

# Introduction to Quasi-variational Inequalities in Hilbert Spaces

## Motivations, and elliptic problems



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# Introduction, motivation, and lack of energy formulation

# Introduction - The Dirichlet principle

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Consider the deformation of an elastic string on  $[0, 1]$ . Suppose that

- ▶ displacement is given by  $y : (0, 1) \rightarrow \mathbb{R}$ .
- ▶ endpoints are clamped, i.e.,  $y(0) = y(1) = 0$ .
- ▶ the string is loaded by a normal uniform force  $f : (0, 1) \rightarrow \mathbb{R}$ .

# Introduction - The Dirichlet principle

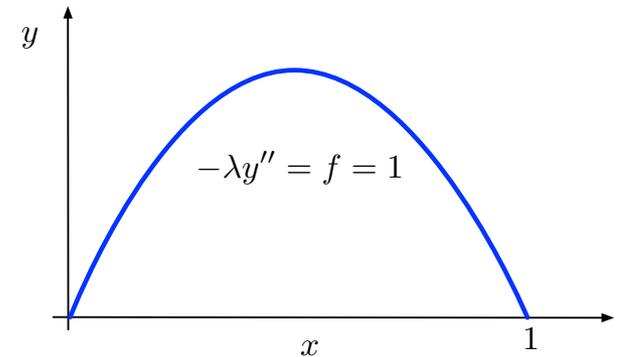
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The displacement satisfies the problem

$$\begin{aligned} -\lambda y''(x) &= f(x), \quad \text{in } (0, 1) \\ y(s) &= 0 \quad \text{for } s = 0, 1; \end{aligned} \quad (\text{BVP})$$

- ▶  $\lambda > 0$  depends on elastic material properties.



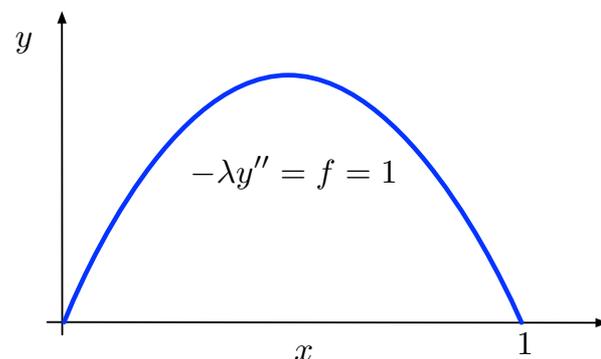
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- ▶  $\lambda > 0$  depends on elastic material properties.

Problem (BVP) is equivalent to minimizing the potential energy:

$$\begin{aligned} \min_v \quad & \frac{\lambda}{2} \int_0^1 |v'(x)|^2 dx - \int_0^1 f(x)v(x) dx; \\ \text{subject to (s.t.)} \quad & v(0) = v(1) = 0; \end{aligned}$$

# Introduction - The Dirichlet principle with obstacle

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Suppose in addition that

- ▶ there is an obstacle  $\phi : (0, 1) \rightarrow \mathbb{R}$  such that  $y(x) \leq \phi(x)$  a.e. in  $(0, 1)$ .

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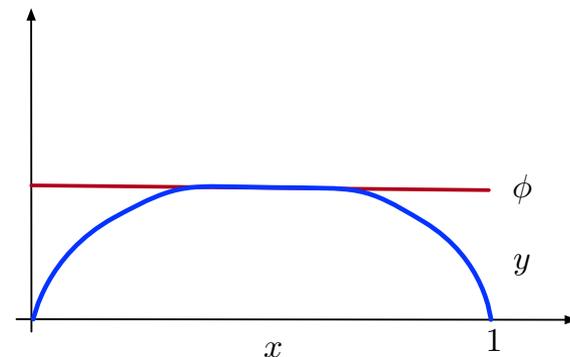
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The displacement satisfies  $y \leq \phi$  and

$$\lambda \int_0^1 y'(v' - y') dx \geq \int_0^1 f(v - y) dx, \quad (\text{VI})$$

$$y(s) = 0 \quad \text{for } s = 0, 1;$$

For all  $v$  smooth s.t.  $v(0) = v(1) = 0$  and  $v \leq \phi$ .



