Optimal Control and Hamilton-Jacobi-Bellman Equations

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"Quasi-Variational Inequalities: Theory, Algorithms, and Applications"
Fall School, Würzburg, September 23 - 25, 2019

ENSTA IP PARIS

Some steady Hamilton-Jacobi QVIs

$$\max\left(\lambda v(x) + H(x, Dv(x)), v(x) - \mathcal{M}v(x)\right) = 0 \quad x \in \mathbb{R}^d$$

Time-dependant equations

$$- \quad \begin{cases} \max\left(\partial_t v(t,x) + H(t,x,Dv(t,x)), v(t,x) - \mathcal{M}v(t,x)\right) = 0 \\ v(0,x) = v_o(x) \end{cases}$$

- Where the Hamiltonian H is defined by

$$H(x,p) := \sup_{q \in F(x)} \left(-q \cdot p - \ell(x,q)\right)$$

where $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a multi-valued function, and $\ell : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.

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$$\max\left(\lambda v(x) + H(x, Dv(x)), v(x) - g(x)\right) = 0 \quad x \in \mathbb{R}^d$$

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Outline

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Controlled systems

- y state of the system
- u control input

Find a control law and its corresponding trajectory that comply with prescribed constraints (physical or economical constraints on the control and/or the state) and/or optimize some performances of the system.

We assume that the system is described by an ODE:

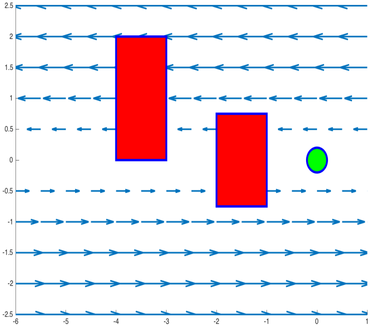
$$\begin{cases} \dot{y}(s) = f(y(s), u(s)) & s \in (0, T), \\ y(0) = x_0, \end{cases}$$

where $f : \mathbb{R}^d \times \mathbb{R}^m$ is a Lipschitz continuous function.

- **Open-loop Control**: the control $u(\cdot)$ is a (possibly discontinuous) function of time taking values in a compact set U. The system can be rewritten as a (DI), with
$$\Gamma(x) := f(x, U)$$
.
- **Closed-loop (feedback) Control**: the control $u(\cdot)$ is a (possibly discontinuous) function of space. Yields to a (DI) with
$$\Gamma(x) := f(x, u(x))$$
.
- **Controlled problem with uncertainties** : $\Gamma(x) := f(x, u(x)) + E(x)$.

Example 1 (An open-loop control problem)

$$\begin{cases} \dot{\mathbf{y}}_1(s) = \cos \mathbf{u}(s) - 0.25 * \tanh(\mathbf{y}_2(s)) \\ \dot{\mathbf{y}}_2(s) = \sin \mathbf{u}(s) \\ \mathbf{y}_1(0) = -5, \quad \mathbf{y}_2(0) = 0 \\ \mathbf{u}(t) \in [-\pi, \pi] \end{cases}$$



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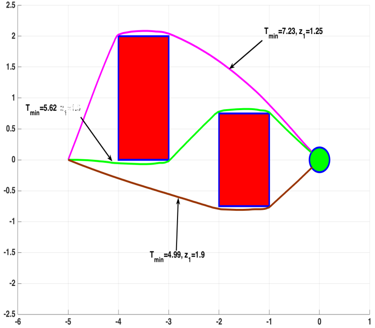
$$J_1(T; \mathbf{y}, \mathbf{u}) := T,$$

$$J_2(T; \mathbf{y}, \mathbf{u}) := \int_0^T |\mathbf{y}_2(s) - 2.4| ds.$$



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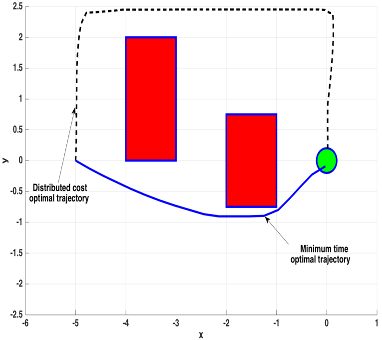


Example 2 (A closed loop control problem)

- Double integrator system:

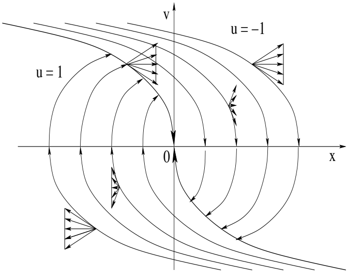
$$\ddot{\mathbf{x}}(s) = \mathbf{u}(s), \quad \mathbf{u}(s) \in [-1, 1]$$

with $\mathbf{x}(0) = \begin{bmatrix} x \\ v \end{bmatrix}$.

- An equivalent formulation:

$$\begin{cases} \dot{\mathbf{y}}_1(s) = \mathbf{y}_2(s) \\ \dot{\mathbf{y}}_2(s) = \mathbf{u}(s) \end{cases} \quad \text{with} \quad \begin{cases} \dot{\mathbf{y}}_1(0) = x \\ \dot{\mathbf{y}}_2(0) = v \end{cases}$$

- Find a feedback control $u = u(x, v)$ steering the system to the origin in minimum time:



Trajectory optimization problem for a space launcher

Aim

Maximize the payload m_0 to be steered from the Earth (Kourou) to a prescribed Orbit (SSO, GEO, ...).

- The physical model involves 6+1 state variables, the position $\overrightarrow{X}$ of the launcher in the 3D space, its velocity $\overrightarrow{V}$ and its mass M:

$$\mathbf{y} := (\mathbf{X}, \mathbf{V}, \mathbf{M}).$$

- The forces acting on the rocket are: Gravity $M\overrightarrow{g}$, Thrust $\overrightarrow{F_T}$, Drag $\overrightarrow{F_D}$, and Coriolis forces.

- Newton's Law:

$$\frac{d\overrightarrow{X}}{dt} = \overrightarrow{V},$$
$$\frac{d\overrightarrow{V}}{dt} = \overrightarrow{g} + \frac{\overrightarrow{F_T}}{M} + \frac{\overrightarrow{F_D}}{M} - 2\overrightarrow{\Omega} \wedge \overrightarrow{V} - \overrightarrow{\Omega} \wedge (\overrightarrow{\Omega} \wedge \overrightarrow{X}),$$

- The launcher is controlled by means of:

  - *launch parameters* $\Pi = (\psi, \omega) \in \mathbb{R}^2$
  - *incidence and sideslip angles* $\alpha(\cdot), \delta(\cdot)$

- Physical state-constraints (intermediate times, end-point, along the time interval)

A simplified problem

Example (Goddard problem)

$$\dot r(s) = v(s)$$
$$\dot v(s) = \frac{1}{m}(F_T\, u(s) - F_D(r,v)) - \frac{1}{r(s)^2}$$
$$\dot m(s) = -\beta\, F_T\, u(s)$$
$$r(0)=1,\ v(0)=0,\ m(0)=1$$

$r(\cdot)$: altitude
$v(\cdot)$: velocity
$m(\cdot)$: masse

$u(\cdot)$: % of the thrust

- The ratio $u(t)$ is subject to the following constraint: $0 \le u(t) \le 1$.
- The rocket's mass satisfies the final constraint: $r(T) \ge r^*$.

