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Diagonal stability of matrices with cyclic structure and the secant condition

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ABSTRACT

An existence result on diagonal solutions of a linear matrix inequality is used to study diagonal Hurwitz and Schur stability and to derive the secant condition for systems with cyclic structure.

1. Introduction

The starting point of this paper is the stability result in Theorem 1 below, which deals with real matrices of the form

$$\tilde{A} = \begin{pmatrix} -\theta_1 & 0 & 0 & \dots & 0 & \tilde{\beta}_n \\ \tilde{\beta}_1 & -\theta_2 & 0 & \dots & 0 & 0 \\ 0 & \tilde{\beta}_2 & -\theta_3 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & -\theta_{n-1} & 0 \\ 0 & 0 & 0 & \dots & \tilde{\beta}_{n-1} & -\theta_n \end{pmatrix},$$

$$\theta_j > 0,\ \tilde{\beta}_j \neq 0,\ j = 1, \dots, n. \tag{1}$$

Such matrices of cyclic structure play a role in the study of biochemical reactions. Tyson and Othmer [1] and Thron [2] considered negative feedback cyclic systems of the form

$$\begin{aligned} \dot{x}_1 &= -f_1(x_1) - g_n(x_n) \\ \dot{x}_2 &= -f_2(x_2) + g_1(x_1) \\ &\vdots \\ \dot{x}_n &= -f_n(x_n) + g_{n-1}(x_{n-1}), \end{aligned} \tag{2}$$

assuming that the functions $f_j(\cdot), j = 1, \dots, n$, and $g_j(\cdot), j = 1, \dots, n-1$ are increasing and $g_n(\cdot)$ is decreasing. A linearization of (2) at the equilibrium yields a matrix of the form (1) as a Jacobian matrix. The following result is due to Arcak and Sontag [3, p.1532].

Theorem 1. *Let $\tilde{A}$ be a real $n \times n$ matrix given by (1) and assume $\tilde{\beta}_n < 0, \tilde{\beta}_j > 0, j = 2, \dots, n-1$. Then there exists a positive definite diagonal matrix X such that*

$$\tilde{A}^{\mathrm{T}} X + X\tilde{A} < 0$$

if and only if the condition

$$\left(\frac{\tilde{\beta}_1 \cdots \tilde{\beta}_{n-1} |\tilde{\beta}_n|}{\theta_1 \cdots \theta_{n-1}\theta_n} \right)^{1/n} < \sec \frac{\pi}{n} \tag{3}$$

is satisfied. If (3) holds then the system $\dot{x}(t) = \tilde{A}x(t)$ is asymptotically stable.

The *secant condition* (3) goes back to Tyson and Othmer [1] and Thron [2]. According to [4] an application of the secant condition to mathematical biology can also be found in [5], where Kholodenko analyses a model of negative feedback around mitogen-activated protein kinase cascades. The connection of (3) with passivity was made by Sontag [6]. Diagonal stability was in the focus of [3], where Theorem 1 is a tool to obtain results on global stability of nonlinear systems.

Let $\dot{x}_j(t) = -\theta_j x_j(t), j = 1, \dots, n$, be n asymptotically stable scalar systems, which are interconnected through

$$\begin{aligned} \dot{x}_1(t) &= -\theta_1 x_1(t) + \tilde{\beta}_n x_n(t), \\ \dot{x}_j(t) &= -\theta_j x_j(t) + \tilde{\beta}_{j-1} x_j(t), \quad j = 2, \dots, n. \end{aligned} \tag{4}$$

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Define $\Theta = \operatorname{diag}(\theta_1, \dots, \theta_n)$ and

$$\Delta = \begin{pmatrix} 0 & \dots & 0 & \tilde{\beta}_n \\ \tilde{\beta}_1 & \dots & 0 & 0 \\ . & \dots & . & . \\ 0 & \dots & \tilde{\beta}_{n-1} & 0 \end{pmatrix}.$$