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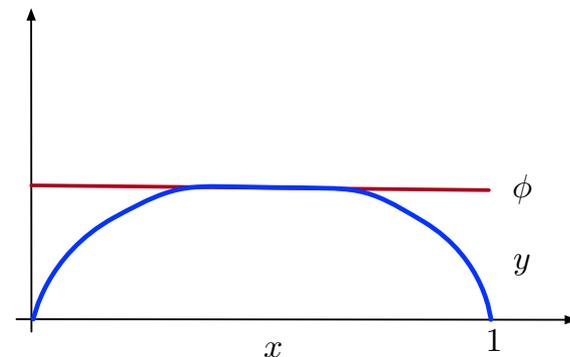
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$$y(s) = 0 \quad \text{for } s = 0, 1;$$

For all  $v$  smooth s.t.  $v(0) = v(1) = 0$  and  $v \leq \phi$ .



Problem (VI) is equivalent to minimizing the (constrained) potential energy:

$$\min_v \frac{\lambda}{2} \int_0^1 |v'(x)|^2 dx - \int_0^1 f(x)v(x) dx;$$

$$\text{subject to (s.t.) } v(0) = v(1) = 0;$$

$$v \leq \phi$$

# Introduction - Lack of Dirichlet principle with implicit obstacle

---

Suppose in addition that

- ▶ there is an **implicit** obstacle  $\phi : (0, 1) \rightarrow \mathbb{R}$  depending on  $y$ , i.e.,  $\phi = \Phi(y)$ .

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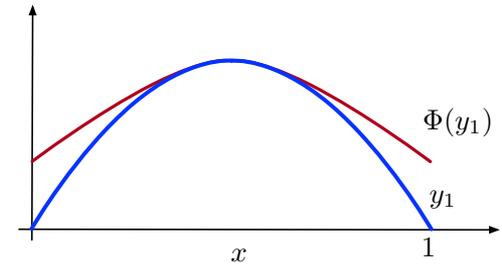
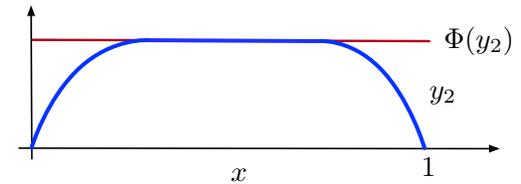
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$$\lambda \int_0^1 y'(v' - y') \, dx \geq \int_0^1 f(v - y) \, dx, \quad (\text{QVI})$$

$$y(s) = 0 \quad \text{for } s = 0, 1;$$

$$\forall v \text{ smooth s.t. } v(0) = v(1) = 0 \text{ and } v \leq \Phi(y).$$



# Introduction - Lack of Dirichlet principle with implicit obstacle

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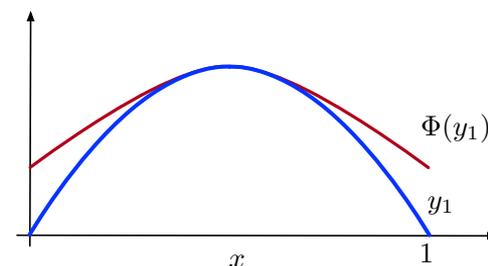
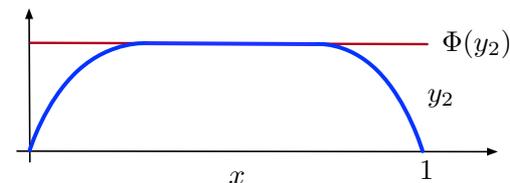
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$$y(s) = 0 \quad \text{for } s = 0, 1;$$

$\forall v$  *smooth* s.t.  $v(0) = v(1) = 0$  and  $v \leq \Phi(y)$ .



Problem (QVI) is **NOT** (in general) equivalent to minimizing the (implicitly constrained) potential energy:

$$\min_v \frac{\lambda}{2} \int_0^1 |v'(x)|^2 \, dx - \int_0^1 f(x)v(x) \, dx;$$

subject to (s.t.)  $v(0) = v(1) = 0$ ;

$$v \leq \Phi(v)$$

# The general elliptic QVI problem

# The class of elliptic QVIs

Let  $A : V \rightarrow V'$  and  $f \in V'$  for some (real) Hilbert space  $V$ . Consider

$$\text{Find } y \in \mathbf{K}(y) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y) \quad (\text{QVI})$$

where

$$\mathbf{K}(w) := \{z \in V : \Psi(Gz) \leq \Phi(w)\}.$$

- ▶  $V \in \{H_0^1(\Omega), H^1(\Omega), L^2(\Omega), \dots\}$ , and part of a Gelfand triple  $(V, H, V')$ .
- ▶  $A$  is Lipschitz continuous and strongly monotone, i.e.,

$$\langle A(u) - A(v), u - v \rangle \geq c \|u - v\|_V^2.$$

- ▶  $G : V \rightarrow L^p(\Omega)^d$  is linear, and  $\Psi : \mathbb{R}^d \rightarrow \mathbb{R}$  is convex.
- ▶  $\Phi(v) : \Omega \rightarrow \mathbb{R}$  is measurable.