The optimal control problem is the following (for $T > 0$):

$$\left\| \begin{array}{l} \max\ m(T) \\ u(t) \in [0,1], \quad r(T) \ge r^*. \end{array} \right.$$

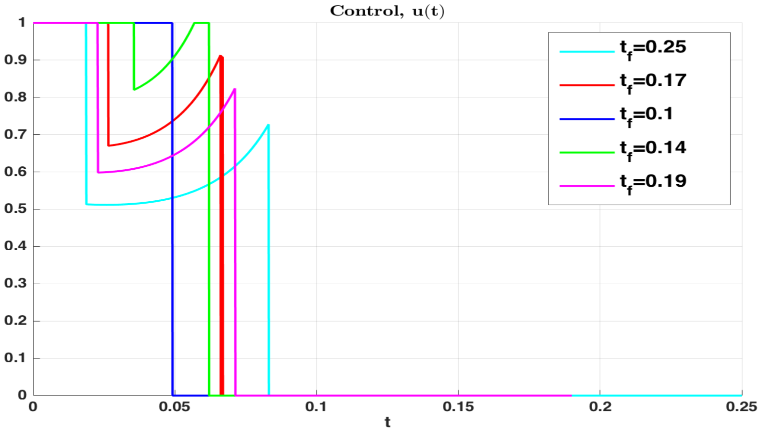
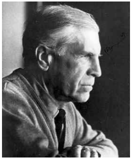


Figure: Optimal control laws for different final times $T = t_f$.

Major development since the 50s



- Pontryagin's principle: necessary optimality conditions (Euler-Lagrange and Weierstrass conditions)

- Bellman's principle: dynamic programming principle (Verification theorem)
- Important breakthroughs **in the 80s**: nonsmooth analysis, viscosity theory.

Mathematical formulation of the Control problem

- For a given non-empty compact subset A of $\mathbb{R}^k$, define the set of admissible controls as:

$$\mathcal{A} := \left\{ \alpha : (0,+\infty) \to \mathbb{R}^k, \text{ measurable, } \alpha(t) \in A \text{ a.e } \right\}.$$

- Consider the following control system:

$$\begin{cases} \dot{y}(s) := f(y(s), \alpha(s)), & \text{a.e } s \in [0,t], \\ y(0) := x, \end{cases} \tag{1}$$

where $f : \mathbb{R}^d \times A \to \mathbb{R}^d$ is continuous, loc. Lipschitz continuous w.r.t y, and there existe $C > 0$ such that $|f(y,a)| \leq C(1 + |y|)$ for all $y \in \mathbb{R}^d$.

- Define the set of trajectories:

$$\mathcal{S}_{[0,t]}(x) := \{ y_x^\alpha \in W^{1,1}(0,t;\mathbb{R}^d), \ \ y_x^\alpha \text{ satisfies (1) for some } \alpha \in \mathcal{A} \},$$

The multi-application: $x \rightsquigarrow \mathcal{S}_{[0,t]}(x)$ is Lipschitz continuous; i.e.,

$$\exists L > 0, \mathcal{S}_{[0,t]}(x) \subset \mathcal{S}_{[0,t]}(z) + L|x - z|\mathcal{B}_{W^{1,1}} \quad \forall x, z \in \mathbb{R}^d.$$

Example (1)

$$\dot{y}(s) = u(s) \quad s \in (0,1),$$
$$y(0) = 0,$$
$$u(s) \in \{-1, 1\}$$

- $u_n(s) = \begin{cases} 1 & \text{sur } (\frac{2k}{2n}, \frac{2k+1}{2n}) \\ -1 & \text{sur } (\frac{2k+1}{2n}, \frac{2k+2}{2n}) \end{cases}$
- $y_n(s) =$ $\begin{cases} s - \frac{k}{n} & \text{sur } (\frac{2k}{2n}, \frac{2k+1}{2n}) \\ -s + \frac{(k+1)}{n} & \text{sur } (\frac{2k+1}{2n}, \frac{2k+2}{2n}) \end{cases}$

No optimal solution

$y_n \to 0, \qquad y \equiv 0$ is not an admissible trajectory!!

$\|u_n\|_{L^\infty, L^2} = 1 \not\to 0$

Example (1')

$$\dot{y}(s) = u(s) \quad s \in (0,1),$$
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The DI with "convexified dynamics" admits a closed set of trajectories

$y_n \to 0, \qquad y \equiv 0$ is admissible

$\|u_n\|_{L^\infty, L^2} = 1 \not\to 0$

Set of constrained trajectories

- Assume that $f(x, A) := \{f(x, a),\ a \in A\}$ is a convex set.
  Then, by **Filippov's theorem**, the set of trajectories $\mathcal{S}_{[0,t]}(x)$ is a compact set of $W^{1,\infty}([0, t])$ endowed with the C^0-topology.

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- Introduce the set of feasible trajectories:

$$\mathcal{S}^g_{[0,t]}(x) := \{y \in \mathcal{S}_{[0,t]}(x) \mid \ g(y(s)) \leq 0 \ \ \forall s \in [0, t]\}$$

  This set is a compact subset of $W^{1,\infty}([0, t])$ ... when it is non-empty!

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- Inward pointing (IP) condition: Assume g smooth and

$$\exists \beta > 0, \quad \forall x \text{ s.t. } g(x) = 0, \quad \min_{a \in A} f(x, A) \cdot \nabla g(x) < -\beta.$$

  Then, for $x \in \mathcal{K}$, $\mathcal{S}^g_{[0,t]}(x) \neq \emptyset$, and $x \longmapsto \mathcal{S}^g_{[0,t]}(x)$ is Lipschitz.

  Ref: Arutyunov'84, Soner'86, Rampazzo-Vinter'99, Vinter-Frankowska'00, Clarke-Rifford-Stern'02, Hermosilla-Vinter-HZ'18 ...

