The Algebraic Riccati Equation:
Conditions for the Existence and Uniqueness of Solutions

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ABSTRACT

Unmixed solutions of the matrix equation $XDX + XA + AX^* - C = 0$, $D > 0$ are studied.

1. INTRODUCTION, BACKGROUND

Let $A$, $C$, $D$, and $X$ be complex $n \times n$ matrices such that $C$, $D$, and $X$ are hermitian and $D \geq 0$ (positive semidefinite). The algebraic Riccati equation

$$XDX + XA + AX^* - C = 0 \quad (1.1)$$

is important in control theory. The optimal control in the quadratic regulator problem on the infinite time interval requires a solution $X$ of (1.1) such that the “feedback matrix” $A + DX$ is stable (see e.g. [9]). In this note we study the more general situation where $A + DX$ and $(A + DX)^*$ have at most pure imaginary eigenvalues in common. It is known from [3] and [8] that the existence of solutions $X$ with this property depends on those eigenvalues $\lambda$ of $A$ which are not $D$-controllable [i.e. for which $\text{rank}(A - \lambda I, D) < n$]. It is standard to associate to (1.1) the hamiltonian matrix

$$M = \begin{pmatrix} A & D \\ C & -A^* \end{pmatrix}.$$
If $X$ is a solution then

$$
\begin{pmatrix}
I & 0 \\
X & I
\end{pmatrix}^{-1}
M
\begin{pmatrix}
I & 0 \\
X & I
\end{pmatrix}
= 
\begin{pmatrix}
A + DX & D \\
0 & -(A + DX)^*
\end{pmatrix}.
$$

Hence any solution $X$ yields a factorization of the characteristic polynomial $\chi(M)$ of $M$ as

$$
\chi(M) = (-1)^n \chi(A + DX) \chi[(A + DX)^*],
$$

which indicates that the structure of $M$ is crucial. We will relate the solvability of (1.1) to factorizations of the rational matrix

$$
(I_n, 0)(zI - M)^{-1}(I_n)
$$

and also establish a link between results in [1], [3], [6], and [8]. As an example of the type of results obtained in Section 3 we mention the following: There exists a unique solution $X_+$ of (1.1) such that all eigenvalues of $A + DX_+$ have nonnegative real part if and only if all elementary divisors corresponding to pure imaginary eigenvalues of $M$ have even degree and $\text{rank}(A - \mu I, D) = n$ for all $\mu$ with $\text{Re} \mu \leq 0$ (i.e. the pair $\{-A, D\}$ is stabilizable).

Some of the auxiliary results which we put together in this section are contained in [8]. For concepts and facts from systems theory we refer to [9].

For a complex polynomial $p(z) = \sum_{r=0}^{n} a_r z^r$ let $\tilde{p}$ be defined by

$$
\tilde{p}(z) = \sum_{r=0}^{n} a_r (-z)^r = \tilde{p}(-z).
$$

If we put $q(z) = \chi(A + DX)$ for the characteristic polynomial of $A + DX$, then (1.2) can be written as $\chi(M) = (-1)^n q(z) \tilde{q}(z)$. For an $n \times n$ matrix $H = (h_{ij})$ of complex rational functions we define $\hat{H}$ by

$$
\hat{H}(z) = (\hat{h}_{ij}(-z)).
$$

By $\delta(H)$ we denote the least common denominator of all minors of $H$.

Let $S \in \mathbb{C}^{n \times n}$ be nonsingular, and put

$$
\hat{A} = SAS^{-1}, \quad \hat{C} = (S^*)^{-1}CS, \quad \hat{D} = SDS^*, \quad \hat{X} = (S^*)^{-1}XS^{-1}.
$$

(1.3)
Then $X$ is a solution of (1.1) if and only if $\dot{X}$ is a solution of

$$
\dot{X}D\dot{X} + \dot{X}A + \dot{A}^*\dot{X} - \dot{C} = 0.
$$

(1.4)

The hamiltonian matrix

$$
\hat{M} = \begin{pmatrix}
\hat{A} & \hat{D} \\
\hat{C} & -\hat{A}^*
\end{pmatrix}
$$

of (1.4) is related to $M$ by $\dot{M} = Z^{-1}MZ$, $Z = \text{block diag}(S^{-1}, S^*)$.

