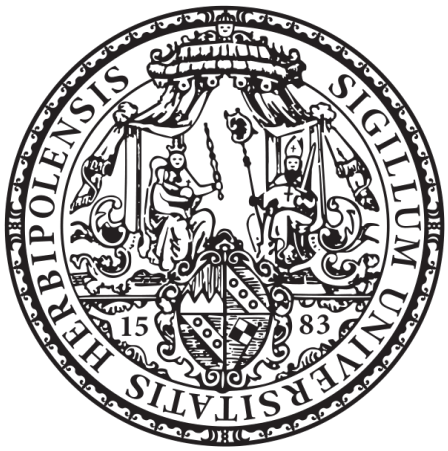
ADMM-Type Methods for Optimization and Generalized Nash Equilibrium Problems in Hilbert Spaces



Dissertation zur Erlangung
des naturwissenschaftlichen Doktorgrades
der Julius-Maximilians-Universität Würzburg

vorgelegt von

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Würzburg, 2020

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Abstract

This thesis is concerned with a certain class of algorithms for the solution of constrained optimization problems and generalized Nash equilibrium problems in Hilbert spaces. This class of algorithms is inspired by the alternating direction method of multipliers (ADMM) and eliminates the constraints using an augmented Lagrangian approach. The alternating direction method consists of splitting the augmented Lagrangian subproblem into smaller and more easily manageable parts.

Before the algorithms are discussed, a substantial amount of background material, including the theory of Banach and Hilbert spaces, fixed-point iterations as well as convex and monotone set-valued analysis, is presented. Thereafter, certain optimization problems and generalized Nash equilibrium problems are reformulated and analyzed using variational inequalities and set-valued mappings. The analysis of the algorithms developed in the course of this thesis is rooted in these reformulations as variational inequalities and set-valued mappings.

The first algorithms discussed and analyzed are one weakly and one strongly convergent ADMM-type algorithm for convex, linearly constrained optimization. By equipping the associated Hilbert space with the correct weighted scalar product, the analysis of these two methods is accomplished using the proximal point method and the Halpern method.

The rest of the thesis is concerned with the development and analysis of ADMM-type algorithms for generalized Nash equilibrium problems that jointly share a linear equality constraint. The first class of these algorithms is completely parallelizable and uses a forward-backward idea for the analysis, whereas the second class of algorithms can be interpreted as a direct extension of the classical ADMM-method to generalized Nash equilibrium problems.

At the end of this thesis, the numerical behavior of the discussed algorithms is demonstrated on a collection of examples.

Zusammenfassung

Die vorliegende Arbeit behandelt eine Klasse von Algorithmen zur Lösung restringierter Optimierungsprobleme und verallgemeinerter Nash-Gleichgewichtsprobleme in Hilberträumen. Diese Klasse von Algorithmen ist angelehnt an die Alternating Direction Method of Multipliers (ADMM) und eliminiert die Nebenbedingungen durch einen Augmented-Lagrangian-Ansatz. Im Rahmen dessen wird in der Alternating Direction Method of Multipliers das jeweilige Augmented-Lagrangian-Teilproblem in kleinere Teilprobleme aufgespaltet.

Zur Vorbereitung wird eine Vielzahl grundlegender Resultate präsentiert. Dies beinhaltet entsprechende Ergebnisse aus der Literatur zu der Theorie von Banach- und Hilberträumen, Fixpunktmethoden sowie konvexer und monotoner mengenwertiger Analysis. Im Anschluss werden gewisse Optimierungsprobleme sowie verallgemeinerte Nash-Gleichgewichtsprobleme als Variationsungleichungen und Inklusionen mit mengenwertigen Operatoren formuliert und analysiert. Die Analysis der im Rahmen dieser Arbeit entwickelten Algorithmen bezieht sich auf diese Reformulierungen als Variationsungleichungen und Inklusionsprobleme.

Zuerst werden ein schwach und ein stark konvergenter paralleler ADMM-Algorithmus zur Lösung von separablen Optimierungsaufgaben mit linearen Gleichheitsnebenbedingungen präsentiert und analysiert. Durch die Ausstattung des zugehörigen Hilbertraums mit dem richtigen gewichteten Skalarprodukt gelingt die Analyse dieser beiden Methoden mit Hilfe der Proximalpunktmethode und der Halpern-Methode.

Der Rest der Arbeit beschäftigt sich mit Algorithmen für verallgemeinerte Nash-Gleichgewichtsprobleme, die gemeinsame lineare Gleichheitsnebenbedingungen besitzen. Die erste Klasse von Algorithmen ist vollständig parallelisierbar und es wird ein Forward-Backward-Ansatz für die Analyse genutzt. Die zweite Klasse von Algorithmen kann hingegen als direkte Erweiterung des klassischen ADMM-Verfahrens auf verallgemeinerte Nash-Gleichgewichtsprobleme aufgefasst werden.

Abschließend wird das Konvergenzverhalten der entwickelten Algorithmen an einer Sammlung von Beispielen demonstriert.

Acknowledgments

My foremost thanks goes to my supervisor, Prof. Dr. Christian Kanzow, for offering me a doctoral position and for his consistent mentorship and support throughout the last years. From the beginning of my studies, I have benefited from his profound knowledge of many areas of applied mathematics, and his uncanny habit of recognizing connections and similarities between various problems which we have encountered. In addition, his door has always been open and we have enjoyed a myriad of fruitful discussions.

At the same time, I want to express my gratitude to Prof. Dr. Radu Ioan Boţ, who agreed to co-referee this thesis.

In addition, I would like to thank all the members of the Institute of Mathematics at the Julius-Maximilians-Universität Würzburg, in particular Dr. Richard Greiner, Silke Korbl, Dr. Jens Jordan, Dr. Robert Hartmann, Andreas Seeg, Dr. Florian Möller, Ursula Radler, Anita Schmid, Patricia Pelz and Susann Schmitt, who have supported me overcoming all kinds of organizational and technical challenges throughout the years.

Furthermore, I want to thank my scientific colleagues at the chair of numerical mathematics and optimization. The relaxed, almost family-like atmosphere at the institute has made my time here very enjoyable. Special thanks go to Prof. Dr. Daniel Wachsmuth, Dr. Frank Pörner, and Dr. Daniel Steck for their aid in many questions of functional analysis and optimal control theory. Apart from this, I would like to specifically mention my colleagues Veronika Karl, Theresa Lechner, Tobias Geiger, Carolin Natemeyer, and PD Dr. Christian Zillober and thank them for their camaraderie and friendship during the last years.

For their constant support throughout my studies and life, I also want to express my gratitude to my family, namely Rainer and Martina Koopmann as well as Beatrice Börgens.

Finally, I would like to thank my friends for their camaraderie and friendship throughout my time here, especially Nicolas Braune, Yannik Gleichmann, Tobias Geiger and Rosina Ziegler, who have helped me develop new perspectives through extensive discussions. Among these, Rosina Ziegler deserves special mention, as she spent several days correcting all kinds of typos and grammatical inconsistencies.

Further, I would also like to acknowledge the financial support by the German Research Foundation (DFG) under the grant number KA 1296/24-1.

Contents

Abbreviations and Notations

Abbreviations

a.e.	almost everywhere
ADMM	alternating direction method of multipliers
cf.	confirm
e.g.	exempli gratia (for example)
etc.	et cetera (and so forth)
GNEP	generalized Nash equilibrium problem
i.e.	id est (that is)
KKT	Karush–Kuhn–Tucker conditions
lsc	lower semicontinuous
NEP	Nash equilibrium problem
PDE	partial differential equation

Basic Sets and Relations

$\mathbb{N}$	natural numbers (without zero)
$\mathbb{N}_0$	natural numbers with zero
$\mathbb{R}$	real numbers
(a, b), $[a, b]$	open and closed intervals, respectively
$\mathbb{R}^n$	space of n-dimensional real vectors
$\mathbb{R}^{m\times n}$	space of $m \times n$ matrices
$\leq$	partial ordering on $\mathbb{R}^n$ induced by componentwise comparison
$\subset$	subset (or equal)
$\subseteq$	subset with emphasis that it can be equal
$\subsetneq$	subset and not equal
I_n	identity matrix of dimension $n \times n$
A^{-1}	inverse of a nonsingular square matrix A
A^T	transpose of a matrix A

Banach Spaces

X, Y	Banach spaces

$\mathcal{H}$	a Hilbert space
$\mathcal{K}$	a Hilbert space, usually the constraint space
$2^{\mathcal{H}}$	power set of $\mathcal{H}$
$\|\cdot\|_X$	norm on the space X
$\|\cdot\|$	original norm where the space is clear from the context
$\|\cdot\|_Q$	norm induced by a self-adjoint, strongly monotone, bounded, linear operator Q
$\dim(X)$	dimension of the space X
$\mathcal{L}(X,Y)$	space of bounded linear operators between Banach spaces X and Y
$\mathcal{L}(X)$	space of bounded linear operators from the Banach space X to itself
X^*	dual space of a Banach space X, i.e. $X^* = \mathcal{L}(X,\mathbb{R})$
$\|\cdot\|_{\mathcal{L}(X,Y)}$	norm in the space of linear operators $\mathcal{L}(X,Y)$
$\|\cdot\|_{X\to Y}$	norm in the space of linear operators $\mathcal{L}(X,Y)$, alternative notation
I	identity mapping on a Banach space, which is clear from the context
$I_{X\to Y}$	identity or isometric isomorphism from X to Y
$A^\star$	adjoint of an operator $A \in \mathcal{L}(X,Y)$
A^*	Hilbert space adjoint of an operator $A \in \mathcal{L}(\mathcal{H})$
$\langle\cdot \mid \cdot\rangle_{X^*,X}$	duality pairing of a Banach space
$\langle\cdot \mid \cdot\rangle_{\mathcal{H}}$	scalar product of a Hilbert space $\mathcal{H}$
$\langle\cdot \mid \cdot\rangle_{\mathcal{H}\times\mathcal{K}}$	scalar product of a Hilbert space $\mathcal{H}\times\mathcal{K}$
$\langle\cdot \mid \cdot\rangle$	original scalar product of a Hilbert space, which is clear from the context
$\langle\cdot \mid \cdot\rangle_Q$	scalar product induced by a self-adjoint, strongly monotone, bounded, linear operator Q
$X \hookrightarrow Y$	embedding between Banach spaces X and Y
$X \overset{d}{\hookrightarrow} Y$	dense embedding between Banach spaces X and Y
$B_r(x)$	closed ball with radius r around some point x in a Banach space
$\{x^k\}_{k\in\mathbb{N}} \subset X$	sequence of vectors in a Banach space X
$\{\rho^k\}_{k\in\mathbb{N}} \subset \mathbb{R}$	sequence of scalars
$x^k \to x$	convergence of a sequence in a Banach space
$x^k \rightharpoonup x$	weak convergence of a sequence in a Banach space
$\phi^k \rightharpoonup^* \phi$	weak-* convergence in the dual of a Banach space
$\rho^k \searrow 0$	convergence to zero of a non-negative, scalar sequence $\{\rho^k\}$
$x^k = O(\rho^k)$	Landau symbol for $\{x^k\} \subset X$ satisfying $\|x^k\|_X \leq C\rho^k$ with $C > 0$
$x^k = o(\rho^k)$	Landau symbol for $\{x^k\} \subset X$ satisfying $\|x^k\|_X \leq z_k\rho^k$ with $z_k \downarrow 0$
$\{x^k\}_{k\in\mathcal{I}}$	subsequence of $\{x^k\}_{k\in\mathbb{N}}$ corresponding to $\mathcal{I} \subseteq \mathbb{N}$

$x^k \to_{\mathcal{I}} x$	convergence of the subsequence $\{x^k\}_{k\in\mathcal{I}}$ to x

Geometry and Set Operations

2^A	power set a set A
$\mathrm{cl}(A)$	closure of a set A in a topological space
$\mathrm{int}(A)$	interior of a set A in a topological space
∂A	boundary of a set A in a topological space
A°	polar cone of a set A in a Hilbert space
$A^\perp$	orthogonal complement of a set A in a Hilbert space
$A + B$	Minkowski sum of sets in a Banach space
αA	α-multiple of a set in a Banach space, $\alpha \in \mathbb{R}$
d_C, $\mathrm{dist}(\cdot, C)$	distance function to a nonempty set C in a Banach space
Proj_C	projection onto a nonempty, closed, convex set C in a Hilbert space

Functions and Derivatives

$f : X \to Y$	mapping between Banach spaces X and Y
$\mathrm{epigraph}(f)$	epigraph of a function $f : X \to (-\infty, +\infty]$
$\mathrm{dom}(f)$	domain of a function $f : X \to (-\infty, +\infty]$, the set of $x \in X$ with $f(x) \neq +\infty$
f'	first Fréchet-derivative of $f : X \to Y$
∇f	Riesz representative of f' for a function $f : \mathcal{H} \to \mathbb{R}$
∂f	convex subdifferential of a function $f : \mathcal{H} \to \mathbb{R}$
$T : \mathcal{H} \to 2^{\mathcal{H}}$	set-valued mapping between a Hilbert space $\mathcal{H}$ and its power set $2^{\mathcal{H}}$
$\mathrm{graph}(T)$	graph of $T : \mathcal{H} \to 2^{\mathcal{H}}$
$\mathrm{dom}(T)$	domain of a maximally monotone operator $T : \mathcal{H} \to 2^{\mathcal{H}}$
$\mathrm{dom}(\partial f)$	domain of the maximally monotone operator ∂f
$\mathrm{zer}(T)$	set of zeros of $T : \mathcal{H} \to 2^{\mathcal{H}}$
$\mathrm{fix}(F)$	set of fixed points of $F : \mathcal{H} \to \mathcal{H}$

Function Spaces

Ω	domain in $\mathbb{R}^d$
$\overline{\Omega}$	closure of $\Omega \subseteq \mathbb{R}^d$
$\partial\Omega$	boundary of $\Omega \subseteq \mathbb{R}^d$
$A \subset\subset \Omega$	A is compactly contained in Ω
$\mathrm{supp}(u)$	the support of a function $u : \Omega \to \mathbb{R}$
$L^p(\Omega)$	Lebesgue space of p-integrable functions $u : \Omega \to \mathbb{R}$, $p \in [1, \infty]$
$C(\overline{\Omega})$	space of bounded, uniformly continuous functions $u : \Omega \to \mathbb{R}$

$C^k(\overline{\Omega})$	space of functions $u : C(\overline{\Omega})$ whose derivatives up to k-th exist and are in $C(\overline{\Omega})$
$C_0^\infty(\Omega)$	space of infinitely differentiable functions $u : \Omega \to \mathbb{R}$ with compact support
$H^k(\Omega)$	Sobolev space of functions $u : \Omega \to \mathbb{R}$ whose weak derivatives up to k-th order exist and belong to $L^2(\Omega)$
$H_0^k(\Omega)$	closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$
$H^{-1}(\Omega)$	dual space of $H_0^1(\Omega)$
∂_{x_i}	(weak) derivative of a function $u : \Omega \to \mathbb{R}$ in x_i-direction
∇	(weak) gradient operator
Δ	Laplace operator
$\mathrm{trace}(u)$	trace of a function $u \in H^1(\Omega)$ on the whole boundary $\partial\Omega$
$\mathrm{trace}_{\Gamma_{i,j}}(u)$	trace of a function $u \in H^1(\Omega)$ on the boundary part $\Gamma_{i,j} \subset \partial\Omega$
$\partial_n u$	normal derivative of a function $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$
$L^2(\partial\Omega)$	Lebesgue space of all square-integrable functions on the boundary of Ω
$H^{1/2}(\partial\Omega)$	image of $H^1(\Omega)$ under the trace operator
$H^{-1/2}(\partial\Omega)$	dual space of $H^{1/2}(\partial\Omega)$

Optimization and Nash Problems

x	whole primal variable of an optimization problem or GNEP
N	number of block components of x or the corresponding number of players in a GNEP
x_i	i-th block component of x in an optimization problem
x_ν, $x_{-\nu}$	ν-th player variable and its complement in a GNEP
μ	Lagrange multiplier of an optimization problem or GNEP
w	aggregate primal-dual variable (x, μ) of an optimization problem or GNEP
f_i	part of the objective function of an optimization problem that only depends on x_i, which can be non-smooth
f	objective function of an optimization problem and the sum of all f_i
L	Lagrange function of an optimization problem or GNEP
L_β	augmented Lagrange function of an optimization problem or GNEP
θ_ν	part of the objective function of player ν in a GNEP that depends on all players' variables x
φ_ν	part of the objective function of player ν in a GNEP that depends only on the player's variable x_ν, which can be non-smooth
φ	sum of all φ_ν; dependent on the whole primal variable x

ψ	φ interpreted as a function that depends on the aggregate primal-dual variable $w = (x, \mu)$
∂_{x_ν}	subdifferential operator with respect to x_ν
∇_{x_ν}	partial derivative operator with respect to x_ν
$\widehat{P}_\theta$	pseudo gradient $(\nabla_{x_\nu}\theta_\nu)_{\nu=1}^N$; dependent on the primal variable x
P_θ	pseudo gradient, interpreted as a function that depends on the aggregate primal-dual variable $w = (x, \mu)$

Chapter 1

Introduction

A variety of problems in engineering, economics, physics, statistics, machine learning, and other practical sciences can be expressed as optimization problems. Often, their solutions cannot be computed analytically and therefore have to be approximated. Due to the rise of computers and the associated possibility of automating these approximations, optimization has emerged as a very fruitful discipline in mathematics, whereas its roots date back to Fermat, Newton, and Lagrange. The latter has introduced the concept of Lagrange multipliers, which lies at the heart of constrained optimization and is used in most associated algorithms, such as augmented Lagrangian methods, which are explained later in more detail. In the early days of computers, even relatively high-performance servers often did not have enough computational capacity to solve certain optimization problems at once. As a consequence, it was necessary to split these problems into smaller and easily manageable often even analytically solvable subproblems. These subproblems were then solved consecutively, i.e. the information resulting from the computations regarding the previous subproblem was used to solve the current subproblem.

Due to the use of the latest information, methods using consecutive splitting tend to have an advantage over the use of a parallelization approach with respect to the speed of convergence. Even nowadays it often makes sense to use such consecutive splitting methods, especially when the subproblems can be solved analytically or much faster than the original problem.

However, in the last two decades large data centers have been established where many cores or servers are used to compute a solution for one large task. In order to be able to benefit from this computational power, the (optimization) problem has to be split up into smaller tasks that can be performed in parallel or even fully decentralized and distributed among individual workers.

Such methods that split up the problem into smaller subproblems are called splitting methods, and, as motivated by the above considerations, they have been

studied since the rise of computers. This is epitomized by a vast amount of literature around this topic, including [8, 13, 26, 51, 56, 59, 85] and many more.

In the mid-1970s R. Glowinski and A. Marrocco [58] as well as D. Gabay and B. Mercier [55] introduced an algorithm that consecutively splits up the primal problem tackled in the augmented Lagrangian algorithm into two smaller dimensional subproblems. This algorithm trades today under the name alternating direction method of multipliers, or ADMM for short, and is elaborated in the next subsection. The ADMM turned out to be very efficient for convex, separable, linear equality constrained optimization problems which enabled its successful application in many areas, including artificial intelligence, internet applications, computational biology, medicine, finance, marketing, network analysis, and logistics. The survey paper [26] is dedicated entirely to the study of applications of the ADMM from a present point of view. After the theory of the standard ADMM had evolved from the 1970s to the 1990s, new applications gave impetus to further improve this method and apply ADMM-type methods to more general problems. Hence, splitting methods, in particular the ADMM, have recently attracted attention again in the research community and among potential users.

Another fruitful concept in applied mathematics is the notion of non-cooperative N-person games, also known as Nash equilibrium problems (NEPs), which were initially developed by John F. Nash in the 1950s, see [83]. Over the years, Nash's theory has been extended and broadly applied to many fields in biology, economics and engineering, see for instance [4, 46, 53, 81, 86, 87] and the references therein. Consequently, the demand for numerical methods tackling these kinds of problems rose. Since Nash and generalized Nash problems are intertwined optimization problems, the optimization theory and certain algorithms were extended to these problem classes. For an introduction to the theory and algorithmic of generalized Nash problems, see for example [46].

The structure of these (generalized) Nash problems suggests that a suitable generalization of the splitting methods described above could yield efficient algorithms, which could serve as tools to solve these kinds of problems. Furthermore, it is desirable to develop such splitting methods because they tend to resemble the structure of applications where each player decides on her or his own how to react.

In this thesis, in addition to presenting an ADMM-type parallel splitting method for optimization, we therefore focus on the application of ADMM-type methods to generalized Nash problems.

Now, we want to discuss the above described methods and problems in more detail. Therefore we first introduce an optimization problem, to which the ADMM-type methods are applied, and discuss some literature around this topic. Thereafter, we discuss the class of generalized Nash problems this thesis is concerned with.

1.1 The Alternating Direction Method of Multipliers and the Augmented Lagrangian Method

Here we want to discuss the alternating direction method of multipliers in more detail and for this sake the augmented Lagrangian method as well. To do so, we consider the general problem

$$\min_{x\in\mathcal{H}_1\times\cdots\times\mathcal{H}_N} \sum_{i=1}^N f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^N A_i x_i = b, \quad x_i \in \mathcal{X}_i \ (i=1,\dots,N), \tag{Opt}$$

where $\mathcal{H}_i$ and $\mathcal{K}$ are Hilbert spaces, $f_i : \mathcal{H}_i \to \mathbb{R}$ are lower semi-continuous, convex functions, $\mathcal{X}_i \subset \mathcal{H}_i$ are closed convex sets, the A_i are linear operators from $\mathcal{H}_i$ to $\mathcal{K}$, i.e. $A_i \in \mathcal{L}(\mathcal{H}_i, \mathcal{K})$, and $b \in \mathcal{K}$ is a vector.

For the sake of notational simplicity, we use the abbreviations

$$\begin{aligned} \mathcal{H} &:= \mathcal{H}_1 \times \cdots \times \mathcal{H}_N, \qquad \mathcal{X} := \mathcal{X}_1 \times \ldots \times \mathcal{X}_N \subseteq \mathcal{H} \\ x &:= \big(x_1,\dots,x_N\big) \in \mathcal{H}, \qquad f(x) := \sum_{i=1}^N f_i(x_i), \qquad Ax := \sum_{i=1}^N A_i x_i. \end{aligned}$$

Canonically, $\mathcal{H}$ becomes a Hilbert space with the scalar product $\langle x \mid y\rangle := \langle x_1 \mid y_1\rangle + \cdots + \langle x_N \mid y_N\rangle$, the scalar product in the space $\mathcal{H} \times \mathcal{K}$ is defined analogously. The symbol $\|\cdot\|$ always denotes the norm induced by the corresponding scalar product (in $\mathcal{H}_i, \mathcal{H}, \mathcal{K}$, or $\mathcal{H} \times \mathcal{K}$); the meaning should be clear from the context.

Using this notation, we can rewrite (Opt) as

$$\min_x \ f(x) \quad \text{s.t.} \quad Ax = b, \quad x \in \mathcal{X}. \tag{1.1}$$

Let

$$\begin{aligned} L(x,\mu) &:= f(x) + \langle \mu \mid Ax - b\rangle, \\ L_A(x,\mu) &:= f(x) + \langle \mu \mid Ax - b\rangle + \frac{\beta}{2}\|Ax-b\|^2 \end{aligned}$$

denote the Lagrangian and the augmented Lagrangian of (1.1), respectively, where $\beta > 0$ is the penalty parameter. Then a standard optimization technique for solving optimization problems of this kind is the augmented Lagrangian or multiplier-penalty method, in the following abbreviated by ALM. The basic iteration of ALM applied to (1.1) is given by

$$\begin{aligned} x^{k+1} &:= \operatorname*{arg\,min}_{x\in\mathcal{X}} L_A(x,\mu^k), \\ \mu^{k+1} &:= \mu^k + \beta(Ax^{k+1} - b), \end{aligned}$$

provided that a minimum of the augmented Lagrangian exists and can be computed (hopefully) easily, cf. [12, 23, 71, 84].

Unfortunately, when applied to the separable problem (Opt), the quadratic term in the augmented Lagrangian destroys the separable structure and, therefore, ALM cannot take advantage of the separability in the computation of the new iterate x^{k+1}. This observation is the main motivation for the alternating direction method of multipliers, ADMM for short. This method computes the block components x_i^{k+1} essentially again by minimizing $L_A(x, \mu^k)$, but with the full-dimensional vector x being replaced either by $\big(x_1^k, \ldots, x_{i-1}^k, x_i, x_{i+1}^k, \ldots, x_N^k\big)$ or by $\big(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_i, x_{i+1}^k, \ldots, x_N^k\big)$, so that the subproblems are minimization problems in x_i alone. More precisely, discarding some constant terms, the former approach leads to a parallel Jacobi-type ADMM-method with x_i^{k+1} being computed by

$$x_i^{k+1} := \operatorname*{arg\,min}_{x_i \in \mathcal{X}_i} \Big\{ f_i(x_i) + \langle \mu^k \mid A_i x_i \rangle + \frac{\beta}{2} \big\| A_i x_i + \sum_{l \neq i} A_l x_l^k - b \big\|^2 \Big\} \tag{1.2}$$

for all $i = 1, \ldots, N$, whereas the latter approach yields a consecutive Gauss-Seidel-type ADMM-method

$$x_i^{k+1} := \operatorname*{arg\,min}_{x_i \in \mathcal{X}_i} \Big\{ f_i(x_i) + \langle \mu^k \mid A_i x_i \rangle + \frac{\beta}{2} \big\| \sum_{l<i} A_l x_l^{k+1} + A_i x_i + \sum_{l>i} A_l x_l^k - b \big\|^2 \Big\}. \tag{1.3}$$

Both methods coincide for the case $N = 1$ and reduce to the standard ALM approach, whereas the classical ADMM-method, introduced by Glowinski, Marrocco, Gabay, and Mercier in the manuscripts [55, 58], corresponds to the Gauss-Seidel-type iteration (1.3) with $N = 2$ blocks.

Note that the two schemes (1.2) and (1.3) have different properties. The former is implementable completely in parallel, whereas the latter is not, but uses the newer information and is therefore often faster convergent in terms of the number of outer iterations. In any case, both methods have the major advantage that they can fully exploit the separable structure of (Opt) and often yield small-dimensional subproblems that are easy to solve (sometimes even analytically). Unfortunately, however, without any further assumptions, these subproblems might not have solutions, and even if they have, none of the two schemes necessarily converge. In fact, while there is a satisfactory global convergence theory for the Gauss-Seidel-type ADMM-scheme for the special case $N = 2$, see [14, 26, 43], the recent paper [32] shows that convergence cannot be expected, in general, for $N > 2$. For the Jacobi-type scheme the situation is even worse since [63] provides a counterexample for the case of $N = 2$ blocks.

It therefore comes with no surprise that there exist a couple of modifications of the two basic iterations (1.2) and (1.3) in the literature.

