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UNMIXED SOLUTIONS OF THE DISCRETE-TIME ALGEBRAIC RICCATI EQUATION*

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Abstract. The algebraic Riccati equation of the optimal control problem associated with the discrete-time system $x(k+1)=Fx(k)+Gu(k)$ is studied. It is shown that in the case of a controllable system, there exist solutions with prescribed unmixed characteristic polynomial of the corresponding closed-loop matrix. Existence of solutions will also be proved under the weaker condition of modulus-controllability. Maximal solutions are discussed.

Key words. discrete-time algebraic Riccati equation, symplectic pencils, unmixed solutions, modulus-controllability, maximal solution

AMS(MOS) subject classifications. 15A24, 93C55

1. Introduction. A solution X of the continuous-time algebraic Riccati equation (CARE)

$$F^*X+XF-XGG^*X-Q=0 \tag{1.1}$$

is called *unmixed* [11] if the closed-loop matrix $F-GG^*X$ and the matrix $-(F-GG^*X)^*$ have only purely imaginary eigenvalues in common. According to Shayman [11], [12], such solutions share properties of maximal and minimal solutions of the CARE (1.1). In this paper, we are concerned with the algebraic Riccati equation that is the discrete-time counterpart of (1.1). We consider the discrete-time algebraic Riccati equation (DARE)

$$\mathfrak{R}(X)=X-F^*XF+F^*XG(I+G^*XG)^{-1}G^*XF-Q=0, \tag{1.2}$$

where F, G, Q are complex matrices of sizes $n\times n$, $n\times p$, and $n\times n$, respectively, and Q is positive semidefinite ($Q\geqq 0$). Only Hermitian matrices that satisfy (1.2) will be regarded as solutions. We prove an existence and uniqueness result for unmixed solutions of (1.2) and discuss maximal solutions. Our approach is based on the associated symplectic pencil

$$M-zL=\begin{pmatrix} F & 0\\ -Q & I\end{pmatrix}-z\begin{pmatrix} I & \Gamma\\ 0 & F^*\end{pmatrix},\qquad \Gamma=GG^*. \tag{1.3}$$

Notation. Let $\sigma(F)$ denote the spectrum of F. We write $|\sigma(F)|=1$, respectively, $|\sigma(F)|\leqq 1$ if all eigenvalues of F lie on the unit circle, respectively, in the closed unit disc. A complex number λ is a *characteristic root* of the pencil $M-zL$ if $\det(M-\lambda L)=0$. Let $g(z)=\prod_{\nu=1}^n(\lambda_\nu-z)$ be a complex polynomial. Put

$$\tilde g(z)=\prod_{\nu=1}^{n}(1-\bar\lambda_\nu z).$$

We call g an *unmixed polynomial* if g and $\tilde g$ have only zeros α in common (if any) with $|\alpha|=1$. In other words, if $g(\lambda)=0$ and $|\lambda|\neq 1$, $\lambda\neq 0$, then $g(\bar\lambda^{-1})\neq 0$. In particular, g is unmixed if all its roots lie in the closed unit disc. We say that X is an *unmixed*

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solution of (1.2) if the characteristic polynomial $\det(F_X - zI)$ of its associated closed-loop matrix

$$F_X = (I+\Gamma X)^{-1}F = F - G(I+G^*XG)^{-1}G^*XF \tag{1.4}$$

is unmixed. Given the pencil (1.3), we see later that it is possible to factorize its determinant into

$$\det(M - zL) = cg(z)\tilde{g}(z), \qquad c \in \mathbb{C}, \tag{1.5}$$

if all unimodular eigenvalues of F are G-controllable. We call (1.5) an *unmixed factorization* if the polynomial g is unmixed.

Let $K = K(F, G) = \operatorname{Im}(G, FG, \cdots, F^{n-1}G)$ be the (F, G)-controllable subspace of $\mathbb{C}^n$. Put $\bar{K} = \mathbb{C}^n/K$. Since the matrix F leaves K invariant, it induces an endomorphism $\bar{F}$ on $\bar{K}$. Define

$$h(z) = \det(zI - \bar{F}). \tag{1.6}$$

With respect to an appropriate basis of $\mathbb{C}^n$, the matrices F and G have the form

$$F = \begin{pmatrix} F_1 & 0 \\ F_{21} & F_2 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix}, \tag{1.7}$$

where the pair (F_2, G_2) is controllable. Then F_1 is a matrix representation of $\bar{F}$, and we have

$$h(z) = \det(zI - F_1). \tag{1.8}$$

In the case of the CARE (1.1), the pair (F, G) is called sign-controllable (see, e.g., [4]) if the polynomials $h(z)$ and $\bar{h}(-z)$ are coprime or, equivalently, if $\operatorname{rank}(F - \lambda I, G) < n$ implies $\operatorname{rank}(F^* + \bar{\lambda} I, G) = n$. The counterpart to sign-controllability in the case of the DARE (1.2) will be described in the following definition.