Obstacle type

$$G = \text{id, and } \Psi(x) = x.$$

Gradient type

$$G = \nabla, \text{ and } \Psi(x) = \|x\|_{\mathbb{R}^n}.$$

# The class of elliptic QVIs

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Many contributors [Adly](#), [Alphonse](#), [Aubin](#), [Aussel](#), [Barrett](#), [Bensoussan](#), [Bergounioux](#), [Biroli](#), [Caffarelli](#), [Facchinei](#), [Friedman](#), [Frehse](#), [Fukao](#), [Fukushima](#), [Gwinner](#), [Hanouzet](#), [Harker](#), [Hintermüller](#), [Joly](#), [Kano](#), [Kanzow](#), [Kenmochi](#), [Lions](#), [Mignot](#), [Mordukhovich](#), [Mosco](#), [Murase](#), [Outrata](#), [Pang](#), [Prigozhin](#), [Rodrigues](#), [Santos](#), [Tartar](#), [Yousept](#), . . . .

- [A. Bensoussan, J.-L. Lions](#) , In a series of papers concerned with impulse control (1973/1974).

# The class of elliptic QVIs : Prototypicals $V$ , $A$ , and $\mathbf{K}$

The typical setting is given by

▶  $(V, H, V') = (H_0^1(\Omega), L^2(\Omega), H^{-1}(\Omega)).$

▶ The map  $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  is given

$$\langle Av, w \rangle = \int_{\Omega} \left( \sum_{i,j} a_{ij}(x) \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_i} + \sum_i a_i(x) \int_{\Omega} \frac{\partial v}{\partial x_j} w + a_0(x)vw \right) dx,$$

with usual assumptions over coefficients. Also fractional powers  $A^s$  for  $s \in (0, 1)$  are suitable.

▶  $\Phi : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is

▲ A *superposition operator*, i.e.,  $\Phi(y)(x) = \varphi(y(x))$  for some  $\varphi$ .

▲ A *solution operator coming from a PDE*, e.g.,  $\Phi(y) = (-\Delta)^{-1}y + \phi_0$ .

# The class of elliptic QVIs

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## Main difficulties:

- ▶ The problem is non-smooth and non-convex.
- ▶ In general there are multiple solutions. The solution set

$$\mathbf{Q}(f)$$

might be of any cardinality!

- ▶ In general, the problem does not arise as first order condition of an optimization problem.

# The class of elliptic QVIs - Cardinality of $\mathbf{Q}(f)$

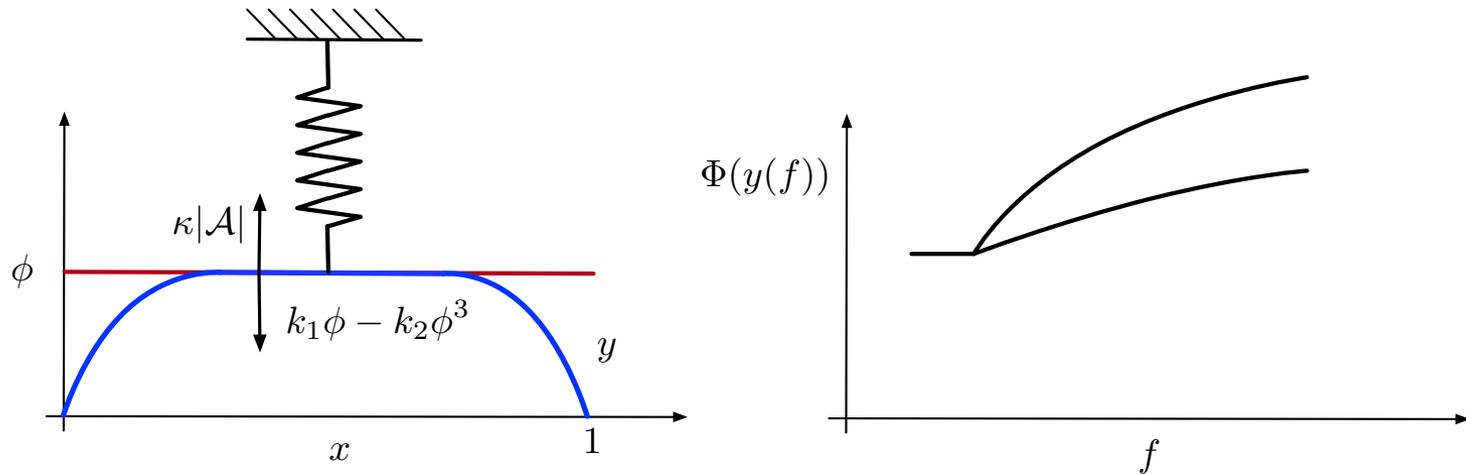
$$\text{Find } y \leq \Phi(y) : \langle Ay - f, v - y \rangle \geq 0, \quad \forall v \leq \Phi(y). \quad (\text{QVI})$$

