USC Viterbi School of Engineering

Four Lectures on Generalized Nash Equilibrium Problems in Finite Dimensions

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presented at

Autumn School on Quasi-Variational Inequalities:

Theory, Algorithms, and Applications

Universität Würzburg, Germany

Monday September 23 through Wednesday September 25, 2019

Foreword: Brief history of generalized Nash games

Lecture I: Formulations and a recent model

Lecture II: Existence results with and without convexity

Lecture III: Exact penalization theory

Lecture IV: Best-response algorithms

A brief historical account

Gerard Debreu. A social equilibrium existence theorem. *Proceedings of the National Academy of Sciences* 38(10): 886–893, **1952**.

Kenneth J. Arrow and Gerard Debreu. Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society* 22(3): 265–290, **1954**.

Hukukane Nikaido and Kazuo Isoda. Note on noncooperative convex games. *Pacific Journal of Mathematics* 5(Suppl. 1): 807-815, **1955**.

J. Ben Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society* 33(3): 520–534, **1965**.

Alain Bensoussan. Points de Nash dans le cas de fontionnelles quadratiques et jeux differentiels linéaires a N personnes. *SIAM Journal on Control* 12(3): 460–499, **1974**.

A brief historical account (cont.)

PATRICK T HARKER. Generalized Nash games and quasi-variational inequalities. *European Journal of Operational Research* 54(1): 81–94, **1991**.

JONG-SHI PANG AND MASAO FUKUSHIMA. Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. *Computational Management Science* 2: 21-56, **2005**. [Erratum: same journal 6: 373-375, 2009.]

FRANCISCO FACCHINEI AND CHRISTIAN KANZOW. Generalized Nash equilibrium problems. *Annals of Operations Research* 175(1):177–211, **2007**.

FRANCISCO FACCHINEI AND JONG-SHI PANG. Nash equilibria: the variational approach. Chapter 12 in Daniel P. Palomar and Yonica C. Eldar, editors. *Convex optimization in signal processing and communications*, Cambridge University Press, pages 443–493, **2010**.

FRANCISCO FACCHINEI. Computation of generalized Nash equilibria: Recent advances. Part II in Roberto Cominetti, Francisco Facchinei and Jean-Bernard Lasserre. Editors and authors. *Modern Optimization Modeling Techniques*. Birkhauser Springer Basel, pp. 133–184, **2012**.

Some recent references

Extensive papers by our host: **Christian Kanzow**
https://www.mathematik.uni-wuerzburg.de/optimization/team/kanzow-christian

Q. Ba and J.S. Pang. Exact penalization of generalized Nash equilibrium problems. *Operations Research* (accepted August 2019) in print.

D.A. Schiro, J.S. Pang and U.V. Shanbhag. On the solution of affine generalized Nash equilibrium problems with shared constraints by Lemke's method. *Mathematical Programming, Series A* 142(1–2), 1–46 (2013).

Modelling

From single decision maker $\longrightarrow$ multiple agents

From optimization $\longrightarrow$ equilibrium

From cooperation $\longrightarrow$ non-cooperation

From centralized decision making $\longrightarrow$ distributed responses

Mathematics

From Weierstrass/Cauchy $\longrightarrow$ Brouwer/Banach

From smooth calculus $\longrightarrow$ variational analysis

From coordinated descent $\longrightarrow$ asynchronous computation

From potential function $\longrightarrow$ non-expansion of mappings

Lecture I: Formulations and a recent model

9:30 – 10:30 AM Monday September 23, 2019

The Abstract Generalized Nash Equilibrium Problem (GNEP)

N decision makers each (labeled $\nu = 1, \cdots, N$) with

- a moving strategy set $\Xi^\nu(x^{-\nu}) \subseteq \mathbb{R}^{n_\nu}$, and
- a cost function $\theta_\nu(\bullet, x^{-\nu}) : \mathbb{R}^{n_\nu} \to \mathbb{R}$,

both dependent on the rivals' strategy tuple
$x^{-\nu} \triangleq \left(x^{\nu'}\right)_{\nu' \neq \nu} \in \mathbb{R}^{-\nu} \triangleq \prod_{\nu' \neq \nu} \mathbb{R}^{n_{\nu'}}$.

Anticipating rivals' strategy $x^{-\nu}$, player ν solves:

$$\underset{x^\nu \in \Xi^\nu(x^{-\nu})}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu})$$

A Nash equilibrium (NE) is a strategy tuple $\mathbf{x}^* \triangleq (x^{*,\nu})_{\nu=1}^N$ such that

$$x^{*,\nu} \in \underset{x^\nu \in \Xi^\nu(x^{*,-\nu})}{\text{argmin}} \quad \theta_\nu(x^\nu, x^{*,-\nu}), \quad \forall\, \nu = 1, \cdots, N.$$

In words, no player can improve individual objective by unilaterally deviating from an equilibrium strategy.

Notation:

- $\mathbf{x} \triangleq (x^\nu)_{\nu=1}^N$;
- $\boldsymbol{\Xi}(\mathbf{x}) \triangleq \prod_{\nu=1}^{N} \Xi^\nu(x^{-\nu})$;
- $\mathsf{FIX}_{\boldsymbol{\Xi}} \triangleq \{\mathbf{x} : \mathbf{x} \in \boldsymbol{\Xi}(\mathbf{x})\}$;
- the GNEP $(\boldsymbol{\Xi}, \theta)$.

Necessarily, a NE $\mathbf{x}^*$ must belong to $\mathsf{FIX}_{\boldsymbol{\Xi}}$, the "feasible set" of the GNEP.

The GNEP in action

- basic case: $\Xi^\nu(x^{-\nu}) \triangleq X^\nu$ is independent of $x^{-\nu}$ for all ν
- finitely representable with convexity and constraint qualifications:

$$\Xi^\nu(x^{-\nu}) \triangleq \left\{ x^\nu \in \mathbb{R}^{n_\nu} : \boxed{\begin{array}{l} h^\nu(x^\nu) \leq 0 \\ \text{private constraints} \end{array}} \; \boxed{\begin{array}{l} g^\nu(x^\nu, x^{-\nu}) \leq 0 \\ \text{coupled constraints} \end{array}} \right\}$$

where $h^\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}^{\ell_\nu}$ is C^1 and for each $i = 1, \cdots, \ell_\nu$, the component function h_i^ν is convex, and $g^\nu : \mathbb{R}^n \to \mathbb{R}^{m_\nu}$ is C^1 and for each $i = 1, \cdots, m_\nu$ and every $x^{-\nu}$, the component function $g_i^\nu(\bullet, x^{-\nu})$ is convex

- joint convexity: graph of $\Xi^\nu \triangleq \mathbf{C} \times \widehat{\mathbf{X}}$, where $\widehat{\mathbf{X}} \triangleq \prod_{\nu=1}^{N} X^\nu$ and $\mathbf{C} \subset \mathbb{R}^n$,

where $n \triangleq \sum_{\nu=1}^{N} n_\nu$ are closed and convex;

- roles of multipliers: these can be different for same constraints in different players' problems;
- common multipliers for $\mathbf{C} \triangleq \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 0\}$, where $g : \mathbb{R}^n \to \mathbb{R}^m$ is C^1, each component function g_i is convex, and Lagrange multipliers for the constraint $g(\mathbf{x}) \leq 0$ is the same for all players, leading to variational equilibria.

Quasi variational inequality (QVI) formulation

- If $\theta_\nu(\bullet, x^{-\nu})$ is convex and Ξ^ν is convex-valued, then $\mathbf{x}^* \triangleq (x^{*,\nu})_{\nu=1}^N$ is a NE if and only if for every $\nu = 1, \cdots, N$, $\exists\, a^{*,\nu} \in \partial_{x^\nu}\theta_\nu(\mathbf{x}^*)$ such that

$$(\, x^\nu - x^{*,\nu}\,)^\top a^{*,\nu} \,\geq\, 0, \quad \forall\, x^\nu \,\in\, \Xi^\nu(x^{*,-\nu}),$$

or equivalently, $\exists\, \mathbf{a}^* \in \boldsymbol{\Theta}(\mathbf{x}^*) \,\triangleq\, \prod_{\nu=1}^N \partial_{x^\nu}\theta_\nu(\mathbf{x}^*)$ such that

$$(\,\mathbf{x} - \mathbf{x}^*\,)^\top \mathbf{a}^* \,=\, \sum_{\nu=1}^N (\, x^\nu - x^{*,\nu}\,)^T a^{*,\nu} \,\geq\, 0,\ \forall\, \mathbf{x} \,\triangleq\, (x^{*,\nu})_{\nu=1}^N \,\in\, \boldsymbol{\Xi}(\mathbf{x}^*).$$

- If in addition $\theta_\nu(\bullet, x^{-\nu})$ is differentiable, then $\boldsymbol{\Theta}(\mathbf{x}) = (\,\nabla_{x^\nu}\theta_\nu(\mathbf{x})\,)_{\nu=1}^N$ is a single-valued map.