DEFINITION 1.1. Let the polynomial h be defined as in (1.6) or (1.8). We call the pair (F, G) *modulus-controllable* if $(h, \tilde{h}) = 1$ or, equivalently, if $|\lambda\mu| = 1$ implies $\operatorname{rank}(F - \lambda I, G) = n$ or $\operatorname{rank}(F - \mu I, G) = n$.

If the pair (F, G) is stabilizable, then we have $|\sigma(F_1)| < 1$; hence (F, G) is modulus-controllable.

The main result of the paper is the following theorem.

THEOREM 1.2. *Let*

$$M - zL = \begin{pmatrix} F - zI & -z\Gamma \\ -Q & I - zF^* \end{pmatrix}, \qquad \Gamma = GG^* \tag{1.3$'$}$$

be the pencil associated to the DARE

$$X - F^*XF + F^*XG(I+G^*XG)^{-1}G^*XF - Q = 0 \tag{1.2$'$}$$

and let

$$F_X = (I+\Gamma X)^{-1}F \tag{1.4$'$}$$

be the closed-loop matrix corresponding to the solution X. Assume that (F, G) *is modulus-controllable and let the polynomial h be defined as in* (1.6) *or* (1.8). *Then there exists an unmixed factorization*

$$\det(M - zL) = cg(z)\tilde{g}(z) \tag{1.5$'$}$$

such that

$$(h, \tilde{g}) = 1. \tag{1.9}$$

To each unmixed factorization (1.5)′ *satisfying* (1.9), *there exists a unique solution* X *with* $\det(F_X - zI) = g(z)$.

For the CARE (1.1), a result analogous to Theorem 1.2 is available [13]. In the case of the DARE (1.2)′, stabilizability of (F, G) is the weakest assumption known to guarantee the existence of a solution [3].

2. Reduction to the controllable case. The following lemma, together with the remarks at the end of this section, should make it clear why the concept of modulus-controllability will play an important role for existence and uniqueness of solutions.

LEMMA 2.1. *Assume that* F *and* G *are of the form* (1.7)

$$F = \begin{pmatrix} F_1 & 0 \\ F_{21} & F_2 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix},$$

and put $\Gamma_2 = G_2 G_2^*$ *such that* $\Gamma = \operatorname{diag}(0, \Gamma_2)$. *Let*

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^* & Q_2 \end{pmatrix}$$

be partitioned accordingly. Define

$$M_2 - zL_2 = \begin{pmatrix} F_2 - zI & -z\Gamma_2 \\ -Q_2 & I - zF_2^* \end{pmatrix} \tag{2.1}$$

and put

$$\hat{F}_2 = (I + \Gamma_2 X_2)^{-1} F_2. \tag{2.2}$$

Then

$$\det(M - zL) = \det(F_1 - zI)\det(I - zF_1^*)\det(M_2 - zL_2). \tag{2.3}$$

A matrix

$$X = \begin{pmatrix} X_1 & X_{12} \\ X_{12}^* & X_2 \end{pmatrix} \tag{2.4}$$

is a solution of (1.2) *if and only if it consists of blocks that satisfy the following set of equations*:

$$\mathfrak{R}_2(X_2) = X_2 - F_2^* X_2 F_2 + F_2^* X_2 G_2 (I + G_2^* X_2 G_2)^{-1} G_2^* X_2 F_2 - Q_2 = 0, \tag{2.5a}$$

$$X_{12} - F_1^* X_{12} \hat{F}_2 = B, \tag{2.5b}$$

$$X_1 - F_1^* X_1 F_1 = C, \tag{2.5c}$$

where

$$B = Q_{12} + F_{21}^* X_2 \hat{F}_2 \tag{2.6}$$

and

$$C = F_{21}^* X_{12}^* F_1 + (F_1^* X_{12} + F_{21}^* X_2)[-\Gamma_2 X_{12}^* (I + \Gamma_2 X_2)^{-1} F_1 + \hat{F}_2] + Q_1. \tag{2.7}$$

For the closed-loop matrix associated to (2.4), *we have*

$$\det(zI - F_X) = \det(zI - F_1)\det(zI - \hat{F}_2). \tag{2.8}$$

Proof. The factorization (2.3) is obvious. Using the matrix identity $I - G(I + G^*XG)^{-1}G^*X = (I + \Gamma X)^{-1}$ we can write (1.2) as

$$\mathfrak{R}(X) = X - F^*X(I + \Gamma X)^{-1}F - Q = X - F^*XF_X - Q = 0. \tag{2.9}$$

