Inertia Theorems for Matrices, Controllability, and Linear Vibrations

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ABSTRACT

If $H$ is a Hermitian matrix and $W = AH + HA^*$ is positive definite, then $A$ has as many eigenvalues with positive (negative) real part as $H$ has positive (negative) eigenvalues \cite{5}. Theorems of this type are known as inertia theorems. In this note the rank of the controllability matrix of $A$ and $W$ is used to derive a new inertia theorem. As an application, a result in \cite{8} and \cite{4} on a damping problem of the equation $M\ddot{x} + (D + G)\dot{x} + Kx = 0$ is extended.

1. INTRODUCTION

The inertia of an $n \times n$ matrix $A$ with complex elements is defined as the integer triple $I(A) = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)$, $\nu(A)$, and $\delta(A)$ are, respectively, the number of eigenvalues of $A$ with positive, negative and zero real part.

Generalizing a theorem of Lyapunov, Ostrowski, and Schneider \cite{5} and Taussky \cite{7} relate the inertia of a matrix to the matrix inequality $AH + HA^* > 0$ (positive definite). $H$ shall always denote a Hermitian matrix.

**Inertia Theorem** \cite{5}. If $AH + HA^* > 0$, then $I(A) = I(H)$ and $\delta(A) = \delta(H) = 0$.

The case $AH + HA^* \geq 0$ (positive semidefinite) is discussed by Carlson and Schneider \cite{1}. We note the following result.

Theorem 1 [1]. If $H$ is nonsingular and $A$ has no eigenvalues on the imaginary axis (i.e., $\delta(A) = \delta(H) = 0$), then $AH + HA^* \succeq 0$ implies $\ln A = \ln H$.

In this note we use the concept of a controllable pair of matrices to investigate the inertia of $A$. Our results will be applied to the equation $M\ddot{x} + (D + G)x + Kx = 0$ and will provide an extension of a theorem of Zajac [8] and Müller [4] on the pervasive damping of linear mechanical systems.

2. SOME LEMMAS OF CONTROL THEORY

For the basic definitions and lemmas of linear control theory in this section we refer to [2] and [3]. Let $B$ be a complex $n \times r$ matrix. The controllability matrix $C(A|B)$ of $A$ and $B$ is defined as the $n \times nr$ matrix

$$C(A|B) = (B, AB, A^2B, \ldots, A^{n-1}B).$$

The pair $(A, B)$ is called controllable, if rank $C(A|B) = n$.

Lemma 1 [2]. $(A, B)$ is controllable, if and only if

$$\text{rank}(A - \lambda I, B) = n$$

for each eigenvalue $\lambda$ of $A$.

Lemma 2 [3, p. 99]. Given the pair $(A, B)$ there exists a nonsingular $S$ such that

$$S^{-1}AS = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad S^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

and $(A_{11}, B_1)$ is controllable; rank $C(A|B) = \text{rank } C(A_{11}|B_1)$.

Lemma 3 [2]. If $(A, B)$ is controllable, then for every $r \times n$ matrix $C$ the pair $(A + BC, B)$ is controllable.

3. INERTIA THEOREMS

Theorem 2. If $AH + HA^* = W$, $W \succeq 0$ and $(A, W)$ is controllable, then $\delta(A) = \delta(H) = 0$ and $\ln A = \ln H$. 
Proof. We show by contradiction that our assumptions imply \( \delta(A) = 0 \)
and \(|H| \neq 0\). In \( A = \ln H \) follows from Theorem 1.

Let \( A \) have an eigenvalue \( \im \alpha \) with zero real part and let \( u^* \) be a corresponing left eigenvector. Then \( u^*A = \im \alpha u^* \) and \( u^*(AH + HA^*)u = u^*Wu = 0 \). \( W \geq 0 \) implies \( u^*W = 0 \). Thus there exists a \( u \neq 0 \) such that \( u^*(A - \im \alpha I) = 0 \) and \( u^*W = 0 \). By Lemma 1 \( (A, W) \) is not controllable. We assume now that \( H \) is singular, so there is a \( v \neq 0 \) with \( v^*H = 0 \). Suppose \( v^*A^kH - 0 \) for some \( k \). Then \( v^*A^kW. A^kW \) implies \( v^*A^kW = 0 \) and again \( W \geq 0 \) implies \( v^*A^kW = 0 \). We prove by induction

\[
v^*A^kH = 0, \quad k = 0, 1, 2, \ldots \quad (1)
\]

By the assumption on \( v \) (1) is true for \( k = 0 \). If (1) holds for \( k = r \), then

\[
0 = v^*A^rW = v^*A^{r+1}H + v^*A^rHA^* = v^*A^{r+1}H.
\]

From (1) we deduce by the argument above \( v^*A^kW = 0 \) for \( k \geq 0 \) or

\[
v^*(W, AW, \ldots, A^{n-1}W) = 0
\]

which means \( (A, W) \) is not controllable.