Let a pair $\{A, B\}$ of matrices be given, $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times r}$. An eigenvalue $\lambda$ of $A$ is called $B$-controllable if

$$
\text{rank}(A - \lambda I, B) = n. \quad (1.5)
$$

The pair $\{A, B\}$ is called stabilizable if $\text{rank}(A - \lambda I, B) < n$ implies $\text{Re} \lambda < 0$. $\{A, B\}$ is called controllable if (1.5) holds for all eigenvalues $\lambda$ of $A$. Let $C(A, B)$ denote the $(A, B)$-controllable subspace of $\mathbb{C}^n$,

$$
C(A, B) = \text{Im}(B, AB, \ldots, A^{n-1}B).
$$

Then $\{A, B\}$ is controllable if and only if $\dim C(A, B) = n$, or equivalently if

$$
\text{row rank}_C(A - zI)^{-1}B = n.
$$

The matrix $S$ in (1.3) can be chosen such that

$$
\hat{A} = \begin{pmatrix}
A_1 & 0 \\
A_2 & A_2
\end{pmatrix}, \quad \hat{D} = \begin{pmatrix}
0 & 0 \\
0 & D_2
\end{pmatrix}, \quad (1.6a)
$$

where

$$
D_2 \geq 0 \quad \text{and} \quad \{A_2, D_2\} \text{ is controllable.} \quad (1.6b)
$$

Put

$$
h: = \chi(A_1). \quad (1.6c)
$$

Since $C(A, D)$ is invariant under $A$, the polynomial $h$ can be defined
independently of (1.6a) by

\[ X(A) = hX(A|_{C(A, D)}). \quad (1.7) \]

An eigenvalue \( \lambda \) of \( A \) is not \( D \)-controllable if and only if it is a zero of \( h \). Let \( M_2 \) be given by

\[ M_2 = \begin{pmatrix} A_2 & D_2 \\ C_2 & -A_2^* \end{pmatrix}. \quad (1.8) \]

Then

\[ X(M) = (-1)^i hX(M_2). \quad (1.9) \]

A matrix \( W = (w_{ij}) \in \mathbb{C}^{n \times k}(z) \) is said to be proper rational if \( w_{ij} = f_{ij}/g_{ij} \) such that \( f_{ij} = 0 \) or \( \deg f_{ij} < \deg g_{ij} \). A factorization

\[ W(z) = L(zI - F)^{-1}K \quad (1.10) \]

with \( L \in \mathbb{C}^{n \times r}, F \in \mathbb{C}^{r \times r}, \) and \( K \in \mathbb{C}^{r \times k} \) is called a realization of \( W \). (1.10) is a minimal realization (i.e., the size \( r \) of \( F \) is minimal) if and only if the pairs \( \{F, K\} \) and \( \{F^T, L^T\} \) are controllable, or equivalently if \( X(F) = \delta(W) \). If \( L_i(zI - F_i)^{-1}K_i, i = 1,2, \) are two minimal realizations, then \( F_1 \) and \( F_2 \) are similar.

The solvability of (1.1) will be related to factorizations of the \( n \times n \) matrix

\[ T(z) = (I_n \ 0)(zI - M)^{-1} \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \quad (1.11) \]

**Lemma 1.1.** The following statements are equivalent.

(i) The pair \( \{A, D\} \) is controllable.

(ii) The realization (1.11) of \( T \) is minimal.

(iii) The rows and columns of \( T \) are linearly independent over \( \mathbb{C} \).

**Proof.** (i) \( \iff \) (iii): Suppose \( Tq = 0 \) for some nonzero \( q \in \mathbb{C}^n \). Let

\[ (zI - M)^{-1} = \begin{pmatrix} R & T \\ S & U \end{pmatrix} \quad (1.12) \]
be partitioned into \( n \times n \) blocks. Then \((zI - A)T - DU = 0\) and \(-CT + (zI + A^*)U = I\) imply \(DUq = 0\) and \((zI + A^*)Uq = q\). Hence

\[
q^*(zI + A)^{-1}D = 0, \tag{1.13}
\]

which means that \(\{A, D\}\) is not controllable. Conversely (1.13) implies \(Tq = 0\). Because of \(\tilde{T}(z) = T(-z)\), the first part of the proof is complete.

(i) \(\Leftrightarrow\) (ii): If (1.11) is not a minimal realization, then

\[
\text{rank}\left(\lambda I - M, \begin{pmatrix} 0 \\ I \end{pmatrix}\right) < 2n
\]

for some \(\lambda \in \mathbb{C}\), or equivalently

\[
\text{rank}(A - \lambda I, D) < n.
\]

The following fact about pure imaginary eigenvalues will be used.

**Lemma 1.3 [6].** Let the pair \(\{F, K\}\) be controllable. Then the elementary divisors of

\[
\begin{pmatrix} F & KK^* \\ 0 & -F^* \end{pmatrix}
\]

which belong to pure imaginary eigenvalues have even degree.