Outline

Consider the following control problems:

- Mayer's problem:

$$V(x,t) = \inf_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t))$$

- Time minimum problem ($\mathcal{C}$ closed set in $\mathbb{R}^d$):

$$\mathcal{T}(x) = \inf\left\{t;\ \ y_x(t) \in \mathcal{C},\ y_x \in \mathcal{S}_{[0,t]}(x)\right\}$$

- Supremum cost (the notation $a \bigvee b$ stands for $\max(a,b)$):

$$V^g(x,t) = \inf_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t)) \bigvee \sup_{\theta \in [0,t]} g(y_x(\theta))$$

- Infinite horizon problem :

$$V^\infty(x) = \inf_{y_x^u \in \mathcal{S}_{[0,+\infty[}(x)} \int_0^{+\infty} e^{-\lambda t} \ell(y_x^u(t), u(t))\, dt$$

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$$
\begin{aligned}
|V(x,t) - V(x',t)| &\leq \sup_{y_x \in \mathcal{S}_{[0,t]}(x)} |\Phi(y_x(t)) - \Phi(y_{x'}(t))| \\
&\leq L_\Phi \sup_{y_x \in \mathcal{S}_{[0,t]}(x)} |y_x(t) - y_{x'}(t)| \\
&\leq L_\Phi e^{L_f T} |x - x'|.
\end{aligned}
$$

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- If $\Phi : \mathbb{R}^d \to \mathbb{R}$ is lsc , then V and V^g are lsc.

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- When the target $\mathcal{C}$ is closed, the minimum time function $\mathcal{T}$ is lsc.

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- When the target $\mathcal{C}$ is closed, the minimum time function $\mathcal{T}$ is lsc.

- If the function ℓ is Lipschitz continuous and if the discount factor $\lambda > 0$ is big enough, then V^∞ is Lipschitz continuous.

Optimal control problems. HJB equations

Now, consider again the following control problems:

- Mayer's problem:

$$V(x,t) = \inf_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t))$$

- Time minimum problem ($\mathcal{C}$ closed set in $\mathbb{R}^d$):

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- Supremum cost:

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- Infinite horizon:

$$V^\infty(x) = \inf_{y_x^u \in \mathcal{S}_{[0,+\infty[}(x)} \int_0^{+\infty} e^{-\lambda t} \ell(y_x^u(t), u(t))\, dt$$

Dynamic programming principle: HJB equation

Mayer's Problem

$$
\begin{aligned}
V(x,t) &= \min_{y_x \in \mathcal{S}_{[0,h]}(x)} V(y_x(h), t-h) \qquad h \in (0,t), \\
V(x,0) &= \Phi(x)
\end{aligned}
$$

Dynamic programming principle: HJB equation

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\end{aligned}
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Minimum time problem:

$$
\begin{aligned}
\mathcal{T}(x) &= \min_{y_x \in \mathcal{S}_{[0,h]}(x)} \mathcal{T}(y_x(h)) + h \qquad h < \mathcal{T}(x),\ x \notin \mathcal{C},\\
\mathcal{T}(x) &= 0 \quad x \in \mathcal{C};
\end{aligned}
$$

Dynamic programming principle: HJB equation

Supremum cost

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V^g(x,t) &= \min_{y_x \in \mathcal{S}_{[0,h]}(x)} V^g(y_x(h), t-h) \bigvee \sup_{\theta \in [0,h]} g(y_x(\theta)) \\
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Infinite horizon

$$
V^\infty(x) = \min_{y_x^u \in \mathcal{S}_{[0,h]}(x)} e^{-\lambda h} V^\infty(y_x^u(h)) + \int_0^h e^{-\lambda s} \ell(y_x^u(s), u(s))\, ds
$$

$$V(x,t) = \min_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t))$$

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- ***Sub-optimality:***

$$\forall y_x \in \mathcal{S}_{[0,t]}(x), \quad s \longmapsto V(y_x(s), t-s) \text{ is increasing,}$$

- **Super-optimality**

$$\exists y_x^* \in \mathcal{S}_{[0,t]}(x), \quad s \longmapsto V(y_x^*(s), t-s) \text{ is constant}$$

An infinitesimal version of the DPP: Hamilton-Jacobi-Bellman equation (HJB).

If the value function is C^1 in a neighborhood of (x, t), then

- $\partial_t V(x,t) + \mathcal{H}(x, D_x V(x,t)) = 0, \quad x \in \mathbb{R}^d,\ t > 0;$

- $\mathcal{H}(x, D\mathcal{T}(x)) = 1, \quad x \notin \mathcal{C},\ \mathcal{T}(x) < +\infty;$

- $\min(\partial_t V^g(x,t) + \mathcal{H}(x, DV^g(x,t)), V^g(x,t) - g(x)) = 0,$

  $x \in \mathbb{R}^d, t > 0;$

- $-\lambda V^\infty(x) + \mathcal{H}^\infty(x, DV^\infty(x)) = 0,\ x \in \mathbb{R}^d;$

where

$$\mathcal{H}(x,q) := \max_{a \in A}(-f(x,a)\cdot q), \qquad \mathcal{H}^\infty(x,q) := \max_{a \in A}(-f(x,a)\cdot q - \ell(x,a))$$

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  $V(x,0) = \Phi(x)$ Time-dependent HJB equation

- $\mathcal{H}(x, D\mathcal{T}(x)) = 1, \quad x \notin \mathcal{C},\ \mathcal{T}(x) < +\infty;$
  $\mathcal{T}(x) = 0$ on $\mathcal{C}$ Steady HJB equation

- $\min(\partial_t V^g(x,t) + \mathcal{H}(x, DV^g(x,t)), V^g(x,t) - g(x)) = 0,$

  $V^g(x,0) = \Phi(x) \vee g(x)$ HJB-VI ineqation

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  Steady HJB equation

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.

Verification Theorem - Mayer Problem

Let $w \in C^1_b(\mathbb{R}^d \times [0,T])$. We say that w is a classical verification function if it satisfies

$$\partial_t w(x,t) + \mathcal{H}(x, D_x w(x,t)) \leq 0 \quad x \in \mathbb{R}^d, t \in [0,T],$$
$$w(x,0) = \Phi(x);$$

with the Hamiltonian function

$$\mathcal{H}(x,q) = \max_{a \in A}(-f(x,a) \cdot q).$$

Verification Theorem - Mayer Problem

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$$w(x,0) = \Phi(x);$$

Proposition

Let $y_x^* \in \mathcal{S}_{[0,T]}(x)$ be a feasible arc. Assume that there exists a classical verification function w, s.t.

$$w(x, T) = \Phi(y_x^*(T)),$$

then y_x^* is optimal on $[0, T]$.

Proof.

