Rellich's Perturbation Theorem on Hermitian Matrices of Holomorphic Functions

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The principal axis theorem has a remarkable generalization in the perturbation theory of linear operators. Rellich [5] proved that a symmetric operator $A(\varepsilon)$ which depends analytically on a real parameter $\varepsilon$ has an orthonormal basis of eigenvectors depending also analytically on $\varepsilon$. In this note we give a short proof of Rellich's theorem based on the fact that the ring $H(\Omega)$ of complex functions which are holomorphic in a region $\Omega$ is an elementary divisor domain.

Let $J$ be a real interval and let $A$ denote the functions which are holomorphic on $J$, $A = \bigcup \{H(\Omega), J \subset \Omega\}$. For $W = (w_{ij}(z)) \in A^{k \times n}$ define

$$W* := (\overline{w_{ji}(\overline{z})}).$$

We call a matrix $M \in A^{n \times n}$ hermitian if $M = M^*$ holds.

**Theorem** (Rellich [5, pp. 33–34; 4, p. 122]). If $M \in A^{n \times n}$ is hermitian then there exists a $U \in A^{n \times n}$ and functions $\mu_i \in A, i = 1, 2, \ldots, n$, such that

$$M(z) = U^*(z) \text{diag}(\mu_1(z), \ldots, \mu_n(z)) U(z),$$

$$U^*(z) U(z) = I$$

hold for all $z$ in some region $\Omega$ with $J \subset \Omega$.

We recall that for any region $\Omega$ the ring $H(\Omega)$ is an elementary divisor domain ([1, 2, 3], see also [6] for a proof). This means that for a matrix $Q \in H(\Omega)^{n_i \times n_2}$ of rank $r$ there exist unimodular matrices $F_i \in H(\Omega)^{n_i \times n_i}, i = 1, 2$, i.e., matrices $F_i$ which are invertible in $H(\Omega)^{n_i \times n_i}$, such that

$$F_1 Q F_2 = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

$$D = \text{diag}(d_1, \ldots, d_r), \quad d_i, d_{i+1}, \quad i = 1, \ldots, r - 1.$$
Note the following two special cases:

(a) If the elements of a vector \( v \in H(\Omega)^n \), \( v \neq 0 \), have no common zeros in \( \Omega \), then there exists a unimodular matrix \( R \in H(\Omega)^{n \times n} \) such that

\[
Rv = (1, 0, \ldots, 0)^T.
\]

(b) If the column rank of \( Q \in H(\Omega)^{k \times n} \) is not maximal then there exists an \( h \in H(\Omega)^n \), such that \( h(z) \neq 0 \) for \( z \in \Omega \) and \( Qh = 0 \). The observation is not trivial, as \( H(\Omega) \) is not a field.

For a set \( B \subseteq \mathbb{C} \) let \( \bar{B} = \{ \bar{z} \mid z \in B \} \) denote its reflection at the real axis. We observe that for an \( f \in H(\Omega) \) the function \( \overline{f(z)} \) is in general not holomorphic in \( \Omega \). However the function of \( z \) with values \( f(z) \) is holomorphic in \( \Omega \). Thus if \( \Omega = \overline{\Omega} \) then for \( M \in H(\Omega)^{k \times n} \) the matrix \( M^* \) is also defined on \( \Omega \) and \( M^* \in H(\Omega)^{n \times k} \). In the sequel \( \Omega \) shall always be a region with \( \Omega = \overline{\Omega} \) and \( J \subseteq \Omega \). Since \( M \in A^{k \times n} \) implies \( M \in H(\Omega)^{k \times n} \) for a suitable \( \Omega \), the ring \( A \) is also an elementary divisor domain.

Two lemmas will be needed.

**Lemma 1** [5, p. 31]. The eigenvalues \( \mu_i \) of a hermitian matrix \( M \in A^{n \times n} \) can be arranged in such a way that they are holomorphic on \( J \). Hence the characteristic polynomial of \( M \) can be factored as

\[
\det(\lambda I - M) = \prod_{i=1}^{n} (\lambda - \mu_i) \tag{2}
\]

with \( \mu_i \in A \), \( i = 1, \ldots, n \).

**Lemma 2.** Let \( v \in H(\Omega)^n \) be a vector which is normed in \( \Omega \), i.e.,

\[
v^*(z) v(z) = 1 \quad \text{for} \quad z \in \Omega. \tag{3}
\]

Then there exists a matrix \( V \in H(\Omega)^{n \times n} \) with first column \( v \) and

\[
V(z)^* V(z) = I \quad \text{for} \quad z \in \Omega. \tag{4}
\]

**Proof.** Because of (3) the elements of \( v \) have no common zero in \( \Omega \) and rank \( v = 1 \). From (a) we have a unimodular matrix \( Y \in H(\Omega)^{n \times n} \) such that \( v^* Y = (1, 0, \ldots, 0) \). Let \( y_i \) denote the \( i \)th column of \( Y \). Because of \( y_i(z) \neq 0 \) for \( z \in \Omega \) we can apply an orthonormalization process and assume \( y_i(z)^* y_k(z) = \delta_{ik} \) for \( z \in \Omega \) and \( i, k = 2, \ldots, n \). Then \( V = (v, y_2, \ldots, y_n) \) has the desired properties.

The diagonalization result (1) follows now readily from the preceding lemmas.
Proof of Rellich's Theorem. Let \( \det(\lambda I - M) \) be given by (2) and let \( \Omega \) be such that \( M \in H(\Omega)^{n \times n} \) and \( \mu_i \in H(\Omega), \ i = 1, \ldots, n \). Then \( L := M - \mu_i I \in H(\Omega)^{n \times n} \). According to (b) there exists a \( v_1 \in H(\Omega)^n \) such that \( Lv_1 = 0 \) and \( v_1(z)^* v_1(z) = 1 \) for \( z \in \Omega \), and because of Lemma 2 we find a matrix \( V = (v_1, v_2, \ldots, v_n) \in H(\Omega)^{n \times n} \) which satisfies (4). Hence

\[
V(z)^* L(z) V(z) = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & L_2(z)
\end{pmatrix}
\]

and \( L_2 \in H^{(n-1) \times (n-1)}(\Omega), L_2^* = L_2 \). An induction argument completes the proof.  

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References