Then (4), that is $\dot{x} = \tilde{A}x$, can be rewritten as $\dot{x}(t) = (-\Theta + \Delta)x(t)$. The gain matrix of (4) is

$$\Gamma = \begin{pmatrix} 0 & \dots & 0 & \dfrac{|\tilde{\beta}_n|}{\theta_1} \\ \dfrac{|\tilde{\beta}_1|}{\theta_2} & \dots & 0 & 0 \\ . & \dots & . & . \\ 0 & \dots & \dfrac{|\tilde{\beta}_{n-1}|}{\theta_n} & 0 \end{pmatrix},$$

and

$$\rho(\Gamma) = \left(\frac{|\tilde{\beta}_1 \cdots \tilde{\beta}_n|}{\theta_1 \cdots \theta_n} \right)^{1/n}$$

is the spectral radius of Γ. It is known (see e.g. [7, p.113] for a more general result) that $\rho(\Gamma) < 1$ implies asymptotic stability of (4). Thus Theorem 1 is complemented by the following result from [6].

Theorem 2. *The system $\dot{x}(t) = \tilde{A}x(t)$ is asymptotically stable if*

$$\frac{|\tilde{\beta}_1 \cdots \tilde{\beta}_n|}{\theta_1 \cdots \theta_n} < 1.$$

In this context we also refer to the argument used to study diagonal stability of the matrix E_b in [8, p. 13, Lemma 1].

It is the purpose of this paper to give a unified proof of the two preceding theorems. Essentially, they represent two different cases. If the product $\tilde{\beta}_1 \cdots \tilde{\beta}_n$ is negative then one has the secant condition, if the product is positive one has a small gain theorem. A corresponding result will be derived for discrete-time systems. There is a common generalization of parts of the continuous- and discrete-time results which involves diagonal solutions of a general matrix inequality.

With the exception of (7) all matrices in the following are complex (or real) of size $n \times n$, and X shall always be Hermitian (or symmetric). We write $X > 0$, resp. $X < 0$, if X is positive definite, resp. negative definite. Let $\sigma(A)$ denote the spectrum of A. With regard to asymptotic stability of continuous-time systems $\dot{x}(t) = Ax(t)$ or of discrete-time systems $x(t+1) = Ax(t)$ we say that a matrix A is *c-stable* (or *Hurwitz stable*) if $\operatorname{Re} \lambda < 0$ for all eigenvalues λ of A, whereas A will be called *d-stable* (or *Schur stable*) if $|\lambda| < 1$ for all $\lambda \in \sigma(A)$. The existence of a positive definite solution X of the Lyapunov inequality

$$A^*X + XA < 0 \tag{5}$$

is equivalent to c-stability. Analogously, A is d-stable if and only if the Stein inequality

$$X - A^*XA > 0 \tag{6}$$

is solvable for $X > 0$. Many problems, e.g. sliding mode control [9], the analysis of Volterra–Lotka models of population dynamics [10, Chapter 15.3] or robust stabilization of digital filters [11], require a quadratic Lyapunov function of the form $V(x) = V(x_1, \dots, x_n) = x^{\mathrm{T}}Xx = \sum c_i x_i^2$. Thus the following refinements of the concepts of Hurwitz or Schur stability have been useful. We call a matrix A *diagonally c-stable* resp. *diagonally d-stable*, if (5), resp. (6), holds for some positive definite diagonal matrix X. We refer to Kaszkurewicz and Bhaya [12] for a comprehensive study of diagonal stability.