With the matrices

$$F_X = (I+\Gamma X)^{-1}F = \begin{pmatrix} F_1 & 0 \\ -\Gamma_2 X_{12}(I+\Gamma_2 X_2)^{-1}F_1 + \hat{F}_2 & \hat{F}_2 \end{pmatrix}$$

and

$$F^*X = \begin{pmatrix} F_1^*X_1 + F_{21}^*X_{12} & F_1^*X_{12} + F_{21}^*X_2 \\ F_2^*X_{12}^* & F_2^*X_2 \end{pmatrix}$$

at hand, it is not difficult to verify that (2.9) is equivalent to (2.5) □

Suppose that a basis of $\mathbb{C}^n$ is chosen such that F and G are as in (1.7) and the pair

$$(F_2, G_2) \text{ is controllable.} \tag{2.10}$$

Starting from (2.5a), i.e., from the Riccati equation $\mathfrak{R}_2(X_2) = 0$ that fulfils hypothesis (2.10), a solution X of (1.2) can be obtained in two steps by solving the linear matrix equations (2.5b) and (2.5c). It will be seen that unique solvability of (2.5b) is equivalent to $(\tilde{h}, g) = 1$. Given the blocks X_2 and X_{12}, there exists a unique solution X_1 of (2.5c) if $1 \notin \sigma(F_1^*)\sigma(F_1)$, which is equivalent to the condition $(h, \tilde{h}) = 1$ of modulus-controllability of (F, G).

3. Basic facts of the DARE and the associated symplectic pencil. The pencil $M - zL$ given by (1.3) plays a crucial role in the study of the DARE. Most of the statements of the following lemma are well known. We refer to [9], [3], [8], and [15].

LEMMA 3.1. (i) *Let R and S be nonsingular complex $2n \times 2n$ matrices such that*

$$(M - zL)R = S\begin{pmatrix} \Lambda - zI & -zD \\ 0 & I - z\Lambda^* \end{pmatrix}, \qquad D = D^*. \tag{3.1}$$

Let

$$R = \begin{pmatrix} R_1 & \cdot \\ R_{21} & \cdot \end{pmatrix} \tag{3.2}$$

be partitioned into $n \times n$ blocks. If R_1 is nonsingular and $X = R_{21}R_1^{-1}$ is Hermitian, then $I + \Gamma X$ (and hence $I + G^*XG$) is nonsingular [15] *and X is a solution of* (1.2). *Suppose that* $\det(\Lambda - zI) = g(z)$ *holds; then* (1.5) *holds, and* $F_X = R_1 \Lambda R_1^{-1}$ *implies* $\det(F_X - zI) = g(z)$.

(ii) *Let X be a solution of* (1.2). *Then*

$$(M - zL)\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} I + \Gamma X & 0 \\ F^*X & I \end{pmatrix}\begin{pmatrix} F_X - zI & -zD \\ 0 & I - zF_X^* \end{pmatrix}, \tag{3.3}$$

where $D = D^* = (I + \Gamma X)^{-1}\Gamma = G(I + G^*XG)^{-1}G^*$. Furthermore,

$$\det(M - zL) = c\det(F_X - zI)\det(I - zF_X^*). \tag{3.4}$$

The main feature of $M - zL$ is the relation

$$MJM^* = LJL^*, \tag{3.5}$$

where J is given by $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Property (3.5) characterizes symplectic pencils. Elementary divisors of $M - zL$ corresponding to a characteristic root λ appear in pairs $(\lambda - z)^\nu$ and $(1 - z\bar{\lambda})^\nu$ if $\lambda \neq 0$ and $|\lambda| \neq 1$. A pairing also exists between elementary divisors of the form z^ν and infinite elementary divisors [9]. Hence an unmixed factorization of

$\det(M - zL)$ exists if and only if all unimodular characteristic roots have even algebraic multiplicity. If $\det(M - zL) \neq 0$ whenever $|\alpha| = 1$, then (1.5) is an unmixed factorization if and only if $(g, \tilde{g}) = 1$. In that case, there exist matrices R and S such that (3.1) holds with $D = 0$ and $g(z)$ is the characteristic polynomial of Λ.

4. Characteristic roots on the unit circle. In this section, we see that the proof of Theorem 1.2 can be reduced to the case of a pencil $M - zL$ without unimodular characteristic roots.

Notation. The generalized eigenspace corresponding to an eigenvalue λ of F will be denoted by $E_\lambda(F)$, i.e., $E_\lambda(F) = \operatorname{Ker}(F - \lambda I)^n$. Let

$$V = V(F, Q) = \operatorname{Ker}\begin{pmatrix} Q \\ QF \\ \vdots \\ QF^{n-1} \end{pmatrix}$$

be the weakly unobservable subspace of $\mathbb{C}^n$. For $\lambda \in \sigma(F)$, put

$$V_\lambda = V_\lambda(F, Q) = E_\lambda(F) \cap V.$$

Since V is invariant under F, we have

$$V = \oplus[E_\lambda(F) \cap V], \qquad \lambda \in \sigma(F),$$

and we could define V_λ as the maximal F-invariant subspace of $\mathbb{C}^n$ contained in $E_\lambda(F) \cap \operatorname{Ker} Q$.

In [3] the existence of unimodular characteristic roots of $M - zL$ is related to the rank conditions (4.2) and (4.3) below.