- If further $\Xi^\nu(x^{*,-\nu}) = X^\nu$, then get the VI $(\boldsymbol{\Theta}, \widehat{\mathbf{X}})$:

$$(\,\mathbf{x} - \mathbf{x}^*\,)^\top \boldsymbol{\Theta}(\mathbf{x}) \,\geq\, 0, \quad \forall\, \mathbf{x} \,\in\, \widehat{\mathbf{X}}.$$

Complementarity formulation

- Each $\theta_\nu(\bullet, x^{-\nu})$ is differentiable and

$$\Xi^\nu(x^{-\nu}) = \{ x^\nu \in \mathbb{R}^{n_\nu}_+ : g^\nu(x^\nu, x^{-\nu}) \le 0 \}$$

Karush-Kuhn-Tucker conditions (KKT) of the GNEP

$$\left\{ \begin{array}{l} 0 \le x^\nu \perp \nabla_{x^\nu} \theta_\nu(\mathbf{x}) + \sum_{i=1}^{m_\nu} \nabla_{x^\nu} g_i^\nu(\mathbf{x}) \, \lambda_i^\nu \ge 0 \\ 0 \le \lambda^\nu \perp g^\nu(\mathbf{x}) \le 0 \end{array} \right\} \nu = 1, \cdots, N.$$

yielding the nonlinear complementarity problem (NCP) formulation, under suitable constraint qualifications:

$$0 \le \underbrace{\begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix}}_{\text{denoted } \mathbf{z}} \perp \underbrace{\begin{pmatrix} \left(\nabla_{x^\nu} \theta_\nu(\mathbf{x}) + \sum_{i=1}^{m_\nu} \nabla_{x^\nu} g_i^\nu(\mathbf{x}) \, \lambda_i^\nu \right)_{\nu=1}^{N} \\ -\left(g^\nu(\mathbf{x}) \right)_{\nu=1}^{N} \end{pmatrix}}_{\text{denoted } \mathbf{F}(\mathbf{z})} \ge 0.$$

One recent model in transportation e-hailing

Contributions

- realistic modeling of a topical research problem
- novel approach for proof of solution existence
- awaiting design of convergent algorithm
- opportunity in model refinements and abstraction

J. Ban, M. Dessouky, and J.S. Pang. A general equilibrium model for transportation systems with e-hailing services and flow congestion. *Transportation Research, Series B* (2019) in print.

Model background: Passengers are increasingly using e-hailing as a means to request transportation services. Goal is to obtain insights into how these emergent services impact traffic congestion and travelers' mode choices.

Formulation as a GNE/MOPEC/Multi-VI

• e-HSP companies' choices in making decisions such as where to pick up the next customer to maximize the profits under given fare structures;

• travelers' choices in making driving decisions to be either a solo driver or an e-HSP customer to minimize individual disutility;

• network equilibrium to capture traffic congestion as a result of everyone's travel behavior that is dictated by Wardrop's route choice principle;

• market clearance conditions to define customers' waiting cost and constraints ensuring that the e-HSP OD demands are served.

Network set-up

$\mathcal{N}$	set of nodes in the network
$\mathcal{A}$	set of links in the network, subset of $\mathcal{N} \times \mathcal{N}$
$\mathcal{K}$	set of OD pairs, subset of $\mathcal{N} \times \mathcal{N}$
$\mathcal{O}$	set of origins, subset of $\mathcal{N}$; $\mathcal{O} = \{ \mathrm{O}_k : k \in \mathcal{K} \}$
$\mathcal{D}$	set of destinations, subset of $\mathcal{N}$; $\mathcal{D} = \{ \mathrm{D}_k : k \in \mathcal{K} \}$ besides being the destinations of the OD pairs where customers are dropped off, these are also the locations where the e-HSP drivers initiate their next trip to pick up other customers
O_k, D_k	origin and destination (sink) respectively of OD pair $k \in \mathcal{K}$
$\mathcal{M}$	labels of the e-HSPs; $\mathcal{M} \triangleq \{1, \cdots, M\}$
$\mathcal{M}_+$	union of the solo driver/customer label (0) and the e-HSP labels

Overall model is to determine:

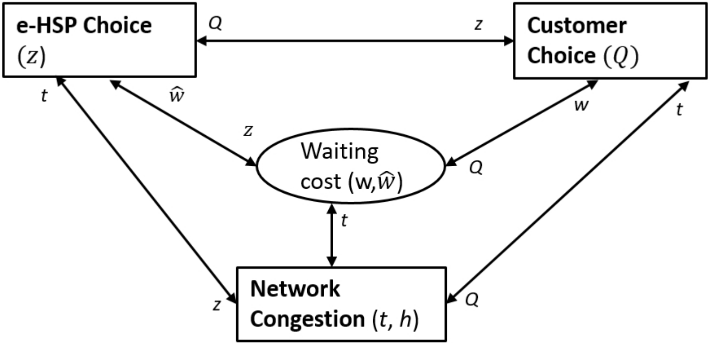
- e-HSP vehicle allocations: $\left\{z_{jk}^m : m \in \mathcal{M}; j \in \mathcal{D}; k \in \mathcal{K}\right\}$ for each type m, destination j, and OD pair k;

- travel demands: $\{Q_k^m : m \in \mathcal{M}; k \in \mathcal{K}\}$ for each e-HSP type m and OD pair k;

- travel times: $\{t_{ij} : (i, j) \in \mathcal{N} \times \mathcal{N}\}$ of shortest path from node i to j;

- vehicular flows: $\{h_p : p \in \mathcal{P}\}$ on paths in network.

Interaction of modules in model



The e-HSP module

The per-customer (or per-pickup) profit of an e-HSP$_m$ trip, for $m \in \mathcal{M}$, at location j who plans to serve OD pair k can be modeled as:

$$R_{jk}^m \triangleq \widehat{R}_{jk}^m - \beta_3^m \underbrace{\widehat{w}_k^m}_{\text{e-HSP}_m \text{ waiting time incurring loss of revenue}}, \quad \text{where}$$

$$\widehat{R}_{jk}^m \triangleq F_{\mathrm{O}_k}^m - \underbrace{\beta_1^m \left(t_{j\mathrm{O}_k} + t_{\mathrm{O}_k\mathrm{D}_k} \right)}_{\text{travel time based cost}} - \underbrace{\beta_2^m \left(d_{j\mathrm{O}_k} + d_{\mathrm{O}_k\mathrm{D}_k} \right)}_{\text{travel distance based cost}}$$

$$+ \underbrace{\alpha_1^m \left(t_{\mathrm{O}_k\mathrm{D}_k} - f_{\mathrm{O}_k\mathrm{D}_k}^0 \right)}_{\text{time based revenue}} + \underbrace{\alpha_2^m d_{\mathrm{O}_k\mathrm{D}_k}}_{\text{distance based revenue}}$$

Anticipating R_{jk}^m as a parameter, e-HSP$_m$ company decides on the allocation z_{jk}^m of vehicles from location j to serve OD pair k by solving

e-HSP's choice module of profit maximization: for $m \in \mathcal{M}$,

$$\underset{z_{jk}^m \geq 0}{\text{maximize}} \quad \underbrace{\sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{D}} \widehat{R}_{jk}^m \, z_{jk}^m}_{\text{average trip profit}} \underbrace{- \beta_3^m \left[N^m - \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{D}} z_{jk}^m t_{jO_k} - \sum_{k \in \mathcal{K}} Q_k^m \, t_{O_k D_k} \right]}_{\text{cost due to waiting}}$$

$$\text{subject to} \quad \underbrace{\sum_{k \in \mathcal{K}} z_{jk}^m = \sum_{k' : j = \mathrm{D}_{k'}} Q_{k'}^m}_{\text{available e-HSP vehicles for service}} \qquad \text{for all } j \in \mathcal{D}$$

$$\underbrace{\sum_{j \in \mathcal{D}} z_{jk}^m \geq Q_k^m}_{\text{OD demands served by e-HSP}_m} \qquad \text{for all } k \in \mathcal{K}$$

$$\text{and} \quad \underbrace{\sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{D}} z_{jk}^m \, t_{jO_k}}_{\substack{\text{trip hours of vehicles} \\ \text{en route to service calls}}} + \underbrace{\sum_{k \in \mathcal{K}} Q_k^m \, t_{O_k D_k}}_{\substack{\text{trip hours of vehicles} \\ \text{serving travel demands}}} \leq \underbrace{N^m}_{\substack{\text{e-HSP}_m \text{ vehicle} \\ \text{availability}}} .$$

The passenger module:

An e-HSP$_m$ customer's disutility V_k^m for $m \in \mathcal{M}$:

$$F^m_{\mathrm{O}_k} + \underbrace{\alpha_1^m \left(t_{\mathrm{O}_k\mathrm{D}_k} - f^0_{\mathrm{O}_k\mathrm{D}_k} \right)}_{\text{time based fare}} + \underbrace{\alpha_2^m \, d_{\mathrm{O}_k\mathrm{D}_k}}_{\text{distance based fare}} + \underbrace{\gamma_1^m \, t_{\mathrm{O}_k\mathrm{D}_k}}_{\text{in-vehicle based disutility}}$$

$+ w_k^m$ (disutility due to waiting for e-HSPs' pick up).

