties (3.6) and (3.7) on $[-1, i]$. Such an integer exists, namely $i=0$. From the hypothesis (3.6) follows in particular

$$X(i)\leq\tilde{X}(i). \tag{3.10}$$

According to Lemma 2.1 the assumptions

$$H(i+1)\leq\tilde{H}(i+1) \quad\text{and}\quad I+G^*(i+1)X(i)G(i+1)>0$$

imply $I+\tilde{G}^*(i+1)X(i)\tilde{G}(i+1)>0$. From (3.10) we obtain

$$I+\tilde{G}^*(i+1)\tilde{X}(i)\tilde{G}(i+1)>0, \tag{3.11}$$

which establishes (3.7) for $t=i+1$. Therefore $\tilde{\mathscr{A}}_{i+1}(\tilde{X}(i))$ is well-defined and we can continue the solution $\tilde{X}(t)$ from $[-1, i]$ one step further to $t=i+1$. Put $\Delta(t)=\tilde{X}(t)-X(t)$. Recall (3.10), i.e. $\Delta(i)\geq 0$. We want to show that $\Delta(i+1)\geq 0$. Obviously

$$\begin{aligned}\Delta(i+1)&=\tilde{\mathscr{A}}_{i+1}(\tilde{X}(i))-\mathscr{A}_{i+1}(X(i))\\&=\left[\tilde{\mathscr{A}}_{i+1}(\tilde{X}(i))-\mathscr{A}_{i+1}(\tilde{X}(i))\right]+\left[\mathscr{A}_{i+1}(\tilde{X}(i))-\mathscr{A}_{i+1}(X(i))\right].\end{aligned}$$

From Lemma 2.1 follows immediately

$$\tilde{\mathscr{A}}_{i+1}(\tilde{X}(i))-\mathscr{A}_{i+1}(\tilde{X}(i))\geq 0. \tag{3.12}$$

The inequalities

$$I+G^*(i+1)X(i)G(i+1)>0 \tag{3.13}$$

and (3.8) yield $I+G^*(i+1)\tilde{X}(i)G(i+1)>0$. Hence we can define

$$\Phi=F(i+1)-G(i+1)\left[I+G^*(i+1)\tilde{X}(i)G(i+1)\right]^{-1}G^*(i+1)\tilde{X}(i)F(i+1).$$

We see from (2.4) that

$$\begin{aligned}&\mathscr{A}_{i+1}(\tilde{X}(i))-\mathscr{A}_{i+1}(X(i))\\&\quad=\Phi^*\Delta(i)\Phi+\Phi^*\Delta(i)G(i+1)[I+G^*(i+1)X(i)G(i+1)]^{-1}G^*(i+1)\Delta(i)\Phi.\end{aligned}$$

From (3.13) and $\Delta(i)\geq 0$ we obtain

$$\mathscr{A}_{i+1}(\tilde{X}(i))-\mathscr{A}_{i+1}(X(i))\geq 0. \tag{3.14}$$

Now (3.12) and (3.14) imply $\Delta(i+1)\geq 0$ which completes the proof. □

4. Applications

In this section we give a proof of Theorem 1.1 and derive a monotonicity result for positive-semidefinite solutions of (3.1).

Proof of Theorem 1.1. Recall (1.5). Define

$$\mathscr{G}(X)=F^*XF-F^*XG_1(R+G_1^*XG_1)^{-1}G_1^*XF+Q$$

and assume $R>0$. Put $G=G_1R^{-1/2}$. Then we have $\mathscr{G}(X)=\mathscr{A}(X)$ where $\mathscr{A}$ is given by (2.1), and the associated Hermitian matrix is