LEMMA 4.1. *Assume that* $|\alpha| = 1$. *Then we have*

$$\det(M - \alpha L) = 0 \tag{4.1}$$

if and only if

$$\operatorname{rank}(F - \alpha I, \Gamma) < n \tag{4.2}$$

or

$$\operatorname{rank}\begin{pmatrix} F - \alpha I \\ Q \end{pmatrix} < n. \tag{4.3}$$

Proof. Let $w \in \mathbb{C}^{2n}$, $w \neq 0$, be such that

$$(M - \alpha L)w = 0. \tag{4.4}$$

Put $w^T = (w_1^T, w_2^T)$. It is easy to see that (4.4) and $|\alpha| = 1$ imply $-w_1^*(I - \alpha F^*) - w_2^*\Gamma = 0$, and $-Qw_1 + (I - \alpha F^*)w_2 = 0$, which yields

$$-w_1^*(I - \alpha F^*)w_2 - w_2^*\Gamma w_2 - w_1^*Qw_1 + w_1^*(I - \alpha F^*)w_2 = 0.$$

From $\Gamma \geqq 0$, $Q \geqq 0$, and $w_2^*\Gamma w_2 + w_1^*Qw_1 = 0$, we obtain $\Gamma w_2 = Qw_1 = 0$. Hence

$$\begin{pmatrix} F^* - \bar{\alpha} I \\ \Gamma \end{pmatrix} w_2 = 0 \tag{4.5}$$

and

$$\begin{pmatrix} F - \alpha I \\ Q \end{pmatrix} w_1 = 0. \tag{4.6}$$

From $w \neq 0$ follows (4.2) or (4.3). That (4.2) or (4.3) implies (4.1) is obvious, since (4.5) and (4.6) yield $(0, w_2^*)(M - zL) = 0$ and

$$(M - zL)\begin{pmatrix} w_1 \\ 0 \end{pmatrix} = 0,$$

respectively. □

LEMMA 4.2. *Let $U = (U_1, U_2)$ be a nonsingular $n \times n$ matrix with $U_i \in \mathbb{C}^{n \times n_i}$, $i = 1, 2$. Then the columns of U_1 form a basis of V_α if and only if*

$$U^{-1}FU = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}, \qquad U^*QU = \operatorname{diag}(0, Q_2), \tag{4.7}$$

$$\sigma(A_1) = \{\alpha\}, \tag{4.8}$$

and A_2, Q_2 are of size $n_2 \times n_2$ and

$$\operatorname{rank}\begin{pmatrix} A_2 - \alpha I \\ Q_2 \end{pmatrix} = n_2. \tag{4.9}$$

Proof. Let $\mathfrak{U}$ denote the set of F-invariant subspaces contained in $E_\alpha(F) \cap \operatorname{Ker} Q$. Then (4.7) and (4.8) are equivalent to span $U_1 \in \mathfrak{U}$. Suppose now that

$$\operatorname{rank}\begin{pmatrix} A_2 - \alpha I \\ Q_2 \end{pmatrix} < n_2. \tag{4.10}$$

Then we have $(A_2 - \alpha I)w = 0$, $Q_2 w = 0$, for some $w \neq 0$. Hence span $(U_1, U_2 w) \in \mathfrak{U}$ and span U_1 is not maximal in $\mathfrak{U}$. To prove the converse, suppose that span $(U_1, y) \in \mathfrak{U}$ for some $y \notin U$. We can assume that $y \in \operatorname{span} U_2$ such that $y = U_2 w$, $w \neq 0$. Then $Qy = 0$ and, therefore, $Q_2 w = 0$. From $Fy = FU\binom{0}{w} = U_2 A_2 w$ follows $Fy \in \operatorname{span} U_2 \cap \operatorname{span}(U_1, y) \cap E_\alpha(F)$. Hence we have $Fy = \alpha y$, which implies $A_2 w = \alpha w$. We have found that

$$\begin{pmatrix} A_2 - \alpha I \\ Q_2 \end{pmatrix} w = 0, \qquad w \neq 0,$$

which yields (4.10). □

The notation U^{-*} below is used for the matrix $(U^*)^{-1}$.

LEMMA 4.3. *Consider an eigenvalue α of F such that $|\alpha| = 1$, $E_\alpha(F) \cap \operatorname{Ker} Q \neq 0$, and*

$$\operatorname{rank}(F - \alpha I, \Gamma) = n. \tag{4.11}$$

Let $U = (U_1, U_2)$ be a nonsingular matrix that transforms F and Q as in (4.7) *and* (4.8). *Let*

$$U^{-1}\Gamma U^{-*} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \Gamma_2 \end{pmatrix} \tag{4.12}$$

be partitioned, conforming to (4.7), *and put*

$$M_2 - zL_2 = \begin{pmatrix} A_2 & 0 \\ -Q_2 & I \end{pmatrix} - z\begin{pmatrix} I & \Gamma_2 \\ 0 & A_2^* \end{pmatrix}. \tag{4.13}$$

Then

$$\operatorname{span} U_1 = V_\alpha \tag{4.14}$$

if and only if

$$\det(M_2 - \alpha L_2) \neq 0. \tag{4.15}$$

Proof. We know from Lemma 4.1 that $\det(M_2-\alpha L_2)=0$ holds if and only if the matrices $(A_2-\alpha I, \Gamma_2)$ and

$$\begin{pmatrix} A_2-\alpha I \\ Q_2 \end{pmatrix}$$

do not both have maximal rank. Now (4.11) and $\Gamma \geqq 0$ imply that rank $(A_2-\alpha I, \Gamma_2)=n_2$. Hence (4.15) is equivalent to (4.9), which, according to the previous lemma, is equivalent to (4.14). □

The following result shows that a unimodular characteristic root α yields a subspace V_α, which lies in the kernel of each solution X.