For a solo driver, the disutility can be expressed as:

$$V_k^0 = \underbrace{\gamma_1^0 \, t_{\mathrm{O}_k\mathrm{D}_k}}_{\text{travel time based disutility}} + \underbrace{\beta_2^0 \, d_{\mathrm{O}_k\mathrm{D}_k}}_{\text{distance based disutility}} .$$

Anticipating V_k^m and e-HSP vehicle allocations z_{kj}^m, passengers decide on travel modes (solo or e-hailing) by solving

Passenger mode choice of disutility minimization:

$$\underset{Q_k^m \geq 0}{\text{minimize}} \quad \sum_{m \in \mathcal{M}_+} \sum_{k \in \mathcal{K}} V_k^m \, Q_k^m$$

$$\text{subject to} \quad \boxed{\begin{array}{l} \displaystyle\sum_{j \in \mathcal{D}} z_{jk}^m \geq Q_k^m \\ \text{shared constraints} \end{array}} \quad \text{for all } (k, m) \in \mathcal{K} \times \mathcal{M}$$

$$\sum_{m \in \mathcal{M}_+} Q_k^m = \underbrace{Q_k}_{\text{given}}, \quad \text{for all } k \in \mathcal{K},$$

where $\mathcal{M}_+ = \mathcal{M} \cup \{ \text{ solo mode } \}$.

Network congestion module

Wardrop's principle of route choice: for the determination of travel time t_{ij} from node i to j and traffic flow h_p on path p joining these two nodes:

$$0 \leq t_{ij} \quad \perp \quad \sum_{p \in \mathcal{P}_{ij}} h_p - \left[\sum_{k \in \mathcal{K}} \delta^{\rm OD}_{ijk} \, Q_k + \sum_{(k,\ell) \in \mathcal{K} \times \mathcal{K}} \delta^{\rm e-HSP}_{ijk\ell} \sum_{m \, \in \, \mathcal{M}} z^m_{i\ell} \right] \geq 0$$

$$\text{for all } (i,j) \in \mathcal{N} \times \mathcal{N},$$

$$0 \leq h_p \quad \perp \quad C_p(h) - t_{ij} \geq 0, \quad \text{for all } p \in \mathcal{P}_{ij}, \quad \text{where}$$

$$\delta^{\rm OD}_{ijk} \triangleq \left\{ \begin{array}{ll} 1 & \text{if } i = {\rm O}_k \text{ and } j = {\rm D}_k \\ 0 & \text{otherwise} \end{array} \right\};$$

$$\delta^{\rm e-HSP}_{i'j'k\ell} \triangleq \left\{ \begin{array}{ll} 1 & \text{if } i' = {\rm D}_k \text{ and } j' = {\rm O}_\ell \\ 0 & \text{otherwise} \end{array} \right\}.$$

Customer waiting disutility

$$w_k^m = \gamma_2^m \boxed{\underbrace{\frac{\sum_{j \in \mathcal{D}} z_{jk}^m \, t_{j\mathrm{O}_k}}{\sum_{j \in \mathcal{D}} z_{jk}^m}}_{\text{average travel time to origin of OD-pair } k \text{ from all locations}}} \quad \text{for all } m \in \mathcal{M}$$

Remark: need a complementarity maneuver to rigorously handle the denominator to avoid division by zero.

End model is a highly complex generalized Nash equilibrium problem with side conditions for which state-of-the-art theory and algorithms are not directly applicable.

A penalty approach is employed for proving solution existence.

Lecture II: Existence results with and without convexity

14:30 – 15:30 AM Monday September 23, 2019

Recall, QVI formulation: convex player problems:

If $\theta_\nu(\bullet, x^{-\nu})$ is convex and Ξ^ν is convex-valued, then $\mathbf{x}^* \triangleq (x^{*,\nu})_{\nu=1}^N \in \mathsf{FIX}_{\boldsymbol{\Xi}}$ is a NE if and only if for every $\nu = 1, \cdots, N$, $\exists\, a^{*,\nu} \in \partial_{x^\nu}\theta_\nu(\mathbf{x}^*)$ such that

$$(x^\nu - x^{*,\nu})^\top a^{*,\nu} \geq 0, \quad \forall\, x^\nu \in \Xi^\nu(x^{*,-\nu}),$$

or equivalently, $\exists\, \mathbf{a}^* \in \boldsymbol{\Theta}(\mathbf{x}^*) \triangleq \prod_{\nu=1}^N \partial_{x^\nu}\theta_\nu(\mathbf{x}^*)$ such that

$$(\mathbf{x} - \mathbf{x}^*)^\top \mathbf{a}^* = \sum_{\nu=1}^N (x^\nu - x^{*,\nu})^T a^{*,\nu} \geq 0, \; \forall\, \mathbf{x} \triangleq (x^{*,\nu})_{\nu=1}^N \in \boldsymbol{\Xi}(\mathbf{x}^*),$$

where $\boldsymbol{\Xi}(\mathbf{x}) \triangleq \prod_{\nu=1}^N \Xi^\nu(x^{-\nu})$ and $\mathsf{FIX}_{\boldsymbol{\Xi}} \triangleq \{\mathbf{x} : \mathbf{x} \in \boldsymbol{\Xi}(\mathbf{x})\}$.

Existence results for convex player problems

Icchiishi 1983: with compactness. A NE exists if

- each $\Xi^\nu : \mathbb{R}^{-\nu} \to \mathbb{R}^{n_\nu}$ is a continuous convex-valued multifunction on $\mathrm{dom}(\Xi^\nu)$;
- each $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\theta_\nu(\bullet, x^{-\nu})$ is convex $\forall\, x^{-\nu} \in \mathrm{dom}(\Xi^\nu)$;
- $\exists$ compact convex set $\emptyset \neq \widehat{\mathbf{K}} \subseteq \mathbb{R}^n$ such that $\boldsymbol{\Xi}(\mathbf{y}) \subseteq \widehat{\mathbf{K}}$ for all $\mathbf{y} \in \widehat{\mathbf{K}}$.

Proof.. Apply Kakutani fixed-point theorem to the self-map:

$$\mathbf{y} \in \widehat{\mathbf{K}} \mapsto \prod_{\nu=1}^{N} \operatorname*{argmin}_{x^\nu \in \Xi^\nu(y^{-\nu})} \theta_\nu(x^\nu, y^{-\nu}) \subseteq \widehat{\mathbf{K}}.$$

Assumptions ensure applicability of this theorem. □

Applicable to the jointly convex, compact case with

$$\text{graph}\ \Xi^\nu = \underbrace{\mathbf{C}}_{\text{shared constraints}} \times \underbrace{\widehat{\mathbf{X}}}_{\text{private constraints}}, \quad \text{for all}\ \nu.$$

Facchinei and Pang 2009: no compactness. Instead of the last assumption,

- $\exists$ a bounded open set Ω with $\overline{\Omega} \subseteq \mathrm{dom}(\boldsymbol{\Xi})$, a vector $\mathbf{x}^{\mathrm{ref}} \triangleq \left(x^{\mathrm{ref},\nu} \right)_{\nu=1}^{N} \in \Omega$, and a continuous function $\mathbf{s} : \overline{\Omega} \to \Omega$ such that
  - (continuous selection) $\mathbf{s}(\mathbf{y}) \triangleq (s^{\nu}(\mathbf{y}))_{\nu=1}^{N} \in \boldsymbol{\Xi}(\mathbf{y})$ for all $\mathbf{y} \in \overline{\Omega}$;
  - the open line segment joining $\mathbf{x}^{\mathrm{ref}}$ and $\mathbf{s}(\mathbf{y})$ is contained in Ω for all $y \in \overline{\Omega}$;
  - (an abstract form of weak coercivity) $\boxed{L_{<} \cap \partial\Omega = \emptyset}$ where

$$L_{<} \triangleq \left\{ \begin{array}{c} \mathbf{y} \triangleq (\, y^{\nu} \,)_{\nu=1}^{N} \in \mathsf{FIX}_{\boldsymbol{\Xi}} : \text{ for each } \nu \text{ such that } y^{\nu} \neq s^{\nu}(\mathbf{y}) \\ (\, y^{\nu} - s^{\nu}(\mathbf{y}) \,)^{T} u^{\nu} < 0 \text{ for some } u^{\nu} \in \partial_{x^{\nu}} \theta_{\nu}(\mathbf{y}) \end{array} \right\}.$$

Facchinei and Pang 2009: no compactness. Instead of the last assumption,

• $\exists$ a bounded open set Ω with $\overline{\Omega} \subseteq \mathrm{dom}(\boldsymbol{\Xi})$, a vector $\mathbf{x}^{\mathrm{ref}} \triangleq \left(x^{\mathrm{ref},\nu} \right)_{\nu=1}^{N} \in \Omega$, and a continuous function $\mathbf{s} : \overline{\Omega} \to \Omega$ such that

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• Facchinei and Pang 2003. A NE exists if the solutions (if they exist) of the NCP: $0 \leq \mathbf{z} \perp \mathbf{F}(\mathbf{z}) + \tau \mathbf{z} \geq 0$ over all scalars $\tau > 0$ are bounded.

Nonconvex games with **shared** constraints

- $\theta_\nu(\bullet, x^{-\nu})$ nonconvex, and

- $$\Xi^\nu(x^{-\nu}) \triangleq \left\{ \underbrace{x^\nu \in X^\nu : h^\nu(x^\nu) \leq 0}_{\substack{\text{private constraints} \\ \text{denoted } \mathcal{X}^\nu}}; \underbrace{\underbrace{G(\mathbf{x}) \leq 0}_{\text{nonconvex}}; \underbrace{\widehat{G}(\mathbf{x}) \leq 0}_{\text{polyhedral}}}_{\text{shared}} \right\},$$

where $G : \mathbb{R}^n \to \mathbb{R}^p$ and $\widehat{G} : \mathbb{R}^n \to \mathbb{R}^{\widehat{p}}$. Denote $H(\mathbf{x}) \triangleq -\left(h^\nu(x^\nu) \right)_{\nu=1}^N$.