- We take any $y_x \in \mathcal{S}_{[0,T]}(x)$, we have:

$$\frac{d}{ds}[w(y_x(s), T-s)] = -\partial_t w(y_x(s), T-s) + \dot{y}_x(s) \cdot Dw(y_x(s), T-s) \geq 0.$$

Which means that $s \longmapsto w(y_x(s), T-s)$ is increasing:

$$w(x,T) \leq w(y_x(T), 0) = \Phi(y_x(T)).$$

Therefore, $w(x,T) \leq V(x,T)$.

- $w(x,T) = \Phi(y_x^*(T)) \geq V(x,T)$.

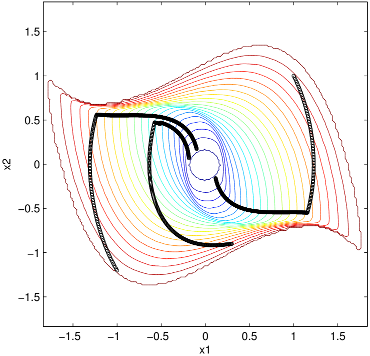
- Therefore $w(x,T) = \Phi(y_x^*(T)) = V(x,T)$, and y_x^* is optimal.

Van der Pol Problem :

$$\begin{cases} \dot{y}_1(t) = y_2 \\ \dot{y}_2(t) = -y_1 + y_2(1 - y_1^2) + a(t) \\ a(t) \in [-1, 1] \end{cases}$$

The final cost function:

$$\Phi(y) := \|y\| - r_0$$



➤ The HJB equation to be solved for this problem is:

$$\partial_t \vartheta(x,t) + \begin{pmatrix} -x_2 \\ x_1 - x_2(1 - x_1^2) \end{pmatrix} \cdot D_x \vartheta(x,t) + |\partial_{x_2} \vartheta(x,t)| = 0$$

$$\vartheta(x,0) = \Phi(x) = \|x\| - r_0.$$

Outline

Definition.

Let $\Omega \subset \mathbb{R}^M$, and $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}$ be continuous. Consider the HJB equation:

$$\mathcal{F}(x, u(x), Du(x)) = 0 \quad \text{in } \Omega. \tag{2}$$

(i) A function $u \in USC(\Omega)$ is a viscosity **sub-solution** of (3) if for any $\varphi \in C^1(\Omega)$ and any local maximum point $x_0 \in \Omega$ of $u - \varphi$,

$$\mathcal{F}(x_0, u(x_0), D\varphi(x_0)) \leq 0.$$

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(iii) A function $u \in C(\Omega)$ is a viscosity **solution** of (3) if it is a sub- and super-solution.

Vanishing Viscosity Limits

Lemma

Let u_ε be a sequence of smooth solutions to the **viscous** equations:

$$\mathcal{F}(x, u_\epsilon(x), Du_\epsilon(x)) - \varepsilon \Delta u_\epsilon(x) = 0 \qquad x \in \Omega.$$

If $u_\varepsilon \to u$ uniformly on Ω, when $\varepsilon \to 0+$, then u is a viscosity solution of

$$\mathcal{F}(x, u(x), Du(x)) = 0.$$

Idea of the proof - 1:

- u_ε is a viscosity sub-solution to the viscous equation.
  Indeed, let $\psi \in C^1(\Omega)$, and let $x \in \Omega$ a local maximum of $u_\varepsilon - \psi$.
  Then
  $$Du_\varepsilon(x) = D\psi(x), \qquad \Delta u_\varepsilon(x) \leq \Delta\psi(x).$$
  Therefore,
  $$\mathcal{F}(x, u_\varepsilon(x), D\psi(x)) - \varepsilon\Delta\psi(x) \leq \mathcal{F}(x, u_\epsilon(x), Du_\epsilon(x)) - \varepsilon\Delta u_\epsilon(x) = 0.$$

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  Therefore,
  $$\mathcal{F}(x, u_\varepsilon(x), D\psi(x)) - \varepsilon\Delta\psi(x) \leq 0.$$

- Let $\varphi \in C^1(\Omega)$, $x \in \Omega$ is a local maximum of $u - \varphi$.
  $\forall \rho > 0$, $\exists \delta \leq \rho$ and there exists $\psi \in C^1(\Omega)$ s.t.

  $$\|\psi - \varphi\|_{C^1(\Omega)} \leq \delta$$

  $u_\varepsilon - \psi$ has a local maximum x_ε inside the ball $B(x, \rho)$.

Idea of the proof - 2:

- Extract a convergent subsequence $x_\varepsilon \to \bar{x} \in B(x,\rho)$. By passing to the limit in the inequality

$$\mathcal{F}(x_\varepsilon, u_\varepsilon(x_\varepsilon), D\psi(x_\varepsilon)) - \varepsilon\Delta\psi(x_\varepsilon) \leq 0,$$

we get:

$$\mathcal{F}(\bar{x}, u(\bar{x}), D\psi(\bar{x})) \leq 0.$$

Idea of the proof - 2:

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  we get:

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- Choosing ρ small enough, we have $|\bar{x} - x|$ and $|D\psi(\bar{x}) - D\varphi(x)|$ as small as we want. So by continuity of $\mathcal{F}$, we finally obtain:

$$\mathcal{F}(x, u(x), D\varphi(x)) \le 0.$$

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- The supersolution can be proved in the same way.

A more general stability result

- A general stability theorem holds for viscosity solutions, without additional requirements on the derivatives.

Theorem

Consider a sequence of continuous functions $(u_n)_{n\geq 1}$, where u_n is a viscosity sub- solutions (resp. super-solutions) to

$$\mathcal{F}_n(x, u_n(x), Du_n(x)) = 0, \quad x \in \Omega.$$

Assume $\mathcal{F}_n \to \mathcal{F}$ uniformly on compact subsets of $\Omega \times \mathbb{R}^d \times \mathbb{R}^d$, when $n \to +\infty$ and $u_n \to u$ uniformly on Ω.
Then u is a viscosity sub-solution (a super-solution) of

$$\mathcal{F}(x, u(x), Du(x)) = 0, \quad x \in \Omega.$$

- The stability results are based on the following "key ingredient":

Lemma

Let $v : \Omega \to \mathbb{R}$ be a upper semi-continuous function that achieves a *strict* local maximum at $\bar{x} \in \Omega$. Let $(v_n)_n$ be a sequence of upper semi-continuous function on Ω.

Assume that $\limsup\limits_{\substack{z \to \bar{x} \\ n \to +\infty}} v_n(z) = v(x_0)$. Then there exists a sequence $(x_n)_n$ in Ω such that for every $n \geq 1$, x_n is a local maximum of v_n and

$$\lim_{n \to +\infty} x_n = \bar{x}, \qquad \lim_{n \to +\infty} v(x_n) = v(\bar{x}).$$

Semi-relaxed limits

Let $(u_n)_n$ be a sequence of functions on Ω.