► If  $\Phi : V \rightarrow V$  is Lipschitz with small Lipschitz constant  $\implies$  *Unique* solution.

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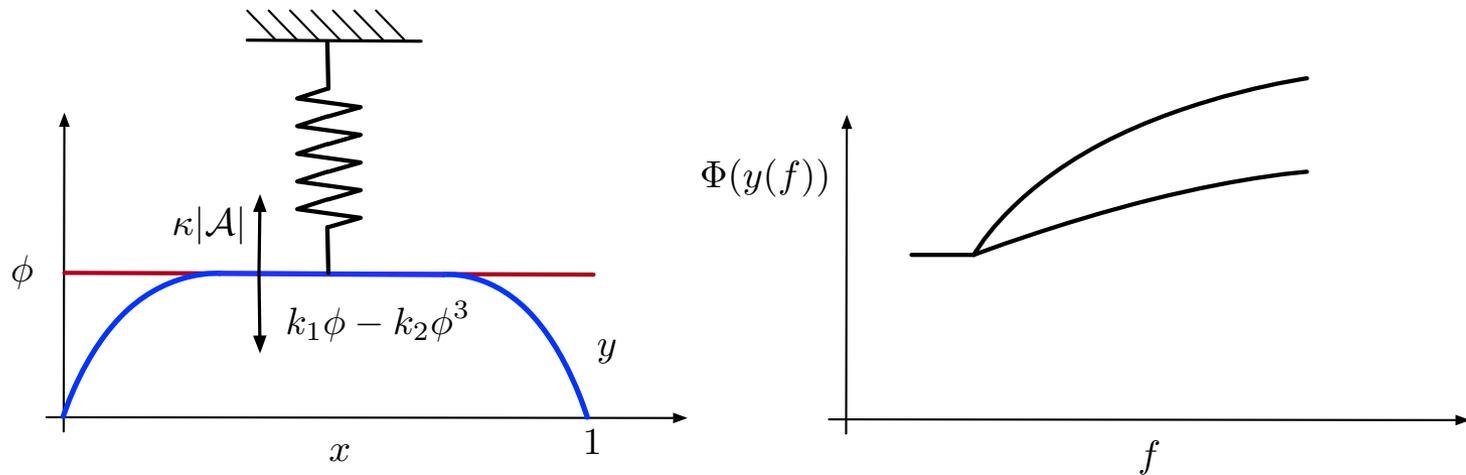
- ▶ If  $\Phi : V \rightarrow V$  is Lipschitz with small Lipschitz constant  $\implies$  *Unique solution.*
- ▶ Let  $A = -\Delta$ ,  $V = H_0^1(\Omega)$ ,  $f \in \mathbb{R}$   $\implies$  *Two solutions or one.*



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- ▶ Let  $\Phi(y) = y$ , then if  $Ay \leq f$  then  $y$  is a solution of (QVI)  $\implies$  *Multiple solutions - Not isolated solutions.*

# Historical note on quasi-variational inequalities

# A little history on QVIs

For  $x \in \mathbb{R}^N$ ,  $\sigma > 0$ ,  $w$  a Wiener process and  $g : \mathbb{R}^N \rightarrow \mathbb{R}^N$  Lipschitz and bounded, consider

$$\begin{aligned} dz &= g(z) dt + \sigma I dw(t) + \sum_i \delta(t - \theta_i) \xi_i \\ z(0) &= x. \end{aligned} \tag{P_x}$$