2. A general matrix inequality

The inequalities (5) and (6) are special cases of a more general linear matrix inequality. Let

$$H = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} \tag{7}$$

be Hermitian and nonsingular. Define

$$f(\lambda) = c_{00} + c_{10}\bar{\lambda} + c_{01}\lambda + c_{11}\bar{\lambda}\lambda. \tag{8}$$

The curve $f(\lambda) = 0$ is a circle in the complex plane if and only if $c_{11} \neq 0$ and $\det H < 0$, and a straight line if and only if $c_{11} = 0$. We define

$$f(A; X) = c_{00}X + c_{10}A^*X + c_{01}XA + c_{11}A^*XA. \tag{9}$$

If $f(\lambda) = -(\bar{\lambda} + \lambda)$ then the inequality $f(A; X) > 0$ gives rise to the Lyapunov inequality (5), and

$$f(\lambda) > 0 \quad \text{for all } \lambda \in \sigma(A) \tag{10}$$

is equivalent to c-stability of A. Similarly, if $f(\lambda) = 1 - \bar{\lambda}\lambda$ then $f(A; X) > 0$ yields the Stein inequality (6). In that case (10) is the same as d-stability.

Lemma 3. *Suppose there exists a nonsingular diagonal matrix D such that $D^{-1}AD$ is normal. Then $Y = (DD^*)^{-1}$ is a positive definite diagonal matrix. The following statements are equivalent.*
(i) *Condition (10) holds.*
(ii) *The matrix Y satisfies $f(A; Y) > 0$.*

Proof. (i) $\Rightarrow$ (ii) Suppose $D^{-1}AD$ is normal and let U be unitary such that

$$\Lambda = U^*D^{-1}ADU = \operatorname{diag}(\lambda_1, \dots, \lambda_n).$$

Then (10) is equivalent to $f(\Lambda; I) > 0$, and this is equivalent to $f(A; Y) > 0$. (ii) $\Rightarrow$ (i) Suppose X is a Hermitian matrix (not necessarily diagonal) such that $X > 0$ and $f(A; X) > 0$. If $Au = \lambda u$ then $u^*f(A; X)u = f(\lambda)u^*Xu$, which implies (10). □

Corollary 4. *Let A be diagonally similar to a normal matrix. Then A is diagonally c-stable (resp. d-stable) if and only if A is c-stable (resp. d-stable).*

Let us show that Lemma 3 applies to a complex matrix of the form

$$A = \begin{pmatrix} \alpha & 0 & 0 & \dots & 0 & \beta_n \\ \beta_1 & \alpha & 0 & \dots & 0 & 0 \\ 0 & \beta_2 & \alpha & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & \beta_{n-1} & \alpha \end{pmatrix},$$

$$\alpha \in \mathbb{R}, \beta_j \neq 0, j = 1, \dots, n. \tag{11}$$

If $B = \operatorname{diag}(\beta_1, \dots, \beta_n)$ and

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{12}$$

then $A = \alpha I + PB$.

Lemma 5. *Let A be the matrix in (11). Set*

$$r = |\beta_1 \dots \beta_n|^{1/n}$$

and let ϵ and ω be given by

$$\beta_1 \dots \beta_n = r^n \epsilon \quad \text{and} \quad \epsilon = e^{i\omega}, \quad 0 \le \omega < 2\pi. \tag{13}$$

If

$$D=\operatorname{diag}\left(1,\frac{\beta_1}{r},\frac{\beta_1\beta_2}{r^2},\ldots,\frac{\beta_1\ldots\beta_{n-1}}{r^{n-1}}\right) \tag{14}$$

then

$$D^{-1}AD=\alpha I+r\operatorname{diag}(\epsilon,1,\ldots,1)P, \tag{15}$$

and $D^{-1}AD$ *is normal, and*

$$\lambda_k=\alpha+re^{\mathrm{i}\left(\frac{\omega}{n}+\frac{2k\pi}{n}\right)},\quad k=0,1,\ldots,n-1, \tag{16}$$

are the eigenvalues of A.

