Morrey’s conjecture for the planar volumetric-isochoric split

We consider Morrey’s open question of whether rank-one convexity already implies quasiconvexity in the planar case. For this, we focus on the volumetric-isochoric split

\[ W(F) = W_{\text{iso}}(F) + W_{\text{vol}}(\det F) = \tilde{W}_{\text{iso}} \left( \frac{F}{\sqrt{\det F}} \right) + W_{\text{vol}}(\det F) \]

in isotropic hyperelasticity and give a precise analysis of rank-one convexity criteria. Starting from the classical two-dimensional criterion by Knowles and Sternberg, we show that the Legendre-Hadamard ellipticity condition separates and simplifies in a suitable sense.

We identify several “least” rank-one convex energies and, in particular, show that for energies with a concave volumetric part, the question of whether rank-one convexity implies quasiconvexity can be reduced to the open question of whether the rank-one convex energy function

\[ W_{\text{magic}}^+(F) = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} - \log \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} + \log \det F = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} - 2 \log \lambda_{\text{min}} \]

is quasiconvex. In addition, we demonstrate that under affine boundary conditions \( W_{\text{magic}}^+(F) \) allows for non-trivial inhomogeneous deformations with the same energy level as the homogeneous solution.