Nonconvex games with **shared** constraints

- $\theta_\nu(\bullet, x^{-\nu})$ nonconvex, and

- $$\Xi^\nu(x^{-\nu}) \triangleq \left\{ \underbrace{x^\nu \in X^\nu : h^\nu(x^\nu) \leq 0}_{\substack{\text{private constraints} \\ \text{denoted } \mathcal{X}^\nu}};\ \underbrace{\underbrace{G(\mathbf{x}) \leq 0}_{\text{nonconvex}};\ \underbrace{\widehat{G}(\mathbf{x}) \leq 0}_{\text{polyhedral}}}_{\text{shared}} \right\},$$

where $G : \mathbb{R}^n \to \mathbb{R}^p$ and $\widehat{G} : \mathbb{R}^n \to \mathbb{R}^{\widehat{p}}$. Denote $H(\mathbf{x}) \triangleq -\left(h^\nu(x^\nu) \right)_{\nu=1}^N$.

Given prices $(\pi, \widehat{\pi})$ and anticipating $x^{-\nu}$, player ν's problem:

$$\underset{x^\nu \in \mathcal{X}^\nu}{\text{minimize}} \left[\underbrace{\theta_\nu(x^\nu, x^{-\nu}) + \sum_{j=1}^{p} \pi_j\, G_j(x^\nu, x^{-\nu}) + \sum_{j=1}^{\widehat{p}} \widehat{\pi}_j\, \widehat{G}_j(x^\nu, x^{-\nu})}_{\text{denoted } L_\nu(\mathbf{x}, \pi, \widehat{\pi})} \right]$$

The game $\mathcal{G}(\Theta, H, G, \widehat{G})$

A **Nash (variational) equilibrium (NE)** is a tuple $(x^*, \pi^*, \widehat{\pi}^*)$ with $x^* \triangleq (\, x^{*,\nu}\,)_{\nu=1}^N$ such that

$$x^{*,\nu} \in \underset{x^\nu \in \mathcal{X}^\nu}{\text{argmin}}\; L_\nu(x^\nu, x^{*,-\nu}, \pi^*, \widehat{\pi}^*), \tag{1}$$

for every $\nu = 1, \cdots, N$, and

$$0 \le \pi^* \perp G(\mathbf{x}^*) \le 0 \quad \text{and} \quad 0 \le \widehat{\pi}^* \perp \widehat{G}(\mathbf{x}^*) \le 0,$$

Multipliers $(\pi^*, \widehat{\pi}^*)$ of the common shared constraints are the same for all players; they are optimal solutions of the market player's problem who, anticipating $\mathbf{x}^*$, solves

$$(\pi^*, \widehat{\pi}^*) \in \underset{(\pi,\widehat{\pi}) \ge 0}{\text{minimize}}\; \pi^\top G(\mathbf{x}^*) + \widehat{\pi}^\top \widehat{G}(\mathbf{x}^*).$$

The game $\mathcal{G}(\Theta, H, G, \widehat{G})$

A **Nash (variational) equilibrium (NE)** is a tuple $(x^*, \pi^*, \widehat{\pi}^*)$ with $x^* \triangleq (x^{*,\nu})_{\nu=1}^N$ such that

$$x^{*,\nu} \in \underset{x^\nu \in \mathcal{X}^\nu}{\text{argmin}}\ L_\nu(x^\nu, x^{*,-\nu}, \pi^*, \widehat{\pi}^*), \qquad (1)$$

for every $\nu = 1, \cdots, N$, and

$$0 \leq \pi^* \perp G(\mathbf{x}^*) \leq 0 \quad \text{and} \quad 0 \leq \widehat{\pi}^* \perp \widehat{G}(\mathbf{x}^*) \leq 0,$$

Multipliers $(\pi^*, \widehat{\pi}^*)$ of the common shared constraints are the same for all players; they are optimal solutions of the market player's problem who, anticipating $\mathbf{x}^*$, solves

$$(\pi^*, \widehat{\pi}^*) \in \underset{(\pi,\widehat{\pi}) \geq 0}{\text{minimize}}\ \pi^\top G(\mathbf{x}^*) + \widehat{\pi}^\top \widehat{G}(\mathbf{x}^*).$$

A **local Nash equilibrium (LNE)** is a tuple $(x^*, \pi^*, \boxed{\widehat{\pi}^*})$ for which an open neighborhood $\mathcal{N}^\nu$ of $x^{*,\nu}$ exists such that (1) is replaced by

$$x^{*,\nu} \in \underset{x^\nu \in \mathcal{X}^\nu \cap \mathcal{N}^\nu}{\text{argmin}}\ L_\nu(x^\nu, x^{*,-\nu}, \pi^*, \widehat{\pi}^*),$$

A **quasi-Nash equilibrium (QNE)** is a a tuple $(x^*, \pi^*, \boxed{\lambda^*})$ solving the game's variational formulation; i.e., the (linearly constrained) VI $(\mathbf{K}, \mathbf{\Phi})$,

where $\mathbf{K} \triangleq \mathbf{\Xi} \times \mathbb{R}^p_+ \times \prod_{\nu=1}^{N} \mathbb{R}^{\ell_\nu}_+$, with

$$\mathbf{\Xi} \triangleq \left\{ \mathbf{x} = (x^\nu)_{\nu=1}^N \in \prod_{\nu=1}^N X^\nu \mid \underbrace{\widehat{G}(\mathbf{x}) \leq 0}_{\text{coupled constraints}} \right\}; \quad \text{polyhedral}$$

$$\mathbf{\Phi}(\mathbf{x}, \pi, \lambda) \triangleq \begin{pmatrix} \left(\nabla_{x^\nu} \theta_\nu(\mathbf{x}) + \sum_{j=1}^p \pi_j \nabla_{x^\nu} G_j(\mathbf{x}) + \sum_{i=1}^{\ell_\nu} \lambda_i^\nu \nabla h_i^\nu(x^\nu) \right)_{\nu=1}^N & \text{VI Lagrangian} \\ \Psi(\mathbf{x}) \triangleq \begin{pmatrix} -G(\mathbf{x}) \\ -H(\mathbf{x}) \end{pmatrix} & \text{nonconvex constraints} \end{pmatrix}$$

Roadmap of the Analysis

For a QNE

• Formulate VI as a system of (nonsmooth) equations and use degree theory to show existence and uniqueness of a QNE.

— need a Slater condition plus a copositivity assumption

• Invoke second-order sufficiency condition in NLP to yield a LNE from a QNE.

For a NE (model remains nonconvex)

• Derive sufficient conditions for best-response map to be single-valued, yielding existence of a NE

— rely on bounds of multipliers and positive definiteness of the Hessian matrices of the Lagrangian.

• Use a distributed Jacobi or Gauss-Seidel iterative scheme to compute a NE, whose convergence is based on a contraction argument.

A Review of VI Theory

Let K be a closed convex subset of $\mathbb{R}^n$ and $F : K \to \mathbb{R}^n$ be a continuous map.

- The solution set of the VI defined by this pair (K, F) is denoted $\mathrm{SOL}(K, F)$.
- The critical cone at $x^* \in \mathrm{SOL}(K, F)$ is

$$\mathcal{C}(K, F; x^*) \triangleq \mathcal{T}(K; x^*) \cap F(x^*)^\perp = \mathcal{T}(K; x^*) \cap (-F(x^*))^*.$$

where $\mathcal{T}(K; x^*)$ is the tangent cone.

- The normal map of the VI (K, F) is

$$F_K^{\mathrm{nor}}(z) \triangleq F(\Pi_K(z)) + z - \Pi_K(z), \quad z \in \mathbb{R}^n,$$

where Π_K is the Euclidean projector onto K.

- $F_K^{\mathrm{nor}}(z) = 0$ implies $x \triangleq \Pi_K(z) \in \mathrm{SOL}(K, F)$; conversely, $x \in \mathrm{SOL}(K, F)$ implies $F_K^{\mathrm{nor}}(z) = 0$, where $z \triangleq x - F(x)$.
- A matrix $M \in \mathbb{R}^{n \times n}$ is copositive on a cone $\mathcal{C} \subseteq \mathbb{R}^n$ if $x^T M x \geq 0$ for all $x \in \mathcal{C}$, strictly copositive if $x^T M x \geq 0$ for all $x \in \mathcal{C} \setminus \{0\}$.

Solution Existence and Uniqueness of VIs

(a) $\mathrm{SOL}(K,F) \neq \emptyset$ if $\exists$ a vector $x^{\mathrm{ref}} \in K$ such that the set

$$L_< \triangleq \{ x \in K \mid (x - x^{\mathrm{ref}})^T F(x) < 0 \} \quad \text{is bounded.}$$

(b) $\mathrm{SOL}(K,F) \neq \emptyset$ and bounded if $\exists$ a vector $x^{\mathrm{ref}} \in K$ such that the set

$$L_\leq \triangleq \{ x \in K \mid (x - x^{\mathrm{ref}})^T F(x) \leq 0 \} \quad \text{is bounded.}$$

(c) $\mathrm{SOL}(K,F)$ is a singleton for a polyhedral K if
(i) the set $L_\leq$ is bounded, and
(ii) for every $x^* \in \mathrm{SOL}(K,F)$, $JF(x^*)$ is copositive on $\mathcal{C}(K,F;x^*)$
and

$$\mathcal{C}(K,F;x^*) \ni v \perp JF(x^*)v \in \mathcal{C}(K,F;x^*)^* \;\Rightarrow\; v = 0,$$

i.e., if $(\mathcal{C}(K,F;x^*), JF(x^*))$ is a R_0 pair .