Without loss of generality we will assume that \(A\) and \(D\) are given in the form (1.6). In the sequel \(q \in \mathbb{C}[z]\) will always be a monic polynomial of degree \(n\) which has at most pure imaginary zeros in common with \(\tilde{q}\).

2. REDUCTION TO THE CONTROLLABLE CASE

If \(\{A, D\}\) is not controllable, then (1.1) decomposes into one quadratic and two linear matrix equations (2.1). Let

\[
X = \begin{pmatrix} X_1 & X_{12} \\ X_{12}^* & X_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 & C_{12} \\ C_{12}^* & C_2 \end{pmatrix}
\]
be partitioned according to (1.6). Write (1.1) as

\begin{align}
X_1 A_1 + A_1^* X_1 &= C_1 - X_{12} D_2 X_{12}^*, \\
X_{12}(A_2 + D_2 X_2) + A_1^* X_{12} &= C_{12} - A_{21}^* X_2^*, \\
X_2 D_2 X_2 + X_2 A_2 + A_2^* X_2 - C_2 &= 0.
\end{align}

(2.1a) (2.1b) (2.1c)

The blocks \(X_1\) and \(X_{12}\) have no influence on

\[ \chi(A + DX) = \chi(A_1) \chi(A_2 + D_2 X_2). \]  

(2.2)

**Lemma 2.1.**

(a) The equation (1.1) has a unique solution with \(\chi(A + DX) = q\) if and only if

\[ \chi(M) = (-1)^n q\tilde{q} \quad \text{and} \quad (h, \tilde{q}) = 1 \]  

(2.3)

hold and (2.1c) has a unique solution with \(\chi(A_2 + D_2 X_2) = q_2, q_2 = q/h\).

(b) If (2.3) holds, then \(h\) has no pure imaginary zeros and the elementary divisors which belong to pure imaginary eigenvalues are the same in \(M\) and \(M_{12}^*\).

**Proof.** (a): Let \(X\) be a unique solution with \(\chi(A + DX) = q\). Then the solution \(X_1\) of the Lyapunov equation (2.1a) is unique. Hence (see e.g. [5]) the matrices \(A_1\) and \(-A_1^*\) have no common eigenvalue, i.e., \(h = \chi(A_1)\) and \(\tilde{h} = (-1)^n \chi(-A_1^*)\) have no common root. Similarly we conclude from (2.1b) that \(A_2 + D_2 X_2\) and \(-A_1^*\) have no eigenvalues in common or \((h, \tilde{q}_2) = 1\). Therefore \((h, h\tilde{q}_2 = q) = 1\), which combined with (1.2) yields (2.3). The preceding arguments can be used to prove the converse statement of (a).

(b): From (2.2) we obtain \(h|\tilde{q}\). Suppose \(h\) had a pure imaginary root \(i\alpha\). Then \(q(i\alpha) = \tilde{q}(i\alpha) = 0\), which contradicts \((h, \tilde{q}) = 1\). Hence \(A_1\) and \(-A_1^*\) have no eigenvalues on the imaginary axis. The pure imaginary eigenvalues of \(M\) and their elementary divisors are determined by \(M_{12}\).

Recommending arguments can be used to prove the converse statement of (a).

The condition \((h, \tilde{q}) = 1\) can be expressed in a different form. As the roots of \(h\) are precisely those eigenvalues which are not \(D\)-controllable, the following fact is obvious.

**Remark.** Let \(\chi(M)\) be factored as \(\chi(M) = (-1)^n q\tilde{q}\). Then \((h, \tilde{q}) = 1\) if and only if all eigenvalues of \(A\) which are zeros of \(\tilde{q}\) (if any) are \(D\)-controllable.
The next lemma shows that for factorizations of $T$ only the part $M_2$ of $M$ is essential. Let $n_2$ be the size of $A_2$ in (1.6), or equivalently $n_2 = \dim C(A, D)$.