$$\begin{pmatrix} Q & F^* \\ F & -G_1R^{-1}G_1^* \end{pmatrix}$$

Theorem 4.1. *Let*

$$\begin{aligned} X(0)&=X_0,\\ X(t)&=\mathscr{A}_t(X(t-1))\\ &=F^*(t)X(t-1)F(t)\\ &\quad -F^*(t)X(t-1)G(t)[I+G^*(t)X(t-1)G(t)]^{-1}G^*(t)X(t-1)F(t)\\ &\quad +Q(t),\quad t\geq 1, \end{aligned}\tag{4.1}$$

be a Riccati difference equation with associated Hermitian matrix

$$H(t)=\begin{pmatrix} Q(t) & F^*(t) \\ F(t) & -G(t)G^*(t) \end{pmatrix}.$$

Assume

(1) $0\leq X_0$,

(2) $X_0\leq\mathscr{A}_1(X_0)$, (4.2)

(3) $H(t)\leq H(t+1)$, $t\in[0,T-1]$, (4.3)

where $T\geq 1$. *Then the solution* $X(t)$ *of* (4.1) *exists on* $[0,T]$ *and has the property*

$$X(t)\leq X(t+1)\tag{4.4}$$

on $[0,T-1]$.

If

$$0\leq X_0\leq Q(1)\tag{4.5}$$

holds then (4.2) *is satisfied.*

Proof. Because of $X(1)=\mathscr{A}_1(X_0)\geq X_0\geq 0$ the theorem is obviously true for $T=1$. Now assume $T\geq 2$ and let $i\in[1,T-1]$ be such that a solution of $X(t)$ of (4.1) exists on the interval $[0,i-1]$ and such that

$$X_0=X(0)\leq X(1)\leq\cdots\leq X(i-1).$$

If we put $\tilde{X}(t)=X(t+1)$ then $\tilde{X}(t)$ satisfies a difference equation

$$\tilde{X}(0)=X(1),\qquad \tilde{X}(t)=\tilde{\mathscr{A}}_t(\tilde{X}(t-1)),\quad t\in[0,i-2],$$

with associated Hermitian matrix $\tilde{H}(t)=H(t+1)$. Because of $X(0)\leq\tilde{X}(0)=X(1)$ and (4.3) we conclude from Theorem 3.1 that $\tilde{X}(i-1)$ is defined and that $X(i-1)\leq\tilde{X}(i-1)=X(i)$, which closes the induction argument.

To show that (4.5) implies (4.2) we use the identity

$$S(X)=X-XG(I+G^*XG)^{-1}G^*X=X(I+GG^*X)^{-1}$$

and note that $S(X) \geq 0$ if $X \geq 0$. Hence

$$X(1) = \mathscr{A}_1(X_0) = F^*(1)X_0[I + G(1)G^*(1)X_0]^{-1}F(1) + Q(1),$$

and (4.5) yields $X(1) \geq Q(1) \geq X_0$. □

In the special case where $H(t+1) = H(t)$ for all $t \geq 0$ we recover a result of [2, p.311].

References

[1] J. Ackermann, *Sampled-Data Control Systems* (Springer-Verlag, Berlin 1985).

[2] R.R. Bitmead, M.R. Gevers and I.R. Petersen, Monotonicity and stabilizability properties of solutions of the Riccati difference equations, *Systems Control Lett.* **5** (1985) 309–315.

[3] S.W. Chan, G.C. Goodwin and K.S. Sin, Convergence properties of the Riccati difference equation in optimal filtering of nonstabilizable systems, *IEEE Trans. Automat. Control* **29** (1984) 110–118.

[4] W.A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics No. 220 (Springer-Verlag, Berlin, 1971).

[5] G. Picci, *Elementi di Elaborazione Statistica del Segnale* (CLEUP Editore, Padova, 1986).

[6] A.P. Sage, *Optimal Systems Control* (Prentice-Hall, Englewood Cliffs, NJ, 1968).

[7] H.K. Wimmer, Monotonicity and maximality of solutions of discrete-time algebraic Riccati equations, *J. Math. Syst. Estim. Control* **2** (1992) 219–235.