Proof by applying degree theory to the normal map $F_K^{\mathrm{nor}}(z)$.

Existence of QNE

The game $\mathcal{G}(\Theta, H, G, \widehat{G})$ has a QNE if $\exists$ a tuple $x^{\text{ref}} \triangleq \left(x^{\text{ref},\nu}\right)_{\nu=1}^{N} \in \boldsymbol{\Xi}$ such that

(Slater condition of nonconvex constraints) $\Psi(x^{\text{ref}}) \triangleq \begin{pmatrix} H(x^{\text{ref}}) \\ G(x^{\text{ref}}) \end{pmatrix} < 0$;

(copositivity of private constraints) the Hessian matrix $\nabla^2 h_i^\nu(x^\nu)$ is copositive on $\mathcal{T}(X^\nu; x^{\text{ref},\nu})$ for every $x^\nu \in X^\nu$ and all $i = 1, \cdots, \ell_\nu$ and every $\nu = 1, \cdots, N$;

(copositivity of coupled constraints) so are $\nabla^2 G_j(x)$ on $\mathcal{T}(\boldsymbol{\Xi}; x^{\text{ref}})$ for all $j = 1, \cdots p$;

(weak coercivity of players' objectives) the set $\left\{ x \in \boldsymbol{\Xi} \mid (x - x^{\text{ref}})^T \Theta(x) < 0 \right\}$ is bounded (possibly empty).

Uniqueness of QNE

For uniqueness, examine the pair $(J\mathbf{\Phi}(x,\pi,\lambda), \mathcal{C}(\mathbf{K},\mathbf{\Phi};(x,\pi,\lambda))$. First,

$$J\mathbf{\Phi}(x,\chi) = \left[\begin{array}{cc} \mathbf{A}(x,\chi) & J\Psi(x)^T \\ -J\Psi(x) & 0 \end{array}\right], \quad \text{where } \chi \triangleq (\pi,\lambda) \quad \text{and}$$

$$\mathbf{A}(x,\chi) \triangleq \underbrace{J\Theta(x) + \sum_{j=1}^{p} \pi_j \nabla^2 G_j(x) + \text{blkdiag}\left[\sum_{i=1}^{\ell_\nu} \lambda_i^\nu \nabla^2 h_i^\nu(x^\nu)\right]_{\nu=1}^{N}}_{\text{Jacobian of VI Lagrangian}}.$$

The critical cone of the VI $(\mathbf{K},\mathbf{\Phi})$ at $y \triangleq (x,\pi,\lambda) \in \mathrm{SOL}(\mathbf{K},\mathbf{\Phi})$:

$$\mathcal{C}(\mathbf{K},\mathbf{\Phi};y) =$$

$$\widehat{\mathcal{C}}(x,\chi) \times \left[\mathcal{T}(\mathbb{R}^p_+;\pi) \cap G(x)^\perp\right] \times \prod_{\nu=1}^{N}\left[\mathcal{T}(\mathbb{R}^{\ell_\nu}_+;\lambda^\nu) \cap h^\nu(x)^\perp\right]$$

where $\widehat{\mathcal{C}}(x,\chi) \triangleq \mathcal{T}(\mathbf{\Xi};x) \cap (\Theta(x) + J\Psi(x)\chi)^\perp$.

Thus, $J\mathbf{\Phi}(x,\chi)$ is copositive on $\mathcal{C}(\mathbf{K},\mathbf{\Phi};y)$ if and only if $\mathbf{A}(x,\chi)$ is copositive on $\widehat{\mathcal{C}}(x,\chi)$.

Thus, $J\mathbf{\Phi}(x,\chi)$ is copositive on $\mathcal{C}(\mathbf{K},\mathbf{\Phi};y)$ if and only if $\mathbf{A}(x,\chi)$ is copositive on $\widehat{\mathcal{C}}(x,\chi)$.

The game $\mathcal{G}(\Theta, H, G, \widehat{G})$ has a unique QNE if in addition to the conditions for existence,

• the set $\left\{ x \in \mathbf{\Xi} \mid (x - x^{\text{ref}})^T \Theta(x) \leq 0 \right\}$ is bounded,

• for every QNE (x^*, π^*, λ^*), the matrix $\mathbf{A}(x^*, \pi^*, \lambda^*)$ is strictly copositive on $\widehat{\mathcal{C}}(x^*, \chi^*)$,

• a strict Mangasarian-Fromovitz constraint qualification holds that ensures the uniqueness of (π^*, λ^*).

When is a QNE a LNE?

Answer: Under a second-order sufficiency condition as in NLP.

Let $L_\nu^{(2)}(x,\pi,\lambda^\nu) \triangleq \nabla^2_{x^\nu}\theta_\nu(x) + \sum_{j=1}^{p} \pi_j \, \nabla^2_{x^\nu} G_j(x) + \sum_{i=1}^{\ell_\nu} \lambda_i^\nu \, \nabla^2 h_i^\nu(x^\nu)$;

note that

$$\mathbf{A}(x,\chi) \, = \, \underbrace{\text{Diag}\left(L_\nu^{(2)}(x,\pi,\lambda^\nu) \right)_{\nu=1}^{N}}_{\text{diagonal blocks}} \, + \, \text{Off-diagonal blocks of } J\Theta(x),$$

The critical cone of player q's optimization problem:

$$\mathcal{C}^{\,\nu}(x^*,\pi^*,\widehat{\pi}^{\,*}) \, \triangleq$$

$$\mathcal{T}(\mathcal{X}^{\,\nu};x^{*,\nu}) \, \cap \, \left[\nabla_{x^\nu}\theta_\nu(x^*) + \sum_{j=1}^{p} \pi_j^* \, \nabla_{x^\nu} G_j(x^*) + \sum_{j=1}^{\widehat{p}} \widehat{\pi}_j^{\,*} \, \nabla_{x^\nu} \widehat{G}_j(x^*) \right]^\perp.$$

When is a QNE a LNE?

Answer: Under a second-order sufficiency condition as in NLP.

Let $L_\nu^{(2)}(x,\pi,\lambda^\nu) \triangleq \nabla^2_{x^\nu}\theta_\nu(x) + \sum_{j=1}^{p} \pi_j \nabla^2_{x^\nu} G_j(x) + \sum_{i=1}^{\ell_\nu} \lambda_i^\nu \nabla^2 h_i^\nu(x^\nu)$;

note that

$$\mathbf{A}(x,\chi) = \underbrace{\text{Diag}\left(L_\nu^{(2)}(x,\pi,\lambda^\nu)\right)_{\nu=1}^{N}}_{\text{diagonal blocks}} + \text{ Off-diagonal blocks of } J\Theta(x),$$

The critical cone of player q's optimization problem:

$$\mathcal{C}^\nu(x^*,\pi^*,\widehat{\pi}^*) \triangleq$$

$$\mathcal{T}(\mathcal{X}^\nu; x^{*,\nu}) \cap \left[\nabla_{x^\nu}\theta_\nu(x^*) + \sum_{j=1}^{p} \pi_j^* \nabla_{x^\nu} G_j(x^*) + \sum_{j=1}^{\widehat{p}} \widehat{\pi}_j^* \nabla_{x^\nu} \widehat{G}_j(x^*)\right]^\perp.$$

- Let $(x^*,\pi^*,\lambda^*) \in \text{SOL}(\mathbf{K},\mathbf{\Phi})$. If $\exists\, \widehat{\pi}^*$ such that for each $\nu = 1,\cdots,N$, $L_\nu^{(2)}(x^*,\pi^*,\lambda^{*,\nu})$ is strictly copositive on $\mathcal{C}^\nu(x^*,\pi^*,\widehat{\pi}^*)$, then $(x^*,\pi^*,\widehat{\pi}^*)$ is a LNE.

Existence and Uniqueness of NE

Recall 2 challenges:

- nonconvexity of players' optimization problems

$$\underset{x^\nu \in X^\nu \text{ and } h^\nu(x^\nu) \leq 0}{\text{minimize}} \quad L_\nu(\mathbf{x}, \pi, \widehat{\pi}). \tag{2}$$

- no explicit bounds on prices π and $\widehat{\pi}$

$$\underset{\widehat{\pi} \geq 0}{\text{minimize}}\ \widehat{\pi}^T \widehat{G}(\mathbf{x}) \quad \text{and} \quad \underset{\pi \geq 0}{\text{minimize}}\ \pi^T G(\mathbf{x}).$$

Remedies:

- impose uniqueness (guaranteeing continuity) of players' best responses for given rivals' strategies $x^{-\nu}$ and prices $(\pi, \widehat{\pi})$: how to justify such uniqueness?
- for $t > 0$, consider a price-truncated game $\mathcal{G}_t(\,\Theta, H, G, \widehat{G}\,)$ with

$$\underset{\widehat{\pi} \in \widehat{S}_t}{\text{minimize}}\ \widehat{\pi}^T \widehat{G}(\mathbf{x}) \quad \text{and} \quad \underset{\pi \in S_t}{\text{minimize}}\ \pi^T G(\mathbf{x}),$$

where $S_t \triangleq \left\{ \pi \geq 0 \mid \sum_{j=1}^{p} \pi_j \leq t \right\}$ and similarly for $\widehat{S}_t$.