- Assume that $(u_n)_n$ are locally uniformly bounded from above Define the *upper relaxed limit* by:

$$u^*(x) := \limsup_{x_n \to x} u_n(x_n).$$

- Assume that $(u_n)_n$ are locally uniformly bounded from below. Define the *lower relaxed limit* by:

$$u_*(x) := \liminf_{x_n \to x} u_n(x_n)).$$

Theorem (Stability by semi-relaxed limits)

Consider a sequence of usc (resp. lsc) functions $(u_n)_{n\geq 1}$, where u_n is a viscosity sub- solutions (resp. super-solutions) to

$$\mathcal{F}_n(x, u_n(x), Du_n(x)) = 0, \quad x \in \Omega.$$

Assume $\mathcal{F}_n \to \mathcal{F}$ uniformly on compact subsets of $\Omega \times \mathbb{R}^d \times \mathbb{R}^d$, when $n \to +\infty$ and $(u_n)_n$ is locally uniformly bounded from above (resp. from below) on Ω.
Then u^* is a viscosity sub-solution (resp u_* is a super-solution) of

$$\mathcal{F}(x, u(x), Du(x)) = 0, \quad x \in \Omega.$$

Viscosity notion: An equivalent definition.

Let $\Omega \subset \mathbb{R}^M$, and $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}$ be continuous. Consider the HJB equation:

$$\mathcal{F}(x, u(x), Du(x)) = 0 \quad \text{in } \Omega, \tag{3}$$

(i) A function $u \in USC(\Omega)$ is a viscosity sub-solution of (3) if

$$\mathcal{F}(x, u(x), q) \leq 0 \quad \forall q \in D^+u(x).$$

$$q \in D^+u(x) \Longleftrightarrow u(y) \leq u(x) + q \cdot (y - x) + o(|y - x|),$$

$$q \in D^-u(x) \Longleftrightarrow u(y) \geq u(x) + q \cdot (y - x) + o(|y - x|).$$

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$$\mathcal{F}(x, u(x), q) \geq 0 \quad \forall q \in D^-u(x).$$

$$q \in D^+u(x) \iff u(y) \leq u(x) + q \cdot (y - x) + o(|y - x|),$$

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(iii) A function $u \in C(\Omega)$ is a viscosity solution of (3) if it is a sub- and super-solution.

$$q \in D^+u(x) \Longleftrightarrow u(y) \leq u(x) + q \cdot (y - x) + o(|y - x|),$$

$$q \in D^-u(x) \Longleftrightarrow u(y) \geq u(x) + q \cdot (y - x) + o(|y - x|).$$

Lemma

Suppose $u \in C(\Omega)$.

(i) $q \in D^+u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = q$ and x is a local minimum of $u - \varphi$.

(ii) $q \in D^-u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = q$ and x is a local maximum of $u - \varphi$.

Some remarks

- ✐ If u is differentiable at $x \in \Omega$, then $D^- u(x) = D^+ u(x) = \{\nabla u(x)\}$.
- ✐ A viscosity solution u of the HJ equation satisfies the equality

  $$\mathcal{F}(x, u(x), \nabla u(x)) = 0$$

  in the classical sense whenever u is differentiable at x.
- ✐ A Lipschitz continuous viscosity solution u is also a solution in the *"almost everywhere"* sense. (Indeed, by Rademacher's theorem, a Lipschitz continuous function is differentiable a.e.)

Comparison principle: Doubling variable techniques (Kruzhkov's method for conservation laws)

Theorem

Let Ω be a bounded open set of $\mathbb{R}^d$. Let $u_1, u_2 \in BUC(\overline{\Omega})$ be respectively, viscosity sub- and super-solutions of

$$u + H(x, Du(x)) = 0 \quad \textit{on}\ \Omega,$$

and $u_1 \leq u_2$ on $\partial\Omega$. Moreover, assume that $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is uniformly continuous w.r.t the x-variable

$$|H(x_1, p) - H(x_2, p)| \leq \omega\left(|x_1 - x_2|(1 + |p|)\right),$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a continous non-deacring function satisfying $\omega(0) = 0$. Then $u_1(x) \leq u_2(x)$ for every $x \in \overline{\Omega}$.

Proof - 1: Case where u_1, u_2 are smooth (C^1 functions)

- Assume $u_1 - u_2$ attains a maximum at $x_0 \in \Omega$. Then $p := Du_1(x_0) = Du_2(x_0)$ and:

$$u_1(x_0) + H(x_0, p) \leq 0 \leq u_2(x_0) + H(x_0, p).$$

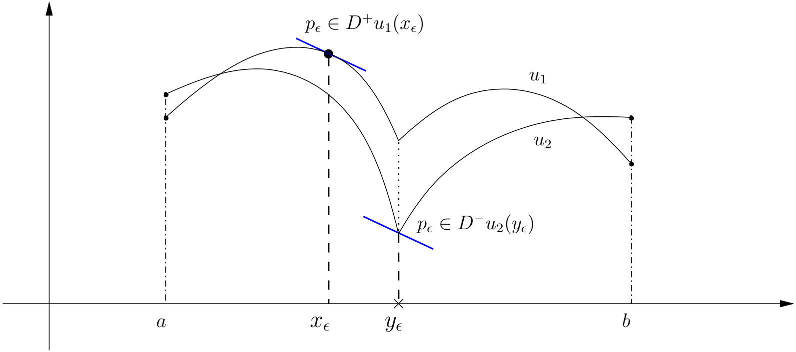
- We conclude that $u_1(x_0) \leq u_2(x_0)$.

Proof - 2: the general case

Assume $u_1 - u_2$ attains a positive maximum at $x_0 \in \Omega$.

- It may happen that $D^+u_1(x_0) \cap D^-u_2(x_0) = \emptyset$. In this case, the HJB equation doesn't give any information at x_0.
- The doubling variables idea consists of building two sequences $(x_\epsilon),\ (y_\epsilon)$ in a neighbourhood of x_0, such that

$$D^+u_1(x_\epsilon) \cap D^-u_2(y_\epsilon) \neq \emptyset$$



$\exists p_\epsilon \in D^+u_1(x_\epsilon) \bigcap D^-u_2(y_\epsilon)$

Proof - 2: the general case

- Assume $u_1 - u_2$ attains a positive maximum at $x_0 \in \Omega$. Set

$$\delta := u_1(x_0) - u_2(x_0) > 0.$$

- Define the function of two variables:

$$\Psi_\varepsilon(x, y) := u_1(x) - u_2(y) - \frac{|x-y|^2}{2\varepsilon}.$$

- Ψ attains its global maximum $(x_\varepsilon, y_\varepsilon)$ on $\overline{\Omega} \times \overline{\Omega}$, and we have:

$$\Psi(x_\varepsilon, y_\varepsilon) \geq \Psi(x_0, x_0) \Longrightarrow \Psi(x_\varepsilon, y_\varepsilon) \geq \delta > 0. \tag{4}$$

- Let $M > 0$ s.t $\|u_1\|_\infty + \|u_2\|_\infty \leq M$. We get:

$$0 < \Psi(x_\varepsilon, y_\varepsilon) \leq M - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon},$$

wich implies that: $|x_\varepsilon - y_\varepsilon| \leq \sqrt{2M\varepsilon}$.