Let $f(\lambda)$ *and* $f(A;X)$ *be defined by* (8) *and* (9). *If* (10) *holds then*

$$Y=\left(\operatorname{diag}\left(1,r|\beta_1|^{-1},r^2|\beta_1\beta_2|^{-1},\ldots,r^{n-1}|\beta_1\cdots\beta_{n-1}|^{-1}\right)\right)^2 \tag{17}$$

is a positive definite diagonal solution of $f(A;Y)>0$.

Proof. We adapt a scaling argument of Arcak and Sontag [3]. If we write D in (14) as $D=\operatorname{diag}(\delta_1,\delta_2,\ldots,\delta_n)$ then

$$\frac{\beta_n\delta_n}{\delta_1}=r\epsilon,\qquad \frac{\beta_{j-1}\delta_{j-1}}{\delta_j}=r,\quad j=2,\ldots,n.$$

Hence

$$\begin{aligned}&D^{-1}(PBP^{-1})(PDP^{-1})\\&\quad=D^{-1}\operatorname{diag}(\beta_n,\beta_1,\ldots,\beta_{n-1})\operatorname{diag}(\delta_n,\delta_1,\ldots,\delta_{n-1})\\&\quad=r\operatorname{diag}(\epsilon,1,\ldots,1).\end{aligned}$$

Put

$$P_\epsilon=\begin{pmatrix}0&0&\ldots&0&\epsilon\\1&0&\ldots&0&0\\.&.&\ldots&.&.\\0&0&\ldots&1&0\end{pmatrix}.$$

Then $\operatorname{diag}(\epsilon,1,\ldots,1)P=P_\epsilon$, and $D^{-1}(PB)D=rP_\epsilon$ yields (15). Because of $\epsilon\bar\epsilon=1$ we have $P_\epsilon P_\epsilon^*=I$, and the matrix $\alpha I+rP_\epsilon$ is normal. Note that $\det(zI-P_\epsilon)=z^n-\epsilon$. Hence

$$\zeta_k=e^{\mathrm{i}\left(\frac{\omega}{n}+\frac{2k\pi}{n}\right)},\quad k=0,1,\ldots,n-1,$$

are the eigenvalues of P_ϵ. Therefore the eigenvalues of A are those given by (16). Note that $Y=(DD^*)^{-1}$. Therefore Lemma 3 implies $f(A,Y)>0$. $\square$

3. Diagonal stability

Let $\tilde B=\operatorname{diag}(\tilde\beta_1,\ldots,\tilde\beta_n)$ and $\Theta=\operatorname{diag}(\theta_1,\ldots,\theta_n)$ be complex diagonal matrices such that $\tilde\beta_i\neq 0$ and $\theta_i\neq 0$, $i=1,\ldots,n$. In this section we are concerned with diagonal stability of the matrix

$$\tilde A=\alpha\Theta+P\tilde B=\begin{pmatrix}\alpha\theta_1&0&\ldots&0&\tilde\beta_n\\\tilde\beta_1&\alpha\theta_2&\ddots&&0\\0&\tilde\beta_2&\alpha\theta_3&\ddots&\vdots\\\vdots&\ddots&\ddots&\ddots&0\\0&\ldots&0&\tilde\beta_{n-1}&\alpha\theta_n\end{pmatrix},\quad \alpha\in\mathbb{R}, \tag{18}$$

where P is the permutation matrix in (12). Define

$$r=\left|\frac{\tilde\beta_1\cdots\tilde\beta_n}{\theta_1\cdots\theta_n}\right|^{1/n},$$

and let ϵ, ω and μ be given by

$$\frac{\tilde\beta_1\cdots\tilde\beta_n}{\theta_1\cdots\theta_n}=r^n\epsilon,\quad \epsilon=e^{\mathrm{i}\omega},\ 0\le\omega<2\pi,$$
$$\mu=\min\{\omega,2\pi-\omega\}. \tag{19}$$

Set $A=\tilde A\Theta^{-1}$ and $B=\tilde B\Theta^{-1}$, and let $B=\operatorname{diag}(\beta_1,\ldots,\beta_n)$. Then $A=\alpha I+PB$, and the numbers r and ϵ in (19) and (13) are identical for $\tilde A$ and A. If $\Theta>0$ then it is easy to see that diagonal c-stability of $\tilde A$ is equivalent to diagonal c-stability of A. Similarly, if $\Theta^*\Theta=I$ then diagonal d-stability of $\tilde A$ is equivalent to diagonal d-stability of A.