**Lemma 2.2.** The matrix

\[ T(z) = (I - zI - M)^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \]

can be factorized as

\[ T = -V \tilde{V}, \quad V \in \mathbb{C}^{n \times n}(z) \]

(2.4)

such that

\[ \delta(V) = q_2 = q/h \]

(2.5)

and

\[ \text{row rank}_C V = n_2 \]

(2.6)

if and only if

\[ T_2(z) = (I_{n_2} - zI - M_2)^{-1} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} \]

(2.7)

has a factorization

\[ T_2 = -V_2 \tilde{V}_2, \quad V_2 \in \mathbb{C}^{n_2 \times n_2}(z) \]

(2.8)

with

\[ \delta(V_2) = q_2 \quad \text{and} \quad \text{row rank}_C V_2 = n_2. \]

**Proof.** Let $T$ be partitioned according to (1.6) into

\[ T = \begin{pmatrix} T_1 & T_{12} \\ T_{21} & T_2 \end{pmatrix}. \]

(2.9)
Then (1.11), i.e.

\[
(zI - M) \begin{pmatrix} R & T \\ S & U \end{pmatrix} = I,
\]

yields \((zI - A_1)(T_1, T_{12}) = (0, 0)\). Hence \(T = \text{block diag}(0, T_2)\) and (2.7) holds. Clearly (2.8) implies (2.4). We assume now (2.4)–(2.6). Then there exists a nonsingular \(K \in \mathbb{C}^{n \times n}\) such that

\[
LV = \begin{pmatrix} 0 \\ W_2 \end{pmatrix}, \quad W_2 \in \mathbb{C}^{n_2 \times n}(z),
\]

row rank \(C_{W_2} = n_2\), and \(\delta(W_2) = \delta(V)\).

Obviously we have \(W_2 \bar{W}_2 = QQ^*, \delta(Q) = \delta(W_2)\) for a suitable \(Q \in \mathbb{C}^{n_2 \times n_2}(z)\). If \(L\) is partitioned as \(T\) in (2.9), then

\[
L \begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -W_2 \bar{W}_2 \end{pmatrix} (L^*)^{-1}
\]

yields \(L_{12} T_2 = 0\). As \(\{A_2, D_2\}\) is controllable, Lemma 1.1 implies \(L_{12} = 0\). Therefore \(T_2 = -L_2^{-1}W_2 \bar{W}_2 (L^{-1})^*\) or \(T_2 = -V_2 \bar{V}_2\) with \(V_2 = L_2^{-1}Q\).

### 3. Existence and Uniqueness

The next theorem contains the main result of this note. Recall that \(h\) is given by (1.6c) and that we assume \(D \geq 0\). If a complex polynomial is denoted by \(q\), it has degree \(n\) and by assumption \(q(z)\) and \(\bar{q}(z) = \bar{q}(-z)\) have at most pure imaginary roots in common. We write \(\chi(G)\) for the characteristic polynomial of a matrix \(G\).

**Theorem 3.1.** Let the following conditions be defined:

\(\text{(EU)}_{M,q}\). There exists a unique hermitian solution \(X\) of

\[
XD X + XA + A^*X - C = 0 \tag{1.1}
\]

such that

\[
\chi(A + DX) = q.
\]
(LR)_M. The elementary divisors of
\[ M = \begin{pmatrix} A & D \\ C & -A^* \end{pmatrix} \]
which belong to pure imaginary eigenvalues have even degree.

(Co)_{M,q} \quad \chi(M) = (-1)^n q\bar{q} \quad \text{and} \quad (h, \bar{q}) = 1.

(F)_{M,q} \quad The matrix
\[ T(z) = (I_n \quad 0)(zI - M)^{-1}\begin{pmatrix} 0 \\ I_n \end{pmatrix} \]
can be factorized as
\[ T = -V\bar{V}, \quad V \in \mathbb{C}^{n \times n}(z) \tag{3.1} \]
such that
\[ \delta(V) = q_2 = q/h, \quad \text{row rank}_C V = \deg q_2. \]

Then
\[ (a) \quad (Co)_{M,q} \land (LR)_{M} \Leftrightarrow (EU)_{M,q}, \]
\[ (b) \quad (Co)_{M,q} \land (F)_{M,q} \Leftrightarrow (EU)_{M,q}. \]

Proof. (a): If \( \{A, D\} \) is controllable then \( (Co)_{M,q} \) means \( \chi(M) = (-1)^n q\bar{q} \). In this case (a) is true [8]. Otherwise Lemma 2.1 implies
\[ (EU)_{M,q} \Leftrightarrow (Co)_{M,q} \land (EU)_{M_2,q_2} \tag{3.2} \]
and
\[ (Co)_{M,q} \land (LR)_{M} \Leftrightarrow (Co)_{M,q} \land (LR)_{M_2} \]
with \( q_2 = q/h \) and \( M_2 \) given by (1.8). With \( \{A_2, D_2\} \) we are back at the controllable case.

