PROBLEM SET
POISSON GEOMETRY AND NORMAL FORMS: A GUIDED TOUR THROUGH EXAMPLES

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(1) Prove that the bracket $\{f,g\} = \omega(X_f, X_g)$ for $f, g \in \mathcal{C}^\infty$ defines a Poisson structure on a symplectic manifold (M^{2n}, ω).

Hint: To check the Jacobi identity, expand $d\omega(X_f, X_g, X_h)$.

(2) **(Poisson surfaces)** Let Π be a bivector field on a surface S.

(a) Prove that Π defines a Poisson structure on S.

(b) Consider the sphere S^2 with Poisson structure $\Pi = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$ with h the height function on the sphere and θ the angular coordinate.

(i) Describe the symplectic foliation of Π.

(ii) Consider the equator on the sphere $E = \{h = 0\}$. Prove that E with the zero Poisson structure is a Poisson submanifold of (S^2, Π).

(iii) Check that the vector field $\frac{\partial}{\partial \theta}$ is a Poisson vector field but it is not a Hamiltonian vector field (indeed it is a vector field transverse to the symplectic foliation that you described above).

(c) Prove that the Poisson structure Π induces a Poisson structure $\overline{\Pi}$ on $\mathbb{R}P^2$. What is the symplectic foliation corresponding to $\overline{\Pi}$?

(d) For any $f \in C^\infty(\mathbb{R}^2)$, describe the symplectic foliation of the Poisson structure $\Pi_f = f(x,y)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

(3) Consider $\mathbb{T}^3$ endowed with angular coordinates $\theta_1, \theta_2, \theta_3$, and consider the bivector field

$$\Pi = \left(\frac{\partial}{\partial \theta_1} + \alpha \frac{\partial}{\partial \theta_3}\right) \wedge \left(\frac{\partial}{\partial \theta_2} + \beta \frac{\partial}{\partial \theta_3}\right).$$

Check that Π defines a Poisson structure and describe its symplectic foliation.

(4) **(Poisson structure on $\mathfrak{g}^*$)** Let $\mathfrak{g}^*$ the dual of a Lie algebra. For any pair of functions $f, g : \mathfrak{g}^* \longrightarrow \mathbb{R}$ we define the bracket at a point $\eta \in \mathfrak{g}^*$,

$$\{f,g\}(\eta) = \langle \eta, [df_\eta, dg_\eta] \rangle$$

where df_η and dg_η are naturally identified with elements of $\mathfrak{g}$ and where $[,]$ denotes the Lie algebra bracket on $\mathfrak{g}$.

(a) Check that this bracket is a Poisson bracket on $\mathfrak{g}^*$.

Hint: To check the Jacobi identity, it suffices to verify that it holds for a choice of coordinate functions.

(b) Prove that the symplectic foliation defined by this Poisson structure coincides with the coadjoint orbits of $\mathfrak{g}^*$.
(c) Study the cases of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3,\mathbb{R})$.

(5) Show that the space of Poisson vector fields and the space of Hamiltonian vector fields are both Lie subalgebras of the space of all vector fields (with respect to the standard Lie bracket of vector fields) on a Poisson manifold. Show that every Hamiltonian vector field is a Poisson vector field, and give two examples of Poisson manifolds for which the space of Hamiltonian vector fields is strictly smaller than the space of Poisson vector fields.

Hint: For a vector field X and a multivector field Y, $[X,Y]=\mathcal{L}_X(Y)$

(6) Let $\mathbb{R}^4$ endowed with coordinates x_1,x_2,y_1,y_2 and let f,g be smooth functions. Consider the bivector field $\Pi_{(f,g)}=\frac{\partial}{\partial y_1}\wedge\frac{\partial}{\partial x_1}+f(y_2)\frac{\partial}{\partial y_2}\wedge\frac{\partial}{\partial x_2}+g(y_2)\frac{\partial}{\partial y_1}\wedge\frac{\partial}{\partial x_2}$.
(a) Check that $[\Pi_{(f,g)},\Pi_{(f,g)}]=0$ for any functions f and g. Describe the symplectic foliation.
(b) Prove that the Hamiltonian vector field of x_1 is tangent to the family of hyperplanes $x_2=c$.
(c) Prove that the distribution $\langle X_{x_1},X_{x_2}\rangle$ is involutive and, thus, it defines a foliation. Check that the leaves of this foliation are submanifolds of the leaves of the symplectic foliation described by $\Pi_{(f,g)}$.