LEMMA 4.4. *Let α be a characteristic root of $M-zL$ with $|\alpha|=1$, which satisfies* (4.11).

(i) *For each solution X of the* DARE (1.2), *we have $V_\alpha \subseteq \operatorname{Ker} X$. Furthermore, $E_\alpha(F_X)=V_\alpha$ and $F_X=F$ on V_α.*

(ii) *Let $U=(U_1, U_2)$ be nonsingular such that* span $U_1=V_\alpha$, *let the pencil M_2-zL_2 be given as in* (4.13), *and assume that α is not a characteristic root of M_2-zL_2. Put*

$$U^{-1}G=\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}. \tag{4.16}$$

A matrix X is a solution of (1.2) *if and only if*

$$X=U^{-*}\operatorname{diag}(0, X_2)U^{-1}, \tag{4.17}$$

and X_2 is a solution of the DARE

$$X_2-A_2^*X_2A_2-A_2^*X_2G_2(I+G_2^*X_2G_2)^{-1}G_2^*X_2A_2-Q_2=0. \tag{4.18}$$

Proof. (i) According to [14], condition (4.11) yields

$$E_\alpha(F_X)\subseteq E_\alpha(F)\cap \operatorname{Ker} Q\cap \operatorname{Ker} X,$$

and $F_X=F$ on $E_\alpha(F_X)$. Hence $E_\alpha(F_X)\in \mathfrak{U}$. Put $k=\dim E_\alpha(F_X)$. Then (3.4) implies that

$$\det(M-zL)=(z-\alpha)^{2k}b(z), \qquad b(\alpha)\neq 0.$$

Let $U=(U_1, U_2)$ be a matrix as in Lemma 4.2 such that span $U_1=V_\alpha$. Recall $n_1=\dim V_\alpha$. From

$$\det(M-zL)=\det(F_1-zI)\det(I-zF_1^*)\det(M_2-zL_2) \tag{4.19}$$

and Lemma 4.3, we obtain $\det(M-zL)=(z-\alpha)^{2n_1}f(z)$, $f(\alpha)\neq 0$. Hence $k=n_1$ and $E_\alpha(F_X)=V_\alpha$.

(ii) It is easy to verify that each X_2 coming from (4.18) yields a solution X given by (4.17). It is not obvious, however, that under hypotheses (4.11) all solutions of (1.2) should be of the form (4.17). We know from part (i) that $V_\alpha=\operatorname{span} U_1\subseteq \operatorname{Ker} X$. Hence $U^*XU=\operatorname{diag}(0, X_2)$, and X_2 is a solution of (4.18). □

5. Auxiliary results, proof of Theorem 1.2. A matrix R in (3.1) and (3.2) yields a solution of (1.2) only if R_1^{-1} exists. To prove nonsingularity of R_1, we use a result on the discrete-time Lyapunov matrix equation.

LEMMA 5.1. *Let Λ and P be complex $n\times n$ matrices such that*

$$1\notin \sigma(\Lambda^*)\sigma(\Lambda) \tag{5.1}$$

and $P\geqq 0$. If Y is a solution of

$$Y-\Lambda^*Y\Lambda=P, \tag{5.2}$$

then $Y = Y^*$ *and*

$$\operatorname{Ker} Y = V(\Lambda, P) = \operatorname{Ker} \begin{pmatrix} P \\ P\Lambda \\ \vdots \\ P\Lambda^{n-1} \end{pmatrix}.$$

Proof. Choose a basis of $\mathbb{C}^n$ such that

$$V(\Lambda, P) = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, x_1 \in \mathbb{C}^{n_1} \right\}.$$

Then

$$\Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_{12} \\ 0 & \Lambda_2 \end{pmatrix}, \qquad P = \operatorname{diag}(0, P_2),$$

and the pair (Λ_2^*, P_2) is controllable. Because of $1 \notin \sigma(\Lambda_2^*)\sigma(\Lambda_2)$, the equation $Y_2 - \Lambda_2^* Y_2 \Lambda_2 = P_2$, $P_2 \geqq 0$ has a unique Hermitian solution Y_2 that is nonsingular [16]. Then $Y = \operatorname{diag}(0, Y_2)$ is a solution of (5.2) with $\operatorname{Ker} Y = V(\Lambda, P)$ and, because of (5.1), the solution is unique. □

The uniqueness statement of Theorem 1.2 will follow from a result of Willems (see [10, p. 197]).