The price-truncated game $\mathcal{G}_t(\,\Theta, H, G, \widehat{G}\,)$ for fixed $t > 0$

Write $\widehat{X} \triangleq \prod_{\nu=1}^{N} X^\nu$ and let $\Lambda_t^\nu \triangleq \left\{ \lambda^\nu \geq 0 \mid \sum_{i=1}^{\ell_\nu} \lambda_i^\nu \leq \xi_t \right\}$, where

$$\xi_t \triangleq \frac{\displaystyle\max_{(\mu,\widehat{\mu}) \in S_t \times \widehat{S}_t} \left\{ \max_{x \in \widehat{X}} \left| \sum_{q=1}^{Q} (\, x^{\mathrm{ref},\nu} - x^\nu \,)^T \nabla_{x^\nu} L_\nu(\mathbf{x}, \mu, \widehat{\mu}) \right| \right\}}{\displaystyle\min_{1 \leq \nu \leq N} \left(\min_{1 \leq i \leq \ell_\nu} (\, -h_i^\nu(x^{\mathrm{ref},\nu})\,) \right)}.$$

Suppose $\widehat{X}$ is bounded and Slater and copositivity hold. Let $t > 0$.

- For each pair $(\pi, \widehat{\pi}) \in S_t \times \widehat{S}_t$, every KKT multiplier λ^q belongs to Λ_t.
- The game $\mathcal{G}_t(\,\Theta, H, G, \widehat{G}\,)$ has a NE if in addition:

(a) a CQ holds at every optimal solution of players' optimization problem (2),

(b) the matrix $L_\nu^{(2)}(x, \mu, \lambda^\nu)$ is positive definite for all $(x, \mu, \lambda^\nu) \in \widehat{X} \times S_t \times \Lambda_t^\nu$.

Proof by Brouwer fixed-point theorem applied to best-response map and price regularization. □

Bounding the prices and removing truncation

There exists a scalar ξ_* such that every KKT tuple $(x^t, \widehat{\pi}^{\,t}, \pi^t, \lambda^t)$ of the game $\mathcal{G}_t(\Theta, H, G, \widehat{G})$ satisfies

$$\sum_{j=1}^{p} \pi_j^t + \sum_{j=1}^{\widehat{p}} \widehat{\pi}_j^{\,t} + \sum_{\nu=1}^{N} \sum_{i=1}^{\ell_\nu} \lambda_i^{t,\nu} \leq \xi_*.$$

If $t > \xi_*$, then the price-truncation constraints $\sum_{j=1}^{\widehat{p}} \widehat{\pi}_j \leq t$ and $\sum_{j=1}^{p} \pi_j \leq t$, are not binding, thus recovering the price complementarity condition.

Existence and Uniqueness of NE

(A) Assume boundedness of $\widehat{X}$, Slater, and copositivity, and that $\exists$ a scalar $t > \xi_*$ satisfying for all $\nu = 1, \cdots, N$

• a CQ holds at every optimal solution of (2) for every $(x^{-\nu}, \pi, \widehat{\pi}) \in X^{-\nu} \times S_t \times \widehat{S}_t$,

• the matrix $L_\nu^{(2)}(x, \pi, \lambda^q)$ is positive definite for all $(x, \pi, \lambda^\nu) \in \widehat{X} \times S_t \times \Lambda_t^\nu$.

Then the game $\mathcal{G}(\,\Theta, H, G, \widehat{G}\,)$ has a NE $(x^*, \pi^*, \widehat{\pi}^{\,*})$ with $\pi^* \in S_t$ and $\widehat{\pi}^{\,*} \in \widehat{S}_t$.

(B) If the matrix $\mathbf{A}(x, \pi, \lambda)$ is positive definite for all $(x, \pi, \lambda) \in \widehat{X} \times S_t \times \prod_{\nu=1}^{N} \Lambda_t^\nu$, then the x-component of a NE of the game $\mathcal{G}(\,\Theta, H, G, \widehat{G}\,)$ is unique.

A special multi-leader-follower game

Setting. There are Q dominant players. Associated with dominant player q is a group $\mathcal{F}_q$ of Nash followers who react non-cooperatively to his/her strategy z^q. Assume that the Nash equilibria of the followers in $\mathcal{F}_q$, denoted $y^q \in \mathbb{R}^{\ell_q}$, are characterized by the linear complementarity problem parameterized by z^q:

$$0 \leq y^q \perp w^q \triangleq r^q + N^q z^q + M^q y^q \geq 0$$

for some constant vector r^q and matrices N^q and M^q with the latter being a square matrix of order ℓ_q.

Dominant player q's optimization problem is:

$$\begin{array}{ll} \underset{z^q,\, y^q,\, w^q}{\text{minimize}} & \theta_q(z^q, y^q, z^{-q}) \\ \text{subject to} & (z^q, y^q) \in Z^q \triangleq \{ (z^q, y^q) \mid A^q z^q + B^q y^q \leq b^q \} \\ & 0 \leq y^q,\ w^q \perp r^q + N^q z^q + M^q y^q \geq 0. \end{array}$$

The overall non-cooperative game among the selfish dominant players is an equilibrium program with equilibrium constraints.

Let $X^q \triangleq \{ (z^q, y^q) \in Z^q \mid y^q \geq 0$ and $r^q + N^q z^q + M^q y^q \geq 0 \}$.

Existence of ε-QNE

For every $\varepsilon > 0$, consider the relaxed problem:

$$\begin{aligned}
\underset{z^q,\,y^q,\,w^q}{\text{minimize}} \quad & \theta_q(z^q, y^q, z^{-q}) \\
\text{subject to} \quad & (\, z^q, y^q \,) \in Z^q \triangleq \{\, (z^q, y^q) \mid A^q z^q + B^q y^q \le b^q \,\} \\
& 0 \le y^q, \; w^q \triangleq r^q + N^q z^q + M^q y^q \ge 0 \\
\text{and} \quad & y_i^q \, w_i^q \le \varepsilon, \quad i = 1, \cdots, \ell_q.
\end{aligned}$$

Suppose that for every $q = 1, \cdots, Q$, $\theta_q(\bullet, \bullet, x^{-q})$ is continuously differentiable on an open set containing X^q and $z^{\mathrm{ref},q}$ exists such that $A^q z^{\mathrm{ref},q} \le b^q$ and $r^q + N^q z^{\mathrm{ref},q} = 0$. Assume further that the set

$$\left\{ \begin{array}{l} (\, z^q, y^q \,)_{q=1}^Q \in \displaystyle\prod_{q=1}^Q X^q \mid \\ \displaystyle\sum_{q=1}^Q \left[(\, z^q - z^{\mathrm{ref},q} \,)^T \nabla_{z^q} \theta_q(z^q, y^q, z^{-q}) + (\, y^q \,)^T \nabla_{y^q} \theta_q(z^q, y^q, z^{-q}) \right] < 0 \end{array} \right\}$$

is bounded (possibly empty). Then, for every $\varepsilon > 0$, an ε-QNE to the multi-leader-follower game exists.

Lecture III: Exact penalization via error bounds and constraint qualifications

11:00 – noon Tuesday September 24, 2019

Recall the GNEP, which we denote $\mathcal{G}(C;X;\theta)$, where player ν solves:

$$\underset{x^\nu \in C^\nu \cap X^\nu(x^{-\nu})}{\text{minimize}} \quad \theta_\nu(x^\nu, x^{-\nu})$$

where C^ν and $X^\nu(x^{-\nu})$ are both closed and convex in $\mathbb{R}^{n_\nu}$. Let $r_\nu(\bullet, x^{-\nu})$ be a C^ν-residual function of the latter set; i.e., $r_\nu(\bullet, x^{-\nu})$ is nonnegative on C^ν and

$$\left[x^\nu \in C^\nu \text{ and } r_\nu(x^\nu, x^{-\nu}) = 0 \right] \Leftrightarrow x^\nu \in C^\nu \cap X^\nu(x^{-\nu}).$$

Let $\theta_{\nu;\rho;X}(x^\nu, x^{-\nu}) = \theta_\nu(x^\nu, x^{-\nu}) + \rho\, r_\nu(x^\nu, x^{-\nu})$. Consider the Nash equilibrium problem, which we denote NEP $(C;\theta_{\rho;X})$, where player ν solves:

$$\underset{x^\nu \in C^\nu}{\text{minimize}} \quad \theta_{\nu;\rho;X}(x^\nu, x^{-\nu})$$

(Partial) exact penalization is concerned with the question: Does $\exists$ a finite $\bar{\rho} > 0$ such that for all $\rho \geq \bar{\rho}$,

$$\mathcal{G}(C;X;\theta) \Leftrightarrow \text{NEP}(C;\theta_{\rho;X})??$$

Review: (partial) exact penalization of optimization problems

Consider

$$\underset{x \in W \cap S}{\text{minimize}} \; f(x),$$

where W and S are two closed sets of constraints with S considered more complex than W. Assume that $\boxed{S \cap W \neq \emptyset}$.

The penalized optimization problem for a $\rho > 0$,

$$\underset{x \in W}{\text{minimize}} \; f(x) + \rho\, r_S(x),$$

where $r_S(x)$ is a W-residual function of the set S.