Remark: Because $\{x_1,x_2\}=0$, the functions x_1 and x_2 define an integrable system. We will study integrable systems in lecture 4 and 5.

(7) Take a holomorphic function on $\mathbb{C}^2$, $F:\mathbb{C}^2\longrightarrow\mathbb{C}$ decompose it as $F=G+iH$ with $G,H:\mathbb{R}^4\longrightarrow\mathbb{R}$.
Cauchy-Riemann equations for F in coordinates $z_j=x_j+iy_j$, $j=1,2$ read as,

$$\frac{\partial G}{\partial x_i}=\frac{\partial H}{\partial y_i},\quad \frac{\partial G}{\partial y_i}=-\frac{\partial H}{\partial x_i}$$

(a) Reinterpret these equations as the equality

$$\{G,\cdot\}_0=\{H,\cdot\}_1\quad \{H,\cdot\}_0=-\{G,\cdot\}_1$$

with $\{\cdot,\cdot\}_j$ the Poisson brackets associated to the real and imaginary part of the symplectic form $\omega=dz_1\wedge dz_2$ $(\omega=\omega_0+i\omega_1)$.
(b) Check that both Poisson structures are compatible.
(c) Check that $\{G,H\}_0=0$ and $\{H,G\}_1=0$.

(8) Let $X_1,\dots,X_{2k}$ be a set of commuting vector fields on a manifold M.
(a) Check that the bivector field $\Pi=X_1\wedge X_2+\dots X_{2k-1}\wedge X_{2k}$ defines a Poisson structure on M.
(b) Consider the vector fields on $\mathbb{R}^3$, $X_1=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}$ and $X_2=-x_1x_3\frac{\partial}{\partial x_1}-x_2x_3\frac{\partial}{\partial x_2}+(x_1^2+x_2^2)\frac{\partial}{\partial x_3}$, check that they define vector fields on

$S^2 = \{(x_1, x_2, x_3), \quad x_1^2 + x_2^2 + x_3^2 = 1\}$ apply the previous strategy to see that $\Pi = X_1 \wedge X_2$ is a Poisson structure. Describe the symplectic foliation.

(c) Describe the symplectic foliation associated to the induced Poisson structure $\Pi = X_1 \wedge X_2$ on S^3 where $X_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$ and $X_2 = x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}$ are vector fields on $\mathbb{R}^4$.

(d) Describe the symplectic foliation associated to the Poisson structure $\Pi = X_1 \wedge X_2 + X_3 \wedge X_4$ induced on S^7 where $X_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$, $X_2 = x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4}$, $X_3 = x_5 \frac{\partial}{\partial x_6} - x_6 \frac{\partial}{\partial x_5}$ and $X_4 = x_7 \frac{\partial}{\partial x_8} - x_8 \frac{\partial}{\partial x_7}$

(9) Consider $\mathbb{R}^2$ with Poisson bracket $\{x, y\} = x$, and $\mathbb{R}^4$ with Poisson bivector $\frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2}$. Prove that the mapping $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ given by $F(q_1, p_1, q_2, p_2) = (q_1, p_1 q_1 + q_2)$ is a surjective Poisson map.

(10) Consider $\mathbb{T}^3$ with angular coordinates $\theta_1, \theta_2, \theta_3$ and the Poisson structure

$$\Pi = \left(\frac{\partial}{\partial \theta_1} + \alpha \frac{\partial}{\partial \theta_3}\right) \wedge \left(\frac{\partial}{\partial \theta_2} + \beta \frac{\partial}{\partial \theta_3}\right)$$

(exercise 2 of the problem set).

(a) Check that the bivector field

$$\Pi_0 = \left(\frac{\partial}{\partial x_1} + \alpha \frac{\partial}{\partial x_3}\right) \wedge \left(\frac{\partial}{\partial x_2} + \beta \frac{\partial}{\partial x_3}\right) + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$$

is of maximal rank and thus $(\Pi_0)^{-1}$ defines a symplectic structure on $\mathbb{R}^4$.