Admissible controls  $\mathbf{U}$  for  $(P_x)$  are given by  $\{(\theta_i, \xi_i)\}_{i=1}^{\infty}$  where

- ▶ Time instants  $\{\theta_i\}$  such that  $0 = \theta_0 \leq \theta_1 \leq \theta_2 \leq \dots$  and  $\lim_i \theta_i = +\infty$
- ▶ State jumps  $\{\xi_i\}$  on some compact subset  $\mathbf{K} \subset \mathbb{R}^N$ ,

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A. Bensoussan and J. L. Lions considered the optimal impulse control problem

Given  $k > 0$  and  $\alpha > 0$ , consider

$$\min_{\nu \in \mathbf{U}} J(x, \nu) := \mathbb{E} \left\{ \int_0^{\infty} e^{-\alpha t} f(z_x(t)) dt + k \sum_i e^{-\alpha \theta_i} \right\}, \tag{P}$$

where  $\nu = \{(\theta_i, \xi_i)\}$ , and  $z_x$  solves  $(P_x)$ .

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Define the value function

$$y^*(x) := \inf_{\nu \in \mathbf{U}} J(x, \nu^*).$$

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## Benoussan-Lions(1974)

Under certain conditions,  $\exists \tilde{y}$  solution to: Find  $y \in H^1(\mathbb{R}^N)$  such  $y \leq My$  and

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where  $My(x) := k + \inf_{\xi \in \mathbf{K}} y(x + \xi)$ .

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- ▶ The value function  $y^*$  is a solution to the QVI above.
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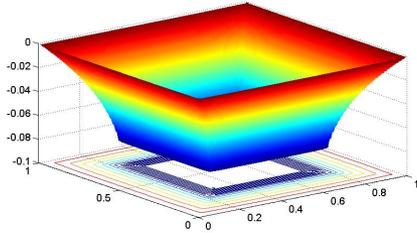
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where  $Mw(x) := k + \inf_{\xi \in \mathbf{K}} u(x + \xi)$ .

- ▶ The value function  $y^*$  is a solution to the QVI above.
- ▶ There exists a solution to  $(\mathbb{P}) \rightarrow$  it can be constructed via a solution  $\tilde{y}$ .

QVIs are powerful models



► **Stationary Magnetization of a superconductor**

Determination of the magnetic field

[Rodrigues, Prigozhin, Yousept]

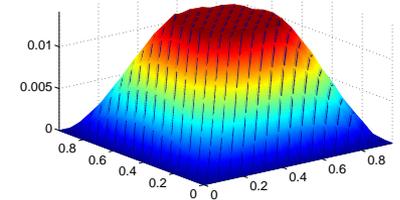
$$A(y) = -\operatorname{div}|\nabla y|^{p-2}\nabla y \quad \mathbf{K}(y) = \{v : |\nabla v| \leq \Phi(y)\},$$

$\Phi$  is a superposition operator.

- **Flow trough semi-permeable membranes/anomalous diffusion.** Pressure of a chemical solution of a slightly incompressible fluid [Antil, R. (2018)].

$$A = (-\Delta)^s \quad \mathbf{K}(y) = \{v : v \leq \Phi(y)\},$$

with  $s \in (0, 1)$ , and  $\Phi$  solution map of a PDE.



- **Thermo-elastic equilibrium of a locking material.** Determination of a the displacement field [Rodrigues-Santos (2019)].

$$\langle A(y), v \rangle = \mu(Dy, Dv) + \lambda(\nabla \cdot y, \nabla \cdot v) \quad \mathbf{K}(y) = \{v \in H_0^1(\Omega) : |Dv| \leq \Phi(y)\}$$

# The compactness approach

# Existence theory - Compactness approaches

$$\text{Find } y \in \mathbf{K}(y) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y). \quad (\text{QVI})$$

Define  $S(f, \mathbf{K})$ , for a fixed closed, and convex set  $\mathbf{K}$  as the unique solution to:

$$\text{Find } z \in \mathbf{K} : \langle A(z) - f, v - z \rangle \geq 0, \quad \forall v \in \mathbf{K}.$$

Then, for non-constant  $v \mapsto \mathbf{K}(v)$ , consider  $T(v) := S(f, \mathbf{K}(v))$  so that

$$y = T(y) \iff y \text{ solves (QVI).}$$

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## Proposition

Since  $V$  is reflexive, if  $T : V \rightarrow V$  is completely continuous, i.e.,

$$v_n \rightharpoonup v \quad \text{implies} \quad T(v_n) \rightarrow T(v),$$

then  $T$  has fixed point à la Schauder ( $T(B_R(0, V)) \subset B_R(0, V)$  is simple to obtain).