Theorem 6. *Assume* $\theta_j\in\mathbb{R}$, $\theta_j>0$, $j=1,\ldots,n$. (i) *The matrix*

$$\tilde A=-\Theta+P\tilde B=\begin{pmatrix}-\theta_1&0&\ldots&0&\tilde\beta_n\\\tilde\beta_1&-\theta_2&\ddots&&0\\0&\tilde\beta_2&-\theta_3&\ddots&\vdots\\\vdots&\ddots&\ddots&\ddots&0\\0&\ldots&0&\tilde\beta_{n-1}&-\theta_n\end{pmatrix}$$

is diagonally c-stable if and only if

$$r<\sec\frac{\mu}{n}. \tag{20}$$

(ii) *If* (20) *holds then*

$$\begin{aligned}\tilde X=\ &\operatorname{diag}(\theta_1,\theta_2,\ldots,\theta_n)\Big(\operatorname{diag}\Big(1,r\theta_1|\tilde\beta_1|^{-1},\ldots,\\&r^{n-1}\theta_1\cdots\theta_{n-1}|\tilde\beta_1\cdots\tilde\beta_{n-1}|^{-1}\Big)\Big)^2\end{aligned}$$

is a positive definite diagonal solution of the Lyapunov inequality

$$\tilde A^*X+X\tilde A<0. \tag{21}$$

Proof. (i) Because of $\Theta>0$ it suffices to deal with diagonal c-stability of $A=\tilde A\Theta^{-1}$. We know from Lemma 5 that A is diagonally similar to a normal matrix. To show that c-stability of A is equivalent to (20) we consider the eigenvalues of A. In this case $\alpha=-1$. Hence the eigenvalues are

$$\lambda_k=-1+re^{\mathrm{i}\left(\frac{\omega}{n}+\frac{2k\pi}{n}\right)},\quad k=0,1,\ldots,n-1.$$

Then $\operatorname{Re}\lambda_k=-1+r\cos(\frac{\omega}{n}+\frac{2k\pi}{n})$. Set

$$s(n,\omega)=\max_{k=0,1,\ldots,n-1}\cos\left(\frac{\omega}{n}+\frac{2k\pi}{n}\right).$$

Because of $0\le\omega<2\pi$ we have $s(n,\omega)=\cos\frac{\omega}{n}$ or

$$s(n,\omega)=\cos\left(\frac{\omega}{n}-\frac{2\pi}{n}\right)=\cos\frac{2\pi-\omega}{n}.$$

Therefore

$$s(n,\omega)=\begin{cases}\cos\dfrac{\omega}{n}, & \text{if } \omega\le 2\pi-\omega,\\[2mm]\cos\dfrac{2\pi-\omega}{n}, & \text{if } 2\pi-\omega\le\omega.\end{cases}$$

Hence $\operatorname{Re}\lambda_k<0$, $k=0,\ldots,n-1$, is equivalent to $-1+r\cos\frac{\mu}{n}<0$.