LEMMA 5.2. *Let* X *and* W *be two solutions of* (1.2); *then*

$$X - W = F_X^*(X - W)F_W. \tag{5.3}$$

Proof. Since [10] seems to be the only reference for (5.3), we include a proof. Recall (2.9) and note that $F = (I + \Gamma W)F_W$ and $F^* = F_X^*(I + X\Gamma)$. Then

$$\begin{aligned} \Re(X) - \Re(W) &= X - W - (F_W^* X F - F^* W F_W) \\ &= X - W - [F_X^* X(I + \Gamma W)F_W - F_X^*(I + X\Gamma)WF_W] \\ &= X - W - F_X^*(X - W)F_W, \end{aligned}$$

which yields (5.3). □

After a reduction to the controllable case that was carried out in § 2, we are able to discard unimodular characteristic roots of $M - zL$. The following lemma will justify such a simplification. Since previous results are extended from V_α to $\oplus\{V_\alpha, |\alpha| = 1\}$, we refer to matrices and equations of the preceding section, making the provision that (4.8) is to be replaced by $|\sigma(A_1)| = 1$.

LEMMA 5.3. *Assume that*

$$|\alpha| = 1 \textit{ implies } \operatorname{rank}(F - \alpha I, \Gamma) = n. \tag{5.4}$$

(i) *There exists a nonsingular matrix* U *such that* (4.7) *holds with* $|\sigma(A_1)| = 1$ *and such that the pencil* $M_2 - zL_2$ *given by* (4.13) *and* (4.12) *has no unimodular characteristic roots.*

(ii) *Put* $f(z) = \det(A_1 - zI)$. *Then* $\det(M - zL) = cg(z)\tilde{g}(z)$ *is an unmixed factorization if and only if*

$$g(z) = f(z)b(z) \tag{5.5}$$

and

$$\det(M_2 - zL_2) = cb(z)\tilde{b}(z), \qquad (b, \tilde{b}) = 1. \tag{5.6}$$

(iii) *A matrix X is a solution of* (1.2) *if and only if it is of the form* (4.17), *where X_2 is a solution of the Riccati equation given by* (4.18) *and* (4.16).

(iv) *A solution X is unmixed with*

$$\det(F_X - zI) = g(z) = f(z)b(z) \tag{5.7}$$

if and only if the matrix X_2 of (4.17) *is an unmixed solution of* (4.18) *such that the closed loop matrix*

$$\Phi_2 = (I + \Gamma_2 X_2)^{-1} A_2 \tag{5.8}$$

satisfies

$$\det(\Phi_2 - zI) = b(z). \tag{5.9}$$

Proof. Parts (i) and (iii) are immediate consequences of Lemmas 4.2 and 4.4. Note that $|\sigma(A_1)| = 1$ is equivalent to $f = \gamma\tilde{f}$, $\gamma \in \mathbb{C}$. Hence (5.5) and (5.6) follow from (4.19). For a solution X of the form (4.17), we have

$$F_X = U\begin{pmatrix} A_1 & \Phi_{12} \\ 0 & \Phi_2 \end{pmatrix} U^{-1},$$

with Φ_2 as in (5.8). Hence $\det(F_X - zI) = f(z)\det(\Phi_2 - zI)$, and (5.7) is equivalent to (5.9). □

THEOREM 5.4. *Assume that all unimodular eigenvalues α of F are G-controllable, i.e., that condition* (5.4) *holds. If* $\det(M - zL) = cg(z)\tilde{g}(z)$ *is an unmixed factorization, then there is at most one solution X of* (1.2) *such that* $\det(F_X - zI) = g(z)$.

Proof. From the preceding lemma we know that the proof involves only a pencil $M_2 - zL_2$ without unimodular characteristic roots. Hence it suffices to prove uniqueness under the assumption $(g, \tilde{g}) = 1$. Suppose that X and W are two solutions such that

$$\det(F_X - zI) = \det(F_W - zI) = g(z).$$

Then $1 \notin \sigma(F_X^*)\sigma(F_W)$, and $\Delta = 0$ is the only solution of $\Delta - F_X^*\Delta F_W = 0$. Thus, according to Lemma 5.2, we have $X - W = 0$. □

THEOREM 5.5. *Suppose that (F, G) is controllable. Then there exists an unmixed factorization of* $\det(M - zL)$. *To each unmixed factorization* $\det(M - zL) = cg(z)\tilde{g}(z)$, *there exists a unique solution X such that*

$$\det(F_X - zI) = g(z). \tag{5.10}$$

Proof. The fact that the controllability hypothesis (5.4) of Lemma 5.3 holds allows us to work with a pencil $M - zL$ that has no characteristic roots of modulus 1 and, accordingly, to proceed under the assumption $(g, \tilde{g}) = 1$. In that case, there exist nonsingular matrices R and S such that

$$(M - zL)R = S\begin{pmatrix} \Lambda - zI & 0 \\ 0 & I - z\Lambda^* \end{pmatrix} \tag{5.11}$$

and $\det(\Lambda - zI) = g(z)$. Let R be partitioned as in (3.2). Then

$$\operatorname{rank}\begin{pmatrix} R_1 \\ R_{21} \end{pmatrix} = n. \tag{5.12}$$