# Existence theory - Compactness approaches

$$\text{Find } y \in \mathbf{K}(y) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y). \quad (\text{QVI})$$

Define  $S(f, \mathbf{K})$ , for a fixed closed, and convex set  $\mathbf{K}$  as the unique solution to:

$$\text{Find } z \in \mathbf{K} : \langle A(z) - f, v - z \rangle \geq 0, \quad \forall v \in \mathbf{K}.$$

Then, for non-constant  $v \mapsto \mathbf{K}(v)$ , consider  $T(v) := S(f, \mathbf{K}(v))$  so that

$$y = T(y) \iff y \text{ solves (QVI).}$$

## Proposition

Since  $V$  is reflexive, if  $T : V \rightarrow V$  is completely continuous, i.e.,

$$v_n \rightharpoonup v \quad \text{implies} \quad T(v_n) \rightarrow T(v),$$

then  $T$  has fixed point à la Schauder ( $T(B_R(0, V)) \subset B_R(0, V)$  is simple to obtain).

Since  $T(v) = S(f, \mathbf{K}(v))$ , we need a concept of set convergence so that

$$S(f, \mathbf{K}_n) \rightarrow S(f, \mathbf{K}).$$

# Mosco convergence

$$\text{Find } y \in \mathbf{K}(y) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y). \quad (\text{QVI})$$

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If  $v_n \rightharpoonup v$  in  $V$  implies  $\mathbf{K}(v_n) \xrightarrow{\text{M}} \mathbf{K}(v)$  then  $T$  is completely continuous.

# Mosco convergence - One Example

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Consider  $\mathbf{K}_n = \{v \in H_0^1(\Omega) : v \leq \phi_n\}$  and  $\phi_n \rightarrow \phi$  in  $H_0^1(\Omega)$ .

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►  $\phi_n \rightarrow \phi$  in  $H_0^1(\Omega)$  is very strong. How can we relax this?

# Mosco convergence for obstacle constraints

$$\mathbf{K}_n = \{v \in V : v \leq \phi_n\}$$

- ▶ For  $V = W_0^{1,p}(\Omega)$ , necessary and sufficient conditions on the convergence  $\phi_n \rightarrow \phi$  for  $\mathbf{K}_n \xrightarrow{\text{M}} \mathbf{K}$  are given in terms of capacity  
([Attouch-Picard (1979-1982), Dal Maso (1985)])

The above is not so useful in applications: It is of theoretical interest though, but hard to prove unless high regularity is available for obstacles!

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- ▶ A useful result is the following ([Boccardo-Murat (1982)]): For  $V = W_0^{1,p}(\Omega)$ ,  
 $\phi_n \rightarrow \phi$  in  $W^{1,q}(\Omega)$  or  $W_0^{1,q}(\Omega)$ , for  $q > p \implies \mathbf{K}_n \xrightarrow{\text{M}} \mathbf{K}$ .

The lesson is that you need only an  $\epsilon > 0$  more of regularity of your obstacles than in your state space. This is simple to prove in applications, e.g., in the case when  $\Phi(v) = \phi$  is given by the PDE

$$-\Delta \phi = g(v) + g_0.$$

# Mosco convergence for gradient constraints

$$\mathbf{K}_n = \{v \in H_0^1(\Omega) : |\nabla v| \leq \phi_n\}$$

- ▶ If  $\phi_n \rightarrow \phi$  in  $C(\overline{\Omega})$  if  $\partial\Omega$  is regular enough then  $\mathbf{K}_n \xrightarrow{M} \mathbf{K}$ . Regularity of  $\partial\Omega$  plays a role if  $\phi$  is not strictly positive.
- ▶ If  $\phi_n \rightarrow \phi$  in  $L^\infty_\nu(\Omega)$  then  $\mathbf{K}_n \xrightarrow{M} \mathbf{K}$ .  
([Santos-Assis (2004), Hintermüller-R. (2015)])

Extension from  $L^\infty(\Omega)$  or  $C(\overline{\Omega})$  to other spaces is rather complex due to the non-Lipschitz dependence of the solution w.r.t. the gradient obstacle.