(b): From Lemma 2.2 we obtain \( (F)_{M,q} \Leftrightarrow (F)_{M_2,q_2} \). Taking (3.2) into account, it suffices to prove (b) for a controllable pair \( \{A, D\} \). Assume first that there exists a solution \( \bar{X} \) with \( \chi(A + DX) = q \). Put
\[ G(z) = zI - (A + DX) \quad \text{and} \quad R = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}. \]
Then
\[
\begin{pmatrix} R^{-1}(zI-M)R \end{pmatrix}^{-1} = \begin{pmatrix} \vdots & -G^{-1}DG^{-1} \end{pmatrix}
\]
or \( T = -G^{-1}DG^{-1} = -VV^T \) with \( V = G^{-1}D^{1/2} \). Together with \( \{ A, D \} \), also the pair \( \{ A + DX, D^{1/2} \} \) is controllable; hence
row rank \( c \) \( V = n \).

Since no poles of \( G^{-1} \) are canceled in \( G^{-1}D^{1/2} \), we have \( \delta(V) = q \). Conversely let us assume now (3.1), \( \delta(V) = q \), and (3.3). The matrix \( V \) is proper rational. Let \( L(zI - F)^{-1}K = V(z) \) be a minimal realization of \( V \). Then \( \chi(F) = \delta(V) = q \), and \( \{ F, K \} \) is controllable. Because of (3.3) we can take \( L = I \).

According to Lemma 1.1 the two realizations
\[
T(z) = (I \ 0)(zI - M)^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} = (I \ 0) \begin{pmatrix} zI - F & -KK^* \\ 0 & zI + F^* \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix}
\]
are minimal. Hence \( M \) is similar to
\[
\begin{pmatrix} F & KK \\ 0 & -F^* \end{pmatrix}.
\]
Therefore \( \chi(M) = (-1)^n q \bar{q} \), and from Lemma 1.3 follows \( (LR)_M \). Now (a) yields \( (EU)_{M, q} \).

When can the condition (Co) be satisfied?

**Theorem 3.2.** There exists a \( q \in \mathbb{C}[z] \) such that

\[ (Co)_{M, q} \quad \chi(M) = (-1)^n q \bar{q}, \quad \text{and} \quad (h, \bar{q}) = 1 \]

hold, if and only if

(\( \alpha \)) all pure imaginary eigenvalues of \( M \) have even algebraic multiplicity and

(\( \beta \)) \( (h, \bar{h}) = 1 \).

The following condition (Ch) is equivalent to (\( \beta \)).
(Ch) If $\lambda$ is an eigenvalue of $A$ which is not $D$-controllable, then

$$\text{rank}(A^* + \lambda I, D) = n.$$  

Proof. We assume first (Co)$_M$. Then (1.9) implies $h|q$ and $\bar{h}|\bar{q}$. As $q$ and $\bar{q}$ have at most pure imaginary roots in common, the same has to be true for $h$ and $\bar{h}$. But $h(ia) = 0$, $a \in \mathbb{R}$, is impossible, since it would imply $q(ia) = \bar{q}(ia) = 0$, which contradicts $(h, \bar{q}) = 1$. Thus $(h, \bar{h}) = 1$. For the converse note that (a) and (b) yield $\chi(M) = (-1)^n h\bar{h}ss$, where $s$ can be chosen such that $(h, s) = 1$. Then $q = hs$ has the desired properties. The equivalence of (b) and (Ch) follows from the definition of $h$. \hfill \blacksquare

We note the special case of Theorem 3.1(a) where all zeros of $h$ are in the right half plane, i.e., where $\{-A, D\}$ is stabilizable.

Corollary 3.3. There exists a unique solution $X_+$ of (1.1) such that all eigenvalues of $A + DX_+$ have nonnegative real part if and only if $\{-A, D\}$ is stabilizable and (LR)$_M$ holds.

It follows from [4] that $X_+$ is a maximal solution, i.e., $X_+ - X > 0$ for any solution $X$ of (1.1).

The condition (Co) is contained in [3], (LR) can be found in [2] and [5], and (Ch) is in [1]. For different results on spectral factorizations see [7].

I would like to thank referees for drawing my attention to the results in [1] and [2] and for valuable comments.

Addendum. We note without proof that there is an explicit description of $T$. Write $D = GG^*$ and put $P(z) = (zI - A)^{-1}G$ and $\Phi(z) = \bar{P}(z)CP(z) + I$. Then

$$T(z) = -P(z)\Phi(z)^{-1}\bar{P}(z),$$

which establishes a connection to Molinari's results (see (4.3) in [7], p. 280).

REFERENCES


4 W. A. Coppel, Algebraic aspects of systems theory, unpublished.

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