JACK AND JULIA

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Abstract. We state and prove a multi–point version of Jack's Lemma for functions not necessarily analytic on the closure of the open unit disc. Our proof in particular does not rely on Julia's lemma.

1. Introduction and statement of the main result

Let $\mathbb{D}$ denote the open unit disc $\{z : |z| < 1\}$ of the complex plane $\mathbb{C}$ and $\mathcal{H}(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. $\overline{\mathbb{D}}$ is the closed disc $\{z : |z| \le 1\}$ and $\mathcal{H}(\overline{\mathbb{D}})$ is the class of functions analytic on some open set containing $\overline{\mathbb{D}}$. The following result was first stated by Jack [9] who attributed its proof to Clunie:

Jack's lemma (smooth version). *Let* $F \in \mathcal{H}(\overline{\mathbb{D}})$ *with* $|F(\zeta)| = \max\limits_{|z|\le 1} |F(z)| = \max\limits_{|z|=1} |F(z)| > 0$ *where* $|\zeta| = 1$. *Then*

$$\zeta \frac{F'(\zeta)}{F(\zeta)} \ge 0$$

and in fact $\operatorname{Re}\left(1 + \zeta \frac{F''(\zeta)}{F'(\zeta)}\right) \ge \zeta \frac{F'(\zeta)}{F(\zeta)} > 0$ *if* F *is non-constant.*

After the publication by Jack, it has been observed that the lemma is indeed valid for functions F in $\mathcal{H}(\mathbb{D})$ also analytic in a neighbourhood of $\zeta \in \partial\mathbb{D}$ and this result goes back to Loewner at least in the 1930's (see [14, p.162]). Under the milder hypothesis, the result has been rediscovered, improved and applied by a number of mathematicians (see for example [7], the book of Miller and Mocanu [12] or the interesting survey by Boas [4]. The survey by Elin et al. [6] also contains relevant information).

The following result is indeed valid (we still call it Jack's lemma in what follows):

Lemma (less smooth Jack's lemma). *Let* $F \in \mathcal{H}(\mathbb{D})$ *with* $F(\mathbb{D}) \subseteq \mathbb{D}$ *and* $\zeta \in \partial\mathbb{D}$. *The following statements are equivalent:*

i) $\liminf\limits_{z\to\zeta} \dfrac{1 - |F(z)|}{1 - |z|} < \infty$.

ii) The radial limits $F(\zeta) := \lim\limits_{r\to 1} F(r\zeta)$ *and* $F'(\zeta) := \lim\limits_{r\to 1} F'(r\zeta)$ *exist with* $|F(\zeta)| = 1$ *and* $|F'(\zeta)| < \infty$.

Moreover, under i) or ii),

$$\zeta \frac{F'(\zeta)}{F(\zeta)} = \lim_{r\to 1} \frac{1 - |F(r\zeta)|}{1 - r} = \lim_{r\to 1} \frac{1 - F(r\zeta)/F(\zeta)}{1 - r} > 0.$$

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The strict positivity of $\zeta\frac{F'(\zeta)}{F(\zeta)}$ above also follows, as observed for example by Tom Ransford [15, p.33], from the lemma of Hopf. We stress the fact that a proof of the less smooth Jack's lemma can be obtained, as in [1] or [8], from the properties of the measure in the representation

$$\frac{1+F(\zeta)}{1-F(\zeta)} = \int_0^{2\pi} \frac{1+e^{-i\theta}\zeta}{1-e^{-i\theta}\zeta} \mathrm{d}\mu(\theta).$$

Such a proof is in particular "horocycle free" and does not rely on Julia's lemma [10] which may be conveniently stated as follows:

Julia's lemma. *Let $F \in \mathcal{H}(\mathbb{D})$ with $F(\mathbb{D}) \subseteq \mathbb{D}$, and $\lim_{z\to\zeta} \frac{1-|F(z)|}{1-|z|} < \infty$ for some $\zeta \in \partial\mathbb{D}$. Then*

$$\zeta\frac{F'(\zeta)}{F(\zeta)} \geq \frac{|1-F(z)\overline{F(\zeta)}|}{1-|F(z)|^2} \, \frac{1-|z|^2}{|1-z\overline{\zeta}|^2}, \quad z \in \mathbb{D}.$$

To the best of our knowledge, the relation between Jack's lemma and Julia's lemma was first made explicit by Ruscheweyh [16].We shall prove that a multipoint version of Julia's lemma can be obtained from the apparently weaker less smooth Jack's lemma. Our main result is the following:

Theorem A. *Let $f \in \mathcal{H}(\mathbb{D})$, $f(\mathbb{D}) \subseteq \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$ such that*

$$\liminf_{z\to\zeta} \frac{1-|f(z)|}{1-|z|} < \infty. \tag{1}$$

Let also $\{z_k\} \subset \mathbb{D}$ and define a (possibly finite) sequence $\{f_k\} \subset \mathcal{H}(\mathbb{D})$ by $f_0 = f$ and

$$f_{k+1}(z) = \frac{1-\overline{z}_k z}{z-z_k} \, \frac{f_k(z)-f_k(z_k)}{1-\overline{f_k(z_k)}f_k(z)}, \quad k \geq 0,$$

provided that f_k is not a unimodular constant. Then

$$\zeta\frac{f'(\zeta)}{f(\zeta)} \geq \sum_{j=0}^{n} \left(\prod_{k=0}^{j} \frac{\left|1-\overline{f_k(z_k)}f_k(\zeta)\right|^2}{1-|f_k(z_k)|^2} \right) \frac{1-|z_j|^2}{|1-\overline{z}_j\zeta|^2}.$$

Equality holds if and only if f is a Blaschke product of degree $n+1$.