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Extension from  $L^\infty(\Omega)$  or  $C(\bar{\Omega})$  to other spaces is rather complex due to the non-Lipschitz dependence of the solution w.r.t. the gradient obstacle.

- ▶ **(Open problem)** Necessary and sufficient conditions on  $\{\phi_n\}$  so that  $\mathbf{K}_n \xrightarrow{M} \mathbf{K}$  holds are not known.

This is a complex task! Let  $|\nabla y| \leq \phi$  and  $\phi_n \rightarrow \phi$  in some topology  $X$ , try to construct  $y_n$  such that  $|\nabla y_n| \leq \phi_n$  and  $y_n \rightarrow y$ !

# Mosco convergence for mixed constraints

---

Let  $L \in V'$  and consider the sequence of sets

$$\mathbf{K}_n = \{v \in V : |v| \leq \phi_n \quad \& \quad L(v) = \alpha_n\}$$

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► **(Open problem)** Nontrivial sufficient conditions on  $\{\phi_n\}$  and  $\{\alpha_n\}$  so that  $\mathbf{K}_n \xrightarrow{\text{M}} \mathbf{K}$  holds are not known.

Unilateral and isoperimetric type constraints do not mix easily.

# The contraction approach

# Contractions

$$\text{Find } y \in \Phi(y) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \Phi(y).$$

► If  $\Phi : V \rightarrow V$  is Lipschitz with small Lipschitz constant  $\implies$  *Unique solution.*

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► If  $\Phi : V \rightarrow V$  is Lipschitz with small Lipschitz constant  $\implies$  *Unique solution.*

$$\text{Find } y : |\nabla y| \leq \Phi(y) \ \& \ \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v : |\nabla v| \leq \Phi(y).$$

Theorem ([Hintermüller-R.(2013), Rodrigues-Santos(2019)])

Let  $V = H_0^1(\Omega)$ ,  $f \in L^2(\Omega)$ , and suppose that  $\Phi : V \rightarrow L^\infty(\Omega)$  is defined as

$$\Phi(y) = \lambda(y)\phi$$

with  $\phi \in L^\infty(\Omega)$  and  $\lambda$  a Lipschitz continuous nonlinear functional on  $V$ . Then, provided  $\|f\|_{L^2}$  or  $L_\lambda$  are sufficiently small, the QVI of interest has a unique solution.

► **(Open problem)** Extension to  $\Phi(y) = \lambda_1(y)\phi_1 + \lambda_2(y)\phi_2$  is not direct nor known.

# The contraction approach

## Some problems with the QVI literature

# Some problems with the literature

- ▶ Unfortunately, a significant amount of the literature on QVIs attempts to extend the famous Lions-Stampacchia result (for existence and design of algorithms) with an extreme assumption on the projection map  $v \mapsto P_{\mathbf{K}(v)}$ .

Suppose  $V$  is a Hilbert space, and consider

$$\text{Find } y \in \mathbf{K}(y) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y), \quad (\text{QVI})$$

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...(QVI) can be equivalently written ( $\rho > 0$ ) as

$$\text{Find } y \in \mathbf{K}(y) : (y - j(iy - \rho(A(y) - f)), v - y) \geq 0, \quad \forall v \in \mathbf{K}(y),$$

where  $i : V \rightarrow V'$  is the duality operator and  $j := i^{-1}$  the Riesz map.

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$$y \text{ solves (QVI)} \iff y = B_\rho(y) := P_{\mathbf{K}(y)}(y - \rho j(A(y) - f)).$$

For a closed, convex  $\mathbf{C} \subset V$

$$\|P_{\mathbf{C}}(v) - v\|_V := \inf_{w \in \mathbf{C}} \|w - v\|.$$

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$\mathbf{K}(v) \equiv \mathbf{K}$  constant

[Lions-Stampacchia (1967)]

For some  $\rho > 0$ ,

▶  $B_\rho$  is contractive

$\implies$  the (VI) has a unique solution.

For a closed, convex  $\mathbf{C} \subset V$

$$\|P_{\mathbf{C}}(v) - v\|_V := \inf_{w \in \mathbf{C}} \|w - v\|.$$

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If  $v \mapsto \mathbf{K}(v)$  not constant, to follow the same approach, the assumption used is

$$\|P_{\mathbf{K}(y)}(w) - P_{\mathbf{K}(z)}(w)\|_V \leq \eta \|y - z\|_V \quad (\mathbb{H})$$

for some  $0 < \eta < 1$  and all  $y, z, w$  in some sufficiently large set in  $V$ .