The following theorem is due to Snyders and Zakai [6, Corollary 4.1]. In this paper a different proof is given.

**Theorem 3.** If

\[
AH + HA^* = W, \quad W \geq 0 \quad \text{and} \quad H > 0,
\]

then \( \nu(A) = 0 \), \( \delta(A) = \rho(A) = n - \text{rank } C(A|W) \), \( \pi(A) = \text{rank } C(A|W) \)
where \( \rho(A) \) is the number of elementary divisors of purely imaginary eigenvalues of \( A \).

**Proof.** From [1] we know that \( \nu(A) = 0 \) and \( \delta(A) = \rho(A) \). Without loss of generality we can assume \( A \) and \( W \) in (2) to be partitioned according to Lemma 2:

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad W = \begin{pmatrix} W_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad W_{11} \geq 0,
\]

where \( (A_{11}, W_{11}) \) is controllable. If \( A_{11} \) is an \( s \times s \) matrix, then \( s = \text{rank } C(A_{11}|W_{11}) = \text{rank } C(A|W) \). Let \( H > 0 \) be partitioned conformably

\[
H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21}^* & H_{22} \end{pmatrix}, \quad H_{11} > 0, \quad H_{22} > 0.
\]
then transforms \( H \) into block-diagonal form
\[
QHQ^* = \begin{pmatrix} \tilde{H}_{11} & 0 \\ 0 & H_{22} \end{pmatrix},
\]
where \( \tilde{H}_{11} = H_{11} - H_{12}H_{22}^{-1}H_{12}^* > 0 \).

From \( Q(AH + HA^*)Q^* = (QAQ^{-1})(QHQ^*) + (QHQ^*)(QAQ^{-1})^* = QWQ^* \)
we get the equations
\[
A_{11}\tilde{H}_{11} + \tilde{H}_{11}A_{11}^* = W_{11}, \quad \tilde{H}_{11} > 0, \quad W_{11} \geq 0, \quad (3)
\]
\[
A_{22}H_{22} + H_{22}A_{22}^* = 0, \quad H_{22} > 0. \quad (4)
\]
In (3) the conditions of Theorem 2 are satisfied, therefore \( \delta(A_{11}) = 0 \).
Because of \( H_{11} > 0 \) all eigenvalues of \( A_{11} \) are in the right half plane: \( \nu(A_{11}) = 0, \pi(A_{11}) = \text{rank} \, C(A|W) \). In (4) \( H_{22} > 0 \) implies \( A_{22} \) is similar

to a skew-Hermitian matrix, hence all eigenvalues of \( A_{22} \) are purely
imaginary: \( \nu(A_{22}) = \pi(A_{22}) = 0, \delta(A_{22}) = \rho(A_{22}) = n - \text{rank} \, C(A|W) \).

**Corollary 1.** If \( AH + HA^* = W, \ W \geq 0 \) and \( H > 0 \), then all
eigenvalues of \( A \) have positive real part, if and only if the pair \( (A, W) \) is
controllable.

### 4. Linear Vibrations

Consider the equation
\[
M\ddot{x} + (D + G)x + Kx = 0, \quad (5)
\]
where all matrices are \( n \times n \) and real and where \( M, D, K \) are symmetric
and \( G \) is skew-symmetric. Let \( M \) be nonsingular and \( D \geq 0 \).

We associate with (5) the matrix
\[
F(\lambda) = \lambda^2M + \lambda(D + G) + K. \quad (6)
\]
The latent roots of (6) are defined as the values of \( \lambda \) for which \(|F(\lambda)| = 0\). Nontrivial solutions of \( F(\lambda)q = 0 \) or \( r^TF(\lambda) = 0 \) are known as latent vectors.