Remark. *The functions f_k are the hyperbolic divided differences of the initial function f at the points $\{z_k\}$, cf. the work of Beardon and Minda* [3] *and Baribeau, Rivard and Wegert* [2] *amongst others.*

2. Proof of our main result

Each function f_k belongs to $\mathcal{H}(\mathbb{D})$, $f_k(\mathbb{D}) \subseteq \mathbb{D}$ and satisfies (1) together with

$$\zeta\frac{f'_{k+1}(\zeta)}{f_{k+1}(\zeta)} = \zeta\frac{f'_k(\zeta)}{f_k(\zeta)} \, \frac{1-|f_k(z_k)|^2}{\left|1-\overline{f_k(z_k)}f_k(\zeta)\right|^2} - \frac{1-|z_k|^2}{|1-\overline{z}_k\zeta|^2}$$

for all $k \geq 0$. In particular Jack's lemma yields

$$0 \leq \zeta\frac{f'_1(\zeta)}{f_1(\zeta)} = \zeta\frac{f'(\zeta)}{f(\zeta)} \, \frac{1-|f(z_0)|^2}{\left|1-\overline{f(z_0)}f(\zeta)\right|^2} - \frac{1-|z_0|^2}{|1-\overline{z}_0\zeta|^2}$$

and

$$\zeta\frac{f'(\zeta)}{f(\zeta)} \geq \frac{\left|1-\overline{f(z_0)}f(\zeta)\right|^2}{1-|f(z_0)|^2}\frac{1-|z_0|^2}{|1-\overline{z}_0\zeta|^2}. \tag{2}$$

This is Julia's lemma and clearly equality shall hold in (2) if and only if the function f_1 is constant and therefore if f is a Blaschke product of order 1. An iteration of this procedure shall lead to, for any $n \geq 0$,

$$\zeta\frac{f'(\zeta)}{f(\zeta)} \geq \sum_{j=0}^{n}\left(\prod_{k=0}^{j}\frac{\left|1-\overline{f_k(z_k)}f_k(\zeta)\right|^2}{1-|f_k(z_k)|^2}\right)\frac{1-|z_j|^2}{|1-\overline{z}_j\zeta|^2} \tag{3}$$

and equality holds in (3) if and only if f is a Blaschke product of order $n+1$.

3. Two special cases

Case 1 : Let us take $z_k = 0$ for $k \geq 0$. Then

$$\zeta\frac{f'(\zeta)}{f(\zeta)} \geq \sum_{j=0}^{n}\left(\prod_{k=0}^{j}\frac{\left|1-\overline{f_k(0)}f_k(\zeta)\right|^2}{1-|f_k(0)|^2}\right) \geq \sum_{j=0}^{n}\prod_{k=0}^{j}\frac{1-|f_k(0)|}{1+|f_k(0)|}. \tag{4}$$

It is not difficult to see that the righthand side of (4) is a quantity depending on the $n+1$ first Taylor coefficients $\{\alpha_k\}_{k=0}^{n}$ of $f(\zeta) =: \sum\limits_{j=0}^{\infty}\alpha_j z^j$. The case $n = 0$ is due to Osserman [13] and the case $n = 1$ is due to Lecko and Uzar [11]. The series $\sum\limits_{j=0}^{\infty}\prod_{k=0}^{j}\frac{1-|f_k(0)|}{1+|f_k(0)|}$ is convergent with $\left\{\prod_{k=0}^{j}\frac{1-|f_k(0)|}{1+|f_k(0)|}\right\}$ monotonically decreasing, and hence

$$\lim_{j\to\infty} j\prod_{k=0}^{j}\frac{1-|f_k(0)|}{1+|f_k(0)|} = 0.$$

We recall that according to a result of Boyd [8, p.175]

$$\prod_{k=0}^{\infty} 1-|f_k(0)| = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\ln\left(1-|f(e^{i\theta})|^2\right)\mathrm{d}\theta\right).$$

Case 2. We apply our idea to a function f in $\mathcal{H}(\mathbb{D})$ with $f(\mathbb{D}) \subseteq \mathbb{D}$ satisfying (1) with zeros $\{z_k\}$ in $\mathbb{D}$ and to the sequence defined by $f_{j+1}(z) = f(z)\prod_{k=0}^{j}\frac{1-\overline{z}_k z}{z-z_k}$. We get

$$\zeta\frac{f'_{j+1}(\zeta)}{f_j(\zeta)} = \zeta\frac{f'(\zeta)}{f(\zeta)} - \sum_{k=0}^{j}\frac{1-|z_k|^2}{|1-\overline{z}_k\zeta|^2} \geq 0,$$

and the series $\sum_{k=0}^{\infty}\frac{1-|z_k|^2}{|1-\overline{z}_k\zeta|^2}$ is convergent. We finally recall that in the case where f is a Blaschke product, the convergence of $\sum\limits_{k=0}^{\infty}\frac{1-|z_k|^2}{|1-\overline{z}_k\zeta|^2}$ is necessary and sufficient for the existence of $F(\zeta)$ and $F'(\zeta)$ and in fact $\zeta\frac{f'(\zeta)}{f(\zeta)} = \sum\limits_{k=0}^{\infty}\frac{1-|z_k|^2}{|1-\overline{z}_k\zeta|^2}$. This is a result of Frostman (see [5, p.15]).

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