Moreover, $\dfrac{x_\varepsilon - y_\varepsilon}{\varepsilon} \to 0$, when $\varepsilon \to 0$.

- Since $u_2 \in \mathrm{BUC}(\overline{\Omega})$, there exists $\varepsilon' > 0$ s.t.:

$$|u_2(x) - u_2(y)| \leq \delta/2 \qquad \text{whenever } |x - y| \leq \sqrt{2M\varepsilon'}.$$

For every $\varepsilon < \varepsilon'$, we have: $x_\varepsilon, y_\varepsilon \notin \partial\Omega$. Indeed, if $x_\varepsilon \in \partial\Omega$, then we would have:

$$\begin{aligned}\Psi(x_\varepsilon, y_\varepsilon) &\leq (u_1(x_\varepsilon) - u_2(x_\varepsilon)) + |u_2(x_\varepsilon) - u_2(y_\varepsilon)| - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \\ &\leq 0 + \delta/2 + 0\end{aligned}$$

in contradiction with (4).

- We can check that $p_\varepsilon := \dfrac{x_\varepsilon - y_\varepsilon}{\varepsilon} \in D^+u_1(x_\varepsilon) \cap D^-u_2(y_\varepsilon)$
- From the definition of u_1, u_2, we obtain:

$$u_1(x_\varepsilon) + H(x_\varepsilon, p_\varepsilon) \leq 0 \leq u_2(y_\varepsilon) + H(y_\varepsilon, p_\varepsilon).$$

- Finally, by using the assumption on H, it comes:

$$\delta \leq \Psi(x_\varepsilon, y_\varepsilon) \leq u_1(x_\varepsilon) - u_2(y_\varepsilon) \leq \omega(|x_\varepsilon - y_\varepsilon|(1 + p_\varepsilon)|).$$

This yields a contradiction when ε goes to 0.

Time dependent case.

Theorem

Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous:

$$|H(x,p) - H(y,p)| \leq C(|x-y|(1+|p|)),$$
$$|H(x,p_1) - H(x,p_2)| \leq C|p_1 - p_2|.$$

Let u_1, u_2 are resp., a sub- and super-solution of

$$\partial_t u + H(x, Du) = 0 \quad x \in \mathbb{R}^d,\ t \in (0,T]$$

Assume that $u_1(x,0) \leq u_2(x,0)$, then $u_1 \leq u_2$ on $\mathbb{R}^d \times [0,T]$.

Outline

Optimal control problems. HJB equations

Now, consider again the following control problems:

- Mayer's problem:

$$V(x,t) = \inf_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t))$$

- Time minimum problem ($\mathcal{C}$ closed set in $\mathbb{R}^d$):

$$\mathcal{T}(x) = \inf \left\{ t; \;\; y_x(t) \in \mathcal{C}, \; y_x \in \mathcal{S}_{[0,t]}(x) \right\}$$

- Supremum cost:

$$V^g(x,t) = \inf_{y_x \in \mathcal{S}_{[0,t]}(x)} \Phi(y_x(t)) \bigvee \sup_{\theta \in [0,t]} g(y_x(\theta))$$

Theorem

Assume that the minimum time function is continuous. Then it is a viscosity solution of the steady HJB equation:

$$\mathcal{H}(x, D\mathcal{T}(x)) = 1, \quad x \notin \mathcal{C},\ \mathcal{T}(x) < +\infty,$$
$$\mathcal{T}(x) = 0 \quad x \in \mathcal{C}.$$

Proof. Let $\varphi \in C^1$ is such that $\mathcal{T} - \varphi$ has a local maximum at $x \notin \mathcal{C}$, then for any $y_x \in \mathcal{S}_{[0,s]}(x)$ we have (for small s):

$$\mathcal{T}(y_x(s)) - \varphi(y_x(s)) \leq \mathcal{T}(x) - \varphi(x).$$

This implies that:

$$\varphi(x) - \varphi(y_x(s)) \leq \mathcal{T}(x) - \mathcal{T}(y_x(s)) \leq s,$$

and therefore (for $f(x,a) = \dot{y}_x(0)$, with $a \in A$):

$$-f(x,a) \cdot D\varphi(x) \leq 1,$$

so $\mathcal{T}$ is a subsolution. With similar arguments, we prove that $\mathcal{T}$ is a supersolution as well.

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and therefore :

$$\mathcal{H}(x, D\varphi(x)) = \sup_{a \in A}(- f(x,a) \cdot D\varphi(x)) \leq 1,$$

so $\mathcal{T}$ is a subsolution. With similar arguments, we prove that $\mathcal{T}$ is a supersolution as well.

Assume $\mathcal{C}$ is the closure of asmooth domain, and let η_x be the normal to $\mathcal{C}$.
The minimal time function is Lipschitz continuous **if and only if**

$$\min_{a\in A} f(x,a)\cdot\eta_x < 0, \quad \forall x \in \partial\mathcal{C}.$$

The continuity of $\mathcal{T}$ requires a *controllability* assumption of the system around the target. This important property is not satisfied in several examples.

Introduce the Hamiltonian defined as

$$\mathcal{H}(x,q) := \max_{a\in A}(-f(x,a).q).$$

Theorem

Assume that Φ and g are loc. Lipschitz continuous functions. Then the value functions V and V^∞ are loc. Lipschitz continuous functions as well.

- V *is* **the unique viscosity solution** *of the time-dependent HJB equation:*

$$\partial_t V(x,t) + \mathcal{H}(x, D_x V(x,t)) = 0, \quad x \in \mathbb{R}^d,\ t > 0$$
$$V(x,0) = \Phi(x).$$

- V^g *is* **the unique viscosity solution** *of the HJB inequation:*

$$\min\left(\partial_t V^g(x,t) + \mathcal{H}(x, DV^g(x,t)), V^g(x,t) - g(x)\right) = 0,$$
$$V^g(x,0) = \Phi(x) \vee g(x).$$

(see e.g. Crandall-Lions, Crandall-Evans-Lions, Barles, Bardi-Capuzzo Dolcetta, Ishii, ...)