Concerning the Gauss-Seidel scheme (1.3), there are some noteworthy results that improve the convergence or yield important insight in the proposed method. Daniel Gabay showed in [54] that the classical ADMM (for $N = 2$) is closely related to the (more general) Douglas-Rachford splitting applied to the dual problem.

In the case $N > 2$ it was shown in [103] that the convergence of the Gauss-Seidel scheme can be retained if $N - 2$ of the functions are strongly convex, the A_i have full column rank and β is small enough. An other set of assumptions that guarantees convergence of the Gauss-Seidel scheme for $N > 2$ was introduced in [69]. The manuscripts [64, 65] introduce a reasonable correction of the output of the Gauss-Seidel scheme (1.3) such that convergence can be proven. Also research about the convergence of the ADMM in the absence of convexity or non-linear constraints was deducted and seems to gain more and more relevance, since it widely expands the area of applications, see for example [16, 17, 70] and the references therein.

As already mentioned, the Jacobi-type iteration does not converge in the case $N \geq 2$ without further modification. Therefore it is necessary to modify the iteration. One approach is to rewrite the classical ADMM in order to obtain a parallel algorithm with a certain regularization structure. This was investigated in depth in [43, 60, 66, 108] and is also discussed in Chapter 4. For a further discussion of suitable modifications of the Jacobi-type scheme we refer the reader to the discussion in Chapter 4 and to [34, 63, 101, 102, 107].

Since our aim is to present a modification of the Jacobi-type iteration in Chapter 4, we concentrate our discussion on this class of methods, some more details are given in Chapter 4 after an explicit statement of our algorithm.

To the best of our knowledge, the existing literature (see citations above) that investigates the convergence properties of suitable modifications of the Jacobi-type scheme (1.2) is exclusively written in the finite-dimensional setting. All methods that do not regularize the x_i-subproblems require the suboperators A_i to be injective (full rank assumption) in order to be well-defined and to get convergence of the iterates $\{x^k\}_{k\in\mathbb{N}}$, whereas this assumption is not necessarily needed in an approach that regularizes the subproblems, cf. [2, 34].

The method we present in Chapter 4 is not completely new. In fact, lately we became aware of the recent publication [34] that considers a parallel multi-block ADMM scheme which is essentially the same as the algorithm considered here (see also [101] for a closely related method). Nevertheless, there are some differences which we think are remarkable. First, we present our method and the corresponding theory in a Hilbert space setting, whereas [34] considers finite-dimensional problems. Second, we reduce our convergence theory to a standard proximal-point approach, as opposed to [34] where the authors provide an independent (self-contained) convergence theory. Third, the lower bounds for certain parameters used

here and in [34] seem to be better (smaller) in our theory, which in turn leads to a superior numerical behavior. Finally, we address certain questions, like weak and strong convergence and an application in Hilbert spaces, which do not occur in the finite-dimensional theory.

1.2 Splitting Methods for Generalized Nash Equilibrium Problems

Building on the foundations of the last section, we now introduce the jointly linearly constrained generalized Nash equilibrium problems that were mentioned earlier and this thesis deals mostly with. We further discuss these problems in Section 3.2. We consider the generalized Nash equilibrium problem (GNEP) with N players ν, where the optimization problem of player ν is given by

$$\min_{x_\nu \in \mathcal{H}_\nu} \theta_\nu(x_\nu, x_{-\nu}) + \varphi_\nu(x_\nu) \quad \text{s.t.} \quad \sum_{\nu=1}^{N} A_\nu x_\nu = b, \quad x_\nu \in \mathcal{X}_\nu \tag{GNEP}$$

or, more generally,

$$\min_{x_\nu \in \mathcal{H}_\nu} \theta_\nu(x_\nu, x_{-\nu}) + \varphi_\nu(x_\nu) \quad \text{s.t.} \quad \sum_{\nu=1}^{N} B_\nu x_\nu - b \in \mathcal{C}, \quad x_\nu \in \mathcal{X}_\nu \tag{GNEP$_{conic}$}$$

for all $\nu = 1, \ldots, N$. Here, $\mathcal{H}_\nu$ and $\mathcal{K}$ are given Hilbert spaces, $\varphi_\nu : \mathcal{H}_\nu \to \mathbb{R}$ are proper, convex, and lower semi-continuous functions, $\theta_\nu : \mathcal{H}_1 \times \cdots \times \mathcal{H}_N \to \mathbb{R}$ are continuously Fréchet-differentiable with $\theta_\nu(\cdot, x_{-\nu})$ being convex for any fixed $x_{-\nu}$, $\mathcal{X}_\nu \subset \mathcal{H}_\nu$ are non-empty, closed, and convex sets, $\mathcal{C}$ is a non-empty, closed, convex cone, $A_\nu, B_\nu \in \mathcal{L}(\mathcal{H}_\nu, \mathcal{K})$, and $b \in \mathcal{K}$. Following standard notation in Nash games, we write $x = (x_\nu, x_{-\nu})$, where $x_{-\nu}$ subsumes all the remaining block components x_i with $i \neq \nu$. This notation is used to emphasize the particular role played by the block component x_ν within the entire vector x and does not mean that the components of x are re-ordered. In particular, we therefore have $x = (x_\nu, x_{-\nu}) = (x_1, \ldots, x_N)$ and, similarly, $(y_\nu, x_{-\nu}) = (x_1, \ldots, x_{\nu-1}, y_\nu, x_{\nu+1}, \ldots, x_N)$.

There exist many approaches for the numerical solution of GNEPs, and the interested reader is referred to the survey papers [46, 50] for more details. However, these survey papers consider the finite-dimensional case only. Solution methods in an infinite-dimensional Hilbert space (or Banach space) are still in their infancy. The ideas from the finite-dimenional setting can sometimes be generalized to the Hilbert space setting, like the usage of the Nikaido-Isoda function and the application of Moreau-Yosida-type methods [67,68]. The augmented Lagrangian methods from [73,74] may be viewed as extensions of this approach, but they have to solve a

(standard) Nash equilibrium problem (NEP) in each iteration. Some other methods operating in an infinite-dimensional context are [25, 39, 40, 75], but none of them is a splitting-type method and many of them are situated in an optimal control context.

Splitting-type methods can, in principle, be relatively easily extended to standard NEPs. For example, the gradient-type method would be

$$x_\nu^{k+1} \;=\; \mathrm{Proj}_{\mathcal{X}_\nu}\left(x_\nu^k - t^k \nabla_{x_\nu}\theta_\nu(x^k)\right)$$

for each $\nu = 1, \ldots, N$. Closely related to such gradient methods is the forward-backward method, which is elaborated later in Section 2.5.3. Therefore we interpret algorithms that are based on the forward-backward scheme as extensions of the gradient method.

Splitting methods for GNEPs that are based on such forward-backward methods can be found in [10, 30, 111, 112]. These articles consider splitting-type methods that are based on forward-backward methods; in [30], the authors focus on standard NEPs and show afterwards how to solve certain GNEPs under a cocoercivity assumption. On the other hand, the closely related algorithms considered in [10, 111, 112] (for finite-dimensional problems) are fully distributed, but they use a strong monotonicity and Lipschitz assumption. Further these algorithms are very close to the projected gradient method incorporating the constraints and they are not related to ADMM-methods.

In Chapter 5, we introduce three fully distributed forward-backward based ADMM-type methods for solving the GNEPs from (GNEP) and (GNEP$_{conic}$). The first one turns out to be equivalent to a standard forward-backward method in a Hilbert space endowed with a different scalar product, and is therefore convergent under a cocoercivity condition. This equivalence observation then leads to a second and third method. The second is strongly convergent and the third, at some extra computational costs, turns out to be weakly convergent under a Lipschitz and monotonicity assumption; hence no cocoercivity or strong monotonicity of a certain (pseudo-gradient) mapping is required.

In contrast to these gradient-type or forward-backward methods, another way to split optimization problems is a (block-)coordinate descent approach, which is also known under the name alternating minimization, see [11, 106], which directly extends to the Gauss-Seidel-type best response algorithm for NEPs, see e.g. [49, 79, 100]. Here the update would be

$$x_\nu^{k+1} \;=\; \operatorname*{arg\,min}_{x_\nu \in \mathcal{X}_\nu} \theta_\nu(x_1^{k+1}, \ldots, x_{\nu-1}^{k+1}, x_\nu, x_{\nu-1}^k, \ldots, x_N^k)$$

for each player $\nu = 1, \ldots, N$. Its extension to GNEPs is more delicate since then the feasible set of each player depends on the other players' variables. A complete

convergence theory for such a method is given in [49] for the class of (finite-dimensional) potential games, but it requires an inner semicontinuity assumption on the set-valued feasibility map, which is often violated. This indicates that Gauss-Seidel-, Jacobi- and other splitting-type methods may work quite well for standard NEPs, but cause severe trouble for GNEPs. A simple way to overcome this drawback is to apply a penalty or multiplier-penalty idea in order to get rid of the joint constraints. This avoids the inner semicontinuity assumption and naturally leads to an approach that can be interpreted as a direct extension of the ADMM to (GNEP).

The aim of Chapter 6 is therefore to investigate the convergence properties of a direct extension of ADMM for solving GNEPs. However, the application of ADMM to GNEPs turns out to be significantly more difficult than for standard optimization problems. In particular, we see later that the direct translation of ADMM to GNEPs may not converge even for $N = 2$ players, in contrast to optimization problems with two blocks. We therefore introduce a regularization which is also popular in the optimization context with multiple blocks. The overall method is then well-defined with uniquely determined iterates for arbitrary positive regularization parameters. In order to prove global convergence results, however, the regularization parameters need to be sufficiently large, see also Section 1.1 for related literature on regularization-type approaches.

The paper [99] also presents an approach to solve (finite-dimensional) GNEPs using a method which is called an ADMM technique by the authors. But their method is quite different from a direct extension of the classical ADMM algorithm to GNEPs, and convergence is shown under relatively strong assumptions. Therefore no connection can be drawn to the methods presented in this thesis.

1.3 Structure of the Thesis

This thesis is structured as follows: First, in Chapter 2, we recall the fundamental definitions and results from functional and convex analysis that are required to understand the presented methods. In this context, we pay particular attention to certain closedness and convexity properties in Section 2.2, the crucial notions of non-expansiveness, Fejér-monotonicity and fixed-point iterations in Section 2.3, and maximally monotone operators including their properties in Section 2.4. These maximally monotone operators and their zeros are closely related to certain non-expansive operators and their fixed points; Section 2.5 is devoted to this connection. Finally, in Section 2.6, fundamental equalities and inequalities are discussed.

The theory of the above introduced problems (Opt), (GNEP) and (GNEP$_{conic}$) is elaborated in Chapter 3, which is based on [20, 21]. Therefore, Section 3.1 is devoted to the notion of KKT pairs of (Opt) and their relation to certain maximally

monotone operators. Inspired by this, variational equilibria, KKT-type conditions and their connection with certain maximally monotone operators of (GNEP) are discussed, and the relation of (GNEP_{conic}) to (GNEP) is elaborated in Section 3.2. In addition, it is shown that KKT-type points of (GNEP) are equal to KKT-type points of a suitable reformulation of (GNEP_{conic}). This enables us to apply algorithms designed for equality constrained problems to conically constrained ones as well.

In Chapter 4, which is based on [21], a parallel ADMM-type splitting method for (Opt) is introduced and one strongly convergent modification is presented. The convergence analysis is heavily based on the theory of maximally monotone operators and the main method turns out to be a proximal point algorithm in a certain scalar product.

Based on [20], we present three parallel ADMM-type algorithms for problems of the structure (GNEP) in Chapter 5. Here, the convergence analysis is based on forward-backward methods and the averagedness of a certain operator. First, the basic algorithm that is weakly convergent under a cocoercivity assumption is presented. This iteration is then modified in order to obtain a strongly convergent algorithm, again using a cocoercivity assumption. Thereafter, a modification is presented that is, at extra computational cost, weakly convergent under a Lipschitz assumption only.

Two Gauss-Seidel-type ADMM-methods for the problem class (GNEP) are introduced in Chapter 6, which is based on [19]. The first method uses a fixed regularization parameter, whereas the second increases the regularization parameter if necessary. Further it is shown that in the proposed methods the regularization cannot be omitted. The convergence analysis is self-contained and, to the best of our knowledge, cannot be reduced to a known method.

Finally, in Chapter 7, the methods presented in the Chapters 4, 5 and 6 are applied to examples, which are mainly drawn from optimal control.

Chapter 2

Background Material

This preliminary chapter establishes some fundamental notions which are indispensable for the remainder of this thesis. Most of the material presented here is simply a careful collection of results from the literature, structured and presented in a way which hopefully makes the theory as clear as possible.

The following is an outline of the chapter. In Section 2.1, we present the fundamentals of functional analysis, including linear operators, dual spaces, weak and strong convergence, as well as basic results regarding function spaces and partial differential equations. Thereafter, elementary convex analysis and closedness properties are recalled in Section 2.2. The following Section 2.3 is devoted to giving a recap of the theory of non-expansive mappings, different fixed-point iterations and also the closely related concept of Fejér-monotonicity. In the ensuing Sections 2.4 and 2.5, (maximally) monotone operators including their zeros and sums are introduced and discussed. We see that these zeros are closely linked to fixed points of certain non-expansive operators, to which the fixed-point iterations are applied, yielding standard operator splitting methods. At the end of this chapter, fundamental and useful equalities and inequalities are summarized in Section 2.6.

2.1 Hilbert and Banach Space Theory

In this section, we present basic results of functional analysis. They can be found in every book on this topic, e.g. in [9, 33, 35, 97, 113] and are not cited explicitly.

It is well known that a Banach space is a complete normed vector space and a Hilbert space is a Banach space whose norm is induced by a scalar product. In this thesis, we are exclusively concerned with real Banach and Hilbert spaces. The symbols $\mathcal{H}$ and $\mathcal{K}$ always denote real Hilbert spaces. If X is a Banach or Hilbert space, we denote the norm of an $x \in X$ by $\|x\|_X$. Whenever the norm is clear from the context, we omit the subscript and write $\|x\|$. The scalar product

of x, y in a Hilbert space $\mathcal{H}$ is denoted by $\langle x \mid y \rangle_{\mathcal{H}}$. If the Hilbert space is clear from the context, the subscript is omitted as well. Using the scalar product $\langle x \mid y \rangle := \langle x_1 \mid y_1 \rangle + \cdots + \langle x_N \mid y_N \rangle$, the Cartesian product $\mathcal{H}_1 \times \cdots \times \mathcal{H}_N$ of N Hilbert spaces canonically again becomes a Hilbert space.

Suppose that X is a Banach space with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. These norms are called *equivalent norms* if there are constants $C, c > 0$ such that $c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$ holds for all $x \in X$. These two norms induce the same topology and the same notion of convergence. Therefore, if X endowed with the norm $\|\cdot\|_1$ is a Banach space, so is X endowed with $\|\cdot\|_2$. Furthermore, we call a Banach space Y *(continuously) embedded* in a Banach space X if Y is a subspace of X and the identity mapping $I : Y \to X,\ I(x) = x$ is continuous. We call the embedding *dense* if $I(Y)$ is dense in X. We denote this by $Y \hookrightarrow X$ and $Y \overset{d}{\hookrightarrow} X$, respectively.

2.1.1 Linear Operators

Continuous linear operators between Banach or Hilbert spaces are often of interest and therefore we briefly discuss them in this section. The space of all continuous linear operators between the Banach spaces X and Y is denoted by $\mathcal{L}(X, Y)$, and in the case $Y = X$, we define $\mathcal{L}(X) := \mathcal{L}(X, X)$. This space of linear operators $\mathcal{L}(X, Y)$ between two Banach spaces X, Y is itself a Banach space with the norm

$$\|A\|_{\mathcal{L}(X,Y)} := \|A\|_{X \to Y} := \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ax\|_Y}{\|x\|_X}. \tag{2.1}$$

Again, the subscript is often omitted. Further, it is easy to see that $\|Ax\| \leq \|A\|\|x\|$ for all $x \in X$ and all $A \in \mathcal{L}(X, Y)$ holds. It is well known that the continuity of a linear operator $A : X \to Y$ is equivalent to A being bounded, i.e. there is a constant $C > 0$ such that $\|Ax\| \leq C\|x\|$ for all $x \in X$. This, in turn, is equivalent to the supremum in (2.1) being finite.

The space of all real valued continuous linear functionals $\mathcal{L}(X, \mathbb{R})$ of a Banach space X is called *dual space* and is denoted by X^*. The application of a $\mu \in X^*$ to an $x \in X$ is denoted by the duality pairing $\langle \mu \mid x \rangle_{X^*,X} =: \mu(x)$. As opposed to the duality pairing, the scalar product only carries one or no subscript, i.e. it is denoted by $\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$ or $\langle \cdot \mid \cdot \rangle$. The dual space $(X^*)^*$ of the dual space X^* is called the *bidual space* and is denoted by X^{**}. It can be easily seen that every element $x \in X$ induces a functional in X^{**} by defining $\langle x \mid \mu \rangle_{X^{**},X^*} := \langle \mu \mid x \rangle_{X^*,X}$, and hence X is continuously embedded in X^{**}. If this (canonical) embedding is surjective, X is called *reflexive*.

Given two Banach spaces X, Y, a linear map $A : X \to Y$ is called an *isomorphism* if A is bijective, and it is called an *isometric isomorphism* if A is an

isomorphism and $\|Ax\|_Y = \|x\|_X$ holds for all $x \in X$. The next theorem shows that a Hilbert space is isometrically isomorph to its dual; therefore it is always reflexive.

Theorem 2.1 (Riesz representation theorem, [113, Section III.6]). *Suppose that $\mathcal{H}$ is a real Hilbert space. For every linear functional $f \in \mathcal{H}^*$ there is a uniquely defined $\mu_f \in \mathcal{H}$ such that*

$$f(x) = \langle \mu_f \mid x \rangle_{\mathcal{H}}$$

for all $x \in \mathcal{H}$. Further $\|f\|_{\mathcal{H}^*} = \|\mu_f\|_{\mathcal{H}}$. The mapping $x \mapsto \langle x \mid \cdot \rangle_{\mathcal{H}}$ is an isometric isomorphism from $\mathcal{H}$ onto $\mathcal{H}^*$. In particular, every Hilbert space is reflexive.

If Y is a Banach space and $\mathcal{H}$ is a Hilbert space such that $Y \hookrightarrow \mathcal{H}$, it is obvious that $\mathcal{H}^* \hookrightarrow Y^*$. Identifying $\mathcal{H}$ with its dual space yields $Y \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow Y^*$, where $(Y, \mathcal{H}, Y^*)$ is called *Gelfand triple*. Note that if Y is a Hilbert space as well, the identification of Y with its dual space Y^* is no longer possible in this setting because the scalar product used is the one from $\mathcal{H}$, but not the one from Y. Hence, we need to use the duality pairing in Y.

An extension of the Riesz representation theorem is the theorem of Lax-Milgram, which deals with in general not necessarily symmetric bilinear forms.

Theorem 2.2 (Lax-Milgram Theorem, [113, Section III.7]). *Let $\mathcal{H}$ be a real Hilbert space and $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ a bilinear form with the following properties:*

(a) *There exists a constant $c_1 > 0$ such that $|b(x, y)| \leq c_1 \|x\| \|y\|$ for all $x, y \in \mathcal{H}$.*

(b) *There exists a constant $c_2 > 0$ such that $b(x, x) \geq c_2 \|x\|^2$ for all $x \in \mathcal{H}$.*

Then the mapping $T_b(x) := b(x, \cdot)$ is a continuous isomorphism from $\mathcal{H}$ onto $\mathcal{H}^*$ with $\|T_b\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}^*)} \leq c_1$ and $\|T_b^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}^*)} \leq c_2^{-1}$.

Given a linear operator $A \in \mathcal{L}(X, Y)$, we define the *adjoint operator* $A^\star \in \mathcal{L}(Y^*, X^*)$ by $\langle \mu \mid Ax \rangle_{X^*, X} = \langle A^\star \mu \mid x \rangle_{X^*, X}$ for all $x \in X$ and all $\mu \in Y$. This adjoint operator can be shown to be uniquely determined and it holds that $\|A\| = \|A^\star\|$.

If A is a continuous linear operator that maps a Hilbert space $\mathcal{H}$ into itself, i.e. $A \in \mathcal{L}(\mathcal{H})$, then $A^\star \in \mathcal{L}(\mathcal{H}^*)$. We define the *Hilbert space adjoint* $A^* := I_{\mathcal{H}^* \to \mathcal{H}} \circ A \circ I_{\mathcal{H} \to \mathcal{H}^*}$, where $I_{\mathcal{H}^* \to \mathcal{H}}$ and $I_{\mathcal{H} \to \mathcal{H}^*}$ are the isometric isomorphisms between $\mathcal{H}$ and $\mathcal{H}^*$. Note that $\langle Ax \mid x \rangle_{\mathcal{H}} = \langle x \mid A^* x \rangle_{\mathcal{H}}$ and thus the adjoint operator and the Hilbert space adjoint only differ in the representation of the dual space. Therefore all results for the adjoint directly transfer to the Hilbert space adjoint.

We call a linear operator $A \in \mathcal{L}(X, Y)$ *compact* if it maps bounded sets in X to precompact sets in Y. The adjoint or Hilbert space adjoint of a compact operator is again compact.

The next theorem is closely related to the Banach closed range theorem, which is one of the most important results of functional analysis, and it is used to derive some useful statements about KKT points. A proof can be found in [35, Thm. 8.11] and all the results are, by dint of the same proof, true for the Hilbert space adjoint as well, cf. [35, Thm. 8.16].

Theorem 2.3. *Let X, Y be two real Banach spaces and $A \in \mathcal{L}(X,Y)$. Then*

$$\begin{aligned} \{\mu \in Y^* \mid A^\star \mu = 0\} &= \{\mu \in Y^* \mid \langle \mu \mid Ax\rangle_{Y^*,Y} = 0 \quad \forall x \in X\}, \\ \{x \in X \mid Ax = 0\} &= \{x \in X \mid \langle A^\star \mu \mid x\rangle_{X^*,X} = 0 \quad \forall \mu \in Y^*\} \end{aligned}$$

and

$$\begin{aligned} &\operatorname{cl}\big(\{Ax \in Y \mid x \in X\}\big) \\ &\quad = \{x \in X \mid \langle \lambda \mid x\rangle_{X^*,X} = 0 \text{ for all } \lambda \in X^* \text{ such that } A^\star \lambda = 0\}. \end{aligned}$$

In particular, if $A^\star$ injective, then the range $AX = \{Ax \mid x \in X\}$ is dense in Y.

If $A \in \mathcal{L}(\mathcal{H})$ and $A = A^*$, we call A *self-adjoint*. The operator norm of a self-adjoint continuous linear operator A can be equally represented as

$$\|A\|_{\mathcal{L}(\mathcal{H})} = \sup_{\substack{x \in \mathcal{H} \\ x \neq 0}} \frac{|\langle Ax | x\rangle|}{\|x\|^2}, \tag{2.2}$$

cf. [33, Thm. 4.10-1].

We call $A \in \mathcal{L}(\mathcal{H})$ *monotone* if $\langle Ax \mid x\rangle \geq 0$ for all $x \in \mathcal{H}$, and if there is a constant $c > 0$ such that $\langle Ax \mid x\rangle \geq c\|x\|^2$ for all $x \in \mathcal{H}$, we say that A is *strongly monotone*. These notions of monotonicity are special cases of the notions introduced later in Section 2.4. A strongly monotone, self-adjoint, continuous, linear operator $A \in \mathcal{L}(\mathcal{H})$ induces a scalar product

$$\langle x \mid y\rangle_A := \langle Ax \mid y\rangle \tag{2.3}$$

and therefore a norm

$$\|x\|_A^2 := \langle Ax \mid x\rangle. \tag{2.4}$$

In this thesis the scalar product and norm induced by a self-adjoint, strongly monotone, continuous, linear operator will be indexed with the operator, i.e. they are denoted by $\langle \cdot \mid \cdot \rangle_A$ and $\|\cdot\|_A$. Whereas, the original scalar product and original norm do not carry an index, i.e. they are denoted by $\langle \cdot \mid \cdot \rangle$ and $\|\cdot\|$. In the sequel a self-adjoint, strongly monotone, continuous, linear operator from a Hilbert space into itself that induces a particular norm, is denoted by Q, therefore the next lemma is stated with a linear operator Q. The following result summarizes some properties of self-adjoint and strongly monotone operators in Hilbert spaces.

Lemma 2.4. *Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \cdot \mid \cdot \rangle$, and let $Q \in \mathcal{L}(\mathcal{H})$ be self-adjoint and strongly monotone. Then the following statements hold:*

(a) *The corresponding norms $\|\cdot\|$ and $\|\cdot\|_Q$, as defined in (2.4), are equivalent, in particular $\mathcal{H}$ endowed with $\langle \cdot \mid \cdot \rangle_Q$ is again a Hilbert space.*

(b) *$\langle Q^2 x \mid x \rangle \leq \|Q\| \, \langle Qx \mid x \rangle$ for all $x \in \mathcal{H}$.*

(c) *The inverse Q^{-1} exists and is also self-adjoint and strongly monotone with modulus $1/\|Q\|$.*

(d) *$\|x\|^2 \leq \|Q^{-1}\| \, \|x\|_Q^2$ for all $x \in \mathcal{H}$, where $\|x\|_Q$ denotes the norm induced by the scalar product $\langle x \mid y \rangle_Q := \langle Qx \mid y \rangle$.*

Proof. (a) Since Q is linear, bounded, and strongly monotone, there exists a constant $c > 0$ such that $c\|x\|^2 \leq \langle Qx \mid x \rangle \leq \|Qx\|\|x\| \leq \|Q\|\|x\|^2$. This immediately yields statement (a).

(b) Let $B := Q/\|Q\|$. Then we have $\langle Bx \mid x \rangle \geq 0$, $\langle (I - B)x \mid x \rangle = \|x\|^2 - \langle Bx \mid x \rangle \geq 0$ by the Cauchy-Schwarz inequality, and both B and $I - B$ are self-adjoint. Since $B - B^2 = B(I - B)B + (I - B)B(I - B)$, we therefore obtain

$$\begin{aligned} \langle (B - B^2)x \mid x \rangle &= \langle B(I - B)Bx \mid x \rangle + \langle (I - B)B(I - B)x \mid x \rangle \\ &= \langle (I - B)Bx \mid Bx \rangle + \langle B(I - B)x \mid (I - B)x \rangle \geq 0. \end{aligned}$$

Hence $\langle B^2 x \mid x \rangle \leq \langle Bx \mid x \rangle$ for all $x \in \mathcal{H}$. Thus (b) follows from the definition of B.