Classical.. Let f be Lipschitz on W with constant $\text{Lip}_f > 0$. If $r_S(x)$ is a W-residual function of the set S satisfying a W-Lipschitz error bound for the set S with constant $\gamma > 0$, i.e., $r_S(x) \geq \gamma \text{dist}(x; S \cap W)$ for all $x \in W$, then for all $\rho > \text{Lip}_f/\gamma$,

$$\underset{x \in S \cap W}{\text{argmin}} \; f(x) = \underset{x \in W}{\text{argmin}} \, f(x) + \rho\, r_S(x).$$

Exact penalization of games: Preliminary result

• If $\mathbf{x}^*$ is an equilibrium solution of penalized NEP$(C; \theta_{\rho;X})$, then $\mathbf{x}^*$ is a NE of $\mathcal{G}(C; X; \theta)$ if and only if $\mathbf{x}^* \in \mathsf{FIX}_{\boldsymbol{\Xi}}$.

• Suppose $\exists$ positive constants Lip$_\nu$ and γ_ν such that for every $x^{-\nu} \in C^{-\nu}$,

– $\boxed{C^\nu \cap X^\nu(x^{-\nu}) \neq \emptyset}$ $\longleftarrow$ main weakness;

– $\theta_\nu(\bullet, x^{-\nu})$ is Lipschitz continuous with constant Lip$_\nu$ on the set C^ν, and

– $r_\nu(\bullet, x^{-\nu})$ provides a C^ν-Lipschitz error bound with constant γ_ν for the set $X^\nu(x^{-\nu})$.

Then for all $\rho > \max\limits_{\nu \in [N]} \text{Lip}_\nu / \gamma_\nu$, every equilibrium solution of the penalized NEP $(C; \theta_{\rho;X})$ is an equilibrium solution of the GNEP $(C, X; \theta)$ and vice versa.

Example.. Consider the shared-constrained GNEP with 2 players:

$$\textbf{Player 1} \quad \underset{x_1 \in \mathbb{R}}{\text{minimize}} \quad x_2 \,(\, x_1 - 1 \,)^2 \quad \text{subject to} \quad x_1 + x_2 \,\leq\, 1 \qquad \Bigg| \qquad \textbf{Player 2} \quad \underset{x_2 \in \mathbb{R}_+}{\text{minimize}} \quad (\, x_2 - 1/2 \,)^2 \quad \text{subject to} \quad x_1 + x_2 \,\leq\, 1.$$

The Karush-Kuhn-Tucker (KKT) conditions of this GNEP are:

$$\begin{aligned}
0 &= 2\, x_2 \,(\, x_1 - 1 \,) + \lambda_1 \\
0 &\leq x_2 \perp 2\,(\, x_2 - \tfrac{1}{2} \,) + \lambda_2 \geq 0 \\
0 &\leq \lambda_1 \perp 1 - x_1 - x_2 \geq 0, \\
0 &\leq \lambda_2 \perp 1 - x_1 - x_2 \geq 0,
\end{aligned}$$

This GNEP has no variational equilibrium, i.e., solutions with $\lambda_1 = \lambda_2$. Partial exact penalization is valid for this GNEP. In particular, the NE $\left(\frac{1}{2}, \frac{1}{2}\right)$ is not a normalized equilibrium in the sense of Rosen.

Thus penalization can yield solutions that are otherwise not obtainable by Rosen's normalization approach.

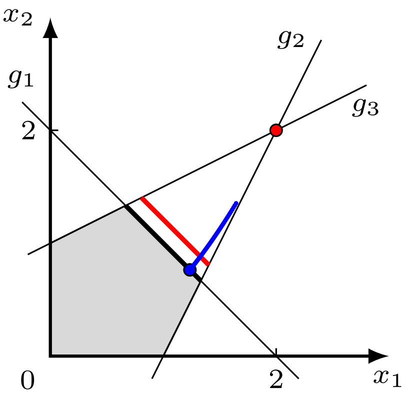
Example.. Consider the shared-constrained GNEP with 2 players:

$$C_1 == C_2 = [0,4] \quad \text{private constraints}$$

common constraints $\left.\begin{array}{l} X_1(x_2) = \{x_1 \mid x_1 + x_2 \leq 2,\ 2x_1 - x_2 \leq 2,\ -x_1 + 2x_2 \leq 2\} \\ X_2(x_1) = \{x_2 \mid x_1 + x_2 \leq 2,\ 2x_1 - x_2 \leq 2,\ -x_1 + 2x_2 \leq 2\} \end{array}\right.$

(A) $\theta_1(x_1,x_2) = -x_1$ and $\theta_2(x_1,x_2) = -x_2$

(B) $\theta_1(x_1,x_2) = -\frac{1}{2}\, x_1\, x_2$ and $\theta_2(x_1,x_2) = -\dfrac{1}{4}\, x_1\, x_2$.



- ℓ_1 **penalization is not exact:**

$$r_1(x_1, x_2) = (x_1 + x_2 - 2)_+ + (2x_1 - x_2 - 2)_+ + (-x_1 + 2x_2 - 2)_+.$$

- **Squared** ℓ_2 **penalization is not exact:**

$$r_2(x_1, x_2)^2 =$$

$$[(x_1 + x_2 - 2)_+]^2 + [(2x_1 - x_2 - 2)_+]^2 + [(-x_1 + 2x_2 - 2)_+]^2.$$

- ℓ_2 **penalization is exact.**

- ℓ_1 **penalization is exact for variational equilibria (VE).**

- ℓ_1 **penalization is exact when** C **is restricted** by redefining $\widetilde{C} = \widetilde{C}_1 \times \widetilde{C}_2$ with

$$\widetilde{C}_1 \triangleq \{ x_1 \in C_1 \mid \exists\, x_2 \in C_2 \text{ such that } x_1 \in X_1(x_2) \}$$

and similarly for $\widetilde{C}_2$. Further research is needed.

- ℓ_2**-penalized NE is not VE.**

The jointly convex GNEP $(\mathbf{C}; \mathbf{D}; \theta)$:

- $\mathbf{C} \triangleq \prod_{\nu=1}^{N} C^{\nu}$ is the product of the private constraint sets;

- for some closed convex subset $\mathbf{D}$ of $\mathbb{R}^n$,

$$\text{graph } X^{\nu} = \mathbf{D}, \quad \text{for all } \nu.$$

- Let r_D be a directionally differentiable $\mathbf{C}$-residual function of the set $\mathbf{D}$, yielding, for $\rho > 0$, the penalized NEP $(\mathbf{C}; \theta + \rho\, r_D)$.

Notation:

Let $\mathcal{T}(\bar{x}; S)$ denote the tangent cone of the set S at $\bar{x} \in S$.

Let $\varphi'(\bar{x}; dx) \triangleq \lim_{\tau \downarrow 0} \dfrac{\varphi(\bar{x} + \tau\, dx) - \varphi(\bar{x})}{\tau}$ be the directional derivative of φ at $\bar{x}$ along the direction dx.

Theorem.. Suppose

• each $\theta_\nu(\bullet, x^{-\nu})$ is directionally differentiable and locally Lipschitz continuous on C^ν for all $x^{-\nu} \in C^{-\nu}$;

• for every $x = (x^\nu)_{\nu=1}^N \in C \setminus D$, either one of the following holds:

– for some $\bar{\nu}$ and some nonzero $d^{\bar{\nu}} \in \mathcal{T}(x^{\bar{\nu}}; C^{\bar{\nu}}) \subseteq \mathbf{R}^{n_{\bar{\nu}}}$,

$$r_D(\bullet, x^{-\bar{\nu}})'(x^{\bar{\nu}}; d^{\bar{\nu}}) \leq -\alpha' \, \| \, d^{\bar{\nu}} \, \|_2;$$

– for some nonzero tangent vector $d \in \mathcal{T}(x; \mathbf{C})$,

$$\underbrace{\sum_{\nu \in [N]} r_D(\bullet, x^{-\nu})'(x^\nu; d^\nu) \leq r_D'(x; d)}_{\text{sum property of directional derivative}} \leq -\alpha \, \| \, d \, \|_2,$$

then $\exists$ a finite number $\bar{\rho}$ such that for every $\rho > \bar{\rho}$, every equilibrium solution of the NEP $(C; \theta + \rho r_D)$ is an equilibrium solution of the GNEP $(C, D; \theta)$.

Linear metric regularity

Two closed convex subsets C and D of $\mathbb{R}^n$ with a nonempty intersection are linearly metrically regular if there exists a constant $\gamma' > 0$ such that

$$\text{dist}(x; C \cap D) \leq \gamma' \max(\text{dist}(x; C), \text{dist}(x; D)), \quad \forall x \in \mathbb{R}^n.$$

This holds if either

- $C \cap D$ is bounded and $\text{rint}(C) \cap \text{rint}(D) \neq \emptyset$; or
- C and D are polyhedra.

Corollary.. Exact penalization holds for shared constrained GNEP if

- $\mathbf{C}$ and $\mathbf{D}$ are linear metrically regular;
- r_D is convex on $\mathbf{C}$, satisfies a $\mathbf{C}$-Lipschitz error bound for $\mathbf{D}$, and differentiable on $\mathbf{C} \setminus \mathbf{D}$.