Non-expansiveness

$$\|P_{\mathbf{C}}(w_1) - P_{\mathbf{C}}(w_2)\|_V \leq \|w_1 - w_2\|_V, \quad \forall w_1, w_2 \in V,$$

holds but it is not the above one!

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## THEOREM ([Attouch-Wets])

Let  $V$  be a Hilbert space and  $\mathbf{K}_1, \mathbf{K}_2$  any two closed, convex, non-empty subsets of  $V$ . For  $y_0 \in V$ , we have that

$$\|P_{\mathbf{K}_1}(y_0) - P_{\mathbf{K}_2}(y_0)\|_V \leq \rho^{1/2} \text{Haus}_\rho(\mathbf{K}_1, \mathbf{K}_2)^{1/2}$$

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with  $\rho := \|y_0\| + d(y_0, \mathbf{K}_1) + d(y_0, \mathbf{K}_2)$ .

- ▶ The  $1/2$  exponent in the right hand side expression is optimal.
- ▶ Examples (even in finite dimensions) can be found where equality holds
- ▶ In Banach spaces like  $L^p(\Omega)$  or  $\ell^p(\mathbb{N})$ , the exponent further degrades: it is  $1/p$  if  $2 < p < +\infty$  and  $1/p'$  if  $1 < p < 2$ .

# Some problems with the literature

- ▶ What does the Attouch-Wets result imply for real applications?

## Example: Gradient constraints

Let  $\Omega = (0, 1)$  and  $V = \{v \in H^1(\Omega) : v(0) = 0\}$ .

Let  $\mathbf{K}_i := \{v \in V : |v'| \leq \phi_i\}$  with  $\phi_2, \phi_1 > 0$  constant, so that

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for sufficiently large  $\rho > 0$ . So that for

$$\mathbf{K}(w) = \{z \in V : |z'| \leq \Phi(w)\},$$

and Attouch-Wets imply if  $\Phi : V \rightarrow \mathbb{R}$  is Lipschitz continuous

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Are there examples for which

$$\|P_{\mathbf{K}(y)}(w) - P_{\mathbf{K}(z)}(w)\|_V \leq \eta \|y - z\|_V,$$

holds?

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Yes, there are, e.g., some obstacle cases with  $\Phi : V \rightarrow V$  Lipschitz with

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Q: Is it worthwhile to use  $(\mathbb{H})$  in the QVI setting?

Let  $c > 0$  and  $C > 0$  be the coercivity and the bound for  $A : V \rightarrow V'$ , i.e.,

$$c\|v\|_V^2 \leq \langle Av, v \rangle \quad \text{and} \quad \|Av\|_{V'} \leq C\|v\|_V, \quad \forall v \in V.$$

The use of  $(\mathbb{H})$  provides existence, uniqueness and fixed point iteration for (QVI). As a consequence, we have that

$$2\frac{C}{c}L_\Phi < 1.$$

Contraction factor is  $\geq 2\frac{C}{c}L_\Phi$ .

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The change of variables  $\tilde{y} = y - \Phi(y)$ , transforms (QVI) to a (VI). Existence, uniqueness, and a fixed point iteration if

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# Some problems with the literature

Yes, there are, e.g., some obstacle cases with  $\Phi : V \rightarrow V$  Lipschitz with

$$\mathbf{K}(w) := \{z \in V : z \leq \Phi(w)\}.$$

Q: Is it worthwhile to use  $(\mathbb{H})$  in the QVI setting?

Let  $c > 0$  and  $C > 0$  be the coercivity and the bound for  $A : V \rightarrow V'$ , i.e.,

$$c\|v\|_V^2 \leq \langle Av, v \rangle \quad \text{and} \quad \|Av\|_{V'} \leq C\|v\|_V, \quad \forall v \in V.$$

The use of  $(\mathbb{H})$  provides existence, uniqueness and fixed point iteration for (QVI). As a consequence, we have that

$$2\frac{C}{c}L_\Phi < 1.$$

Contraction factor is  $\geq 2\frac{C}{c}L_\Phi$ .

The change of variables  $\tilde{y} = y - \Phi(y)$ , transforms (QVI) to a (VI). Existence, uniqueness, and a fixed point iteration if

$$\frac{C}{c}L_\Phi < 1.$$

Contraction factor is  $\frac{C}{c}L_\Phi$ .

A: Not exactly...

Thanks for your attention!