(c) That Q is continuously invertible can be easily seen with the Lax-Milgram theorem, Theorem 2.2, and $c\|x\|^2 \leq \langle Qx \mid x \rangle \leq \|Q\|\|x\|^2$ for all $x \in \mathcal{H}$, since $\langle Q\cdot \mid \cdot \rangle$ defines a bilinear form. That Q^{-1} is self-adjoint can be seen through

$$\begin{aligned} (Q^{-1})^* = Q^{-1} &\iff (Q^{-1})^* x = Q^{-1} x && \forall x \in \mathcal{H} \\ &\iff (Q^{-1})^* (Qy) = Q^{-1}(Qy) && \forall y \in \mathcal{H} \\ &\iff (Q^{-1})^* Qy = y && \forall y \in \mathcal{H} \\ &\iff (Q^{-1})^* Q = I. \end{aligned}$$

This shows $(Q^{-1})^* = Q^{-1}$, by the uniqueness of the inverse.

Now we need to show that Q^{-1} is strongly monotone. To this end, take an arbitrary $y \in \mathcal{H}$. Applying (b) with $x := Qy$, we obtain

$$\|x\|^2 = \langle x \mid x \rangle = \langle Qy \mid Qy \rangle = \langle Q^2 y \mid y \rangle \leq \|Q\| \langle Qy \mid y \rangle = \|Q\| \langle x \mid Q^{-1} x \rangle;$$

hence Q^{-1} is strongly monotone with modulus $1/\|Q\|$.

(d) Applying (b) to $x := Q^{-1}y$ and using (c) yields

$$\begin{aligned}\|x\|^2 &= \langle x \mid x\rangle = \langle Q^{-1}y \mid Q^{-1}y\rangle = \langle Q^{-2}y \mid y\rangle \leq \|Q^{-1}\|\langle Q^{-1}y \mid y\rangle \\ &= \|Q^{-1}\|\langle x \mid Qx\rangle = \|Q^{-1}\|\langle x \mid x\rangle_Q = \|Q^{-1}\|\|x\|_Q^2.\end{aligned}$$

Noting that this holds for all $x \in \mathcal{H}$ gives the desired result. □

Note that statement (b) of Lemma 2.4 already holds for monotone operators Q.

An important kind of linear operators is derived through the affine linear approximation of a function $f : X \to Y$ and leads directly to the notion of differentiability.

Definition 2.5 ((Fréchet-)Differentiability)**.** Let X, Y be real Banach spaces. A function $f : X \to Y$ is called *(Fréchet-)differentiable in* $x \in X$ if there is a bounded linear operator $f'(x) \in \mathcal{L}(X, Y)$ such that

$$f(x + h) = f(x) + f'(x)h + o(\|h\|_X)$$

for all $\|h\|_X \to 0$. If f is differentiable in every $x \in X$, it is called *(Fréchet-) differentiable*. If f is differentiable and f' is continuous as a map $f' : X \to \mathcal{L}(X, Y)$, then f is called *continuously (Fréchet-)differentiable*.

If $X = \mathcal{H}$ is a real Hilbert space and $f : \mathcal{H} \to \mathbb{R}$, then we denote the Riesz representative of $f'(x) \in \mathcal{H}^*$ by $\nabla f(x) \in \mathcal{H}$.

2.1.2 Weak Convergence

In the infinite-dimensional context, the notion of norm convergence is often too strong. Therefore the above discussed continuous linear functionals are used to define a weaker notion of convergence. Consider a sequence $\{x^k\}_{k\in\mathbb{N}} \subset X$ in a Banach space X. We call $\{x^k\}_{k\in\mathbb{N}}$ *weakly convergent* to an $\bar{x} \in X$ if for all functionals $\mu \in X^*$ the duality pair $\langle \mu \mid x^k\rangle_{X^*,X}$ converges (in $\mathbb{R}$) to $\langle \mu \mid \bar{x}\rangle_{X^*,X}$ and we denote this by $x^k \rightharpoonup \bar{x}$. A sequence $\{\mu^k\}_{k\in\mathbb{N}} \subset X^*$ is called *weak-* convergent* to some $\bar{\mu}$ if for all $x \in X$ the sequence $\langle \mu^k \mid x\rangle_{X^*,X}$ converges to $\langle \bar{\mu} \mid x\rangle_{X^*,X}$ and we write $\mu^k \rightharpoonup^* \bar{\mu}$. It can be seen easily that weak convergence in X^* implies weak-* convergence. In reflexive spaces these two notions coincide. In order to distinguish the convergence in the norm from the weak convergence, we often call the former *strong convergence*. In finite-dimensional Banach spaces the notions of strong and weak convergence coincide. By the continuity of the elements in X^*, strong convergence implies weak convergence.

Now the question arises whether the duality pairing or the scalar product has any convergence properties. This question is answered in the next lemma.

Lemma 2.6. *Assume that X is a real Banach space, $\{x^k\}_{k\in\mathbb{N}} \subset X$ and $\{\mu^k\}_{k\in\mathbb{N}} \subset X^*$. Further suppose that either $x^k \rightharpoonup \bar{x}$ and $\mu^k \to \bar{\mu}$, or $x^k \to \bar{x}$ and $\mu^k \rightharpoonup^* \bar{\mu}$. Then*

$$\langle \mu^k \mid x^k \rangle_{X^*,X} \to \langle \bar{\mu} \mid \bar{x} \rangle_{X^*,X}.$$

If $\mathcal{H}$ is a Hilbert space and $\{x^k\}_{k\in\mathbb{N}}, \{\mu^k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ with $x^k \rightharpoonup \bar{x}$ and $\mu^k \to \bar{\mu}$, then

$$\langle \mu^k \mid x^k \rangle_{\mathcal{H}} \to \langle \bar{\mu} \mid \bar{x} \rangle_{\mathcal{H}}.$$

In finite dimensions any bounded sequence has a convergent subsequence, whereas this result has to be attenuated in infinite dimensions.

Theorem 2.7. *Let X be a reflexive real Banach space and $\{x^k\}_{k\in\mathbb{N}} \subset X$ a bounded sequence. Then there is an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that $\{x^k\}_{k\in\mathcal{I}}$ is weakly convergent.*

On the other hand, every weakly convergent sequence is bounded. Whenever $\{x^k\}_{k\in\mathbb{N}}$ converges weakly on a subsequence $\mathcal{I} \subseteq \mathbb{N}$ to an $\bar{x} \in X$, we denote this by $x^k \rightharpoonup_{\mathcal{I}} \bar{x}$. Analogously, if $\{\mu^k\}_{k\in\mathbb{N}}$ converges weak-* on a subsequence $\mathcal{I} \subseteq \mathbb{N}$ to a $\bar{\mu} \in X^*$, we denote this by $\mu^k \rightharpoonup^*_{\mathcal{I}} \bar{\mu}$.

It is well known that a set $C \subseteq X$ is (topologically) closed if every limit $\bar{x}$ of a convergent sequence $\{x^k\}_{k\in\mathbb{N}} \subset C$ lies again in C. We call a set $C \subseteq X$ *weakly sequentially closed* if every weak limit $\bar{x}$ of a weakly convergent sequence $\{x^k\}_{k\in\mathbb{N}} \subset C$ lies again in C.

As already mentioned, equivalent norms define the same notion of strong convergence. The next lemma clarifies that even the notions of weak convergence coincide.

Lemma 2.8. *Consider the Banach space X with the norm $\|\cdot\|$, and let $\|\cdot\|_2$ be an equivalent norm. Let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence in X. Then the following statements are equivalent*

(a) *x^k converges weakly to $\bar{x}$ in X endowed with the norm $\|\cdot\|$.*

(b) *x^k converges weakly to $\bar{x}$ in X endowed with the norm $\|\cdot\|_2$*

Proof. Since both norms are equivalent, the same linear functionals are continuous in X equipped with either $\|\cdot\|$ or $\|\cdot\|_2$. Thus the notions of weak convergence coincide. □

By the identification of a Hilbert space with its dual, the above definitions and results also hold true in Hilbert spaces using the scalar product instead of the duality pairing.

In the sequel, we will often use two different scalar products (and therefore two different induced norms) for a Hilbert space $\mathcal{H} \times \mathcal{K}$. In order to be able to apply the known convergence results of fixed-point methods, it is highly important in our setting that both strong and weak convergence are identical concepts in both scalar products. Formally, this is stated in the following result, which is just an important special case of Lemma 2.8 when the equivalent norm is induced by a strongly monotone, self-adjoint, continuous, linear operator.

Lemma 2.9. *Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces. Consider the Hilbert space $\mathcal{H} \times \mathcal{K}$ with the usual scalar product $\langle \cdot \mid \cdot \rangle := \langle \cdot \mid \cdot \rangle_{\mathcal{H}} + \langle \cdot \mid \cdot \rangle_{\mathcal{K}}$, and let $\langle \cdot \mid \cdot \rangle_Q$ be defined as in* (2.3) *for $Q \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$ self-adjoint and strongly monotone. Then the following statements hold:*

(a) *The corresponding induced norms $\|\cdot\|$ and $\|\cdot\|_Q$ are equivalent, in particular $\mathcal{H}$ endowed with $\|\cdot\|_Q$ is again a Hilbert space.*

(b) *Weak convergence with respect to $\langle \cdot \mid \cdot \rangle$ is equivalent to weak convergence with respect to $\langle \cdot \mid \cdot \rangle_Q$.*

Proof. (a) This was already shown in Lemma 2.4(a).

(b) This follows straight from Lemma 2.8 and part (a). Alternatively, let $\{x^k\}_{k\in\mathbb{N}} \subset \mathcal{H} \times \mathcal{K}$ be a sequence with weak limit point $\bar{x}$. Then the statement follows from

$$\begin{aligned} \langle x^k \mid y \rangle_Q \to \langle \bar{x} \mid y \rangle_Q \quad \forall y \quad &\iff \langle Qx^k \mid y \rangle \to \langle Q\bar{x}^k \mid y \rangle \quad \forall y \\ &\iff \langle x^k \mid Qy \rangle \to \langle \bar{x} \mid Qy \rangle \quad \forall y \\ &\iff \langle x^k \mid z \rangle \to \langle \bar{x} \mid z \rangle \quad \forall z, \end{aligned}$$

where the last two equivalences exploit that Q is self-adjoint and invertible, cf. Lemma 2.4. □

2.1.3 Function Spaces and Partial Differential Equations

After the brief introduction to abstract Banach and Hilbert spaces, it is appropriate to introduce Banach and Hilbert spaces that are relevant in practice. As most of our examples come from an optimal control setting, function spaces are the most frequently used Banach spaces in this manuscript. The results and definitions of this section can be found in every book on this topic, for instance [1, 27, 35, 44, 104, 109]. In order to introduce these function spaces, let $\Omega \subset \mathbb{R}^d$ be a bounded domain, i.e. a bounded, open, and connected subset of $\mathbb{R}^d$, $d \in \mathbb{N}$. By $\partial\Omega$ we denote the boundary of Ω, whereas $\overline{\Omega}$ denotes the closure of Ω. If $\partial\Omega$ is a Lipschitz manifold, we call Ω a *Lipschitz domain*. We call a set $\Omega_0 \subset \Omega$ *compactly*

contained in Ω if $\overline{\Omega}_0 \subset \Omega$ and $\overline{\Omega}_0$ is compact; we write this as $\Omega_0 \subset\subset \Omega$. The closure of the area where a function $u : \Omega \to \mathbb{R}$ is not equal to zero (Lebesgue almost everywhere) is called *support of* u and is denoted by $\mathrm{supp}(u)$.

The space of all bounded uniformly continuous functions on a domain $\Omega \subset \mathbb{R}^d$ is denoted by $C(\overline{\Omega})$. If equipped with the norm $\|u\|_{C(\overline{\Omega})} := \sup_{x\in\Omega} |u(x)|$, this space becomes a Banach space. By $C^\infty(\Omega)$ we denote the space of all infinitely many times differentiable functions from Ω to $\mathbb{R}$, whereas the space of all infinitely many times differentiable $C(\overline{\Omega})$ functions whose derivatives are again in $C(\overline{\Omega})$ is written as $C^\infty(\overline{\Omega})$. The space $C_0^\infty(\Omega)$ is the space of all infinitely many times differentiable functions $u : \Omega \to \mathbb{R}$ whose support is compactly contained in Ω.

Further the space of square Lebesgue integrable functions

$$L^2(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \mid u \text{ is Lebesgue measurable and } \int_\Omega u(x)^2 dx < \infty \Big\}$$

becomes a Hilbert space if endowed with the scalar product

$$\langle u \mid v \rangle_{L^2(\Omega)} := \int_\Omega u(x)v(x)dx.$$

The above definitions are directly transferable to $L^2(\partial\Omega)$, where the integral has to be understood in the sense of basic differential geometry.

In many cases the notion of differentiability needs to be weakened. We call a function $u : \Omega \to \mathbb{R}$ *weakly differentiable* if there is a Lebesgue measurable function $v : \Omega \to \mathbb{R}^d$ such that for every open compactly contained subset $\Omega_0 \subset\subset \Omega$ it is valid that $\sum_{i=1}^d \int_{\Omega_0} |v_i| < \infty$ and

$$\int_\Omega u(x)\partial_{x_i}\varphi(x)dx = -\int_\Omega v_i(x)\varphi(x)dx$$

for all $i = 1, \ldots, n$ and all $\varphi \in C_0^\infty(\Omega)$. The function v is called *weak gradient* and is denoted by ∇u. In the sequel it is clear from the context whether we talk about weak or usual gradients, and thus we use the same notation for these notions.

The space of all $L^2(\Omega)$ functions whose weak gradients belongs to $L^2(\Omega)^d$ is called $H^1(\Omega)$, i.e.

$$H^1(\Omega) := \Big\{ u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega)^d \Big\}.$$

This space becomes a Hilbert space with the scalar product

$$\begin{aligned} \langle u \mid v \rangle_{H^1(\Omega)} &:= \langle u \mid v \rangle_{L^2(\Omega)} + \langle \nabla u \mid \nabla v \rangle_{L^2(\Omega)^d} \\ &= \int_\Omega \Big(u(x)v(x) + \nabla u(x)^T \nabla v(x) \Big) dx. \end{aligned} \tag{2.5}$$

The closure of $C_0^\infty(\overline{\Omega})$ in the $H^1(\Omega)$ norm is denoted by $H_0^1(\Omega)$ and is a closed subspace of $H^1(\Omega)$. Using the Poincaré inequality, we will later see that the norm induced by the scalar product

$$\langle u \mid v \rangle_{H_0^1(\Omega)} \;:=\; \langle \nabla u \mid \nabla v \rangle_{L^2(\Omega)^d} \;:=\; \int_\Omega \nabla u(x)^T \nabla v(x) dx \tag{2.6}$$

is equivalent to the norm induced by the scalar product (2.5) in the space $H_0^1(\Omega)$. If not stated differently, the space $H_0^1(\Omega)$ is always equipped with the scalar product (2.6) in the sequel. We denote the dual space of $H_0^1(\Omega)$ by $H^{-1}(\Omega)$, the norm in $H^{-1}(\Omega)$ is the by (2.6) induced dual norm. Further, let $\Gamma \subset \partial\Omega$ be an open boundary part of Ω (in the subspace topology of $\partial\Omega$). We denote the closure of the set

$$C_{0,\Gamma}^\infty(\Omega) := \Big\{ v \in C^\infty(\Omega) \cap H^1(\Omega) \mid v = 0 \text{ in a neighbourhood of } \Gamma \Big\}$$

in the $H^1(\Omega)$ norm by $H_\Gamma^1(\Omega)$, which is again a closed subspace of $H^1(\Omega)$.

As already mentioned, the next theorem reveals a way to define an equivalent norm on $H_0^1(\Omega)$.

Theorem 2.10 (Poincaré Inequality, [27, Thm. 1.5])**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain contained in a d-dimensional cube with side length c. Then*

$$\|u\|_{L^2(\Omega)} \;\le\; c\|\nabla u\|_{L^2(\Omega)^d} \;=\; c\Big(\sum_{i=1}^d \|\partial_{x_i} u\|_{L^2(\Omega)}^2 \Big)^{1/2} \tag{2.7}$$

for all $u \in H_0^1(\Omega)$.

Note that in the previous theorem the so-called Poincaré-constant c is estimated explicitly. There exist many versions of the Poincaré inequality with analogous statements as in (2.7), also for $H_\Gamma^1(\Omega)$ functions and for $H^1(\Omega)$ functions with $\int_\Omega u = 0$, see for instance [1, 27, 35, 44, 109].

The formulation of elliptic partial differential equations ususally contains a boundary condition, i.e. for a given $g \in C(\partial\Omega)$ a solution y of the partial differential equation is supposed to satisfy $y = g$ on $\partial\Omega$. Since the boundary of Ω has Lebesgue measure zero, boundary values are not defined for $L^2(\Omega)$ functions. The next theorem shows that $H^1(\Omega)$ functions can be uniquely extended to the boundary; thus the requirement of attaining certain boundary values is well defined for these functions.

Theorem 2.11 (Trace Theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then there is a unique continuous linear operator* $\operatorname{trace} : \mathcal{H}^1(\Omega) \to L^2(\partial\Omega)$*, such that* $\operatorname{trace}(u) = u|_{\partial\Omega}$ *holds for all $u \in C^\infty(\overline{\Omega})$.*

It can be shown that the spaces $H^1_\Gamma(\Omega)$ and $H^1_0(\Omega)$ coincide with the $H^1(\Omega)$ functions that have trace equal to zero on Γ and $\partial\Omega$, respectively, c.f. [1, 35, 44, 109].

In Section 7.1, we apply the trace theorem to decompose the domain of a partial differential equation into smaller parts. A way of estimating the norm of the linear trace operator on a rectangle is discussed in Lemma 7.3.

Now we want to take a look at the basic theory of partial differential equation (PDE) theory. Let us consider the *Laplace equation*

$$-\triangle y = u \qquad \text{in } \Omega, \tag{2.8a}$$

$$y = 0 \qquad \text{in } \partial\Omega, \tag{2.8b}$$

where $u \in L^2(\Omega)$ and

$$\triangle y := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i \partial x_i} y \tag{2.9}$$

denotes the Laplace operator. For the definition of the Laplace operator in (2.9) it is necessary that y is twice differentiable. This differentiability requirement is often too restrictive. Therefore we reinterpret the Laplace operator based on the divergence theorem, also known as partial differentiation. In order to operate on the much weaker space $H^1(\Omega)$, or a subspace thereof. We define the operator $-\triangle : H^1_0(\Omega) \to H^{-1}(\Omega)$ by

$$\langle -\triangle u \mid v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} := \langle \nabla u \mid \nabla v \rangle_{L^2(\Omega)^d} \tag{2.10}$$

for all $u, v \in H^1_0(\Omega)$. Then the Laplace equation (2.8) can be interpreted as an equation in $H^{-1}(\Omega)$ in the following manner: For a given $u \in H^{-1}(\Omega)$, we have to find a $y \in H^1_0(\Omega)$ such that $-\triangle y = u$ holds as an equation in $H^{-1}(\Omega)$. Since $L^2(\Omega)$ is continuously embedded into $H^{-1}(\Omega)$ by $u \mapsto \langle u \mid \cdot \rangle_{L^2(\Omega)}$, the solution concept for this equation is weaker than the one discussed earlier. In addition, every twice differentiable solution of this reinterpreted equation satisfies (2.8). The Lax-Milgram theorem, Theorem 2.2, shows that for every $u \in H^{-1}(\Omega)$ the Laplace equation $-\triangle y = u$ has a unique solution $y \in H^1_0(\Omega)$. We call the mapping $S : H^{-1}(\Omega) \to H^1_0(\Omega)$ that maps $u \in H^{-1}(\Omega)$ to the unique solution of the Laplace equation $y \in H^1_0(\Omega)$ the *solution operator (of the Laplace equation)*. The solution operator S is a continuous linear operator. If $\Omega \subset \mathbb{R}^d$ with $d \in \{1, 2, 3\}$ and $u \in L^2(\Omega)$, then the solution of the Laplace equation $y = Su$ is even continuous, i.e. $Su \in H^1_0(\Omega) \cap C(\overline{\Omega})$, cf. [104]. The Laplace operator and the solution operator are obviously inverse to each other and they are the isometric isomorphisms between the spaces $H^1_0(\Omega)$ (endowed with the scalar product (2.6)) and $H^{-1}(\Omega)$; hence it holds that $\| - \triangle \|_{H^1_0(\Omega) \to H^{-1}(\Omega)} = \|S\|_{H^{-1}(\Omega) \to H^1_0(\Omega)} = 1$. On the other hand,

$\|S\|_{L^2(\Omega)\to H_0^1(\Omega)}$ and $\|S\|_{L^2(\Omega)\to L^2(\Omega)}$ do not need to be equal to one. An estimation of the norm of the solution operator S as a mapping from $L^2(\Omega)$ to $H_0^1(\Omega)$ or $L^2(\Omega)$ can be found in Section 7.2.

The solvability of more sophisticated partial differential equations can be examined by applying the Lax-Milgram theorem, stated in Theorem 2.2, however, the approach is the same as above, see for instance [1, 35, 44] for a more detailed discussion of PDE theory.

2.2 Convex Analysis and Notions of Closedness

> *"...the great watershed in optimization isn't between linearity and nonlinearity, but between convexity and nonconvexity."*
>
> — R. T. Rockafellar [95]

In optimization theory, the notions of closedness and convexity play a very important role, for example projections onto closed, convex sets always exist, the necessary optimality conditions are already sufficient for convex problems, and generalized derivatives can be easily defined. We start our recap of convex analysis with notions for sets and continue with considering convexity and closedness properties of functions.

In the following we restrict our considerations to real Hilbert spaces. However, the extension to real Banach spaces is easily possible, see for example the monographs [6, 7]. Most of the presented definitions and results can be found in [9] in a Hilbert space setting or in [6, 7] in a Banach space setting; therefore, we do not cite these results explicitly.

Let $\mathcal{H}$ be a real Hilbert space in this whole section. We call a set $C \subseteq \mathcal{H}$ of a real Hilbert space $\mathcal{H}$ *weakly sequentially closed* if every weak limit of a sequence in C lies again in C. The set C is called *convex* if $tx + (1-t)y \in C$ for all $x, y \in C$ and all $t \in [0, 1]$. An elementary separation argument shows that every non-empty, closed, convex set is weakly sequentially closed. For a non-empty, closed, convex set $C \subset \mathcal{H}$ we define the *distance function to* C as

$$d_C : \mathcal{H} \to \mathbb{R}, \qquad d_C(\bar{x}) \;:=\; \inf_{x\in C} \|x - \bar{x}\|. \tag{2.11}$$

It is well known that the squared distance function d_C^2 is continuously differentiable in Hilbert spaces, cf. [9, Cor. 12.31].

Proposition 2.12 (Projection Theorem, [9, Thm. 3.16])**.** *Let $C \subset \mathcal{H}$ be a non-empty, closed, convex subset of a real Hilbert space $\mathcal{H}$. The* projection

$$\mathrm{Proj}_C(\bar{x}) \;:=\; \operatorname*{arg\,min}_{p\in C} \frac{1}{2}\|p - \bar{x}\|^2 \tag{2.12}$$

of $\bar{x} \in \mathcal{H}$ onto C is uniquely defined for every $\bar{x} \in \mathcal{H}$ and it holds that

$$p = \mathrm{Proj}_C(\bar{x}) \quad \Longleftrightarrow \quad \big[p \in C \quad \textit{and} \quad \langle y - p \mid \bar{x} - p \rangle \leq 0 \quad \forall y \in C\big].$$

Moreover, we have $d_C(\bar{x}) = \|\bar{x} - \mathrm{Proj}_C(\bar{x})\|$.

A set $C \subset \mathcal{H}$ is called a *cone* if $tx \in C$ for all $t \geq 0$ and all $x \in C$. We define the *polar cone* K° of a convex cone K as

$$K^\circ := \{s \in \mathcal{H} \mid \langle s \mid x \rangle \leq 0 \quad \forall x \in K\}.$$

Now we recall the basic definitions of convex and lower semi-continuous functions. We call a function $f : \mathcal{H} \to (-\infty, +\infty]$ *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in \mathcal{H}$ and all $t \in [0, 1]$. Sometimes it is necessary to have a stronger notion than ordinary convexity at hand. We call $f : \mathcal{H} \to (-\infty, +\infty]$ *strongly convex* if there is a $c > 0$ such that

$$f(tx + (1-t)y) + t(1-t)\frac{c}{2}\|x - y\|^2 \leq tf(x) + (1-t)f(y)$$

for all $x, y \in \mathcal{H}$ and all $t \in [0, 1]$. Since we want to exclude the trivial function $f \equiv +\infty$ from our considerations, we call a function $f : \mathcal{H} \to (-\infty, +\infty]$ *proper* if there is an $x \in \mathcal{H}$ such that $f(x) < +\infty$.

One of the most important results about convex functions is that local minimizers are actually global ones.

Proposition 2.13. *Let $\mathcal{H}$ be a real Hilbert space and $f : \mathcal{H} \to (-\infty, +\infty]$ is convex. Then every local minimizer x^* of f is also a global minimizer and the set of minimizers is convex.*

In order to prove the existence of minimizers of a function $f : \mathcal{H} \to (-\infty, +\infty]$, it is necessary that for any sequence $\{x^k\}_{k \in \mathbb{N}} \subset \mathcal{H}$ that converges (weakly) to some $\bar{x}$ the inequality $\liminf_{k \to \infty} f(x^k) \geq f(\bar{x})$ holds. We call functions with this property (weakly sequentially) lower semi-continuous.

Definition 2.14. Let $\mathcal{H}$ be a real Hilbert space. We call a function $f : \mathcal{H} \to (-\infty, +\infty]$ *lower semi-continuous in $\bar{x}$* if for every sequence $x^k \to \bar{x}$, the inequality $\liminf_{k \to \infty} f(x^k) \geq f(\bar{x})$ holds. Further, we want to call $f : \mathcal{H} \to (-\infty, +\infty]$ *weakly sequentially lower semi-continuous in $\bar{x}$* if for every sequence $x^k \rightharpoonup \bar{x}$, the inequality $\liminf_{k \to \infty} f(x^k) \geq f(\bar{x})$ holds.
We call $f : \mathcal{H} \to (-\infty, +\infty]$ *(weakly sequentially) lower semi-continuous ((wslsc) lsc)* or *(weakly sequentially) closed* if it is (weakly sequentially) lower semi-continuous in all $x \in \mathcal{H}$.

The next proposition clarifies that for convex functions, it is not necessary to distinguish between the notions of weak and strong lower semi-continuity.

Proposition 2.15 ([9, Thm. 9.1])**.** *Let $\mathcal{H}$ be a real Hilbert space and $f : \mathcal{H} \to (-\infty, +\infty]$ a proper convex function. Then f is weakly sequentially lower semi-continuous if and only if f is lower semi-continuous.*

While we cannot guarantee the existence and uniqueness of minimizers for solely convex functions, strong convexity yields both.