Shared finitely representable sets

Let $\mathbf{C}$ be convex and compact, and

$$\mathbf{D} = \{ \mathbf{x} \in \mathbf{R}^n \mid g(\mathbf{x}) \leq 0 \text{ and } h(\mathbf{x}) = 0 \},$$

where h is affine and each g_j is convex and differentiable. Let

$$r_q(x) \triangleq \left\| \begin{pmatrix} \max(\, g(x), 0\,) \\ h(x) \end{pmatrix} \right\|_q, \quad x \in \mathbb{R}^n,$$

for a given $q \in [1, \infty)$, which is differentiable at $x \notin \mathbf{D}$.

Theorem.. Suppose each $\theta_\nu(\bullet, x^{-\nu})$ is convex and Lipschitz continuous on $C^{\,\nu}$ with the same Lipschitz constant for all $x^{-\nu} \in C^{\,-\nu}$. Then

- for every $\rho > 0$, the NEP $(C; \theta + \rho\, r_q)$ has an equilibrium solution.

Assume further that a vector $\bar{x} \in \text{rint}(C)$ exists such that $h(\bar{x}) = 0$ and $g(\bar{x}) < 0$. Then $\bar{\rho} > 0$ exists such that for every $\rho > \bar{\rho}$

$$\text{NEP } (C; \theta + \rho\, r_q) \Rightarrow \text{ GNEP } (C, D; \theta).$$

Non-shared finitely representable sets

Similar setting but with

$$X^{\nu}(x^{-\nu}) \triangleq \left\{ \begin{array}{ccc} x^{\nu} \in \mathbb{R}^{n_\nu} & | & g^{\nu}(x^{\nu}, x^{-\nu}) \leq 0 \text{ and} \\ & | & \underbrace{h^{\nu}(x) = 0}_{\text{linear equalities allowed}} \end{array} \right\},$$

where each $h^{\nu} : \mathbb{R}^n \to \mathbb{R}^{p_\nu}$ is an affine function and each $g_j^{\nu} : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, \cdots, m_\nu$ is such that $g_j^{\nu}(\bullet, x^{-\nu})$ is convex and differentiable for every $x^{-\nu} \in C^{-\nu}$. Let

$$r_{\nu;q}(x) \triangleq \left\| \begin{pmatrix} \max(\, g^{\nu}(x), 0\,) \\ h^{\nu}(x) \end{pmatrix} \right\|_q, \quad x \in \mathbb{R}^n,$$

for a given $q \in [1, \infty)$, be the ℓ_q-residual function for the set $X^{\nu}(x^{-\nu})$. Let $r_{X;q}(x) \triangleq (\, r_{\nu;q}(x)\,)_{\nu=1}^N$.

Theorem.. Suppose there exists $\alpha > 0$ such that for every $x \in C \setminus \text{FIX}_{\Xi}$, there exists $\widehat{x} \in C$ satisfying for some $\widehat{\nu}$ with $x^{\widehat{\nu}} \notin X^{\widehat{\nu}}(x^{-\widehat{\nu}})$,

- for all $i \in \{1, \cdots, m_\nu\}$,

$$\underbrace{g_i^{\widehat{\nu}}(x) > 0 \Rightarrow \nabla_{x^{\widehat{\nu}}} g_i^{\widehat{\nu}}(x^{\widehat{\nu}}, x^{-\widehat{\nu}})^T (\widehat{x}^{\widehat{\nu}} - x^{\widehat{\nu}}) \leq -\alpha}_{\text{uniform descent condition at infeasible } x}$$

- for all $j \in \{1, \cdots, p_\nu\}$,

$$\underbrace{h_j^{\widehat{\nu}}(x) \neq 0 \Rightarrow \left[\text{sgn } h_j^{\widehat{\nu}}(x)\right] \nabla_{x^{\widehat{\nu}}} h_j^{\widehat{\nu}}(x^{\widehat{\nu}}, x^{-\widehat{\nu}})^T (\widehat{x}^{\widehat{\nu}} - x^{\widehat{\nu}}) \leq -\alpha\,.}_{\text{uniform descent condition at infeasible } x}$$

Then $\bar{\rho} > 0$ exists such that for every $\rho > \bar{\rho}$, every equilibrium solution of the NEP $(C; \theta + \rho\, r_{X,q})$ is an equilibrium solution of the GNEP $(C, X; \theta)$.

The Extended Mangasarian-Fromovitz Constraint Qualification (EMFCQ) for equalities only:

$$X^{\nu}(x^{-\nu}) \triangleq \{ x^{\nu} \in \mathbb{R}^{n_\nu} \mid g^{\nu}(x^{\nu}, x^{-\nu}) \leq 0 \} \quad \text{no equalities.}$$

For every $x \in \mathbf{C}$ and every $\nu \in \{1, \cdots, N\}$, there exists $\widehat{x} \in \mathbf{C}$ such that

$$g_i^{\nu}(x) \geq 0 \Rightarrow \nabla_{x^\nu} g_i^{\nu}(x^{\nu}, x^{-\nu})^T (\widehat{x}^{\nu} - x^{\nu}) < 0.$$

Remark. Imposed condition applies to all $x \in \mathbf{C} \cap \text{FIX}_{\Xi}$, which is to ensure the validity of the KKT conditions at a solution.

EMFCQ is more restrictive than required for the validity of exact penalization.

Lecture IV: Best-response algorithms

9:30 – 10:30 AM Wednesday September 25, 2019

A fixed-point (best-response) scheme

Returning to the GNEP $(\boldsymbol{\Xi}, \theta)$, we define the proximal response map derived from the regularized Nikaido-Isoda function:

$$\mathbf{y} \triangleq (y^{\nu})_{\nu=1}^{N} \mapsto \widehat{\mathbf{x}}(\mathbf{y}) \triangleq \underset{\mathbf{x} \in \boldsymbol{\Xi}(\mathbf{y})}{\text{argmin}} \sum_{\nu=1}^{N} \left[\theta_{\nu}(x^{\nu}, y^{-\nu}) + \tfrac{1}{2} \| x^{\nu} - y^{\nu} \|^{2} \right]$$

Fact: $\mathbf{y}$ is a NE if and only if $\mathbf{y}$ is a fixed point of the proximal response map $\widehat{\mathbf{x}}$.

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Fact: $\mathbf{y}$ is a NE if and only if $\mathbf{y}$ is a fixed point of the proximal response map $\widehat{\mathbf{x}}$.

A fixed-point iteration: $\mathbf{y}^{k+1} \triangleq \widehat{\mathbf{x}}(\mathbf{y}^k)$; $k = 0, 1, 2, \cdots$.

Theoretically, when is this a contraction, a non-expansion?

Computationally, allow distributed player optimization:

$$\underset{x^\nu \in \Xi^\nu(y^{k,-\nu})}{\text{minimize}} \quad \theta_\nu(x^\nu, y^{k,-\nu}) + \tfrac{1}{2} \| x^\nu - y^{k,\nu} \|^2, \quad \text{for } \nu = 1, \cdots, N,$$

each of which is a strongly convex optimization problem.

Convergence theory

Basic version requires

- $\Xi^\nu(x^{-\nu}) = X^\nu$ for all ν, and
- the second derivatives $\nabla^2_{x^{\nu'} x^\nu} \theta_\nu(\mathbf{x}) \triangleq \left[\frac{\partial^2 \theta_\nu(\mathbf{x})}{\partial x_j^{\nu'} x_i^\nu} \right]_{(i,j)=1}^{(n_\nu, n_{\nu'})}$ exist and are

bounded on $\widehat{\mathbf{X}} \triangleq \prod_{\nu=1}^N X^\nu$. Weakening to a 4-point condition of first derivatives is possible (Hao 2018 Ph.D. thesis).

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A matrix-theoretic criterion. Let

$$\zeta^\nu_{\min} \triangleq \inf_{\mathbf{x}\in\widehat{\mathbf{X}}} \text{ smallest eigenvalue of } \nabla^2_{x^\nu}\theta_\nu(\mathbf{x})$$

$$\xi^{\nu\nu'}_{\max} \triangleq \sup_{\mathbf{x}\in\widehat{\mathbf{X}}} \left\| \nabla^2_{x^\nu x^{\nu'}}\theta_\nu(\mathbf{x}) \right\|$$

Define the $N \times N$ Z-matrix (all off-diagonal entries are non-positive):

$$\Upsilon \triangleq \begin{bmatrix} \zeta_{\min}^{1} & -\xi_{\max}^{12} & -\xi_{\max}^{13} & \cdots & -\xi_{\max}^{1N} \\ -\xi_{\max}^{21} & \zeta_{\min}^{2} & -\xi_{\max}^{23} & \cdots & -\xi_{\max}^{2N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\xi_{\max}^{(N-1)\,1} & -\xi_{\max}^{(N-1)\,2} & \cdots & \zeta_{\min}^{N-1} & -\xi_{\max}^{(N-1)\,N} \\ -\xi_{\max}^{N\,1} & -\xi_{\max}^{N\,2} & \cdots & -\xi_{\max}^{N\,N-1} & \zeta_{\min}^{N} \end{bmatrix}.$$

• If Υ is a P-matrix (all principal minors are positive), then the proximal response map is a contraction; moreover the NE is unique and can be obtained by the fixed-point iteration.

• If $|||\Upsilon||| \leq 1$ for a matrix norm induced by some monotonic vector norm, then the proximal response map is nonexpansive; in this case, an averaging scheme can compute a NE if one exists.