(ii) We have seen that (20) holds if and only if $A=\tilde A\Theta^{-1}$ satisfies (10) with $f(\lambda)=-(\bar\lambda+\lambda)$. The diagonal entries of $B=\Theta^{-1}\tilde B$ are $\beta_j=\tilde\beta_j/\theta_j$. The matrix Y in (17) is a solution of

$$f(A;Y)=-(A^*Y+YA)>0.$$

Therefore $\tilde X=\Theta Y$ satisfies (21). $\square$

If $\tilde{A}$ is the real matrix in (1) then $\tilde{\beta}_1 \cdots \tilde{\beta}_n < 0$ implies $\epsilon = -1$, $\mu = \pi$, and $\sec \frac{\mu}{n} = \sec \frac{\pi}{n}$. On the other hand, $\tilde{\beta}_1 \cdots \tilde{\beta}_n > 0$ implies $\epsilon = 1$, $\mu = 0$, and $\sec \frac{\mu}{n} = 1$. Hence we have recovered Theorems 1 and 2 as special cases of Theorem 6.

We turn to d-stability. Consider

$$x_1(t+1) = \alpha\theta_1 x_1(t) + \tilde{\beta}_n x_n(t),$$
$$x_j(t+1) = \alpha\theta_j x_j(t) + \tilde{\beta}_{j-1} x_j(t), \quad j = 2, \ldots, n, \tag{22}$$

as a cyclically interconnected discrete-time linear system. If $0 < |\alpha\theta_j| < 1, j = 1, \ldots, n$, then the scalar systems $x_j(t+1) = \alpha\theta_j x_j(t)$ are asymptotically stable. In accordance with [7, Section 4.4] the corresponding gain matrix is

$$\Gamma = \begin{pmatrix} 0 & \ldots & 0 & \frac{|\tilde{\beta}_n|}{1-|\alpha\theta_1|} \\ \frac{|\tilde{\beta}_1|}{1-|\alpha\theta_2|} & \ldots & 0 & 0 \\ \cdot & \ldots & \cdot & \cdot \\ 0 & \ldots & \frac{|\tilde{\beta}_{n-1}|}{1-|\alpha\theta_n|} & 0 \end{pmatrix},$$

and

$$\rho(\Gamma) = \left(\frac{|\tilde{\beta}_1| \cdots |\tilde{\beta}_n|}{(1-|\alpha\theta_1|)\cdots(1-|\alpha\theta_n|)} \right)^{1/n}.$$

If $|\theta_1| = \cdots = |\theta_n| = 1$ then $r = \left(|\tilde{\beta}_1| \cdots |\tilde{\beta}_n|\right)^{1/n}$, and the condition $\rho(\Gamma) < 1$ is equivalent to $r < 1 - |\alpha|$. The following result is a counterpart of Theorem 6.

Theorem 7. *Assume*

$$|\theta_j| = 1, \quad j = 1, \ldots, n, \quad \text{and} \quad 0 < \alpha < 1. \tag{23}$$

(i) *The matrix $\tilde{A}$ in (18) is diagonally d-stable if and only if*

$$r < -\alpha \cos \frac{\mu}{n} + \sqrt{1 - \alpha^2 \sin^2 \frac{\mu}{n}}. \tag{24}$$

(ii) *If (24) holds then*

$$\tilde{X} = \left(\operatorname{diag}\left(1, r|\tilde{\beta}_1|^{-1}, \ldots, r^{n-1}|\tilde{\beta}_1 \cdots \tilde{\beta}_{n-1}|^{-1}\right)\right)^2$$

is a positive definite diagonal solution of the Stein inequality

$$X - \tilde{A}^* X \tilde{A} > 0.$$

Proof. (i) Because of $\Theta^*\Theta = I$ we can work with the matrix A in (11) instead of $\tilde{A} = A\Theta$. Let λ_k be an eigenvalue of A. Then (16) implies

$$1 - \bar{\lambda}_k \lambda_k = (1 - \alpha^2) - 2\alpha r \cos\left(\frac{\omega}{n} + \frac{2k\pi}{n}\right) - r^2.$$

Because of $0 < \alpha$ we have $|\lambda_k|^2 - 1 < 0, k = 0, \ldots, n-1$, if and only if

$$r^2 + 2\alpha r \max_{k=0,1,\ldots,n-1} \cos\left(\frac{\omega}{n} + \frac{2k\pi}{n}\right) - (1 - \alpha^2) < 0. \tag{25}$$