Proposition 2.16. *Let $\mathcal{H}$ be a real Hilbert space and let $f : \mathcal{H} \to (-\infty, +\infty]$ be proper, lower semi-continuous, and strongly convex. Then f admits an unique minimizer.*

As already seen, convex functions have a lot of useful properties. One of the most important ones is that it is easy to define a generalized derivative. The *subdifferential* of a convex function $f : \mathcal{H} \to (-\infty, +\infty]$ at a point $\bar{x}$ is defined by

$$\partial f(\bar{x}) := \big\{ s \in \mathcal{H} \mid f(x) - f(\bar{x}) \geq \langle s \mid x - \bar{x} \rangle \quad \forall x \in X \big\}. \tag{2.13}$$

The elements of $\partial f(x)$ are called *subgradients* of f at x. For a non-empty, closed, convex set C, we define the *indicator function* $\chi_C : \mathcal{H} \to [0, +\infty]$ as

$$\chi_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{else} \end{cases}.$$

It can be seen easily that this function is convex. Its subdifferential is called the *normal cone* $N_C(x) := \partial \chi_C(x)$ and has the representation

$$N_C(x) := \begin{cases} \big\{ s \in \mathcal{H} \mid \langle s \mid y - x \rangle \leq 0 \ \forall y \in C \big\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases} \tag{2.14}$$

The convex subdifferential and the normal cone can be interpreted as set-valued mappings, i.e. they are mappings from $\mathcal{H}$ to $2^{\mathcal{H}}$. Whenever we want to emphasize their interpretation as a mapping, we call them *subdifferential map* and *normal cone map*.

The next theorem explains why the convex subdifferential is of such importance for convex optimization. Its proof follows straight from the definition of the subdifferential.

Theorem 2.17 (Fermat's rule)**.** *Let $\mathcal{H}$ be a real Hilbert space and $f : \mathcal{H} \to (-\infty, +\infty]$ be convex. Then x^* is a minimizer of f if and only if $0 \in \partial f(x^*)$.*

The next proposition shows that the subdifferential is actually a generalized derivative; it can be seen as a combination of various results from [89].

Proposition 2.18. *Let $\mathcal{H}$ be a real Hilbert space and $f : \mathcal{H} \to \mathbb{R}$ a continuous convex function. Then $\partial f(x)$ is non-empty for every $x \in \mathcal{H}$. Moreover, f is Fréchet-differentiable in $x \in \mathcal{H}$ if and only if the following two properties hold:*

(a) *$\partial f(x)$ is the singleton $\{\nabla f(x)\}$, and*

(b) *whenever $x^k \to x$ in $\mathcal{H}$ and $d^k \in \partial f(x^k)$, then $d^k \to \nabla f(x)$.*

The last proposition also shows that a convex differentiable function is always continuously differentiable.

Recall that the *domain of a function* $f : \mathcal{H} \to (-\infty, +\infty]$ is defined as

$$\operatorname{dom}(f) := \{x \in \mathcal{H} \mid f(x) \neq +\infty\}, \tag{2.15}$$

and the *domain of the subdifferential* is defined as

$$\operatorname{dom}(\partial f) := \{x \in \mathcal{H} \mid \partial f(x) \neq \emptyset\}. \tag{2.16}$$

The next result shows that the subdifferential is non-empty on the interior of $\operatorname{dom}(f)$.

Proposition 2.19 ([7, Cor. 2.38])**.** *Let $\mathcal{H}$ be a real Hilbert space and $f : \mathcal{H} \to (-\infty, +\infty]$ a proper, lower semi-continuous, and convex function. Then*

$$\operatorname{int}\big(\operatorname{dom}(f)\big) \subset \operatorname{dom}(\partial f).$$

It is well known that the differential of the sum of two differentiable functions equals the sum of the individual differentials. Under weak assumptions, this can be extended to subdifferentials.

Proposition 2.20 ([7, Cor. 2.63])**.** *Let $\mathcal{H}$ be a real Hilbert space and $f, g : \mathcal{H} \to (-\infty, +\infty]$ two proper, lower semi-continuous, and convex functions. Suppose that*

$$\operatorname{dom}(f) \cap \operatorname{int}\big(\operatorname{dom}(g)\big) \neq \emptyset. \tag{2.17}$$

Then

$$\partial(f+g)(x) \;=\; \partial f(x) + \partial g(x) \;:=\; \{s_f + s_g \mid s_f \in \partial f(x), s_g \in \partial g(x)\}$$

for all $x \in \mathcal{H}$.

In particular, (2.17) is satisfied if $f, g : \mathcal{H} \to (-\infty, +\infty]$ are proper, convex, lower semi-continuous, and g is real-valued., i.e. $g : \mathcal{H} \to \mathbb{R}$. The condition of the above Proposition 2.20 can be weakened, see [9, 24]. However, the above formulation is enough for our purposes.

2.3 Fixed-Point Iterations

The Banach fixed-point theorem shows that the fixed-point iteration $x^{k+1} := F(x^k)$ converges to the unique fixed point if $F : X \to X$ is a contraction, where X is a Banach space. As described in this section, we can expect at least weak convergence for certain kinds of fixed-point iterations if F is less than a contraction and admits a fixed point. Since the convergence analysis of these methods heavily depends on the scalar product, we restrict the following considerations to Hilbert spaces. Therefore, we assume F to be a self-map of a subset $D \subset \mathcal{H}$ where $\mathcal{H}$ is a real Hilbert space, i.e. $F : D \to D$. For notational convenience, we often write Fx instead of $F(x)$ and denote the set of all fixed points by $\operatorname{fix}(F) := \{x \in D \mid x = F(x)\}$.

2.3.1 Notions of Non-Expansiveness

First it is necessary to introduce some notions that imply the convergence of certain fixed-point iterations if a fixed point exists.

Definition 2.21. Let $\mathcal{H}$ be a real Hilbert space and let $D \subset \mathcal{H}$ non-empty, closed, convex. A map $F : D \to \mathcal{H}$ is called

(a) *non-expansive (on D)* if

$$\|F(x) - F(y)\| \leq \|x - y\| \qquad \forall x, y \in D,$$

(b) *firmly non-expansive (on D)* if

$$\|F(x) - F(y)\|^2 + \|(I - F)x - (I - F)y\|^2 \leq \|x - y\|^2 \qquad \forall x, y \in D,$$

(c) *α-averaged (on D)* with $\alpha \in (0, 1)$ if F is non-expansive and there is a non-expansive operator $R : D \to \mathcal{H}$ such that

$$F = (1 - \alpha)I + \alpha R.$$

Note that, by definition, averaged and firmly non-expansive mappings are in particular non-expansive. First, we want to see that the fixed-point set of a non-expansive operator is convex.

Proposition 2.22 ([9, Cor. 4.24])**.** *Let D be a non-empty, closed, convex subset of a real Hilbert space $\mathcal{H}$ and let $F : D \to \mathcal{H}$ be a non-expansive operator. Then the set* $\operatorname{fix}(F) := \{x \in D \mid x = F(x)\}$ *is closed and convex.*

The next proposition characterizes firmly non-expansive operators.

Proposition 2.23 ([9, Prop. 4.4])**.** *Let D be a non-empty, closed, convex subset of a real Hilbert space $\mathcal{H}$ and let $F : D \to \mathcal{H}$. Then the following statements are equivalent:*

(a) *F is firmly non-expansive,*

(b) *F is $1/2$-averaged,*

(c) *$I - F$ is firmly non-expansive,*

(d) *$2F - I$ is non-expansive,*

(e) *for all $x, y \in D$ it holds that $\|Fx - Fy\|^2 \leq \langle Fx - Fy \mid x - y \rangle$,*

(f) *for all $x, y \in D$ it holds that $0 \leq \langle Fx - Fy \mid (I - F)x - (I - F)y \rangle$,*

(g) *for all $x, y \in D$ and all $\alpha \in [0, 1]$ it holds that $\|Fx - Fy\| \leq \|\alpha(x - y) + (1 - \alpha)(Fx - Fy)\|$.*

One of the most important examples of a firmly non-expansive operator is the projection onto a non-empty, closed, convex set.

Example 2.24. Let C be a non-empty, closed, convex subset of a real Hilbert space $\mathcal{H}$. Then it is well known that the projection of an $x \in \mathcal{H}$ onto C, defined as

$$\mathrm{Proj}_C(x) := \operatorname*{arg\,min}_{y \in C} \frac{1}{2}\|y - x\|^2,$$

is uniquely defined and firmly non-expansive, cf. Proposition 2.12. ◇

For further examples of (firmly) non-expansive mapping we refer the reader to Section 2.4 and Chapter 3. The next result is a characterization of averagedness.

Proposition 2.25 ([9, Prop. 4.35])**.** *Let D be a non-empty, closed, convex subset of a real Hilbert space $\mathcal{H}$, let $F : D \to \mathcal{H}$ be non-expansive, and let $\alpha \in (0, 1)$. Then the following statements are equivalent:*

(a) *F is α-averaged,*

(b) *$(1 - 1/\alpha)I + (1/\alpha)F$ is non-expansive,*

(c) *for all $x, y \in D$ it holds that $\|Fx - Fy\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|(I - F)x - (I - F)y\|^2$,*

(d) *for all $x, y \in D$ it holds that $\|Fx - Fy\|^2 + (1 - 2\alpha)\|x - y\|^2 \leq 2(1 - \alpha)\langle x - y \mid Fx - Fy \rangle$.*

A useful property of averaged mappings, which is of major importance for the later introduced splitting methods, is that their composition is again an averaged mapping.

Proposition 2.26 ([9, Prop. 4.44])**.** *Let D be a non-empty, closed, convex subset of a real Hilbert space $\mathcal{H}$, let $\alpha_1, \alpha_2 \in (0,1)$. Suppose that $F_1 : D \to D$ is α_1-averaged and $F_2 : D \to D$ is α_2-averaged. Then $F : D \to D$ defined as $F := F_2 \circ F_1$ is $\tilde{\alpha}$-averaged with*

$$\tilde{\alpha} = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2} \in (0,1).$$

The next theorem shows that non-expansive mappings have a certain, very important continuity property linking strong convergence in the image space of $I - F$ and weak convergence in the preimage space.

Theorem 2.27 (Browder's demi-closedness principle, [9, Thm. 4.27, Cor. 4.28])**.** *Let D be a non-empty, closed, convex subset of the real Hilbert space $\mathcal{H}$, let $F : D \to \mathcal{H}$ be a non-expansive operator, let $\{x^k\}_{k\in\mathbb{N}} \subset D$ be a sequence in D, and $\bar{x} \in D$. Suppose that $x^k \rightharpoonup \bar{x}$ and that $(I - F)x^k = x^k - Fx^k \to 0$. Then $\bar{x}$ is a fixed point of F, i.e. $\bar{x} = F\bar{x}$.*

2.3.2 Fejér-Monotonicity

Closely linked to the above notions of non-expansiveness is the notion of Fejér-monotonicity, since the fixed-point iteration of a non-expansive operator can be shown to be be Fejér-monotone. As Fejér-monotonicity is a property of several iterations in convex analysis, this notion repeatedly appears in the course of this thesis.

Definition 2.28 (Fejér-monotonicity)**.** Suppose that $S \subset \mathcal{H}$ is a non-empty subset of a Hilbert space $\mathcal{H}$. Then a sequence $\{x^k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ is called *Fejér-monotone with respect to S* if the inequality $\|x^{k+1} - x\| \leq \|x^k - x\|$ holds for all $k \in \mathbb{N}$ and all $x \in S$.

The next result captures an interesting property of Fejér-monotone sequences, see [9, Thm. 5.33(iv)] for a proof.

Proposition 2.29. *Let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence in a real Hilbert space $\mathcal{H}$ and let S be a non-empty subset of $\mathcal{H}$ such that $\{x^k\}_{k\in\mathbb{N}}$ is Fejér-monotone with respect to S and such that every weak accumulation point of $\{x^k\}_{k\in\mathbb{N}}$ belongs to S. Then the whole sequence converges weakly to a point $\bar{x} \in S$.*

2.3.3 Krasnoselsky-Mann Iteration

Now we return to the initial question under which conditions fixed-point iterations are convergent. First we review the Krasnoselsky-Mann iteration and consequences

thereof. These iteration types are seen to be weakly convergent. Since it is often desired that a method is strongly convergent, the Halpern iteration is introduced afterwards. The classical and well known fixed-point iteration is given by

$$x^{k+1} := Fx^k. \tag{2.18}$$

Krasnoselsky and Mann blended in a bit of identity into (2.18) by using the iteration

$$x^{k+1} := (1-\rho^k)x^k + \rho^k Fx^k \qquad \forall k \in \mathbb{N} \tag{2.19}$$

and were able to show better convergence properties for this iteration scheme. In this section, we investigate the (weak) convergence properties of (2.19) for non-expansive, firmly non-expansive, and α-averaged operators. Whenever it is possible to choose $\rho^k \equiv 1$, we obviously obtain (weak) convergence of the classical fixed-point iteration (2.18) as well.

As stated in the following theorem the iteration (2.19) converges weakly to a fixed-point of F if such a fixed-point exists and ρ^k is chosen appropriately.

Theorem 2.30 (Groetsch's Theorem, [9, Thm. 5.15])**.** *Let D be a non-empty, closed, convex subset of a real Hilbert space $\mathcal{H}$ and let $F : D \to D$ be a non-expansive operator such that $\operatorname{fix}(F) \neq \emptyset$. Let $\{\rho^k\}_{k\in\mathbb{N}} \subset [0,1]$ be a sequence such that $\sum_{k\in\mathbb{N}} \rho^k(1-\rho^k) = +\infty$, and let $x^0 \in D$. Then the iterates x^k generated by* (2.19) *satisfy:*

(a) *the sequence $\{x^k\}_{k\in\mathbb{N}}$ is Fejér-monotone with respect to $\operatorname{fix}(F)$,*

(b) *the sequence $\{(I-F)x^k\}_{k\in\mathbb{N}}$ converges strongly to 0,*

(c) *the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges weakly to an element in $\operatorname{fix}(F)$.*

From the above theorem, the next proposition about the fixed-point iteration applied to averaged operators can easily be derived through rescaling, see [9, Prop. 5.16].

Proposition 2.31 ([9, Prop. 5.16])**.** *Let $\mathcal{H}$ be a real Hilbert space, let $\alpha \in (0,1)$ and let $F : \mathcal{H} \to \mathcal{H}$ be an α-averaged operator such that $\operatorname{fix}(F) \neq \emptyset$. Let $\{\rho^k\}_{k\in\mathbb{N}} \subset [0, 1/\alpha]$ be a sequence such that $\sum_{k\in\mathbb{N}} \rho^k(1-\alpha\rho^k) = +\infty$, and let $x^0 \in \mathcal{H}$. Then the iterates x^k generated by* (2.19) *satisfy:*

(a) *the sequence $\{x^k\}_{k\in\mathbb{N}}$ is Fejér-monotone with respect to $\operatorname{fix}(F)$,*

(b) *the sequence $\{(I-F)x^k\}_{k\in\mathbb{N}}$ converges strongly to 0,*

(c) *the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges weakly to an element in $\operatorname{fix}(F)$.*

Since every firmly non-expansive operator is 1/2-averaged, we can apply the last proposition also to firmly non-expansive operators and obtain the following corollary.

Corollary 2.32. *Let $\mathcal{H}$ be a real Hilbert space $\mathcal{H}$ and let $F : \mathcal{H} \to \mathcal{H}$ be a firmly non-expansive operator such that $\operatorname{fix}(F) \neq \emptyset$. Let $\{\rho^k\}_{k\in\mathbb{N}} \subset [0, 2]$ be a sequence such that $\sum_{k\in\mathbb{N}} \rho^k(2 - \rho^k) = +\infty$, and let $x^0 \in \mathcal{H}$. Then the iterates x^k generated by (2.19) satisfy:*

(a) *the sequence $\{x^k\}_{k\in\mathbb{N}}$ is Fejér-monotone with respect to $\operatorname{fix}(F)$,*

(b) *the sequence $\{(I - F)x^k\}_{k\in\mathbb{N}}$ converges strongly to 0,*

(c) *the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges weakly to an element in $\operatorname{fix}(F)$.*

In particular, we obtain that the classical fixed-point iteration (2.18) converges weakly to a fixed point for α-averaged and firmly non-expansive operators if such a fixed point exists.

The Classical Fixed-Point Iteration in the Inconsistent Case

The above discussion always assumed that a fixed point of F exists. Now we will investigate how the classical fixed-point iteration (2.18) behaves if we omit this assumption. These considerations are helpful for investigating how constrained optimization algorithms behave if there is a minimizer that is no KKT-point. The analysis, which is summarized in the sequel, is essentially taken from [3,88,91,92]. We also include self-contained proofs because the above references prove different parts of the desired results and it is quite challenging to gather all the required manuscripts. We use the notation $F^k x$ to denote that F is k-times consecutively applied to x, i.e $F^k x := F \circ \cdots \circ F \circ Fx$.

Lemma 2.33 (Reich and Shafrir)**.** *Let $\mathcal{H}$ be a real Hilbert space and $D \subseteq \mathcal{H}$ non-empty, closed, and convex. Suppose that $F : D \to D$ is firmly non-expansive, and define $x^{k+1} := F^{k+1}x := Fx^k$ for an arbitrary $x^0 = x \in D$. Then*

$$\lim_{k\to\infty} \|F^{k+1}x - F^k x\| = \lim_{k\to\infty} \frac{\|F^{k+l}x - F^k x\|}{l} = \lim_{k\to\infty} \frac{\|F^k x\|}{k},$$

for all $l \in \mathbb{N}$; in particular the limits exist.

Proof. (1) By using the non-expansiveness of F, we get

$$\begin{aligned} 0 \leq \|F^{k+l}x - F^k x\| &\leq \|F^{k+l-1}x - F^{k-1}x\| \leq \cdots \leq \|F^l x - x\| \\ &\leq \|F^l x - F^{l-1}x\| + \cdots + \|Fx - x\| \\ &\leq l\|Fx - x\|. \end{aligned} \tag{2.20}$$

This shows that the sequence $\{\|F^{k+l}x - F^k x\|/l\}_{k\in\mathbb{N}}$ is bounded and monotonically decreasing for every $l \in \mathbb{N}$; thus the limits $L := \lim_{k\to\infty} \|F^{k+1}x - F^k x\|$ and $R_l := \lim_{k\to\infty} \|F^{k+l}x - F^k x\|$ exist. By the non-expansiveness and the triangle inequality, we obtain $R_l \le lL$ for all $l \in \mathbb{N}$. Therefore we have to show $R_l \ge lL$ for all $l \in \mathbb{N}$. To this end we use induction over $l \in \mathbb{N}$. The case $l = 1$ is obvious. For arbitrary but fixed $l \in \mathbb{N}$, we assume that $R_j = jL$ for all $j = 1, \dots, l$, and we now consider $R_{l+1}/(l+1)$. For this l, let $\varepsilon > 0$ be given; then we can find $k_0 \in \mathbb{N}$ such that

$$L - \varepsilon \;<\; \frac{\|x^{k+j} - x^k\|}{j} \;<\; L + \varepsilon$$

for all $j = 1, \dots, l$ and $k \ge k_0$ by the induction hypothesis. Since F is firmly non-expansive, we also have

$$\begin{aligned} \|x^{k+1} - x^{k+l+1}\| &= \|Fx^k - Fx^{k+l}\| \\ &\overset{\text{Prop. 2.23 (g)}}{\le} \frac{1}{2}\|x^k - x^{k+l} + x^{k+1} - x^{k+l+1}\| \\ &\le \frac{1}{2}\|x^k - x^{k+l+1}\| + \frac{1}{2}\|x^{k+1} - x^{k+l}\|. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \|x^k - x^{k+l+1}\| &\ge 2\|x^{k+1} - x^{k+l+1}\| - \|x^{k+1} - x^{k+l}\| \\ &> 2l(L - \varepsilon) - (l-1)(L + \varepsilon) \\ &= (l+1)L - (3l-1)\varepsilon. \end{aligned}$$

This implies the first equality of the claim.
(2) Now we prove the second equality. We see from (2.20), the first equality and the monotonicity of $\|F^{k+l}x - F^k x\|/l$ in k that

$$\lim_{k\to\infty} \|F^{k+1}x - F^k x\| = \lim_{k\to\infty} \frac{\|F^{k+l}x - F^k x\|}{l} \le \frac{\|F^{k+l}x - F^k x\|}{l} \le \frac{\|F^l x - x\|}{l}$$

for all $l \in \mathbb{N}$. Since $\|F^{l+1}x - F^l x\|$ is monotonically decreasing and bounded from below it is convergent. That the sequence of arithmetic means, also known as Cesaro-means, converges with the same limit is well known and can be easily proved. Therefore we get

$$\lim_{l\to\infty} \|F^{l+1}x - F^l x\| = \lim_{l\to\infty} \frac{1}{l}\sum_{i=0}^{l-1} \|F^{i+1}x - F^i x\| = \limsup_{l\to\infty} \frac{1}{l}\sum_{i=0}^{l-1} \|F^{i+1}x - F^i x\|.$$

Thus we see

$$\begin{aligned}
\lim_{k\to\infty} \|F^{k+1}x - F^k x\| &\leq \liminf_{l\to\infty} \frac{\|F^l x - x\|}{l} \\
&\leq \limsup_{l\to\infty} \frac{\|F^l x - x\|}{l} \\
&\leq \limsup_{l\to\infty} \frac{1}{l} \sum_{i=0}^{l-1} \|F^{i+1}x - F^i x\| \\
&= \lim_{l\to\infty} \frac{1}{l} \sum_{i=0}^{l-1} \|F^{i+1}x - F^i x\| \\
&= \lim_{l\to\infty} \|F^{l+1}x - F^l x\| \\
&= \lim_{k\to\infty} \|F^{k+1}x - F^k x\|.
\end{aligned}$$

This implies the second equality. □

Now we can prove the next result that will be important in the context of analyzing the convergence properties if there are no solutions or KKT pairs of an optimization problem. The analysis heavily depends on results provided by Simeon Reich et al. Since the proof is split upon [92, Thm. 1], [3, Thm. 2.1], and [91, Prop. 2.1 and Thm. 2.3], we will also state a complete proof here.

Theorem 2.34 (Pazy, Reich et al.)**.** *Suppose that $D \subset \mathcal{H}$ is a non-empty, closed, and convex subset of a real Hilbert space $\mathcal{H}$, $F : D \to D$ is firmly non-expansive, $x_0 \in D$, and that the sequence $\{x^k\}_{k\in\mathbb{N}}$ is generated by the fixed-point iteration $x^{k+1} = Fx^k$. Then*

$$\|Fx^k - x^k\| \xrightarrow{k\to\infty} \inf_{x\in D}\{\|Fx - x\|\}.$$

Proof. Consider another initial point $v^0 \in D$ together with its associated sequence $v^k = F^k v^0$ for $k \in \mathbb{N}$. Using the non-expansiveness of F and the iteration rule, we then see that

$$\|v^{k+2} - v^{k+1}\| = \|Fv^{k+1} - Fv^k\| \leq \|v^{k+1} - v^k\|$$

for all $k \in \mathbb{N}$. Applying this inequality and the triangle inequality yields

$$\|v^{k+1} - v^0\| \leq \|v^{k+1} - v^k\| + \|v^k - v^{k-1}\| + \cdots + \|v^1 - v^0\| \leq (k+1)\|Fv^0 - v^0\|. \tag{2.21}$$

Again by the non-expansiveness of F, we obtain

$$\|x^{k+1} - v^{k+1}\| = \|Fx^k - Fv^k\| \leq \|x^k - v^k\| \leq \cdots \leq \|x^0 - v^0\|.$$

Therefore, by the triangle inequality it holds that

$$\begin{aligned}\|x^{k+1} - x^0\| &\leq \|x^{k+1} - v^{k+1}\| + \|x^0 - v^0\| + \|v^{k+1} - v^0\| \\ &\leq 2\|x^0 - v^0\| + (k+1)\|Fv^0 - v^0\|.\end{aligned}$$

Dividing this by $k+1$ and taking into account that for an arbitrary $\varepsilon > 0$, v^0 can be chosen such that $\|Fv^0 - v^0\| \leq \inf_{x\in D}\{\|Fx - x\|\} + \varepsilon$, we obtain

$$\limsup_{k\to\infty} \left\| \frac{x^{k+1}}{k+1} \right\| = \limsup_{k\to\infty} \left\| \frac{x^{k+1} - x^0}{k+1} \right\| \leq \inf_{x\in D}\{\|Fx - x\|\}.$$

By Lemma 2.33 and $x^{k+1} = F^{k+1}x$, the above lim sup are actual limits and coincide with $\lim_{k\to\infty} \|x^{k+1} - x^k\|$. Further it is obvious that $\|x^{k+1} - x^k\| = \|Fx^k - x^k\| \geq \inf_{x\in D}\{\|Fx - x\|\}$. Thus we have

$$\begin{aligned}\inf_{x\in D}\{\|Fx - x\|\} &\leq \lim_{k\to\infty} \|x^{k+1} - x^k\| \\ &= \lim_{k\to\infty} \left\| \frac{x^{k+1}}{k+1} \right\| \\ &= \lim_{k\to\infty} \left\| \frac{x^{k+1} - x^0}{k+1} \right\| \\ &\leq \inf_{x\in D}\{\|Fx - x\|\}.\end{aligned}$$

This completes the proof. □

2.3.4 Halpern's Method

In the above section we have seen that the fixed-point iterations (2.18) and (2.19) converge weakly to a fixed point of F under mild conditions. On the other hand, it is often appreciated to have a strongly convergent algorithm at hand, since strongly convergent algorithms possess better approximation properties. To this end, we note that there exist some modifications of the fixed-point iteration (2.19) that are known to give strongly convergent iterates. One of these modifications is Halpern's method, see [61] for the original reference or the discussion in [9]. In order to describe the simple idea of Halpern's method, consider first the classical fixed-point iteration

$$x^{k+1} = Fx^k \tag{2.22}$$

that converges weakly if F is firmly non-expansive and has at least one fixed point, see Corollary 2.32. In order to be able to obtain weak convergence of only

non-expansive operators F, Krasnoselsky and Mann blended a bit of the identity into (2.22) by using the iteration

$$x^{k+1} = \rho^k x^k + (1 - \rho^k) F x^k, \tag{2.23}$$

where $\rho^k \in [0, 1]$ and $\sum_{k=1}^{\infty} \rho^k (1 - \rho^k) = +\infty$, see Theorem 2.30. Now the idea of Halpern was to replace the vector x^k from the identity map by a fixed vector x. Thus Halpern's iteration is

$$x^{k+1} = \rho^k x + (1 - \rho^k) F x^k, \tag{2.24}$$

where the sequence $\{\rho^k\}_{k \in \mathbb{N}}$ satisfies the conditions

$$\rho^k \to 0, \quad \sum_{k=1}^{\infty} \rho^k = +\infty, \quad \sum_{k=1}^{\infty} |\rho^{k+1} - \rho^k| < \infty, \tag{2.25}$$

which, in particular, hold for the choice $\rho^k := 1/k$. This method is known to be strongly convergent to the particular solution $\mathrm{Proj}_{\mathrm{fix}\, F}\, x$. The next theorem summarizes the convergence properties of (2.24).