To obtain a solution X in the form $X = R_{21}R_1^{-1}$, we must make sure that R_1 is nonsingular. Put $Y = R_{21}^* R_1$. We want to first show that

$$\operatorname{Ker} Y \subseteq \operatorname{Ker} R_{21} \tag{5.13}$$

holds. Since $R_1x=0$ implies $Yx=0$ and (5.13) yields $R_{21}x=0$, we would obtain $x=0$ from (5.12). Hence as soon as we have established (5.13), we know that R_1 is nonsingular.

The subsequent argument that yields the discrete-time Lyapunov equation (5.16) can be found in [3]. From (5.11) follows

$$\begin{pmatrix} F & 0 \\ -Q & I \end{pmatrix}\begin{pmatrix} R_1 \\ R_{21} \end{pmatrix}=\begin{pmatrix} I & \Gamma \\ 0 & F^* \end{pmatrix}\begin{pmatrix} R_1 \\ R_{21} \end{pmatrix}\Lambda,$$

which is equivalent to the pair of equations

$$FR_1=R_1\Lambda+\Gamma R_{21}\Lambda \tag{5.14}$$

and

$$-R_1^*Q+R_{21}^*=\Lambda^*R_{21}^*F. \tag{5.15}$$

Multiplying (5.14) from the left by $\Lambda^*R_{21}^*$ and (5.15) from the right by R_1 are steps that lead to

$$Y-\Lambda^*Y\Lambda=\Lambda^*R_{21}^*\Gamma R_{21}\Lambda+R_1^*QR_1=P. \tag{5.16}$$

Since $(g, \tilde{g})=1$ is equivalent to (5.1), it follows from Lemma 5.1 that Ker $Y=V(\Lambda, P)$. Hence Ker Y is a Λ-invariant subspace spanned by chains of eigenvectors and generalized eigenvectors of Λ, like $x_1, \cdots, x_k$, which satisfy $\Lambda x_i=\lambda x_i+x_{i-1}$, $i=1, \cdots, k$, $x_0=0$, $x_1\neq 0$, and $Px_i=0$. Induction will show that for such a chain, we have

$$x_j\in \text{Ker } R_{21} \tag{5.17}$$

for $j=0, 1, \cdots, k$. Assume that (5.17) holds for $j=i-1$. Then $Px_i=0$, and $\Gamma\geqq 0$, $Q\geqq 0$ imply that

$$\Gamma R_{21}\Lambda x_i=0 \tag{5.18}$$

and

$$QR_1x_i=0. \tag{5.19}$$

From (5.14) and (5.15) we obtain

$$R_{21}x_i=F^*R_{21}\Lambda x_i. \tag{5.20}$$

In the case where $\lambda=0$, we find that $R_{21}x_i=F^*R_{21}x_{i-1}$ and the induction hypotheses yield $R_{21}x_i=0$. In the case where $\lambda\neq 0$, we conclude from (5.18) and (5.15) that $\Gamma R_{21}x_i=0$ and $R_{21}x_i=F^*R_{21}\lambda x_i$. Hence

$$(R_{21}x_i)^*(\bar{\lambda}^{-1}I-F, \Gamma)=0.$$

In this case, controllability of (F, Γ) implies $R_{21}x_i=0$.

To show that $X=R_{21}R_1^{-1}$ is Hermitian, note that because of (5.1) the matrix $Y=R_{21}^*R_1$ is a unique, and hence Hermitian, solution of (5.2). Therefore $Y=R_1^*R_{21}$ and $X=R_1^{-*}YR_1^{-1}$. Hence, X is also Hermitian.

From (5.14) we obtain $F=(I+\Gamma X)R_1\Lambda R_1^{-1}$ and

$$F_X=R_1\Lambda R_1^{-1}. \tag{5.21}$$

Lemma 3.1(i) tells us that we have found a solution X of (1.2) with the desired property (5.10). By Theorem 5.4 such a solution is unique, which completes the proof. □

From (5.16) and (5.21) follows the equation

$$X - F_X^* X F_X = F_X^* X \Gamma X F_X + Q, \tag{5.22}$$

which leads to inertia results for Riccati equations (see [1]).