Outline

A classical setting: differential games problems

Let $\alpha \in \mathbb{A}$ be a controlled input, and $\beta \in \mathbb{B}$ an uncontrolled input (perturbance). Consider the trajectory:

$$\begin{cases} \dot{y}_x(s) = f(y_x(s), \alpha(s), \beta(s)), & s \in (0, \tau), \\ y_x(0) = x, \end{cases}$$

for $x \in \mathbb{R}^d$ and $0 \leq \tau \leq T$ (with T a fixed final horizon).

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- Let $\mathcal{C}$ and $\mathcal{K}$ be two closed sets (of constraints), and consider a zero-sum game involving two players:

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for $x \in \mathbb{R}^d$ and $0 \leq \tau \leq T$ (with T a fixed final horizon).

- Let $\mathcal{C}$ and $\mathcal{K}$ be two closed sets (of constraints), and consider a zero-sum game involving two players:
  - The first player wants to steer the system from the initial position at point x to the target $\mathcal{C}$ and by staying in $\mathcal{K}$ (and using his/her input $\alpha(\cdot) \in \mathbb{A}$);
  - The second player tries to steer the system away from $\mathcal{C}$ or from $\mathcal{K}$ (with his/her input $\beta(\cdot) \in \mathbb{B}$).

Non anticipative strategies (Ref. Elliott and Kalton'72)

The set of control inputs are given by:

$$\begin{aligned} \mathbb{A} &:= \{\alpha : [0, T] \to \mathbb{R}^m \text{ measurable}, \ \alpha(s) \in \mathcal{A}\}, \\ \mathbb{B} &:= \{\beta : [0, T] \to \mathbb{R}^r \text{measurable}, \ \beta(s) \in \mathcal{B}\}, \end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}$ are compact sets.

We define the set of non-anticipative strategies for the first player, as follows:

$$\Gamma := \Big\{ a : \mathbb{B} \to \mathbb{A}, \ \forall(\beta, \tilde{\beta}) \in \mathbb{B} \text{ and } \forall s \in [0, \infty), \\ \Big(\beta(\theta) = \tilde{\beta}(\theta) \ a.e. \ \theta \in [0, s]\Big) \Rightarrow \Big(a[\beta](\theta) = a[\tilde{\beta}](\theta) \ a.e. \ \theta \in [0, s]\Big) \ \Big\}.$$

- Assume that $\mathcal{K}$ is a closed non-empty set.
- Consider a function $g : \mathbb{R}^d \to \mathbb{R}$, Lipschitz continuous, such that

$$\forall x \in \mathbb{R}^d, \quad g(x) \leq 0 \Leftrightarrow x \in \mathcal{K}.$$

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- In particular,

$$\Big[\mathbf{y}(s) \in \mathcal{K}, \ \forall s \in [0, \tau]\Big] \Longleftrightarrow \max_{s \in [0,\tau]} g(\mathbf{y}(s)) \leq 0.$$

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- Consider a function $\varphi : \mathbb{R}^d \to \mathbb{R}$, Lipschitz continuous, such that
$$\forall x \in \mathbb{R}^d, \quad \varphi(x) \leq 0 \Leftrightarrow x \in \mathcal{C}.$$

Formulation of the control problem (Bokanowski-Forcadel-HZ'13, Gammoudi-HZ'19)

- Consider the following two-players control problem ($z \in \mathbb{R}$):

$$w(\tau, x) := \min_{a[.]\in\Gamma} \max_{\beta\in\mathbb{B}} \left\{ \max_{s\in[0,\tau]} g(y_x^{a[\beta],\beta}(s)) \bigvee \varphi(y_x^{a[\beta],\beta}(\tau)) \right\}.$$

where $a \bigvee b$ stands for $\max(b, b)$.

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Theorem

For every $\tau \in [0, T]$, we have:

$$(i) \quad \mathcal{R}(\tau) = \left\{ x : w(\tau, x) \leq 0 \right\},$$

$$(ii) \quad \mathcal{T}(x) = \min \left\{ \tau \in \mathbb{R} \, , \; w(\tau, x) \leq 0 \right\}, \quad \textit{for every } x \in \mathcal{K},$$

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(ii) $\quad \mathcal{T}(x) = \min \Big\{ \tau \in \mathbb{R}\,,\ w(\tau, x) \leq 0 \Big\},$ *for every* $x \in \mathcal{K}$,

(iii) *Under some controllability conditions: for every* $x \in \overset{\circ}{\mathcal{K}}$ *we have:*

$$\mathcal{T}(x) = \tau \iff w(\tau, x) = 0.$$

➤ Define the Hamiltonian as:

$$\mathcal{H}^{\#}(x,p) := \min_{b\in\mathcal{B}} \max_{a\in\mathcal{A}} \big(-f(x,a,b)\cdot p \big) \qquad \forall x,p \in \mathbb{R}^d.$$

Theorem (Altarovici-Bokanowski-HZ'13, Gammoudi-HZ'19)

The value function w is the unique continuous viscosity solution of the following Hamilton-Jacobi (HJ) equation:

$$\min\left(\partial_t w(t,x) + \mathcal{H}^{\#}(x, D_x w),\ w(t,x) - g(x)\right) = 0 \quad]0,T]\times\mathbb{R}^d,$$

$$w(0,x) = g(x) \bigvee \varphi(x), \qquad \mathbb{R}^d \times \mathbb{R}.$$

Reconstruction of optimal trajectories - **Algorithm A.**

- For $n \geq 1$, consider $(t_0 = 0, t_1, ..., t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with $h = \frac{T}{n}$.
- Assume that the second player choose to use a given input $\bar{\beta}(\cdot)$
- Let $\mathbf{y}^n(\cdot)$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$, with $\mathbf{y}^n(0) = x$.

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$$\alpha_k^n \in \arg\min_{\alpha \in \mathcal{A}} \left(w(t_{n-k}, y_k^n + hf_h(y_k^n,\ \alpha, \bar{\beta}(t_k))) \right).$$

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$$\dot{\mathbf{y}}(t) := f(\mathbf{y}(t), \alpha^n(t), \bar{\beta}(t)) \text{ a.e } \ t \in (t_k, t_{k+1}],$$

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Let $\{\mathbf{y}^n(\cdot), \alpha^n(\cdot)\}$ be a sequence generated by algorithm A for $n \geq 1$. Then, the sequence of trajectories $\{\mathbf{y}^n(\cdot)\}_n$ has cluster points with respect to the uniform convergence topology.