(b) Check that the mapping $F : \mathbb{R}^4 \longrightarrow \mathbb{T}^3$ given by the projection onto the first three coordinates, followed by the quotient to $\mathbb{R}^3/\mathbb{Z}^3 \cong \mathbb{T}^3$ is a Poisson submersion.

(11) Consider the Poisson structure $\Pi = X_1 \wedge X_2$ on $\mathbb{R}^4$ with $X_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$ and $X_2 = x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}$.

(a) Prove that the functions $f_1 = x_1^2 + x_2^2$ and $f_2 = x_3^2 + x_4^2$ are in involution and, thus, their Hamiltonian vector fields define an integrable distribution. What are the invariant submanifolds?

(b) Prove that Π induces a Poisson structure on S^3. Do the functions f_1 and f_2 induce an integrable system on S^3?

(12) Fix a smooth function $K \in \mathcal{C}^\infty(\mathbb{R}^3)$ and consider the bracket $\{f, g\}_K = \det(df, dg, dK)$ for $f, g \in \mathcal{C}^\infty$.

(a) Prove that $\{f, g\}_K$ defines a Poisson structure on $\mathbb{R}^3$.

(b) For $H \in C^\infty$, write the hamiltonian vector field X_H in coordinates. Prove that X_H is also Hamiltonian for the Poisson structure, $\{f, g\}_H := \det(df, dg, dH)$. What is the corresponding Hamiltonian function?

(c) Prove that the bivector field associated to $\{,\}_K$ is a cocycle in the Poisson cohomology associated to $\{,\}_H$.

(d) Show that $\{f,g\}_t = \det(df, dg, d((1-t)K + tH))$ defines a Poisson structure for each $t \in \mathbb{R}$.

(e) Prove that the function $(1-t)K + tH$ is a casimir for $\{,\}_t$.

(13) Consider the bivector field $\Pi = x_1\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$ on $\mathbb{R}^3$.

(a) Prove that Π defines a Poisson structure.

(b) Observe that $\Pi = \Pi_0 + \Pi_1$ with $\Pi_0 = (-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}) \wedge \frac{\partial}{\partial x_3}$ and $\Pi_1 = x_3\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$. Check that $[\Pi_0, \Pi_0] = [\Pi_1, \Pi_1] = [\Pi_0, \Pi_1] = 0$ and thus any bivector field on the path $\Pi = (1-t)\Pi_0 + t\Pi_1$ defines a Poisson structure on $\mathbb{R}^3$.

(c) Check that the vector field $X = x_3(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2})$ is Hamiltonian for both Poisson structures Π_0 and Π_1. Check that the Hamiltonians f_0 (Hamiltonian of X with respect to Π_0) and f_1 (Hamiltonian of X with respect to Π_1) define an integrable system on $\mathbb{R}^3$ with respect to the Poisson structure Π.

(d) Prove in general that given any two poisson bivectors Π_0 and Π_1 satisfying $[\Pi_0, \Pi_1] = 0$, the bivector field $\Pi = \Pi_0 + \Pi_1$ is a Poisson structure. Further, if a vector field X is Hamiltonian with respect to both Π_0 and Π_1, this yields two Poisson commuting functions.

(14) Let (M, Π) be a Poisson manifold and let $\{f_1, \ldots, f_{2k}\}$ be a set of pairwise Poisson commuting functions (i.e., $\{f_i, f_j\} = 0$) that are functionally independent (i.e., $df_1 \wedge \cdots \wedge df_{2k} \neq 0$ on a dense set).

(a) Check that the Hamiltonian vector fields X_{f_j} are tangent to the level sets of each f_i.

(b) Prove that the distribution determined by $\langle X_{f_1}, \ldots X_{f_{2k}} \rangle$ is involutive, and that each leaf of the corresponding foliation lies inside a single leaf of the symplectic foliation defined by Π.

(c) Prove that the new bivector field $\overline{\Pi} = X_{f_1} \wedge X_{f_2} + \cdots + X_{f_{2k-1}} \wedge X_{f_{2k}}$ defines a new Poisson structure on M.

(d) Prove that every $4n$-dimensional symplectic manifold admits a Poisson structure with generic rank equal to $2n$. (Hint: Use the fact that every symplectic manifold admits an integrable system).

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