Equation (5) is equivalent to the first order equation \( \dot{y} = \bar{A}y \) with

\[
\bar{A} = \begin{pmatrix}
0 & I \\
- M^{-1}K & - M^{-1}(D + G)
\end{pmatrix}.
\]

The eigenvalues of \( \bar{A} \) are the latent roots of \( F(\lambda) \). We put

\[
V = \frac{1}{2} \begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}, \quad \bar{W} = \begin{pmatrix} 0 & 0 \\ 0 & -D \end{pmatrix},
\]

then

\[
\bar{A}^T V + V \bar{A} = \bar{W}, \quad \bar{W} \preceq 0.
\] (7)

Applying Theorem 2 to (7) (with trivial change of sign) we generalize a result of Zajac [8] and Müller [4].

**Theorem 4.** If the pair \((\bar{A}^T, \bar{W})\) is controllable, then \(F(\lambda)\) has no purely imaginary latent roots, the number of latent roots with negative, resp. positive, real part is equal to \(\pi(M) + \pi(K)\), resp. \(\nu(M) + \nu(K)\).

Because of the given block structure of \( \bar{A} \) and \( \bar{W} \) criteria for controllability of \((\bar{A}^T, \bar{W})\) can be simplified. We also mention that \((\bar{A}^T, \bar{W})\) to be controllable means \((\bar{A}, \bar{W})\) to be observable [3].

**Lemma 4.** The following statements are equivalent:

(a) \((\bar{A}^T, \bar{W})\) is controllable.
(b) \((\bar{A}, \bar{D})\) is controllable where

\[
\bar{A} = \begin{pmatrix} 0 & -K M^{-1} \\ I & G M^{-1} \end{pmatrix} \quad \text{and} \quad \bar{D} = \begin{pmatrix} 0 \\ D \end{pmatrix}.
\]

(c) For each latent root of \( Z(\lambda) = \lambda^2 M - \lambda G + K \)

\[
\text{rank}(Z(\lambda), \lambda D) = n.
\]

**Proof.** From Lemma 3 we infer (a) \(\Leftrightarrow\) (b), since

\[
\hat{A} = \bar{A}^T - \bar{W} \begin{pmatrix} 0 & 0 \\ 0 & M^{-1} \end{pmatrix}.
\]
An easy calculation shows that the eigenvalues of $\hat{A}$ are the latent roots of $Z(\lambda)$ and that $\text{rank}(\hat{A} - \lambda I, \hat{B}) = n + \text{rank}(Z(\lambda), \lambda D)$. Thus (b) $\iff$ (c) by Lemma 2.

We consider (5) now under the additional assumptions $M > 0$ and $K > 0$. If $D > 0$, then all roots of $F(\lambda)$ necessarily lie in the left half plane and the equilibrium $x(t) \equiv 0$ of (5) is asymptotically stable. If $D \geq 0$ is singular, then (5) may have periodic solutions (which means $F(\lambda)$ has purely imaginary roots). The damping $-D\dot{x}$ is called pervasive, if it acts on all components of $x$, so that all solutions of (5) tend to zero as $t \to \infty$.

**Theorem 5 [4].** If $M > 0$, $K > 0$ and $D > 0$, then the damping $-D\dot{x}$ in (5) is pervasive, if and only if the pair $(\hat{A}, \hat{B})$ is controllable.

**Proof.** Because of $M > 0$ and $K > 0$ the matrix $V$ in (7) is positive definite. The theorem is an immediate consequence of Corollary 1 and Lemma 4.

We observe that for pervasive damping we need not require $M$ to be nonsingular.

**Theorem 6.** If $M > 0$, $D \geq 0$, $K > 0$ and $G^T = -G$, then all latent roots of $F(\lambda) = \lambda^2 M + \lambda(D + G) + K$ have negative real part, if and only if there is no latent vector $q$ of $\lambda^2 M + \lambda G + K$ with $Dq = 0$.

**Proof.** Let $\lambda_0$ be a latent value of $F(\lambda)$, then the assumptions imply $\lambda_0 \neq 0$ and $\text{Re} \, \lambda_0 < 0$. Suppose $F(\lambda)$ has an imaginary root $i\alpha$, $\alpha \neq 0$ and $F(i\alpha)q = 0$, $q \neq 0$. Then $0 = q^TF(i\alpha)q = -\alpha^2q^TMq + q^TKq + i\alpha q^TDq$. Separating real and imaginary parts we obtain $Dq = 0$ and $(-\alpha^2 M + i\alpha G + K)q = 0$. The converse follows from the fact that all latent roots of $\lambda^2 M + \lambda G + K$ lie on the imaginary axis.

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**REFERENCES**


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