Proof of Theorem 1.2. We now perform the construction described at the end of § 2. We assume that F and G are given as in (1.7)

$$F = \begin{pmatrix} F_1 & 0 \\ F_{21} & F_2 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ G_2 \end{pmatrix},$$

such that (F_2, G_2) is controllable, and $h(z) = \det(F_1 - zI)$. Modulus-controllability of (F, G) is equivalent to $(h, \tilde{h}) = 1$. Let $\det(M - zL) = cg(z)\tilde{g}(z)$ be an unmixed factorization that satisfies $(\tilde{h}, g) = 1$. Then (2.3) implies $h \mid g$. Put $f = g/h$ and let $M_2 - zL_2$ be the pencil (2.1). Then $\det(M_2 - zL_2) = cf(z)\tilde{f}(z)$ is an unmixed factorization. Since (2.5a) is a Riccati equation where the pair (F_2, G_2) is controllable, we know from Theorem 5.5 that $\Re_2(X_2) = 0$ has a unique solution X_2 such that $\det(\hat{F}_2 - zI) = f(z)$, where $\hat{F}_2$ is given by (2.2). The solution X_2 enters into the definition of B in (2.6). From $(\tilde{h}, g) = 1$ follows $(\tilde{h}, f) = 1$ or, equivalently, $1 \notin \sigma(F_1^*)\sigma(\hat{F}_2)$. Hence (2.5b) has a unique solution X_{12}. Given X_1 and X_{12}, the matrix C in (2.7) is well defined. Now consider (2.5c). Modulus-controllability amounts to $1 \notin \sigma(F_1^*)\sigma(F_1)$. Hence (2.5c) determines X_1 uniquely. The block matrix (2.4) is a solution of (1.2). From (2.8) we obtain $\det(F_X - zI) = h(z)f(z) = g(z)$, and X is the only solution with that property. □

6. Maximal solutions. It is known that (1.2) has a solution if the pair (F, G) is stabilizable [3]. In that case [7], there exists a maximal solution X with the properties $X \geqq 0$ and

$$|\sigma(F_X)| \leqq 1. \tag{6.1}$$

In this section we focus on property (6.1) and its relation with maximality. As a stabilizable pair, (F, G) is necessarily modulus-controllable; the following result is a special case of Theorem 5.5. The existence statement in the subsequent theorem can be found in [3].

THEOREM 6.1. *If* (F, Γ) *is stabilizable, then there exists a unique solution* X *of* (1.2) *such that* $|\sigma(F_X)| \leqq 1$.

A solution X is called *maximal* if $X - W \geqq 0$ holds for all solutions W of (1.2). We see that (6.1) is equivalent to maximality of X, provided that the standing assumption (5.4) holds. Two auxiliary results will be needed.

LEMMA 6.2 (see [2]). *Let* X *and* W *be two solutions of* (1.2). *Then* $\Delta = X - W$ *satisfies the equation*

$$\Delta - F_X^* \Delta F_X = F_X^* \Delta G (I + G^* W G)^{-1} G^* \Delta F_X. \tag{6.2}$$

LEMMA 6.3. *If* X *and* W *are two solutions of* (1.2), *then*

$$\operatorname{In}(I + G^* X G) = \operatorname{In}(I + G^* W G). \tag{6.3}$$

Proof. Relation (6.3) appears in [6] where (1.2) is approached by factorization results of matrices of rational functions under the hypotheses that (F, G) is controllable and $|\sigma(F)| < 1$. Here we use the pencil $M - zL$. It is easy to verify that (3.3) implies that

$$(I \quad 0)(M - zL)^{-1}\begin{pmatrix} 0 \\ I \end{pmatrix} = z(F_X - zI)^{-1}(I + \Gamma X)^{-1}\Gamma(I - zF_X^*)^{-1}. \tag{6.4}$$

Note that $(I+\Gamma X)^{-1}\Gamma = G(I+G^*XG)^{-1}G^*$. Consider (6.4) and the corresponding expression for the solution W and take $z=\alpha$, where $|\alpha|=1$ and $\det(M-zL)\neq 0$. If the symbol $\sim$ denotes congruence, then

$$G(I+G^*XG)^{-1}G \sim G(I+G^*WG)^{-1}G. \tag{6.5}$$

It is not difficult to show that (6.5) implies (6.3). □

THEOREM 6.4. *Assume that* $\operatorname{rank}(F-\alpha I, \Gamma)=n$ *for all* α *with* $|\alpha|=1$. *If* X *is a solution of* (1.2) *that satisfies* $|\sigma(F_X)|\leqq 1$, *then* X *is a maximal solution.*

Proof. According to Lemma 5.3, each solution X of (1.2) is of the form $X = U^{-*}\operatorname{diag}(0, X_2)U^{-1}$, where X_2 is a solution of a Riccati equation whose associated pencil $M_2 - zL_2$ has no unimodular characteristic roots. Hence we can assume for the proof that X is a solution with the property

$$|\sigma(F_X)|<1. \tag{6.6}$$

It is a known application of (5.22) that (6.6) implies $X\geqq 0$. Therefore $I+G^*XG>0$, and by the preceding lemma we have $I+G^*WG>0$ for all solutions W. Put $\Delta = X - W$ and define $S = F_X^*\Delta G(I+G^*WG)^{-1}G^*\Delta F_X$. Then $\Delta - F_X^*\Delta F_X = S$ is (6.2). From (6.6) and $S\geqq 0$ follows $\Delta\geqq 0$; hence X is a maximal solution. □

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