Theorem 2.35 (Halpern's method, [9, Thm. 30.1])**.** *Let D be a non-empty, closed, convex subset of the real Hilbert space $\mathcal{H}$, let $F : D \to D$ be non-expansive such that $\mathrm{fix}(F) \neq \emptyset$, let $x, x_0 \in D$, and let $\{\rho^k\}_{k \in \mathbb{N}} \subset (0, 1)$ satisfy (2.25). Then the iterates $\{x^k\}_{k \in \mathbb{N}}$ generated by (2.24) converge strongly to the fixed point of F that is closest to x, i.e. $x^k \to \mathrm{Proj}_{\mathrm{fix}\, F}\, x$.*

There are other ways than Halpern's method to modify the fixed-point iteration such that the iterates of a fixed-point iteration become strongly convergent. Two of them to be mentioned here are, first, the recently introduced scheme [15], which was inspired by the Krasnoselsky-Mann, Halpern and proximal-Tikhonov algorithm iteration, and, second, the well known Haugazeau scheme, see e.g. [9, Sec. 30.3] and [62] for the original reference, which uses hyperplane projections.

Even though all of the strongly convergent schemes could be used for the applications discussed in the Chapters 4 and 5, we limit ourselves to using Halpern's method since it might be the most convenient one to use.

2.4 Monotone Operators

This section is dedicated to introducing the very general framework of set-valued mappings, that have certain monotonicity properties. In this section $\mathcal{H}$ denotes always a real Hilbert space. All results and definitions can be found in [9] and

therefore will not be cited explicitly; for a Banach space view of this topic see [6, 7].

The kind of set-valued mappings T that we discuss here maps points from the Hilbert space $\mathcal{H}$ to sets contained in $\mathcal{H}$, i.e. $T : \mathcal{H} \to 2^{\mathcal{H}}$. If for all $x \in \mathcal{H}$ the sets $T(x)$ only contain a single element, we call T *single-valued*, and write $T : \mathcal{H} \to \mathcal{H}$. Thus all the notions defined for set-valued mappings can also be applied to ordinary (single-valued) functions.

Given a set-valued mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ from a real Hilbert space $\mathcal{H}$ to its power set $2^{\mathcal{H}}$, we define the *domain of* T as

$$\mathrm{dom}(T) := \{x \in \mathcal{H} \mid T(x) \neq \emptyset\}$$

and the *graph of* T as

$$\mathrm{graph}(T) := \{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in T(x)\}.$$

The inverse T^{-1} of a set-valued operator $T : \mathcal{H} \to 2^{\mathcal{H}}$ is uniquely defined through its graph by

$$\mathrm{graph}(T^{-1}) := \{(u, x) \in \mathcal{H} \times \mathcal{H} \mid u \in T(x)\}. \tag{2.26}$$

By this definition the inverse always exists, maybe with $T^{-1}(u) = \emptyset$. Further it is necessary to define some arithmetic operations for set-valued operators. Let $T_1, T_2 : \mathcal{H} \to 2^{\mathcal{H}}$, and $Q \in \mathcal{L}(\mathcal{H})$. Then scalar multiplication with $\beta \in \mathbb{R}$, the effect of Q, and summation are defined as

$$\begin{aligned} \beta T_1(x) &:= \{\beta u \mid u \in T_1(x)\}, \\ Q T_1(x) &:= \{Q u \mid u \in T_1(x)\}, \\ T_1(x) + T_2(x) &:= \{u_1 + u_2 \mid u_1 \in T_1(x),\ u_2 \in T_2(x)\}. \end{aligned}$$

Definition 2.36. A set-valued mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is called *monotone* if

$$\langle u - v \mid x - y \rangle \geq 0 \qquad \forall (x, u), (y, v) \in \mathrm{graph}(T).$$

It is called *maximally monotone* if it is monotone and there is no monotone operator $\widetilde{T} : \mathcal{H} \to 2^{\mathcal{H}}$ such that $\mathrm{graph}(T) \subsetneq \mathrm{graph}(\widetilde{T})$.
The set-valued mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is called *strongly monotone* if there is a $\rho > 0$ such that

$$\langle u - v \mid x - y \rangle \geq \rho \|x - y\|^2 \qquad \forall (x, u), (y, v) \in \mathrm{graph}(T).$$

Furthermore, we call $T : \mathcal{H}_1 \times \cdots \times \mathcal{H}_N \to 2^{\mathcal{H}_1 \times \cdots \times \mathcal{H}_N}$ *strongly monotone with respect to* $x_{-N} := (x_1, \ldots, x_{N-1})$ if there is a constant $\rho > 0$ such that

$$\langle u - v \mid x - y \rangle \geq \rho \|x_{-N} - y_{-N}\|^2 := \rho \sum_{\nu=1}^{N-1} \|x_\nu - y_\nu\|^2 \qquad \forall (x, u), (y, v) \in \mathrm{graph}\, T.$$

A single-valued operator $T : \mathcal{H} \to \mathcal{H}$ is called *α-cocoercive* for some $\alpha > 0$ if

$$\langle T(x) - T(y) \mid x - y \rangle \geq \alpha \|T(x) - T(y)\|^2.$$

Note that the definition of strong monotonicity with respect to x_{-N} is weaker than plain strong monotonicity. This observation will be important in the application of the method introduced in Chapter 6. Further note that from the Cauchy-Schwarz inequality we obtain that every α-cocoercive operator is $1/\alpha$-Lipschitz continuous. If T is strongly monotone with modulus ρ and Lipschitz continuous with constant L, then T is also ρ/L^2-cocoercive. On the other hand, cocoercivity does not imply strong monotonicity. This observation plays an important role in our analysis is Chapter 5, since a certain operator to be defined later is never strongly monotone but, under suitable conditions, cocoercive.

The above notions of monotonicity yield a useful characterization of (strong) convexity.

Proposition 2.37. *Let $\mathcal{H}$ be a real Hilbert space and $f : \mathcal{H} \to \mathbb{R}$ differentiable. Then f is (strongly) convex if and only if ∇f is (strongly) monotone.*

At first the definition of maximal monotonicity seems to be a bit cumbersome. Thus the next lemma characterizes maximal monotonicity through the solution of a variational inequality that might be easier to handle and its proof follows straight from the original definition.

Lemma 2.38. *A monotone mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone if and only if every solution $(y, v) \in \mathcal{H} \times \mathcal{H}$ of*

$$\langle v - u \mid y - x \rangle \geq 0 \qquad \forall (x, u) \in \operatorname{graph}(T),$$

belongs to $\operatorname{graph}(T)$.

The discussion of maximally monotone operators is motivated by the following result, which states that the convex subdifferential mapping and the convex normal cone mapping are maximally monotone operators.

Proposition 2.39 (Moreau, cf. [6, Thm. 2.8])**.** *Let $\mathcal{H}$ be a real Hilbert space. Then the convex subdifferential mapping $x \mapsto \partial f(x)$ of a proper, lower semi-continuous, and convex function $f : \mathcal{H} \to (-\infty, +\infty]$, defined in* (2.13)*, is maximally monotone.*

In particular, the normal cone mapping $x \mapsto N_C(x)$ of a non-empty, closed, convex set $C \subset \mathcal{H}$, defined in (2.14)*, is also maximally monotone.*

Also we observe that the domain of ∂f, as defined in (2.16), coincides with the domain of the maximally monotone operator ∂f. If f is convex and differentiable,

then Proposition 2.18 shows that ∇f is continuous. Proposition 2.37 states that ∇f is monotone, and therefore we obtain from Proposition 2.39 that ∇f is even maximally monotone. This considerations motivate the hypothesis that every continuous and monotone operator is maximally monotone, which is verified in the next proposition.

Proposition 2.40 ([6, Thm. 2.4])**.** *Let $\mathcal{H}$ be a real Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be monotone and continuous with $dom(T) = \mathcal{H}$. Then T is maximally monotone.*

Maximally monotone operators possess a variety of important properties, some of which are now presented.

Proposition 2.41 ([6, Prop. 2.1])**.** *Let $T : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone then*

(a) $\operatorname{graph}(T)$ *is weak-strong sequentially closed, i.e. if $x^k \rightharpoonup \bar{x}$ and $u^k \in T(x^k)$ with $u^k \to \bar{u}$, then $\bar{u} \in T(\bar{x})$.*

(b) $\operatorname{graph}(T)$ *is strong-weak sequentially closed, i.e. if $x^k \to \bar{x}$ and $u^k \in T(x^k)$ with $u^k \rightharpoonup \bar{u}$, then $\bar{u} \in T(\bar{x})$,*

(c) T^{-1} *is maximally monotone,*

(d) *for each $x \in \operatorname{dom}(T)$ the set $T(x)$ is closed and convex.*

Proof. The statements (a) and (b) follow straight from the fact that

$$\langle u^k - v \mid x^k - y \rangle \geq 0 \qquad \forall (y, v) \in \operatorname{graph}(T).$$

The left-hand side of the last inequality converges, hence

$$\langle \bar{u} - v \mid \bar{x} - y \rangle \geq 0 \qquad \forall (y, v) \in \operatorname{graph}(T).$$

The maximal monotonicity (Lemma 2.38) now implies $\bar{u} \in T(\bar{x})$.
The statement (c) is obvious and (d) can be found in [6, Prop. 2.1]. □

The rest of this section is dedicated to the question under which conditions an operator $T : \mathcal{H} \to 2^{\mathcal{H}}$ and the sum of maximally monotone operators is again maximally monotone.

First, we answer this question with regard of the sum of two operators.

Proposition 2.42 ([6, Thm. 2.6])**.** *Let $\mathcal{H}$ be a real Hilbert space, $T_1, T_2 : \to 2^{\mathcal{H}}$ maximally monotone with $\operatorname{dom}(T_2) = \mathcal{H}$. Then $T_1 + T_2$ is maximally monotone.*

By extending this result to three operators we obtain a theorem that plays a crucial role in this thesis as, in combination with Proposition 2.41, it will be used to show that a certain accumulation point is a solution to the problems mentioned in Chapter 1.

Theorem 2.43. *Let $\mathcal{H}$ be a real Hilbert space, $T_1, T_2 : \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone with $\operatorname{dom}(T_2) = \mathcal{H}$, and let $T_3 : \mathcal{H} \to \mathcal{H}$ with $\operatorname{dom}(T_3) = \mathcal{H}$ be monotone and continuous. Then $T_1 + T_2 + T_3$ is maximally monotone.*

Proof. Apply Proposition 2.42 two times, taking into account Proposition 2.40. □

Recall that the inverse of a self-adjoint, strongly monotone, and linear operator $Q \in \mathcal{L}(\mathcal{H})$ exists and is again self-adjoint as well as strongly monotone, cf. Lemma 2.4. Using the scalar product induced by Q, the "preconditioned" operator $Q^{-1}T$ turns out to be maximally monotone in the Q-scalar product if T is maximally monotone in the original scalar product.

Proposition 2.44 ([9, Prop. 20.24])**.** *Let $\mathcal{H}$ be a real Hilbert space endowed with the scalar product $\langle \cdot \mid \cdot \rangle$, let $Q \in \mathcal{L}(\mathcal{H})$ be self-adjoint and strongly monotone, and let $T : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone. Then $Q^{-1}T$ is maximally monotone in $\mathcal{H}$ endowed with the scalar product $\langle x \mid y \rangle_Q := \langle Qx \mid y \rangle$, which has already been defined in* (2.3)*.*

Proof. This is essentially the result from [9, Prop. 20.24], taking into account part (c) of Lemma 2.4. □

Later the last proposition is of major importance because algorithms for optimization problems or GNEPs can be rewritten using the examined transform in order to see convergence.

The next proposition shows that a single-valued, strongly and maximally monotone operator is a bijection, which can be seen by the Browder–Minty theorem, cf. [6, Cor. 2.2] or [9, Cor. 21.25], and the strong monotonicity.

Proposition 2.45. *Let $\mathcal{H}$ be a real Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be strongly and maximally monotone. Then T is a bijection.*

Proof. The surjectivity follows from [6, Cor. 2.2] or [9, Cor. 21.25] and the injectivity follows straight from the strong monotonicity. □

2.5 Zeros of (Maximally) Monotone Operators

This section deals with several non-expansive mappings that are closely linked to zeros of maximally monotone operators. From these operators we then obtain general and well known algorithms. Why zeros of maximally monotone operators are important can be seen considering a proper, convex, lower semi-continuous function. For such a function, finding a minimizer is equivalent to finding a zero of the maximally monotone operator ∂f, see Theorem 2.17. As we see later in Chapter 3, the optimality conditions for constrained optimization and generalized

Nash problems can be expressed as zeros of maximally monotone operators. Thus it is often desired to find a zero of a maximally monotone operator T. We consider maximally monotone operators in a real Hilbert space $\mathcal{H}$, i.e. $T : \mathcal{H} \to 2^{\mathcal{H}}$. The precise formulation of the problem treated in this section is: Find $x^* \in \mathcal{H}$ such that

$$0 \in T(x^*). \tag{2.27}$$

We denote the set of solutions to the just stated problem (2.27) by $\operatorname{zer}(T)$. In order to solve this problem, we consider particular non-expansive operators that are connected to zeros of T and their fixed-point iterations. We first take a look at the resolvent and the forward operator as well as their associated fixed-point iterations. Based on this, the above root finding problem (2.27) is discussed for the case that the operator T is the sum of two operators. This leads us to the forward-backward and the forward-backward-forward operator and the associated iterations in the Sections 2.5.3 and 2.5.4.

2.5.1 The Resolvent and the Proximal Point Algorithm

We consider the root finding problem (2.27). For $\beta > 0$ it holds that

$$0 \in T(x^*) \iff x^* \in x^* + \beta T(x^*) \iff x^* \in (I + \beta T)^{-1}(x^*), \tag{2.28}$$

with the usual concept of inversion for set-valued operators, see (2.26). Hence finding a zero of T is equivalent to finding a fixed point of the operator $(I + \beta T)^{-1}$, called the *resolvent*.

The next proposition shows that the resolvent is well defined, i.e. for every $x \in \mathcal{H}$ there is at least one $u \in (I + \beta T)^{-1}x$.

Proposition 2.46 (Minty, cf. [9, Thm. 21.1])**.** *Let $\mathcal{H}$ be a real Hilbert space and $T : \mathcal{H} \to 2^{\mathcal{H}}$ monotone. Then T is maximally monotone if and only if* $\operatorname{range}(I + \beta T) = \mathcal{H}$.

Now we will see that there is exactly one $u \in (I + \beta T)^{-1}x$; thus it is possible to write $u = (I + \beta T)^{-1}x$ and the inclusion $x^* \in (I + \beta T)^{-1}x^*$ from (2.28) is an equality. Further it can be shown that the resolvent is maximally monotone and firmly non-expansive.

Proposition 2.47 ([9, Cor. 23.11])**.** *Let $\mathcal{H}$ be a real Hilbert space and $T : \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone. Then the resolvent $(I + \beta T)^{-1}$ is single-valued, maximally monotone, and firmly non-expansive.*

In the sequel we need the resolvent of the "preconditioned" operator $Q^{-1}T$. In Proposition 2.44 it was stated that $Q^{-1}T$ is maximally monotone in $\mathcal{H}$ endowed with the scalar product $\langle Q\cdot \mid \cdot\rangle$, whenever $Q \in \mathcal{L}(\mathcal{H})$ is self-adjoint and strongly monotone. Applying this result in combination with Proposition 2.47 yields the next proposition.

Proposition 2.48. *Let $\mathcal{H}$ be a real Hilbert space, $T : \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone, and let $Q \in \mathcal{L}(\mathcal{H})$ be self-adjoint and strongly monotone. Then $(I+\beta Q^{-1}T)^{-1}$ is single-valued, maximally monotone, and firmly non-expansive in $\mathcal{H}$ endowed with the scalar product $\langle x \mid y\rangle_Q := \langle Qx \mid y\rangle$.*

Proof. By Proposition 2.44 the operator $Q^{-1}T$ is maximally monotone in $\mathcal{H}$ endowed with the scalar product $\langle \cdot \mid \cdot\rangle_Q$. With Proposition 2.47 we obtain the claim. □

In the following example, two special resolvents are considered. First, the resolvent of the convex subdifferential, introduced in (2.13), turns out to be the well known proximal operator. Second, the resolvent of the normal cone mapping yields the projection.

Example 2.49. Let C be closed, convex subset of a real Hilbert space $\mathcal{H}$, let $f : \mathcal{H} \to (-\infty, +\infty]$ be a proper, convex, and lower semi-continuous function, let $\beta > 0$. Using the optimality condition from Theorem 2.17 and Proposition 2.20, it is easy to see that

$$(I + \beta\partial f)^{-1}x := \operatorname*{arg\,min}_{y\in\mathcal{H}} \left(\beta f(y) + \frac{1}{2}\|y - x\|^2\right) =: \operatorname{prox}_{\beta f}(x).$$

Since the subdifferential of the indicator function is the normal cone, we see

$$(I + \beta N_C)^{-1}x = \operatorname{Proj}_C(x).$$

◇

In Proposition 2.48 we have seen that the resolvent is a firmly non-expansive mapping and from Corollary 2.32 it follows that the fixed-point iteration with a firmly non-expansive mapping is weakly convergent. This fixed-point iteration is of major interest and therefore is given the name proximal point algorithm.

Theorem 2.50 (Proximal Point Algorithm)**.** *Let $\mathcal{H}$ be a real Hilbert space, $T : \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone, let $\beta > 0$, and let $Q \in \mathcal{L}(\mathcal{H})$ be self-adjoint and strongly monotone. Suppose that T has a fixed point. Then the iterates generated by*

$$x^{k+1} := (I + \beta Q^{-1}T)^{-1}x^k$$

are uniquely defined, converge weakly to a zero of T, and $x^{k+1} - x^k \to 0$. Further we get the convergence rates $\operatorname{dist}^2(T(x^k), 0) = o(1/k)$ for $k \to \infty$ and $\|x^{k+1} - x^k\|^2 = O(1/k)$.

Proof. By Proposition 2.48 the iterates x^{k+1} are uniquely defined and the operator $F := (I+\beta Q^{-1}T)^{-1}$ is firmly non-expansive in the scalar product $\langle \cdot \mid \cdot \rangle_Q$. Choosing $\rho^k \equiv 1$ in Corollary 2.32, we obtain the weak convergence of the iterates x^{k+1} to a fixed point of the resolvent $(I+\beta Q^{-1}T)^{-1}$ in the space $\mathcal{H}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_Q$. Lemma 2.9 shows that the iterates converge weakly in $\mathcal{H}$ with the original scalar product $\langle \cdot \mid \cdot \rangle$ to a fixed point of the resolvent $(I + \beta Q^{-1}T)^{-1}$. Now we can see, as stated in (2.28), that being a fixed point of $(I + \beta Q^{-1}T)^{-1}$ is equivalent to being a zero of T. The assertion about the convergence rate $\operatorname{dist}^2(T(x^k), 0) = o(1/k)$ follows from [38], which is a recent improvement of a classical result stated in [29, Prop. 8]. The rate $\|w^k - w^{k+1}\|^2 = O(1/k)$ is shown in [37, Thm. 3.1]. Both convergence results can be applied here because of the equivalence of the two norms $\|\cdot\|$ and $\|\cdot\|_Q$. □

There are more sophisticated variants of the proximal point algorithm; first, it is possible to use the Krasnoselsky-Mann iteration (2.19) instead of the classical fixed-point iteration (2.18). Second, it is possible to vary the parameter β in every iteration, see [9, Thm. 23.41]. However, the above proximal point method completely suffices for our purposes.

2.5.2 The Forward Operator and the Gradient Method

Let us again consider the root finding problem (2.27), i.e. find $x^* \in \mathcal{H}$ such that $0 \in T(x^*)$ with a single-valued, maximally monotone operator $T : \mathcal{H} \to \mathcal{H}$, and let $\beta > 0$. In this case we get the equivalences

$$0 = T(x^*) \iff 0 = -\beta T(x^*) \iff x^* = x^* - \beta T(x^*) = (I - \beta T)x^*. \quad (2.29)$$

Thus a zero of T is also a fixed point of the operator $(I - \beta T)$, which we call *forward operator*. It is well known that the forward operator is averaged whenever T is cocoercive and β is sufficient small.

Proposition 2.51 ([9, Prop. 4.39])**.** *Let $\mathcal{H}$ be a real Hilbert space, let $T : \mathcal{H} \to \mathcal{H}$, let $\alpha > 0$, and let $\beta \in (0, 2\alpha)$. Then T is α-cocoercive if and only if $I - \beta T$ is $\beta/(2\alpha)$-averaged.*

When the fixed-point iteration (2.18) is applied to the averaged operator $F := (I - \beta T)$, we obtain a certain kind of gradient method that converges since the forward operator is averaged.

Proposition 2.52 (Gradient method)**.** *Let $\mathcal{H}$ be a real Hilbert space, $x^0 \in \mathcal{H}$, let $\alpha > 0$, let $T : \mathcal{H} \to \mathcal{H}$ be α-cocoercive, and let $\beta \in (0, 2\alpha)$. Then the iterates generated by*

$$x^{k+1} := (I - \beta T)x^k$$

converge weakly to a zero of T.

Proof. The averagedness of the forward operator was shown in Proposition 2.51. Now choose $\rho^k \equiv 1$ in Proposition 2.31 to obtain the weak convergence of x^k to a fixed point of $I - \beta T$. That every fixed point of the forward operator is a zero of T was shown in (2.29). □

Obviously, more sophisticated versions of the gradient method can be obtained using the Krasnoselsky-Mann iteration (2.19) instead of the classical fixed-point iteration (2.18). However, the above gradient method is enough for our purposes.

Under certain conditions, the forward operator is even a contraction; this case is considered next.

Proposition 2.53. *Let $\mathcal{H}$ be a real Hilbert space. Suppose that the operator $T : \mathcal{H} \to \mathcal{H}$ is strongly monotone with modulus ρ and Lipschitz continuous with constant L, and suppose that $\beta \in (0, 2\rho/L^2)$. Then the operator $I - \beta T$ is a contraction, i.e. Lipschitz continuous with constant $1+\beta^2L^2-2\beta\rho < 1$. Furthermore, the method from Proposition 2.52 converges strongly.*

Proof. We see that

$$\begin{aligned}
\|(I - \beta T) - (I - \beta T)y\|^2 &= \|(x - y) - \beta(Tx - Ty)\|^2 \\
&= \|x - y\|^2 + \beta^2\|Tx - Ty\|^2 - 2\beta\langle Tx - Ty \mid x - y\rangle \\
&\leq \|x - y\|^2 + \beta^2\|Tx - Ty\|^2 - 2\beta\rho\|x - y\|^2 \\
&\leq \|x - y\|^2 + \beta^2L^2\|x - y\|^2 - 2\beta\rho\|x - y\|^2 \\
&= (1 + \beta^2L^2 - 2\beta\rho)\|x - y\|^2.
\end{aligned}$$

It is easy to see that $1 + \beta^2L^2 - 2\beta\rho < 1$ for $\beta < 2\rho/L^2$, hence $(I - \beta T)$ is a contraction. The proposed strong convergence follows straight from the Banach fixed-point theorem. □

2.5.3 The Forward-Backward Operator and Iteration

In this section, we want to investigate the root finding problem (2.27) for the case that the operator $T : \mathcal{H} \to 2^{\mathcal{H}}$ is the sum of two operators. We assume that

$T = T_1 + T_2$, where $T_2 : \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone and $T_1 : \mathcal{H} \to \mathcal{H}$ is single-valued and maximally monotone. For $\beta > 0$ we derive the equivalences

$$\begin{aligned} 0 \in T_1 x^* + T_2 x^* &\iff -T_1 x^* \in T_2 x^* \\ &\iff (I - \beta T_1) x^* \in (I + \beta T_2) x^* \\ &\iff x^* \in (I + \beta T_2)^{-1} (I - \beta T_1) x^*. \end{aligned}$$

Assuming T_2 to be maximally monotone implies that the last inclusion is actually an equality, so we obtain the characterization

$$0 \in T_1 x^* + T_2 x^* \iff x^* = (I + \beta T_2)^{-1} (I - \beta T_1) x^*. \tag{2.30}$$

The operator $(I + \beta T_2)^{-1}(I - \beta T_1)$ is called the *forward-backward operator*. Under the conditions from Section 2.5.2 on T_1 and β, the forward operator $(I - \beta T_1)$ is averaged, further, the resolvent $(I + \beta T_2)^{-1}$ is firmly non-expansive, and therefore the forward-backward operator is averaged, cf. Proposition 2.26. This motivates that we can use the fixed-point iterations from Section 2.3 to find zeros of $T_1 + T_2$. First we quantify the averagedness of the forward-backward operator.

Proposition 2.54. *Let $T_2 : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, let $T_1 : \mathcal{H} \to \mathcal{H}$ be single-valued, maximally monotone, and α-cocoercive, let $\beta \in (0, 2\alpha)$. Then the forward-backward operator $(I + \beta T_2)^{-1}(I - \beta T_1)$ is $\tilde{\alpha}$-averaged with $\tilde{\alpha} := 2\alpha/(4\alpha - \beta)$.*

Proof. Combine Proposition 2.26, Proposition 2.47, and Proposition 2.51. □

The last proposition reveals the conditions under which the forward-backward operator is averaged, and thus we can now apply the fixed-point iteration (2.18) or (2.19), which is weakly convergent.

Theorem 2.55 ([9, Thm. 26.14])**.** *Let $\mathcal{H}$ be a real Hilbert space, let $\alpha > 0$, let $T_1 : \mathcal{H} \to \mathcal{H}$ be α-cocoercive, let $T_2 : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, let $\beta \in (0, 2\alpha)$, and set $\delta := 2 - \beta/(2\alpha)$. Further let $\{\rho^k\}_{k \in \mathbb{N}} \subset [0, \delta]$ be a sequence such that $\sum_{k=1}^{\infty} \rho^k(\delta - \rho^k) = +\infty$, and let $x^0 \in \mathcal{H}$. Suppose that $T_1 + T_2$ has a zero. Then the iterates generated by*

$$\tilde{x}^k := (I + \beta T_2)^{-1}(I - \beta T_1) x^k \tag{2.31a}$$

$$x^{k+1} := x^k + \rho^k(\tilde{x}^k - x^k) \tag{2.31b}$$

converge weakly to a zero x^* of $T_1 + T_2$ and $T_1 x^k$ converges strongly to $T_1 x^*$, where the value $T_1 x^*$ is independent of the individual solution x^*. If T_1 or T_2 is strongly monotone, x^k converges strongly to the unique zero of $T_1 + T_2$.