Recall $s(n, \omega) = \cos \frac{\mu}{n}$. Set

$$q(r) = r^2 + 2\alpha r \cos \frac{\mu}{n} - (1 - \alpha^2).$$

Then (25) is equivalent to $q(r) < 0$. Because of $\alpha^2 < 1$ the numbers

$$r_1 = -\alpha \cos \frac{\mu}{n} + \sqrt{1 - \alpha^2 \sin^2 \frac{\mu}{n}} \quad \text{and}$$
$$r_2 = -\alpha \cos \frac{\mu}{n} - \sqrt{1 - \alpha^2 \sin^2 \frac{\mu}{n}}$$

are real, and $r_1 > 0$, $r_2 < 0$. Moreover $q(r) = (r - r_1)(r - r_2)$. Hence $q(r) < 0$ is equivalent to $r - r_1 < 0$, i.e. to (24).

(ii) Condition (24) is equivalent to (10) with $f(\lambda) = 1 - \bar{\lambda}\lambda$ and $A = \tilde{A}\Theta^{-1}$. Hence Y in (17) satisfies the inequality

$$f(A; X) = X - A^* X A > 0,$$

and because of $\Theta^*\Theta = I$ also $X - \tilde{A}^* X \tilde{A} > 0$. From $|\theta_j| = 1$ follows $|\beta_j| = |\tilde{\beta}_j|$. Hence $Y = \tilde{X}$. $\square$

For $\alpha \in (0, 1)$ the function $f(\phi) = -\alpha \cos \phi + (1 - \alpha^2 \sin^2 \phi)^{1/2}$ is strictly increasing in $[0, \frac{\pi}{2}]$. This gives bounds for the right-hand side of (24).

Remark 8. Suppose $0 < \alpha < 1$ and $0 \le \mu \le \pi$. Then

$$1 - \alpha \le -\alpha \cos \frac{\mu}{n} + \sqrt{1 - \alpha^2 \sin^2 \frac{\mu}{n}}$$
$$\le -\alpha \cos \frac{\pi}{n} + \sqrt{1 - \alpha^2 \sin^2 \frac{\pi}{n}} \le \sqrt{1 - \alpha^2} < 1.$$

If $\tilde{A}$ is a real matrix then (23) implies $\theta_i = \pm 1$, and therefore we have either $\epsilon = -1$ and $\mu = \pi$, or $\epsilon = 1$ and $\mu = 0$.

Corollary 9. *Suppose $\tilde{A} \in \mathbb{R}^{n \times n}$. Define*

$$r = |\tilde{\beta}_1 \cdots \tilde{\beta}_n|^{1/n} \quad \text{and} \quad \epsilon = \operatorname{sign}(\tilde{\beta}_1 \cdots \tilde{\beta}_n \cdot \theta_1 \cdots \theta_n).$$

Then $\tilde{A}$ is diagonally d-stable if and only if either $\epsilon = -1$ and

$$r < -\alpha \cos \frac{\pi}{n} + \sqrt{1 - \alpha^2 \sin^2 \frac{\pi}{n}} \tag{26}$$

or

$$\epsilon = 1 \quad \text{and} \quad r < 1 - \alpha. \tag{27}$$

We observe that condition (26) is a discrete-time analogue of the secant criterion (3), whereas the condition $r < 1 - \alpha$ in (27) can be obtained from (22) and a corresponding discrete-time small gain theorem.

4. Concluding remarks

In the class of matrices which are diagonally similar to a normal matrix one can study diagonal Hurwitz and Schur stability in the framework of a general matrix inequality. Matrices with cyclic structure belong to this class. The approach of the paper yields extensions of results of Tyson and Othmer [1], Thron [2], and Sontag [6] on the secant condition, and of results of Arcak and Sontag [3] on diagonal stability.

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