- *In the case of one-player game (i.e, $\mathcal{B} = \{b\}$), for any cluster point $\bar{\mathbf{y}}(\cdot)$ there exists a control law $\bar{\alpha}(\cdot)$ such that $(\bar{\mathbf{y}}(\cdot), \bar{\alpha}(\cdot))$ is optimal for the auxiliary control problem.*
- *In the case of zero-sum game, for any cluster point $\bar{\mathbf{y}}(\cdot)$ there exists a control law $\bar{\alpha}(\cdot)$ such that $(\bar{\mathbf{y}}(\cdot), \bar{\alpha}(\cdot))$ is suboptimal.*

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➤ Let w^Δ be a numerical approximate solution such that,

$$|w^\Delta(t, y) - w(t, y)| \leq E_1(\Delta t, \Delta y),$$

where $E_1(\Delta t, \Delta y) \to 0$ as $\Delta t, \Delta y \to 0$.

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➤ Same convergence result if we replace w by w^Δ.

Worst scenario - **Algorithm B.**

- For $n \geq 1$, consider $(t_0 = 0, t_1, ..., t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with $h = \frac{T}{n}$.
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Let $\{\mathbf{y}^n(\cdot), \alpha^n(\cdot), \beta^n(\cdot)\}$ be a sequence generated by algorithm B for $n \geq 1$. Then, the sequence of trajectories $\{\mathbf{y}^n(\cdot)\}_n$ has cluster points with respect to the uniform convergence topology.

- (α^n, β^n) satisfies a Nash equilibrium.
- Any cluster point is suboptimal soulution (coresponding to the worst case).

A controlled heterogeneous population of cells -

(Carrère-HZ'18)

Ref: Barbolosi-Freyer-Ciccolini-Iliadis'03, Ding et al.'13, ...

$$\begin{cases} \frac{ds}{dt}(t) = \rho s(t)(1 - \frac{s(t)+mr(t)}{K}) - \beta_1(t)s(t)\mathbf{u}(t), \\ \frac{dr}{dt}(t) = \rho r(t)(1 - \frac{s(t)+mr(t)}{K}) - \beta_2(t)s(t)r(t), \\ \frac{dw}{dt}(t) = \rho_w - \mu w(t) - \nu w(t)\max(0, \mathbf{u}(t) - u_{\mathrm{tox}}). \end{cases} \qquad \text{(MHP)}$$

- $s(t)$: *sensitive* cancerous cells population
- $r(t)$: *resistant* cells population
- $w(t)$: *a health indicator* (white blood cells count)
- $\mathbf{u}(t) \in [0, U_{\max}]$: *drug dosage*
- $\beta = (\beta_1(t), \beta_2(t)) \in B$: *uncertainties* on the drug efficiency
- $\rho, \rho_w, \mu, \nu, u_{\mathrm{tox}}$: known parameters
- K is the size of the Petri dish and m the size ratio between sensitive and resistant cells

Problem 1

Let $0 < Q < K$ be a fixed threshold in tumour size. Given $x \in R^3$, does there exist a strategy **u** such that for any perturbation $\beta(\cdot)$, the tumour size never exceeds the size Q:

$$y(t) \in \mathcal{K} := \{(s, r, w) \mid s + mr \leq Q\}.$$

We need to determine the set:

Problem 2

Given a certain time of treatment T, and for $x \in R^3$, does there exist a strategy **u** such that for any perturbation $\beta(\cdot)$, the tumour size can be stabilized under the threshold Q.

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$$\mathcal{N}_{\mathcal{K}} := \{x \in \mathbb{R}^3 \mid \exists \mathbf{u}[\cdot], \quad \forall \beta \in \mathcal{B}, \forall t > 0, \ y_x^{\mathbf{u}[\beta],\beta}(t) \in \mathcal{K}\}.$$

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$$\mathcal{R}_T := \{x \in \mathbb{R}^3 \mid \exists \mathbf{u}[\cdot], \quad \forall \beta \in \mathcal{B}, \quad y_x^{\mathbf{u}[\beta],\beta}(\cdot) \subset \mathcal{D}, \quad \textbf{and } \exists t \in [0, T] y_x^{\mathbf{u}[\beta],\beta}(t) \in \mathcal{N}$$

Problem 1

- Consider the control problem:

$$V(x) := \min_{\mathbf{u}[\cdot]} \max_{\beta \in \mathcal{B}} \int_0^{+\infty} e^{-\lambda t} d_{\mathcal{K}}^{+}(y_x^{\mathbf{u}[\beta],\beta}(t))dt,$$

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Problem 2: For a given time horizon T

- For every $x \in \mathbb{R}^n$ and $t \in [0, T]$, we define

$$W(x,t) := \min_{\mathbf{u}[\cdot]} \max_{\beta \in \mathcal{B}} \left\{ \left(\max_{0 \leq \tau \leq t} d_{\mathcal{D}}(y_x^{\mathbf{u}[\beta],\beta}(\tau)) \right) \bigvee V_Q(y_x^{\mathbf{u}[\beta],\beta}(t)) \right\}.$$

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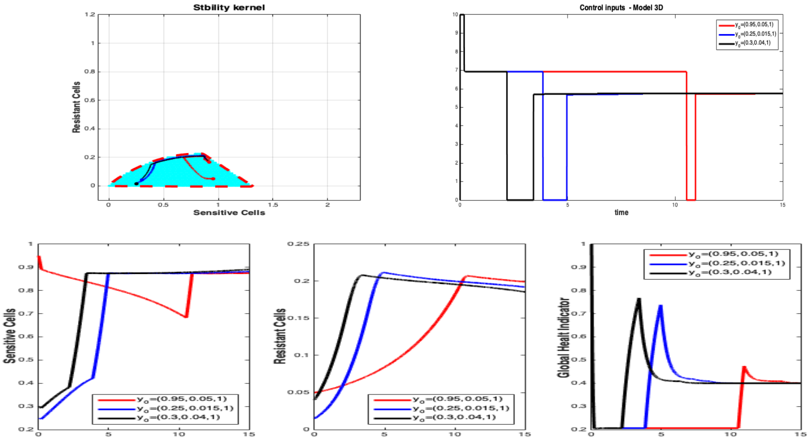
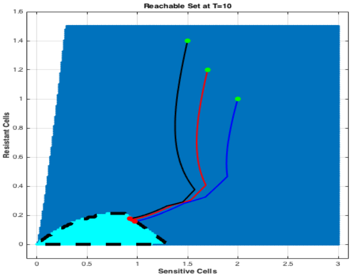
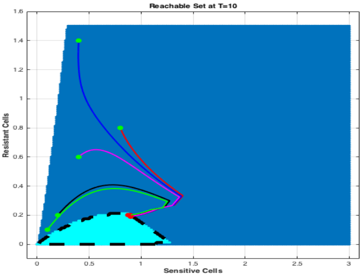
An illustration with $Q = 0.13$, stability kernel and some viable trajectories (without uncertainties)



An illustration with $Q = 0.13$, Reachability and some robust trajectories (without uncertainties)





Stability kernel, reachability set (case with uncertainties)