Proof. The weak convergence follows straight from Proposition 2.31 and Proposition 2.54. The rest of the proof can be found in [9, Thm. 26.14]. □

Note that the assumptions made in Theorem 2.55 allow the choice $\rho^k \equiv 1$.

2.5.4 The Forward-Backward-Forward Iteration

The second splitting method that we recall is due to Tseng [105], see also [9]. For the application we have in mind, we only restate a simplified version of that method. The main advantage of Tseng's approach is that it replaces the cocoercivity assumption for T_1 by a simpler condition. More precisely, assume that T_1 is Lipschitz continuous with constant $1/\alpha$, and let $\beta \in (0, \alpha)$, hence $\beta/\alpha < 1$. Then it is easy to see that $I - \beta T_1$ is strongly monotone and, therefore, a bijection, cf. Propositions 2.45 and 2.53. Consequently, we can further rewrite (2.30) as follows:

$$\begin{aligned} 0 \in T_1 x^* + T_2 x^* &\iff x^* = (I + \beta T_2)^{-1}(I - \beta T_1)x^* \\ &\iff (I - \beta T_1)x^* = (I - \beta T_1)(I + \beta T_2)^{-1}(I - \beta T_1)x^* \\ &\iff x^* = \big((I - \beta T_1)(I + \beta T_2)^{-1}(I - \beta T_1) + \beta T_1\big)x^*. \end{aligned}$$

This motivates the fixed-point iteration

$$x^{k+1} = \big((I - \beta T_1)(I + \beta T_2)^{-1}(I - \beta T_1) + \beta T_1\big)x^k,$$

which can be rewritten as

$$\begin{aligned} y^k &:= x^k - \beta T_1 x^k, && \text{(2.32a)} \\ z^k &:= (I + \beta T_2)^{-1} y^k, && \text{(2.32b)} \\ x^{k+1} &:= z^k + \beta(T_1 x^k - T_1 z^k). && \text{(2.32c)} \end{aligned}$$

This justifies the alternative name *forward-backward-forward splitting method* for Tseng's approach.

Provided that T_2 is maximally monotone, T_1 is monotone and $1/\alpha$-Lipschitz continuous for some $\alpha > 0$, $T_1 + T_2$ is maximally monotone, the parameter β is taken such that $\beta \in (0, \alpha)$, and assuming that $\operatorname{zer}(T_1 + T_2) \neq \emptyset$, one can show that the sequences $\{x^k\}_{k\in\mathbb{N}}$ and $\{z^k\}_{k\in\mathbb{N}}$ generated by this forward-backward-forward scheme converge weakly to a point in $\operatorname{zer}(T_1 + T_2)$. This and further convergence properties are stated in the next theorem.

Theorem 2.56 (Tseng's Method, [9, Thm. 26.17])**.** *Let $\mathcal{H}$ be a real Hilbert space, let $\alpha > 0$, let $T_1 : \mathcal{H} \to \mathcal{H}$ be $1/\alpha$-Lipschitz continuous, let $T_2 : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, let $\beta \in (0, \alpha)$, and let $x^0 \in \mathcal{H}$. Suppose that $T_1 + T_2$ has a zero and is maximally monotone. Let the iterates x^{k+1}, y^k, z^k be generated by (2.32). Then the following assertions hold:*

(a) *$\{x^k - z^k\}_{k\in\mathbb{N}}$ converges strongly to zero.*

(b) *$\{x^k\}_{k\in\mathbb{N}}$ and $\{z^k\}_{k\in\mathbb{N}}$ converge weakly to a point in $\operatorname{zer}(T_1 + T_2)$.*

(c) *If T_1 or T_2 is strongly monotone, then $\{x^k\}_{k\in\mathbb{N}}$ and $\{z^k\}_{k\in\mathbb{N}}$ converge strongly to the unique point in $\operatorname{zer}(T_1 + T_2)$.*

2.6 Fundamental Inequalities

Here we recall and derive some basic inequalities that will be used in our subsequent analysis. To this end, we first restate the well-known Young inequality with ε.

Lemma 2.57 (Young's Inequality)**.** *Suppose that $a, b \in \mathbb{R}$. Then*

$$|a \cdot b| \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2$$

holds for every $\varepsilon > 0$.

As a consequence of the previous result, we obtain the following estimate.

Lemma 2.58. *Let $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$. Then*

$$\left(\sum_{i=1}^{N} \alpha_i \right)^2 \leq N \sum_{i=1}^{N} \alpha_i^2.$$

Proof. Let $e := (1, \ldots, 1)^T$ and $a = (a_1, \ldots a_N)^T$ then the Cauchy-Schwarz inequality yields

$$\left(\sum_{i=1}^{N} \alpha_i \right)^2 = (e^T a)^2 \leq \|e\|_{\mathbb{R}^N}^2 \|a\|_{\mathbb{R}^N}^2 = N \sum_{i=1}^{N} \alpha_i^2.$$

□

Lemma 2.58 immediately yields the following inequality.

Lemma 2.59. *Let $\mathcal{H}$ be a real Hilbert space. For arbitrary $a_1, \ldots, a_N \in \mathcal{H}$, it holds that*

$$\left\| \sum_{i=1}^{N} a_i \right\|^2 \leq N \sum_{i=1}^{N} \|a_i\|^2$$

Proof. We obtain

$$\left\| \sum_{i=1}^{N} a_i \right\|^2 \leq \left(\sum_{i=1}^{N} \|a_i\| \right)^2 \leq N \sum_{i=1}^{N} \|a_i\|^2$$

from the triangle inequality and Lemma 2.58. □

Finally, we restate an identity, which follows directly from expanding $\|w - v\|^2$.

Lemma 2.60 (Polarization Identity)**.** *Let $\mathcal{H}$ be a real Hilbert space. For arbitrary $w, v \in \mathcal{H}$, we have*

$$2\langle w \mid v \rangle = \|w\|^2 + \|v\|^2 - \|w - v\|^2.$$

Proof. This is proved by expanding the right-hand side of the equation. □

Chapter 3

Theory of Optimization and Variational Problems

In this chapter, we recall and discuss the problems that this manuscript is concerned with in more depth. As already mentioned in the introduction, these problems are the separable, convex optimization problem and jointly convex generalized Nash problem, where the joint constraints are linear equalities. Since the following chapters require an extensive understanding of these problems, which were first introduced in Chapter 1, we now discuss the associated theory. The theory presented here is part of the much broader theory of optimization problems or generalized Nash problems, see for instance [7, 18, 46]. Nevertheless, what we present is tailored precisely to the problems that we design algorithms for. We also link the optimality conditions of these problems to zeros of certain maximally monotone operators. This requires the linearity of the constraints (and the separability of the objective function).

First, in Section 3.1, linearly constrained, separable, convex optimization problems are discussed and their KKT conditions are linked to the zeros of a maximally monotone operator. Thereafter, in Section 3.2, generalized Nash equilibrium problems with linear joint constraints are reviewed, and the notions of variational equilibria and variational KKT conditions are introduced. Then these variational KKT conditions are again linked to zeros of a maximally monotone operator. Furthermore, two ways to incorporate linear, conic constraints into this framework are presented.

3.1 Separable Linearly Constrained Optimization

In this section, we consider the separable, linearly constrained optimization problem

$$\min \sum_{i=1}^{N} f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^{N} A_i x_i = b, \quad x_i \in \mathcal{X}_i \ (i = 1, \dots, N). \tag{Opt}$$

Here and throughout this thesis, $\mathcal{H}_i$ and $\mathcal{K}$ are Hilbert spaces, $f_i : \mathcal{H}_i \to \mathbb{R}$ are lower semi-continuous, convex functions, $\mathcal{X}_i \subset \mathcal{H}_i$ are closed, convex sets, A_i are linear operators from $\mathcal{H}_i$ to $\mathcal{K}$, i.e. $A_i \in \mathcal{L}(\mathcal{H}_i, \mathcal{K})$, and $b \in \mathcal{K}$ is a vector.

For the sake of notational simplicity, we use again the same abbreviations as in the introductory Section 1.1:

$$\begin{aligned} &\mathcal{H} := \mathcal{H}_1 \times \cdots \times \mathcal{H}_N, \qquad \mathcal{X} := \mathcal{X}_1 \times \ldots \times \mathcal{X}_N \subseteq \mathcal{H} \\ &x := \big(x_1, \dots, x_N\big) \in \mathcal{H}, \qquad f(x) := \sum_{i=1}^{N} f_i(x_i), \qquad Ax := \sum_{i=1}^{N} A_i x_i. \end{aligned} \tag{3.1}$$

Canonically, $\mathcal{H}$ becomes a Hilbert space with the scalar product $\langle x \mid y \rangle := \langle x_1 \mid y_1 \rangle + \cdots + \langle x_N \mid y_N \rangle$; the scalar product in the space $\mathcal{H} \times \mathcal{K}$ is defined analogously.

First, we recall the well-known notion of KKT pairs.

Definition 3.1. Let the given assumptions for the problem (Opt) hold. A pair $(x^*, \mu^*) \in \mathcal{X} \times \mathcal{K}$ is called a *KKT point* of (Opt) if it satisfies the following *KKT conditions*: $0 \in \partial f(x) + A^*\mu + N_{\mathcal{X}}(x)$ and $0 = b - Ax$, where $A^* : \mathcal{K} \to \mathcal{H}$ denotes the Hilbert space adjoint of A.

Note that a KKT point has to be feasible with respect to the abstract constraints $\mathcal{X}$, whereas they exploit the existence of a multiplier for the equality constraints. This setting is useful for our ADMM-type method where only the linear constraints are penalized, whereas the abstract constraints remain unchanged.

Our aim is to compute a KKT point of the optimization problem (Opt). In many cases, this is equivalent to finding a solution of the minimization problem itself. More precisely, the KKT conditions are always sufficient optimality conditions for convex problems, cf. [18, Prop. 3.3], whereas the necessary part usually requires some constraint qualifications; for example, $b \in \operatorname{sri} A(\mathcal{X})$, see [9, Prop. 27.14], where sri denotes the strong relative interior, see [9, Def. 6.9]. In the finite-dimensional case, the condition $\mathcal{Y} \cap \operatorname{int}(\mathcal{X}) \neq \emptyset$ would be enough for the KKT conditions to be necessary optimality conditions, where $\mathcal{Y} := \{x \mid Ax = b\}$, cf. [96, Cor. 28.2.2] for a more detailed discussion. This constraint qualification holds, in particular, if $\mathcal{X}_i = \mathcal{H}_i$ for all $i = 1, \dots, N$.

In order to rewrite the KKT conditions in a more compact form, let us further introduce the notation

$$\begin{aligned} \mathcal{W} &:= \mathcal{X}_1 \times \ldots \times \mathcal{X}_N \times \mathcal{K}, \\ w &:= \big(x_1, \ldots, x_N, \mu\big), \\ f(w) &:= f(x), \end{aligned} \tag{3.2}$$

where the last expression simply means that, depending on the argument, we either view f as a mapping depending on x only, or depending on the full primal-dual variable $w = (x, \mu)$. Therefore, for the corresponding subdifferentials (with respect to w and x, respectively), depending on the corresponding arguments, we have

$$\partial f(w) = \begin{pmatrix} \partial f(x) \\ \{0\} \end{pmatrix}, \tag{3.3}$$

since f is independent of μ. Finally, let us define $G : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \times \mathcal{K}$,

$$G(w) := \begin{pmatrix} A_1^* \mu \\ A_2^* \mu \\ \vdots \\ A_N^* \mu \\ b - \sum_{i=1}^N A_i x_i \end{pmatrix}. \tag{3.4}$$

The particular structure of G combined with Proposition 2.40 immediately yields the following result.

Lemma 3.2. *Under the given assumptions for the problem* (Opt)*, the mapping G as defined in* (3.4) *satisfies $\big\langle G(w) - G(\bar{w}) \mid w - \bar{w} \big\rangle = 0$ for all $w, \bar{w} \in \mathcal{W}$; in particular, G is a continuous, monotone, and therefore maximally monotone operator.*

The above notation yields the following compact representation of the KKT conditions.

Lemma 3.3. *Under the given assumptions for the problem* (Opt)*, the vector pair $w^* = (x^*, \mu^*) \in \mathcal{X} \times \mathcal{K}$ is a KKT point of* (Opt) *if and only if $w^* \in \mathcal{W}^*$, where $\mathcal{W}^* := \big\{ w \in \mathcal{H} \times \mathcal{K} \mid 0 \in \partial f(w) + G(w) + N_{\mathcal{W}}(w) \big\}$.*

Proof. The proof follows immediately from the previous definitions, taking into account that, due to the Cartesian structure of $\mathcal{W}$, we have

$$N_{\mathcal{W}}(w) = N_{\mathcal{X}_1}(x_1) \times \ldots \times N_{\mathcal{X}_N}(x_N) \times N_{\mathcal{K}}(\mu)$$

and $N_{\mathcal{K}}(\mu) = \{0\}$ since $\mathcal{K}$ is the entire space. Since $N_{\mathcal{W}}(w) = \emptyset$ if and only if $w \notin \mathcal{W}$, it follows that $w \in \mathcal{W} = \mathcal{X} \times \mathcal{K}$. □

Let us define the multifunction

$$T_{Opt}(w) := \partial f(w) + G(w) + N_{\mathcal{W}}(w), \tag{3.5}$$

whose domain is obviously given by the non-empty set $\mathcal{W}$. Then the set $\mathcal{W}^*$ from Lemma 3.3 can be expressed as $\mathcal{W}^* = \{w \in \mathcal{W} \mid 0 \in T_{Opt}(w)\}$. This indicates that the set-valued mapping T_{Opt} plays a central role in our analysis. Its most important property is formulated in the following result.

Proposition 3.4. *Under the given assumptions for the problem* (Opt)*, the set-valued function T_{Opt} defined in* (3.5) *is maximally monotone.*

Proof. Since G is a continuous, monotone function, in view of Lemma 3.2 and $\operatorname{dom} G = \mathcal{H} \times \mathcal{K}$, it follows that $\mathcal{B} := G + N_{\mathcal{W}}$ is a maximally monotone operator, see e.g. Proposition 2.40 and Proposition 2.42. Because $\mathcal{X}$ is non-empty, it follows straight that $\operatorname{dom}(N_{\mathcal{W}}) \neq \emptyset$, and therefore $\operatorname{dom}(\mathcal{B}) \neq \emptyset$. Furthermore, since f is a real-valued, convex function, it is also known by Proposition 2.39 that $\mathcal{A} := \partial f(w)$ is maximally monotone. Since $\operatorname{dom}(\mathcal{A}) = \operatorname{dom}(\partial f) = \mathcal{H} \times \mathcal{K}$, see Proposition 2.19 and $\operatorname{dom}(\mathcal{B}) \neq \emptyset$, it follows again from Proposition 2.42 that $T_{Opt} = \partial f + G + N_{\mathcal{W}} = \mathcal{A} + \mathcal{B}$ is also maximally monotone. □

Another way to verify the maximal monotonicity of T_{Opt} is through the maximal monotonicity of the convex-concave subdifferential, cf. [2].

As already said, the existence of Lagrange multipliers can only be guaranteed through a constraint qualification. The next theorem shows that the existence of an approximating KKT sequence is a necessary optimality condition without the need for any additional constraint qualification. It is a special case of a result presented in [22] and it is used later to investigate the convergence properties of an ADMM-method in the absence of Lagrange multipliers or even solutions.

Theorem 3.5. *Suppose that the given assumptions for the problem* (Opt) *hold and that the problem* (Opt) *admits a solution $\bar{x}$. Then there is a sequence $\{(x^k, \lambda^k)\}_{k\in\mathbb{N}} \subset (\mathcal{H}, \mathcal{K})$ such that*

$$\begin{aligned} \varepsilon_1^k \in \partial f(x^k) + A^*\lambda^k + N_{\mathcal{X}}(x^k) \\ \varepsilon_2^k = Ax^k - b \\ x^k \in \mathcal{X}, \end{aligned}$$

where $(\varepsilon_1^k, \varepsilon_2^k) =: \varepsilon^k \to 0$ in $\mathcal{H} \times \mathcal{K}$ and $x^k \to \bar{x}$.

Proof. Let $r > 0$ be arbitrary. Since the problem is convex and $\bar{x}$ is a minimizer of f, we know that $\bar{x}$ locally minimizes f on $B_r(\bar{x}) \cap \{x \in \mathcal{X} \mid Ax = b\}$. This

localization of the problem is necessary because later in this proof we need certain iterates to be bounded. For $k \in \mathbb{N}$, consider the problem

$$\min_{x \in \mathcal{H}} f(x) + \|x - \bar{x}\|_{\mathcal{H}}^2 + k\|Ax - b\|_{\mathcal{K}}^2 \quad \text{s.t.} \quad x \in B_r(\bar{x}) \cap \mathcal{X}. \tag{3.6}$$

We notice that the squared norm and the composition of the squared norm with a linear function are continuous and convex, and therefore weakly sequentially lower semi-continuous. Further we assumed f to be convex and lsc, thus it is also weakly sequentially lower semi-continuous. Hence, the objective function of the above problem (3.6) is weakly sequentially lower semi-continuous. By the reflexivity of $\mathcal{H}$ we obtain a solution of the problem (3.6). Passing to a subsequence if necessary, we may assume that $x^k \rightharpoonup \hat{x}$ for some $\hat{x}$; by the convexity and closedness of $B_r(\bar{x}) \cap \mathcal{X}$ we see $\hat{x} \in B_r(\bar{x}) \cap \mathcal{X}$. Observe now that

$$f(x^k) + \|x^k - \bar{x}\|_{\mathcal{H}}^2 + k\|Ax - b\|^2 \leq f(\bar{x}) \tag{3.7}$$

for all k by the minimizing property of x^k. By the weak sequential lower semi-continuity we see that $f(x^k)$ is bounded. Dividing the last equation by k and taking the limit $k \to \infty$, it follows by the weak sequential lower semi-continuity of the squared norm that $\|A\hat{x} - b\|_{\mathcal{K}}^2 = 0$, i.e. $\hat{x}$ is feasible. By (3.7), we also obtain $f(\hat{x}) + \|\hat{x} - \bar{x}\|_{\mathcal{H}}^2 \leq f(\bar{x})$. But $f(\bar{x}) \leq f(\hat{x})$, hence $\hat{x} = \bar{x}$ and (3.7) implies that $x^k \to \bar{x}$. In particular, we have $\|x^k - \bar{x}\|_{\mathcal{H}} < r$ for sufficiently large k, and from (3.6) we obtain with $\varepsilon_1^k := -2(x^k - \bar{x}) \to 0$ that

$$0 \in \partial f(x^k) - \varepsilon_1^k + 2kA^*(Ax - b) + N_{\mathcal{X}}(x^k).$$

Moreover, define the sequences $\lambda^k := 2k(Ax^k - b)$ and $\varepsilon_2^k := Ax^k - b$. □

Notice that the sequence $\{\lambda^k\}_{k \in \mathbb{N}}$ is possibly unbounded. This is in particular the case if there is no KKT pair in $\mathcal{H} \times \mathcal{K}$.

3.2 Linearly Constrained Generalized Nash Equilibrium Problems

We consider the generalized Nash equilibrium problem (GNEP) with N players ν, where the optimization problem of player ν is given by

$$\min_{x_\nu \in \mathcal{H}_\nu} \theta_\nu(x_\nu, x_{-\nu}) + \varphi_\nu(x_\nu) \quad \text{s.t.} \quad \sum_{\nu=1}^{N} A_\nu x_\nu = b, \quad x_\nu \in \mathcal{X}_\nu \tag{GNEP}$$

or, more generally,

$$\min_{x_\nu \in \mathcal{H}_\nu} \theta_\nu(x_\nu, x_{-\nu}) + \varphi_\nu(x_\nu) \quad \text{s.t.} \quad \sum_{\nu=1}^{N} B_\nu x_\nu - b \in \mathcal{C}, \quad x_\nu \in \mathcal{X}_\nu \qquad (\text{GNEP}_{conic})$$

for all $\nu = 1, \ldots, N$. Here and throughout this whole thesis, $\mathcal{H}_\nu$ and $\mathcal{K}$ are given Hilbert spaces, $\varphi_\nu : \mathcal{H}_\nu \to \mathbb{R}$ are proper, convex, and lower semi-continuous functions, $\theta_\nu : \mathcal{H}_1 \times \cdots \times \mathcal{H}_N \to \mathbb{R}$ are continuously Fréchet-differentiable with $\theta_\nu(\cdot, x_{-\nu})$ being convex for any fixed $x_{-\nu}$, $\mathcal{X}_\nu \subset \mathcal{H}_\nu$ are non-empty, closed, and convex sets, $\mathcal{C}$ is a non-empty, closed, convex cone, $A_\nu, B_\nu \in \mathcal{L}(\mathcal{H}_\nu, \mathcal{K})$, and $b \in \mathcal{K}$. Following standard notation in Nash games, we write $x = (x_\nu, x_{-\nu})$, where $x_{-\nu}$ subsumes all the remaining block components x_i with $i \neq \nu$. This notation is used to emphasize the particular role played by the block component x_ν within the entire vector x and does not mean that the components of x are re-ordered. In particular, we therefore have $x = (x_\nu, x_{-\nu}) = \big(x_1, \ldots, x_N\big)$ and, similarly, $(y_\nu, x_{-\nu}) = (x_1, \ldots, x_{\nu-1}, y_\nu, x_{\nu+1}, \ldots, x_N)$.

We assume that the generalized Nash equilibrium problems (GNEP) and (GNEP_{conic}) have non-empty feasible sets. Since we have explicit constraints $\mathcal{X}_\nu$, there is essentially no loss of generality in assuming that θ_ν and φ_ν real-valued for all $\nu = 1, \ldots, N$.

For the sake of notational simplicity, we use the abbreviations

$$\begin{aligned} \mathcal{H} &:= \mathcal{H}_1 \times \cdots \times \mathcal{H}_N, & \mathcal{X} &:= \mathcal{X}_1 \times \ldots \times \mathcal{X}_N \subseteq \mathcal{H}, \\ x &:= \big(x_1, \ldots, x_N\big) \in \mathcal{H}, & Ax &:= \sum_{\nu=1}^{N} A_\nu x_\nu, \\ Bx &:= \sum_{\nu=1}^{N} B_\nu x_\nu, & \varphi(x) &:= \sum_{\nu=1}^{N} \varphi_\nu(x_\nu), \end{aligned} \qquad (3.8a)$$

and, depending on the problem, we use either

$$\mathcal{F} := \{x \in \mathcal{X} \mid Ax = b\} \qquad (3.8b)$$

for (GNEP) or

$$\mathcal{F} := \{x \in \mathcal{X} \mid Bx - b \in C\}, \qquad (3.8c)$$

for (GNEP_{conic}); which $\mathcal{F}$ is meant is clear from the context. As above, $\mathcal{H}$ becomes a Hilbert space with the scalar product $\langle x \mid y \rangle := \langle x_1 \mid y_1 \rangle + \cdots + \langle x_N \mid y_N \rangle$; the scalar product in the space $\mathcal{H} \times \mathcal{K}$ is defined analogously. The symbol $\|\cdot\|$ always denotes the norm induced by the corresponding scalar product (in $\mathcal{H}_\nu, \mathcal{H}, \mathcal{K}$, or

$\mathcal{H} \times \mathcal{K}$); it should be clear from the context which norm is used. We omit indexing the norms again in order to keep the notation simple.

In this work, we consider a special kind of solution of the problem (GNEP). To this end, let

$$\Psi(x, y) := \sum_{\nu=1}^{N} \big(\theta_\nu(x_\nu, x_{-\nu}) + \varphi_\nu(x_\nu) - \theta_\nu(y_\nu, x_{-\nu}) - \varphi_\nu(y_\nu)\big)$$

be the *Nikaido-Isoda-function* of (GNEP). Then $x^* \in \mathcal{H}$ is called a *normalized equilibrium* or a *variational equilibrium* of (GNEP) if $\sup_{y \in \mathcal{F}} \Psi(x^*, y) = 0$. Following [46], for example, it is not difficult to see that every variational equilibrium is a generalized Nash equilibrium of the GNEP.

We also introduce the *pseudo-gradient* $\widehat{P}_\theta : \mathcal{H} \to \mathcal{H}$ of the functions θ_ν from (GNEP) as

$$\widehat{P}_\theta(x) := \begin{pmatrix} \nabla_{x_1} \theta_1(x_1, x_{-1}) \\ \vdots \\ \nabla_{x_N} \theta_N(x_N, x_{-N}) \end{pmatrix}. \tag{3.9}$$

Further note that the definition of φ yields

$$\partial\varphi(x) = \begin{pmatrix} \partial_{x_1} \varphi_1(x_1) \\ \vdots \\ \partial_{x_N} \varphi_N(x_N) \end{pmatrix}. \tag{3.10}$$

If all φ_ν are differentiable, we notice that $\partial\varphi = \nabla\varphi = \widehat{P}_\varphi$.

This notation allows us to extend a known result from finite-dimensional GNEPs (see, e.g. [46]) to our Hilbert space setting.

Theorem 3.6. *Under the given assumptions for the problem* (GNEP) *(or* (GNEP_{conic})*), it holds that* x^* *is a variational equilibrium of* (GNEP) *(or* (GNEP_{conic})*) if and only if* $0 \in \partial\varphi(x^*) + \widehat{P}_\theta(x^*) + N_{\mathcal{F}}(x^*)$.

Proof. By definition, x^* is a variational equilibrium of (GNEP) (or (GNEP_{conic})) if and only if $\sup_{y \in \mathcal{F}} \Psi(x^*, y) = 0$ or, equivalently, if $\Psi(x^*, y) \leq 0$ for all $y \in \mathcal{F}$. In turn, this means that x^* solves the problem

$$\min_{y \in \mathcal{F}} \sum_{\nu=1}^{N} \big(\theta_\nu(y_\nu, x^*_{-\nu}) + \varphi(y_\nu)\big). \tag{3.11}$$

Since the objective function is convex as a mapping of y, it follows, using the notation introduced before, that (3.11) is equivalent to $0 \in \partial\varphi(x^*) + \widehat{P}_\theta(x^*) + N_{\mathcal{F}}(x^*)$. □

Note that the previous result remains true for more general convex sets $\mathcal{F}$, not necessarily given as in our framework.

3.2.1 Linear Equality Constrained GNEPs

Under certain regularity conditions, we can characterize the normal cone $N_{\mathcal{F}}(x^*)$ from Theorem 3.6. This leads to a particular notion of a KKT point for which we use the following terminology.

Definition 3.7. Let the given assumptions for problem (GNEP) hold. A pair $(x^*, \mu^*) \in \mathcal{X} \times \mathcal{K}$ is called a *variational KKT point* of (GNEP) if it satisfies the following *KKT-type conditions*: $0 \in \partial\varphi(x) + \widehat{P}_\theta(x) + A^*\mu + N_{\mathcal{X}}(x)$ and $0 = b - Ax$.

As in the optimization case, note that a variational KKT point has to be feasible with respect to the abstract constraints $\mathcal{X}$, whereas it exploits the existence of a multiplier for the equality constraints. This setting is useful for our ADMM-type method where only the linear constraints are penalized, whereas the abstract constraints remain unchanged.

The following result clarifies the relation between variational equilibria and variational KKT points of problem (GNEP).

Theorem 3.8. *Under the given assumptions for the problem* (GNEP)*, the following statements hold:*

(a) *If* $(x^*, \mu^*) \in \mathcal{H} \times \mathcal{K}$ *is a variational KKT pair of* (GNEP)*, then* x^* *is a variational equilibrium.*

(b) *Conversely, assume that* $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *has closed range, and that* $\operatorname{int} \mathcal{X} \cap \{x \in \mathcal{H} \mid Ax = b\} \neq \emptyset$*. If* $x^* \in \mathcal{F}$ *is a variational equilibrium of* (GNEP)*, then there exists a multiplier* μ^* *such that* $(x^*, \mu^*) \in \mathcal{H} \times \mathcal{K}$ *is a variational KKT pair of* (GNEP)*.*

Proof. Recall that $\mathcal{F} = \{x \in \mathcal{X} \mid Ax = b\} = \mathcal{X} \cap \mathcal{Y}$ with $\mathcal{Y} := \{x \mid Ax = b\}$ being the preimage of b under A. Since A is continuous, it follows that $\mathcal{Y}$ is closed and convex. By assumption, $\mathcal{X}$ is a closed, convex set as well. Moreover, an easy calculation shows that $N_{\mathcal{Y}}(x) = \operatorname{Ker}(A)^\perp = \overline{\operatorname{Range}(A^*)} \supseteq \operatorname{Range}(A^*)$ for any $x \in \mathcal{Y}$, where we used a result related to the Banach closed range theorem, cf. Theorem 2.3. Since the inclusion $N_{\mathcal{X}}(x) + N_{\mathcal{Y}}(x) \subseteq N_{\mathcal{F}}(x)$ holds for any $x \in \mathcal{F}$, statement (a) follows from

$$N_{\mathcal{X}}(x^*) + A^*\mu^* \subseteq N_{\mathcal{X}}(x^*) + \operatorname{Range}(A^*) \subseteq N_{\mathcal{X}}(x^*) + N_{\mathcal{Y}}(x^*) \subseteq N_{\mathcal{F}}(x^*)$$

together with Theorem 3.6. Statement (b) can be verified similarly by noting that, under the given assumptions, the equality $N_{\mathcal{X}}(x^*) + N_{\mathcal{Y}}(x^*) = N_{\mathcal{F}}(x^*)$ holds, cf. Proposition 2.20, and $\operatorname{Range}(A^*)$ is closed by the closed range theorem. □

In our subsequent algorithms for the solution of the GNEP from (GNEP), we compute a variational KKT point. Theorem 3.8 shows that this always yields a

variational equilibrium, and that this approach is actually equivalent to finding a variational equilibrium under a certain regularity condition.

In order to rewrite the variational KKT conditions in a more compact form, let us further introduce the notations

$$\begin{aligned} \mathcal{W} &:= \mathcal{X}_1 \times \ldots \times \mathcal{X}_N \times \mathcal{K}, \\ w &:= \big(x_1, \ldots, x_N, \mu\big), \\ \psi(w) &:= \varphi(x), \end{aligned} \tag{3.12}$$

where the last expression is just a formal re-definition of the mapping φ with the only difference being that ψ is viewed as a function of all variables w, whereas φ depends only on x. Hence

$$\partial\psi(w) \;=\; \begin{pmatrix} \partial\varphi(x) \\ \{0\} \end{pmatrix}, \tag{3.13}$$

where the corresponding subdifferentials are taken with respect to w and x, respectively. Moreover, we define the pseudo-gradient as a mapping of the whole vector $w = (x, \mu)$ by

$$P_\theta(w) := \begin{pmatrix} \widehat{P}_\theta(x) \\ \{0\} \end{pmatrix}. \tag{3.14}$$

Finally, let us define $G : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \times \mathcal{K}$ by

$$G(w) := \begin{pmatrix} A_1^*\mu \\ \vdots \\ A_N^*\mu \\ b - \sum_{\nu=1}^N A_\nu x_\nu \end{pmatrix}, \tag{3.15}$$

which is the same as in (3.4), only for the problem (GNEP). The particular structure of G immediately yields again the following result.

Lemma 3.9. *Under the given assumptions for the problem* (GNEP)*, the mapping G as defined in* (3.15) *satisfies $\big\langle G(w) - G(\bar{w}) \mid w - \bar{w} \big\rangle = 0$ for all $w, \bar{w} \in \mathcal{W}$; in particular, G is a continuous, monotone, and therefore maximally monotone operator.*

The above notation enables a compact representation of the variational KKT conditions.

Lemma 3.10. *Under the given assumptions for the problem* (GNEP)*, the vector pair $w^* = (x^*, \mu^*) \in \mathcal{H} \times \mathcal{K}$ is a variational KKT point of* (GNEP) *if and only if $w^* \in \mathcal{W}^*$, where $\mathcal{W}^* := \big\{ w \in \mathcal{H} \times \mathcal{K} \mid 0 \in \partial\psi(w) + P_\theta(w) + G(w) + N_{\mathcal{W}}(w) \big\}$.*

Proof. The proof follows from the previous definitions, taking into account that, due to the Cartesian structure of $\mathcal{W}$, we have $N_{\mathcal{W}}(w) = N_{\mathcal{X}_1}(x_1) \times \ldots \times N_{\mathcal{X}_N}(x_N) \times N_{\mathcal{K}}(\mu)$ and $N_{\mathcal{K}}(\mu) = \{0\}$ since $\mathcal{K}$ is the entire space. Since $N_{\mathcal{W}}(w) = \emptyset$ if and only if $w \notin \mathcal{W}$, it follows that $w \in \mathcal{W} = \mathcal{X} \times \mathcal{K}$. □

Applying the definitions of the normal cone and the convex subdifferential to the statement of Lemma 3.10, we obtain the following characterization of the variational KKT points.

Lemma 3.11. *Under the given assumptions for the problem* (GNEP)*, the vector pair $w^* = (x^*, \mu^*) \in \mathcal{X} \times \mathcal{K}$ is a variational KKT point of* (GNEP) *if and only if there is a $g^* \in \partial\psi(w^*)$ such that*

$$0 \;\leq\; \left\langle g^* + P_\theta(w^*) + G(w^*) \mid w - w^* \right\rangle$$

for all $w \in \mathcal{W}$.

Let us define the multifunctions

$$T_1(w) \;:=\; P_\theta(w), \tag{3.16a}$$
$$T_2(w) \;:=\; \partial\psi(w) + G(w) + N_{\mathcal{W}}(w) \tag{3.16b}$$

and

$$T_{GNEP}(w) \;:=\; T_1(w) + T_2(w). \tag{3.16c}$$

The domains of T_{GNEP} and T_2 are obviously given by the non-empty set $\mathcal{W}$, while the domain of $T_1 = P_\theta$ is the whole space $\mathcal{H} \times \mathcal{K}$. Then the set $\mathcal{W}^*$ from Lemma 3.10 can be expressed as $\mathcal{W}^* = \{w \in \mathcal{H} \times \mathcal{K} \mid 0 \in T_1(w) + T_2(w)\}$, i.e.

$$0 \in T_{GNEP}(w^*) \quad \Longleftrightarrow \quad w^* = (x^*, \mu^*) \text{ is a KKT point of (GNEP).} \tag{3.17}$$

This indicates that the set-valued mappings T_1 and T_2 play a central role in our analysis of algorithms for the problem (GNEP). Their most important properties are formulated in the following result.

Proposition 3.12. *Let the standing assumptions for the problem* (GNEP) *hold and suppose that the pseudo-gradient $\widehat{P}_\theta : \mathcal{H} \to \mathcal{H}$ defined in* (3.9) *is a monotone mapping. Then the pseudo-gradient $P_\theta : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \times \mathcal{K}$ as a mapping of the whole vector defined in* (3.14) *is monotone. Further the three set-valued functions $T_{GNEP}, T_1,$ and T_2 defined in* (3.16) *are maximally monotone.*

Proof. Given that $\widehat{P}_\theta$ is monotone, it follows immediately from the definition that P_θ is monotone.

Using the fact that P_θ is monotone, continuous, and single-valued, it is maximally monotone by Proposition 2.40. Since G is a continuous, monotone function

in view of Lemma 3.9 and $\operatorname{dom} G = \mathcal{H} \times \mathcal{K}$, it follows that $B := G + N_{\mathcal{W}}$ is a maximally monotone operator, see e.g. Proposition 2.42. Since the domain of the operator $A := \partial\psi$ is also equal to the entire space $\mathcal{H} \times \mathcal{K}$ (see Proposition 2.19) and $\operatorname{dom}(B) \neq \emptyset$, it follows again from Proposition 2.42 that $T_2 = \partial\psi + G + N_{\mathcal{W}} = A + B$ is maximally monotone as well. Similarly, it follows that T_{GNEP} is maximally monotone from $\operatorname{dom}(P_\theta) = \mathcal{H} \times \mathcal{K}$ and $\operatorname{dom} T_2 \neq \emptyset$. □

The monotonicity of P_θ or, equivalently, of the pseudo-gradient $\widehat{P}_\theta$, as required in the assumptions of Proposition 3.12, is a standard condition used in the context of GNEPs and typically represents a minimal assumption on the given GNEP in order to prove the convergence of suitable methods to a solution of the GNEP.

3.2.2 Generalization to GNEPs with Conic Constraints

So far, we only considered GNEPs of the form (GNEP) with joint linear equality constraints (linear both because of convexity reasons and in order to have a separable structure). We now want to extend the previous results to the more general class of GNEPs defined by (GNEP_{conic}) with some non-empty, closed, and convex cone $\mathcal{C}$. In particular, this scenario allows us to take into account linear inequalities, a situation that actually occurs in our application in Chapter 7.

The main idea is to transform the GNEP from (GNEP_{conic}) with conical constraints to a GNEP with linear equality constraints, and then extend our previous results to this reformulated problem. To this end, note that the *variational KKT conditions* of (GNEP_{conic}) are given by

$$0 \in \nabla_{x_\nu} \theta_\nu(x^*) + \partial\varphi_\nu(x_\nu^*) + B_\nu^* \lambda^* + N_{\mathcal{X}_\nu}(x_\nu^*), \quad \lambda^* \in N_{\mathcal{C}}\Big(\sum_{i=1}^N B_i x_i^* - b\Big)$$

for all $\nu = 1, \ldots, N$, see [18, Ch. 3.1]. Since $\mathcal{C}$ is a convex cone, the latter condition is equivalent to

$$\sum_{i=1}^N B_i x_i^* - b \in \mathcal{C}, \quad \lambda^* \in \mathcal{C}^\circ, \quad \Big\langle \lambda^* \mid \sum_{i=1}^N B_i x_i^* - b \Big\rangle = 0,$$

where $\mathcal{C}^\circ := \{\lambda \in \mathcal{K} \mid \langle \lambda \mid s \rangle \leq 0 \ \forall s \in \mathcal{C}\}$ denotes the polar cone of $\mathcal{C}$. Under suitable regularity assumptions, these KKT conditions are necessary and sufficient optimality conditions.

Next, we want to rewrite problem (GNEP_{conic}) as an equality constrained GNEP. There are different ways to do this, and in this section, we present two

such reformulations that are probably the most natural ones. The first reformulation uses the optimization problems

$$\min_{\substack{x_\nu \in \mathcal{X}_\nu \\ s_\nu \in \mathcal{C}}} \theta_\nu(x_\nu, x_{-\nu}) + \varphi_\nu(x_\nu) \quad \text{s.t.} \quad \sum_{i=1}^{N} B_i x_i - b - \sum_{i=1}^{N} s_i = 0 \tag{3.18}$$

for all players $\nu = 1, \ldots, N$. The second reformulation uses a GNEP where the first $N-1$ players ν deal with the optimization problems

$$\min_{x_\nu \in \mathcal{X}_\nu} \theta_\nu(x_\nu, x_{-\nu}) + \varphi_\nu(x_\nu) \quad \text{s.t.} \quad \sum_{i=1}^{N} B_i x_i - b - s = 0, \tag{3.19a}$$

whereas the minimization problem of the final player $\nu = N$ is given by

$$\min_{\substack{x_N \in \mathcal{X}_N \\ s \in \mathcal{C}}} \theta_N(x_N, x_{-N}) + \varphi_N(x_N) \quad \text{s.t.} \quad \sum_{i=1}^{N} B_i x_i - b - s = 0. \tag{3.19b}$$

Hence, the first reformulation (3.18) rewrites the conic constraints as an equality constraint by splitting the conic condition among all players, whereas the second reformulation (3.19) uses a slack variable s only for the last player. Note that the splitting used in the first approach is not unique in general. The precise relation between the conical constrained GNEP and these two formulations is discussed in the next two results.

Proposition 3.13. *Under the given assumptions for the problem* (GNEP_{conic})*, the following statements are equivalent:*

(a) $x^* = (x_1^*, \ldots, x_N^*)$ *is a generalized Nash equilibrium of* (GNEP_{conic}).

(b) $(x^*, s^*) = (x_1^*, \ldots, x_N^*, s_1^*, \ldots, s_N^*)$ *is a generalized Nash equilibrium of* (3.18) *for some* $s_\nu^* \in \mathcal{C}$, $\nu = 1, \ldots, N$.

(c) $(x^*, s^*) = (x_1^*, \ldots, x_N^*, s^*)$ *is a generalized Nash equilibrium of* (3.19) *for some* $s^* \in \mathcal{C}$.

Proof. Note that the objective functions of all three GNEPs are identical and independent of s or s_ν. Hence, the statement follows by noting that a feasible point of one GNEP yields a feasible point of the other GNEPs and vice versa.

(a) $\Longrightarrow$ (c): Suppose that x^* is a generalized Nash equilibrium of (GNEP_{conic}). Then set $s^* := \sum B_i x_i^* - b \in \mathcal{C}$. It follows that (x^*, s^*) is feasible for (3.19).

(c) $\Longrightarrow$ (b): Suppose that (x^*, s^*) is a Nash equilibrium of (3.19). Setting $s_\nu^* = 0$ for all $\nu = 1, \ldots, N-1$ and $s_N^* = s^*$ gives a feasible point of (3.18).

(b) $\Longrightarrow$ (a): Since C is a convex cone, it follows that $\sum_{i=1}^{N} s_i = N \sum_{i=1}^{N} (s_i/N) \in \mathcal{C}$, which implies statement (a). □

The previous result states that the reformulated GNEPs have the same solutions in the sense of a generalized Nash equilibrium as the original GNEP from (GNEP_{conic}). The following result shows that the corresponding sets of variational KKT points are also the same, which is of particular interest for our methods since they compute variational KKT points. To this end, recall the close relationship between variational KKT points and variational equilibria described in Theorem 3.8.

Proposition 3.14. *Under the given assumptions for the problem* (GNEP_{conic})*, the following statements are equivalent:*

(a) (x^*, λ^*) *is a variational KKT pair of* (GNEP_{conic})*.*

(b) $\big((x^*, s_1^*, \ldots, s_N^*), \lambda^*\big)$ *is a variational KKT pair of* (3.18) *for some* $s_\nu^* \in \mathcal{C}$, $\nu = 1, \ldots, N$*.*

(c) $\big((x^*, s^*), \lambda^*\big)$ *is a variational KKT pair of* (3.19) *for some* $s^* \in \mathcal{C}$*.*

Proof. (a) $\Longrightarrow$ (c): Let (x^*, λ^*) be a variational KKT pair of (GNEP_{conic}), and define $s^* := \sum_{i=1}^N B_i x_i^* - b$. Then

$$0 \in \nabla_{x_\nu} \theta_\nu(x^*) + \partial \varphi_\nu(x^*) + B_\nu^* \lambda^* + N_{\mathcal{X}_\nu}(x_\nu^*)$$

and

$$\lambda^* \in N_{\mathcal{C}}\Big(\sum_{i=1}^N B_i x_i^* - b\Big) = N_{\mathcal{C}}(s^*)$$

for all $\nu = 1, \ldots, N$. This can be rewritten as

$$0 \in \begin{pmatrix} \widehat{P}_\theta(x^*) + \partial \varphi(x^*) \\ 0 \end{pmatrix} + \begin{pmatrix} B^* \\ -I \end{pmatrix} \lambda^* + N_{\mathcal{X} \times \mathcal{C}}(x^*, s^*)$$

and

$$\sum_{\mu=1}^N B_\mu x_\mu^* - b - s^* = 0,$$

which are exactly the variational KKT conditions of (3.19).

(c) $\Longrightarrow$ (b): Suppose that $\big((x^*, s^*), \lambda^*\big)$ is a variational KKT pair of (3.19), and set $s_1^* := \ldots := s_N^* := \frac{1}{N} s^*$. Then it is easy to see that $\big((x^*, s_1^*, \ldots, s_N^*), \lambda^*\big)$ is a variational KKT pair of (3.18).

(b) $\Longrightarrow$ (a): Let $\big((x^*, s_1^*, \dots, s_N^*), \lambda^*\big)$ be a KKT pair of (3.18), i.e., it satisfies

$$0 \in \begin{pmatrix} \widehat{P}_\theta(x^*) + \partial\varphi(x^*) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} B^* \\ -I \\ \vdots \\ -I \end{pmatrix} \lambda^* + N_{\mathcal{X}\times\mathcal{C}\times\cdots\times\mathcal{C}}(x^*, s_1^*, \dots, s_N^*) \tag{3.20}$$

and $\sum_i B_i x_i^* - b - \sum_i s_i^* = 0$. Using (3.20) and $N_{\mathcal{X}\times\mathcal{C}\times\cdots\times\mathcal{C}}(x, s_1, \dots, s_N) = N_{\mathcal{X}}(x) \times N_{\mathcal{C}}(s_1) \times \cdots \times N_{\mathcal{C}}(s_N)$, we therefore obtain $\lambda^* \in N_{\mathcal{C}}(s_\nu^*)$ for all $\nu = 1, \dots, N$. Hence $\lambda^* \in \bigcap_{\nu=1}^N N_{\mathcal{C}}(s_\nu^*) \subset N_{\mathcal{C}}(\sum_{\nu=1}^N s_\nu^*) = N_{\mathcal{C}}(\sum_{\nu=1}^N B_\nu x_\nu^* - b)$. We therefore have

$$0 \in \widehat{P}_\theta(x^*) + \partial\varphi(x^*) + B^*\lambda^* + N_{\mathcal{X}}(x^*) \quad \text{and} \quad \lambda^* \in N_{\mathcal{C}}\Big(\sum_{i=1}^N B_i x_i^* - b\Big),$$

which are exactly the variational KKT conditions of (GNEP$_{conic}$). □

The previous results allow us to apply our ADMM-type algorithms that are introduced in Chapter 5 and 6 to the more general case of GNEPs with conic constraints. Formally, the corresponding objective functions, and therefore also the resulting mapping $\widehat{P}_\theta$, need to be regarded as functions of (x, s). Note that the latter is never strongly monotone, even if $\widehat{P}_\theta(x)$ is strongly monotone as a function of x alone. Fortunately, we only require a cocoercivity assumption in Chapter 5, and the cocoercivity of $\widehat{P}_\theta$ as a function of x immediately implies the same property of the operator $\widehat{P}_\theta$ as a function of (x, s). In Chapter 6, we require $\widehat{P}_\theta$ to be strongly monotone with respect to x_{-N}; therefore the formulation stated in (3.19) is still applicable.

Remark 3.15. In the last section, we rewrote the problem (GNEP$_{conic}$) as an equality constrained problem of the form (GNEP). In the Chapters 5 and 6, we develop methods for the problem (GNEP) where an upper estimate for the operator norms $\|A_\nu\|$ needs to be known. One should be aware that in the preceding section we changed the operators B_ν from problem (GNEP$_{conic}$) to either an operator $A_\nu(x_\nu, s_\nu) = B_\nu x_\nu - s_\nu$ in the case where we introduced N slack variables, or $A_\nu x_\nu = B_\nu x_\nu$ for $\nu = 1, \dots, N-1$ and $A_\nu(x_\nu, s) = B_\nu x_\nu - s$ in the case where we introduced one slack variable. In order to estimate the norm of these A_ν, we see

that

$$\begin{aligned}
\|A_\nu\| &= \sup_{(x_\nu,s_\nu)\neq 0} \frac{\|B_\nu u_\nu - s_\nu\|_{\mathcal{K}}}{\|(x_\nu,s_\nu)\|_{\mathcal{H}_\nu\times\mathcal{K}}} \\
&\leq \sup_{(x_\nu,s_\nu)\neq 0} \frac{\|B_\nu x_\nu\|_{\mathcal{K}} + \|s_\nu\|_{\mathcal{K}}}{\|(x_\nu,s_\nu)\|_{\mathcal{H}_\nu\times\mathcal{K}}} \\
&= \sup_{(x_\nu,s_\nu)\neq 0} \left(\frac{\|B_\nu x_\nu\|_{\mathcal{K}}}{\|(x_\nu,s_\nu)\|_{\mathcal{H}_\nu\times\mathcal{K}}} + \frac{\|s_\nu\|_{\mathcal{K}}}{\|(x_\nu,s_\nu)\|_{\mathcal{H}_\nu\times\mathcal{K}}} \right) \\
&\leq \sup_{(x_\nu,s_\nu)\neq 0} \left(\frac{\|B_\nu x_\nu\|_{\mathcal{K}}}{\|(x_\nu,s_\nu)\|_{\mathcal{H}_\nu\times\mathcal{K}}} \right) + \sup_{(x_\nu,s_\nu)\neq 0} \left(\frac{\|s_\nu\|_{\mathcal{K}}}{\|(x_\nu,s_\nu)\|_{\mathcal{H}_\nu\times\mathcal{K}}} \right) \\
&= \sup_{x_\nu\neq 0} \left(\frac{\|B_\nu x_\nu\|_{\mathcal{K}}}{\|x_\nu\|_{\mathcal{H}_\nu}} \right) + \sup_{s_\nu\neq 0} \left(\frac{\|s_\nu\|_{\mathcal{K}}}{\|s_\nu\|_{\mathcal{K}}} \right) \\
&= \|B_\nu\| + 1
\end{aligned}$$

in the first case, or $\|A_\nu\| = \|B_\nu\|$ for $\nu = 1,\ldots,N-1$ and $\|A_N\| \leq \|B_N\| + 1$ in the second case. ◇

Chapter 4

Regularized Jacobi-type ADMM-Methods for a Class of Separable Convex Optimization Problems

In this chapter, we introduce and analyze ADMM-type, parallel splitting algorithms for the optimization problem (Opt), which was introduced in Section 1.1 and further discussed in Section 3.1. This chapter is based on the results from [21]. We start with recalling the problem and its assumptions. The problem is given by

$$\min \sum_{i=1}^{N} f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^{N} A_i x_i = b, \quad x_i \in \mathcal{X}_i \ (i = 1, \dots, N). \tag{Opt}$$

As already mentioned in Chapter 3, here and in the whole thesis, $\mathcal{H}_i$ and $\mathcal{K}$ denote Hilbert spaces, $f_i : \mathcal{H}_i \to \mathbb{R}$ are lower semi-continuous, convex functions, $\mathcal{X}_i \subset \mathcal{H}_i$ are non-empty, closed, convex sets, $A_i \in \mathcal{L}(\mathcal{H}_i, \mathcal{K})$, and $b \in \mathcal{K}$. We assume that the optimization problem (Opt) has a non-empty feasible set. Note that all functions f_i are supposed to be convex, but none of them has to be strictly or uniformly convex. Furthermore, no differentiability of f_i is required. Since we have explicit constraints $\mathcal{X}_i$ for each mapping f_i, there is essentially no loss of generality in assuming that f_i is real-valued for every $i = 1, \dots, N$. The assumption that the f_i are real-valued can be circumvented by using a technical condition. Moreover, we do not assume the operators A_i to be injective or surjective, which is a condition that is often used in finite dimensions where the matrices A_i are assumed to have full rank.

For the sake of notational simplicity, we use again the canonical abbreviations from (3.1).

This chapter is organized as follows: Our regularized Jacobi-type ADMM-method is presented in Section 4.1 together with a more detailed discussion regarding some of the related algorithms. The corresponding global convergence analysis is given in Section 4.2. The main idea is to show that, after a linear transformation, the iterates are equal to the elements of a sequence generated by a proximal-point method in a suitable Hilbert space. This transformation is possible for the Jacobi-type iteration and is not directly applicable to the corresponding Gauss-Seidel-version of our approach. Motivated by the proximal-point interpretation of our algorithm, which only yields weak convergence of the iterates unless additional assumptions hold, we present a strongly convergent Halpern-type modification of the Jacobi-type ADMM-method in Section 4.3.

4.1 Regularized Jacobi-type ADMM-Method

The method we consider in this chapter is the following regularized Jacobi-type method for solving the optimization problem (Opt).

Algorithm 4.1. (Regularized Jacobi-type ADMM-Method)

(S.0) Choose a starting point $(x^0, \mu^0) \in \mathcal{X} \times \mathcal{K}$, parameters $\beta, \gamma > 0$, and set $k := 0$.

(S.1) If a suitable termination criterion is satisfied: STOP.

(S.2) For $i = 1, \ldots, N$, compute

$$x_i^{k+1} := \underset{x_i \in \mathcal{X}_i}{\arg\min} \Big\{ f_i(x_i) + \langle \mu^k \mid A_i x_i \rangle + \frac{\beta}{2} \big(\|A_i x_i + \sum_{l \neq i} A_l x_l^k - b\|^2 + \gamma \|x_i - x_i^k\|^2 \big) \Big\}. \tag{4.1}$$

(S.3) Define

$$\mu^{k+1} := \mu^k + \beta \Big(\sum_{l=1}^{N} A_l x_l^{k+1} - b \Big). \tag{4.2}$$

(S.4) Set $k \leftarrow k + 1$, and go to (S.1).

Throughout our convergence analysis, we implicitly assume that Algorithm 4.1 generates an infinite number of iterates. We further note that all subproblems (4.1) are strongly convex for all $i = 1, \ldots, N$ and all iterations $k \in \mathbb{N}$. Hence $x^{k+1} := \big(x_1^{k+1}, \ldots, x_N^{k+1}\big)$ is uniquely defined. Note that this is due to the quadratic regularization term, which does not occur in standard ADMM-methods for two or more components. These standard ADMM-methods are also Gaussian-type methods since they use the newer information $x_1^{k+1}, \ldots, x_{i-1}^{k+1}$ in the computation

of x_i^{k+1} in (4.1). For reasons that become clear during our convergence analysis, we use the above Jacobi-type ADMM-method with its known advantages and disadvantages.

The main computational overhead in Algorithm 4.1 emerges from solving the optimization subproblems in (S.2). However, in contrast to the augmented Lagrangian method discussed in Section 1.1, these subproblems are small-dimensional. Moreover, there are several applications where these subproblems can be solved analytically, in which case each iteration of the algorithm is extremely cheap.

In order to compare our method with some existing ones and to present some suitable modifications, let us denote the outcome of (S.2) also by $\hat{x}_i^k$, i.e.

$$\hat{x}_i^k := \operatorname*{arg\,min}_{x_i \in \mathcal{X}_i} \Big\{ f_i(x_i) + \langle \mu^k \mid A_i x_i \rangle + \frac{\beta}{2} \big(\|A_i x_i + \sum_{l \neq i} A_l x_l^k - b\|^2 + \gamma \|x_i - x_i^k\|^2 \big) \Big\}. \quad (4.3)$$

Furthermore, if there is no partial regularization, we denote the corresponding result by

$$\tilde{x}_i^k := \operatorname*{arg\,min}_{x_i \in \mathcal{X}_i} \Big\{ f_i(x_i) + \langle \mu^k \mid A_i x_i \rangle + \frac{\beta}{2} \big(\|A_i x_i + \sum_{l \neq i} A_l x_l^k - b\|^2 \big) \Big\}. \quad (4.4)$$

This allows us to make the following comments, where we mainly concentrate on some modified updates of the iterates x^k, but it should be clear that corresponding updates are then also needed for the multiplier μ^k.

Remark 4.2. In the following, we discuss some related algorithms from the existing literature; recall that the convergence proofs of all these methods are limited to the finite-dimensional case.

(a) The modified Jacobi-type ADMM-method suggested in [63] uses the iteration $x^{k+1} := x^k + \alpha_k(\tilde{x}^k - x^k)$ for some step size $\alpha_k > 0$. This step size is typically very small and can be computed by a formula in every iteration or is constant and explicitly given. We further note that the paper [63] requires all sub-matrices A_i to be of full column rank.

(b) Motivated by the previous comment, it might also be useful to rewrite Algorithm 4.1 as $x^{k+1} := x^k + \alpha_k(\hat{x}^k - x^k)$ for some step size $0 < c_l \leq \alpha_k \leq c_u < 2$, where c_l and c_u denote some positive constants. Obviously, Algorithm 4.1 corresponds to the case $\alpha_k = 1$, i.e. we can allow much larger step sizes than [63]. This does not automatically guarantee faster convergence, especially since we have the additional regularization term in our method, but indicates that there is some hope for a superior numerical behavior. We get back to this step size later.

(c) Recall that Algorithm 4.1 was already analyzed in [34] for the finite-dimensional case, whereas we deal with the Hilbert space setting and state some further results (e.g., strong convergence). Our results, based on a very simple technique of proof, therefore generalize those from [34]. Furthermore, as noted in (b), our approach also allows under- and overrelaxation of the iterates (x^k, μ^k), whereas the technique in [34] allows to introduce a step size $\tau \in (0, 2)$ only for the dual variable μ^k. The corresponding μ^k-update is

$$\mu^{k+1} = \mu^k + \tau\beta\Big(\sum_{i=1}^{N} A_i x_i^{k+1} - b\Big).$$

A simplified condition on the proximal constant γ_i for the i-th subproblem given in [34, Lem. 2.2] is

$$\gamma_i I \succ \big(\frac{N}{2-\tau} - 1\big) A_i^T A_i,$$

where $A \succ B$ means that $A - B$ is positive definite. In our approach, it is also possible to choose the proximal constant separately for every subproblem or even choose a different equivalent norm for the regularization as in [34], but the added value would be minimal compared to the notational inconvenience. Moreover, taking into account that, in the finite-dimensional case, the matrix $N \cdot \operatorname{diag}(A_1^T A_1, \ldots, A_N^T A_N) - A^T A$ is easily seen to be positive semi-definite, it follows that $(N-1) \cdot \operatorname{diag}(A_1^T A_1, \ldots, A_N^T A_N) \succeq A^T A - \operatorname{diag}(A_1^T A_1, \ldots, A_N^T A_N)$, where $B \succeq C$ means that $B - C$ is positive semi-definite. Consequently, for sufficiently large γ we have

$$\begin{aligned} 0 \;&\preceq\; \gamma I - (N-1) \cdot \operatorname{diag}(A_1^T A_1, \ldots, A_N^T A_N) \\ &\preceq\; \gamma I - (A^T A - \operatorname{diag}(A_1^T A_1, \ldots, A_N^T A_N)), \end{aligned}$$

hence the condition from [34, Lem. 2.2] implies our condition on γ that we introduce later in Section 4.2. Taking A as the identity matrix, we see that our criterion regarding the choice of γ can be indeed significantly weaker.

(d) Another modification of the Jacobi-type iteration is due to [43, 60, 108] and replaces the update from (4.4) by

$$\begin{aligned} \tilde{x}_i^k := \operatorname*{arg\,min}_{x_i \in \mathcal{X}_i} \Big\{ & f_i(x_i) + \langle \mu^k \mid A_i x_i \rangle \\ & + \frac{\beta}{2}\big(\|A_i x_i + \sum_{l \neq i} A_l x_l^k - b\|^2 + (N-1)\|A_i(x_i - x_i^k)\|^2\big) \Big\} \end{aligned}$$

which can also be re-interpreted as a partial regularization method involving the matrix A_i in the regularization term, cf. [60, Alg. 8.1], or as an application of the two function ADMM on a modified problem, see [43, 108]. Theorem 4.1 from [66] shows that these two different approaches yield the same algorithm. Consequently, this modification also requires a full rank assumption on each A_i in order to be well-defined and to get convergence of the iterates x^k.

(e) An algorithm introduced in [107] that is also basically parallel uses the scheme

$$\begin{aligned}
\hat{x}_1^k &:= \operatorname*{arg\,min}_{x_1 \in \mathcal{X}_1} \Big\{ f_1(x_1) + \langle \mu^k \mid A_1 x_1 \rangle + \frac{\beta}{2} \big(\|A_1 x_1 + \sum_{l \neq 1} A_l x_l^k - b\|^2 \big) \Big\}, \\
\hat{\mu}^k &:= \mu^k + \beta (A_1 \hat{x}_1^k + \sum_{l=2}^{N} A_l x_l^k), \\
\hat{x}_i^k &:= \operatorname*{arg\,min}_{x_i \in \mathcal{X}_i} \Big\{ f_i(x_i) + \langle \mu^k \mid A_i x_i \rangle \\
&\qquad + \frac{\beta}{2} \big(\|A_i x_i + A_1 \hat{x}_1^k + \sum_{\substack{l=2 \\ l \neq i}}^{N} A_l x_l^k \|^2 + \|x_i - x_i^k\|_{M_i}^2 \big) \Big\}
\end{aligned}$$

for all $i = 2, \ldots, N$, where the M_i are some positive definite matrices that satisfy a certain condition. The analysis carried out in [107], however, is completely different from ours and requires, similar to [34], a choice of certain parameters related to the matrices M_i. ◇

4.2 Convergence Analysis

The main idea of our convergence analysis is to interpret Algorithm 4.1, after a simple linear transformation, as a proximal-point method applied to a suitable inclusion problem in an appropriate Hilbert space.

To this end, let us introduce the linear operator $M \in \mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ that we define by

$$Mx := \Big(\sum_{\substack{l=1 \\ l \neq i}}^{N} A_i^* A_l x_l \Big)_{i=1}^{N} = \begin{pmatrix} \sum_{l=2}^{N} A_1^* A_l x_l \\ \vdots \\ \sum_{\substack{l=1 \\ l \neq i}}^{N} A_i^* A_l x_l \\ \vdots \\ \sum_{l=1}^{N-1} A_N^* A_l x_l \end{pmatrix}. \tag{4.5}$$

In finite dimensions, the representation matrix of M is given by

$$M := A^T A - \operatorname{diag}(A_1^T A_1, \dots, A_N^T A_N).$$

Further define $Q \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$ by

$$Q \begin{pmatrix} x \\ \mu \end{pmatrix} := \begin{pmatrix} \beta^2(\gamma x - Mx) \\ \mu \end{pmatrix}, \tag{4.6}$$

where β and γ denote the constants from Algorithm 4.1. In finite dimensions, the matrix representation of Q is

$$Q := \begin{pmatrix} \beta^2(\gamma I - M) & 0 \\ 0 & I \end{pmatrix}.$$

The following simple remark regarding some useful properties of Q plays a crucial role in our subsequent convergence analysis.

Remark 4.3. Since M from (4.5) is self-adjoint, it follows that Q from (4.6) is also self-adjoint. Moreover, for all $\gamma > 0$ sufficiently large (say $\gamma > \|M\|$), Q is also strongly monotone. This implies that, in this case, Q is both injective and surjective. Hence the inverse of $Q \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$ exists and is also a linear, continuous, and self-adjoint operator. ◇

We first estimate the norm of the operator M since this is required later.

Lemma 4.4. *It holds that* $\|M\| \leq (N-1) \max_{\nu=1,\dots,N}\{\|A_\nu\|^2\}$.

Proof. The definition of M yields

$$\begin{aligned}
|\langle Mx \mid x\rangle_{\mathcal{H}}| &= \Big|\sum_{\nu=1}^{N}\Big\langle \sum_{\substack{\mu=1\\ \mu\neq\nu}}^{N} A_\nu^* A_\mu x_\mu \mid x_\nu\Big\rangle_{\mathcal{H}_\nu}\Big| \leq \sum_{\nu=1}^{N}\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{N} |\langle A_\nu^* A_\mu x_\mu \mid x_\nu\rangle_{\mathcal{H}_\nu}| \\
&= \sum_{\nu=1}^{N}\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{N} |\langle A_\mu x_\mu \mid A_\nu x_\nu\rangle_{\mathcal{K}}| \overset{Young}{\leq} \sum_{\nu=1}^{N}\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{N} \frac{1}{2}\big(\|A_\nu x_\nu\|_{\mathcal{K}}^2 + \|A_\mu x_\mu\|_{\mathcal{K}}^2\big) \\
&= \sum_{\nu=1}^{N} \frac{N-1}{2}\|A_\nu x_\nu\|_{\mathcal{K}}^2 + \sum_{\nu=1}^{N}\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{N} \frac{1}{2}\|A_\mu x_\mu\|_{\mathcal{K}}^2 \\
&= \sum_{\nu=1}^{N} \frac{N-1}{2}\|A_\nu x_\nu\|_{\mathcal{K}}^2 + \sum_{\nu=1}^{N} \frac{N-1}{2}\|A_\nu x_\nu\|_{\mathcal{K}}^2 \\
&\leq (N-1)\sum_{\nu=1}^{N} \|A_\nu\|^2 \|x_\nu\|_{\mathcal{H}_\nu}^2 \leq (N-1) \max_{\nu=1,\dots,N}\{\|A_\nu\|^2\} \sum_{\nu=1}^{N} \|x_\nu\|_{\mathcal{H}_\nu}^2 \\
&= (N-1) \max_{\nu=1,\dots,N}\{\|A_\nu\|^2\}\|x\|_{\mathcal{H}}^2,
\end{aligned}$$

where the subscripts for the underlying operator norms are omitted for the sake of notational convenience. By (2.2) we have $\|M\| = \sup_{x\neq 0} \frac{|\langle Mx|x\rangle|}{\|x\|^2}$. We therefore obtain the desired estimate. □

Our next result contains a suitable reformulation for the optimality conditions of the subproblems stated in (4.1) and (4.2).

Lemma 4.5. *Under the given assumptions for the problem* (Opt)*, the vector* $w^{k+1} = \big(x^{k+1}, \mu^{k+1}\big)$ *computed in* (4.1) *and* (4.2) *is equivalently characterized by the inclusion*

$$0 \in \beta\partial f(w^{k+1}) + \beta G(w^{k+1}) + Q(w^{k+1} - w^k) + N_{\mathcal{W}}(w^{k+1}), \tag{4.7}$$

where G *is defined as in* (3.4) *and* Q *is defined as in* (4.6)*.*

Proof. Using the optimality conditions for the programs (4.1), it follows that x_i^{k+1} solves these programs if and only if $x_i = x_i^{k+1}$ satisfies the optimality condition

$$0 \in \partial_{x_i}\Big(f_i(x_i) + \langle \mu^k \mid A_i x_i\rangle + \frac{\beta}{2}\big(\|A_i x_i + \sum_{l\neq i} A_l x_l^k - b\|^2 + \gamma\|x_i - x_i^k\|^2\big)\Big) + N_{\mathcal{X}_i}(x_i)$$

for all $i = 1, \ldots, N$. This is equivalent to saying that there exist elements $g_i \in \partial f_i(x_i^{k+1})$ such that

$$-\Big(g_i + A_i^*\mu^k + \beta A_i^*(A_i x_i^{k+1} + \sum_{l\neq i} A_l x_l^k - b) + \beta\gamma(x_i^{k+1} - x_i^k)\Big) \in N_{\mathcal{X}_i}(x_i^{k+1})$$

for all $i = 1, \ldots, N$. By definition of the normal cone, this can be rewritten as

$$\Big\langle g_i + A_i^*\mu^k + \beta A_i^*\big(A_i x_i^{k+1} + \sum_{l\neq i} A_l x_l^k - b\big) + \beta\gamma(x_i^{k+1} - x_i^k) \mid x_i - x_i^{k+1}\Big\rangle \;\geq\; 0 \tag{4.8}$$

for all $x_i \in \mathcal{X}_i$ and all $i = 1, \ldots, N$. Using $\mu^{k+1} = \mu^k + \beta\big(\sum_{l=1}^N A_l x_l^{k+1} - b\big)$, the last inequality is equivalent to

$$\Big\langle g_i + A_i^*\mu^{k+1} + \beta A_i^*\big(\sum_{l\neq i} A_l(x_l^k - x_l^{k+1})\big) + \beta\gamma(x_i^{k+1} - x_i^k) \mid x_i - x_i^{k+1}\Big\rangle \;\geq\; 0$$

for all $x_i \in \mathcal{X}_i$ and all $i = 1, \ldots, N$. Exploiting the definition of M in (4.5), the Cartesian product structure of the set $\mathcal{X}$, and setting $\tilde{g} = (g_1, \ldots, g_N)$, this can be rewritten more compactly as

$$\Big\langle \tilde{g} + A^*\mu^{k+1} + \beta M(x^k - x^{k+1}) + \beta\gamma(x^{k+1} - x^k) \mid x - x^{k+1}\Big\rangle \;\geq\; 0$$

for all $x \in \mathcal{X}$. Since

$$\Big\langle \frac{1}{\beta}(\mu^{k+1} - \mu^k) + \big(b - \sum_{i=1}^{N} A_i x_i^{k+1}\big) \mid \mu - \mu^{k+1} \Big\rangle \;=\; 0 \qquad \forall \mu \in \mathcal{K}$$

in view of (4.2), the previous two formulas can be rewritten as

$$\Big\langle g + G(w^{k+1}) + \begin{pmatrix} \beta(\gamma I - M) & 0 \\ 0 & \frac{1}{\beta} I \end{pmatrix} (w^{k+1} - w^k) \mid w - w^{k+1} \Big\rangle \;\geq\; 0 \quad \forall w \in \mathcal{W},$$

where $g := \big(g_1, \ldots, g_N, 0\big)$. Multiplication with β and taking into account the definition of Q from (4.6) yields

$$\Big\langle \beta g + \beta G(w^{k+1}) + Q(w^{k+1} - w^k) \mid w - w^{k+1} \Big\rangle \;\geq\; 0 \quad \forall w \in \mathcal{W}.$$

Using the definition of the normal cone $N_{\mathcal{W}}$, we can express this as

$$0 \in \beta g + \beta G(w^{k+1}) + Q(w^{k+1} - w^k) + N_{\mathcal{W}}(w^{k+1}).$$

Since $g \in \partial f(w^{k+1})$, we see that this is equivalent to

$$0 \in \beta \partial f(w^{k+1}) + \beta G(w^{k+1}) + Q(w^{k+1} - w^k) + N_{\mathcal{W}}(w^{k+1}).$$

This completes the proof. □

In the following, we use the previous characterization of stationary points to get an equivalent procedure for the computation of the iterates $w^{k+1} := \big(x^{k+1}, \mu^{k+1}\big)$ from Algorithm 4.1. To this end, we assume throughout this chapter that Q is strongly monotone, which is always possible for sufficiently large (and computable) γ, cf. Remark 4.3.

We obtain the following alternative procedure for the computation of w^{k+1} from Algorithm 4.1.

Proposition 4.6. *Let the standing assumptions for the problem* (Opt) *hold and let T_{Opt} be as defined in* (3.5). *Assume that Q, defined in* (4.6), *is self-adjoint and strongly monotone. Given an iterate $w^k = \big(x^k, \mu^k\big)$, the next iterate $w^{k+1} := \big(x^{k+1}, \mu^{k+1}\big)$ generated by Algorithm 4.1 can equivalently be represented by the (single-valued) formula*

$$w^{k+1} := \big(I + \beta Q^{-1} T_{Opt}\big)^{-1} w^k, \tag{4.9}$$

where $Q^{-1}T_{Opt}$ and $\big(I + \beta Q^{-1}T_{Opt}\big)^{-1}$ are maximally monotone and $\big(I + \beta Q^{-1}T_{Opt}\big)^{-1}$ is firmly non-expansive in the Hilbert space $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle x \mid y \rangle_Q := \langle Qx \mid y \rangle$.

Proof. First recall that the iterate w^{k+1} computed by Algorithm 4.1 is uniquely defined. Furthermore, due to convexity, they are fully characterized by the optimality conditions (4.7) from Lemma 4.5. In order to rewrite these optimality conditions, recall that $N_{\mathcal{W}}(w^{k+1})$ is a cone, so we have $\beta N_{\mathcal{W}}(w^{k+1}) = N_{\mathcal{W}}(w^{k+1})$. The definition of the operator T_{Opt} from (3.5) therefore allows us to rewrite the inclusion from (4.7) as

$$\begin{aligned} 0 \in \beta T_{Opt}(w^{k+1}) + Q(w^{k+1} - w^k) &= \big(Q + \beta T_{Opt}\big)(w^{k+1}) - Qw^k \\ \iff \quad Qw^k &\in \big(Q + \beta T_{Opt}\big)(w^{k+1}) \\ \iff \quad w^k &\in \big(I + \beta Q^{-1} T_{Opt}\big)(w^{k+1}) \\ \iff \quad w^{k+1} &\in \big(I + \beta Q^{-1} T_{Opt}\big)^{-1} w^k. \end{aligned}$$

We claim that the last inclusion is actually an equation, from which we then obtain the assertion. To this end, recall that T_{Opt} is a maximally monotone operator in view of Proposition 3.4. That $\big(I + \beta Q^{-1} T_{Opt}\big)^{-1} w^k$ is single-valued as well as the rest of the claim now follows straight from Proposition 2.48. □

Note that Proposition 4.6 uses two different scalar products (and therefore two different induced norms) for our Hilbert space $\mathcal{H} \times \mathcal{K}$. In order to be able to apply the known convergence results of the proximal-point method, it is highly important in our setting that both strong and weak convergence are identical concepts in both settings. Formally, this is stated in Lemma 2.9. This result allows us to re-interpret Algorithm 4.1 as a generalized proximal-point method, thus it inherits its convergence properties from the known convergence properties of proximal-point methods.

Theorem 4.7. *Let the standing assumptions for the problem* (Opt) *hold. Suppose that Q, as defined in* (4.6)*, is self-adjoint, strongly monotone, and that there is at least one KKT point of the optimization problem* (Opt)*. Then the following statements hold:*

(a) *The sequence $\{w^k\}_{k\in\mathbb{N}} = \{(x^k, \mu^k)\}_{k\in\mathbb{N}}$ generated by Algorithm 4.1 converges weakly to a KKT point $w^\infty = (x^\infty, \mu^\infty)$ of* (Opt)*, i.e. $0 \in T_{Opt}(w^\infty)$, where x^∞ is a solution of the optimization problem* (Opt)*.*

(b) *It holds that $\|w^k - w^{k+1}\|^2 = O(1/k)$, in particular, $\|w^k - w^{k+1}\| \to 0$ for $k \to \infty$.*

(c) *We have $\operatorname{dist}^2(T_{Opt}(w^k), 0) = o(1/k)$ for $k \to \infty$ (rate of convergence to KKT-optimality).*

(d) *It holds that $Ax^k - b \to 0$ for $k \to \infty$ and $Ax^\infty = b$ (primal feasibility).*

(e) *It holds that $f(x^k) \to f^*$ for $k \to \infty$, where f^* is the optimal function value of* (Opt)*.*

(f) *We have* $f_i(x_i^k) \to f_i(x_i^\infty)$ *for* $k \to \infty$, *where* x_i^∞ *denotes the weak limit point of* x_i^k, *for all* $i = 1, \dots, N$.

(g) *If* f_i *is strongly convex, then* x_i^k *converges strongly to* x_i^∞ *for* $k \to \infty$.

Proof. (a) - (c) In Proposition 4.6 it was shown that the iterates generated by Algorithm 4.1 can be represented as $w^{k+1} = \big(I + \beta Q^{-1} T_{Opt}\big)^{-1} w^k$. By the maximal monotonicity of T_{Opt}, cf. Proposition 3.4, Theorem 2.50 shows that the iterates $w^{k+1} = (x^{k+1}, \mu^{k+1})$ converge weakly to a zero $w^\infty = (x^\infty, \mu^\infty)$ of T_{Opt}. Further, Theorem 2.50 also shows the convergence rate results (b) and (c). By Lemma 3.3, such zeros of T_{Opt} are KKT points of (Opt). Since $w^\infty = (x^\infty, \mu^\infty)$ is a KKT-pair, it is well known, or can be seen as in Theorem 3.8, that x^∞ is a solution of (Opt).

(d) By definition of μ^{k+1}, we have $Ax^{k+1} - b = (\mu^{k+1} - \mu^k)/\beta$. The first part of the assertion therefore follows from part (b). The second part follows from the fact that $w^\infty = (x^\infty, \mu^\infty)$ is a KKT point, see (a).

(e), (f) Statement (e) is a standard result; however, for the sake of completeness and since it is used to verify assertion (f), we include its proof here.

In view of (a), we may assume that $w^k \rightharpoonup w^\infty$ for some weak limit point $w^\infty = (x^\infty, \mu^\infty)$. Furthermore, using (b) and (c), we obtain

$$\begin{aligned} \|A_i x_i^{k+1} + \sum_{l \neq i} A_l x_l^k - b\| &\leq \|A_i(x_i^{k+1} - x_i^k)\| + \|\sum_{l=1}^N A_l x_l^k - b\| \\ &\leq \|A_i\| \|x_i^{k+1} - x_i^k\| + \|\sum_{l=1}^N A_l x_l^k - b\| \to 0. \end{aligned}$$

Since $\mathcal{X}$ is closed and convex, it is weakly sequentially closed, hence $x^\infty \in \mathcal{X}$. From $x^\infty, x^{k+1} \in \mathcal{X}$, (4.8), and the definition of the subdifferential, we obtain

$$\begin{aligned} &f_i(x_i^\infty) - f_i(x_i^{k+1}) \geq \\ &\quad \Big\langle A_i^* \mu^k + \beta A_i^* \big(A_i x_i^{k+1} + \sum_{l \neq i} A_l x_l^k - b\big) + \beta\gamma(x_i^{k+1} - x_i^k) \mid x_i^{k+1} - x_i^\infty \Big\rangle. \end{aligned}$$

Summation for $i = 1, \dots, N$ yields, taking into account that $x^{k+1} - x^k \to 0$, $A_i x_i^{k+1} + \sum_{l \neq i} A_l x_l^k - b \to 0$, and the boundedness of $x^\infty - x^{k+1}$,

$$\begin{aligned} f(x^\infty) &\geq f(x^{k+1}) + \langle A^* \mu^k \mid x^{k+1} - x^\infty \rangle + \varepsilon^k \\ &= f(x^{k+1}) + \langle \mu^k \mid Ax^{k+1} - Ax^\infty \rangle + \varepsilon^k \\ &= f(x^{k+1}) + \langle \mu^k \mid Ax^{k+1} - b \rangle + \varepsilon^k \\ &= f(x^{k+1}) + \tilde{\varepsilon}^k, \end{aligned}$$

where $\varepsilon^k, \tilde{\varepsilon}^k$ are certain sequences converging to zero. Since f is lsc, it is also weakly sequentially lsc, cf. Proposition 2.15, i.e. $\liminf_{k\to\infty} f(x^k) \geq f(x^\infty)$, and we therefore obtain

$$\limsup_{k\to\infty} f(x^{k+1}) \;\leq\; f(x^\infty) \;\leq\; \liminf_{k\to\infty} f(x^{k+1}).$$

Consequently, we have $f(x^k) \to f(x^\infty) = f^*$. The last equation holds because x^∞ is a minimizer of (Opt). This verifies statement (e). We next exploit that part to show assertion (f). To this end, first note that (e) together with the lsc property of all f_i implies

$$\begin{aligned} f(x^\infty) \;&=\; \sum_{l=1}^N f_l(x_l^\infty) = \sum_{l\neq i} f_l(x_l^\infty) + f_i(x_i^\infty) \\ &\leq\; \sum_{l\neq i} f_l(x_l^\infty) + \liminf_{k\to\infty} f_i(x_i^k) \;\leq\; \sum_{l=1}^N \liminf_{k\to\infty} f_l(x_l^k) \\ &\leq\; \liminf_{k\to\infty} \Big(\sum_{l=1}^N f_l(x_l^k) \Big) \;=\; \liminf_{k\to\infty} f(x^k) \;=\; \lim_{k\to\infty} f(x^k) \;=\; f(x^\infty), \end{aligned}$$

so that equality holds everywhere. In particular, it follows that $\liminf_{k\to\infty} f_i(x_i^k) = f_i(x_i^\infty)$. Together with (e), this further implies

$$\begin{aligned} \limsup_{k\to\infty} f_i(x^k) \;&\leq\; \limsup_{k\to\infty} \Big(\sum_{l=1}^N f_l(x_l^k) \Big) + \limsup_{k\to\infty} \Big(-\sum_{l\neq i} f_l(x_l^k) \Big) \\ &\leq\; f(x^\infty) - \sum_{l\neq i} \liminf_{k\to\infty} \Big(f_l(x_l^k) \Big) \;=\; f_i(x_i^\infty) \;=\; \liminf_{k\to\infty} f_i(x_i^k). \end{aligned}$$

This yields $f_i(x_i^k) \to f_i(x_i^\infty)$.

(g) By assertion (c) we have that $\operatorname{dist}(T_{Opt}(w^k), 0) \to 0$. Suppose that f_i is strongly convex. Then there exists a constant $\nu_i > 0$ such that

$$\begin{aligned} f_i(y_i) - f_i(x_i) \;&\geq\; \langle g_i(x_i) \mid y_i - x_i \rangle + \nu_i \|x_i - y_i\|^2, \\ f_i(x_i) - f_i(y_i) \;&\geq\; \langle g_i(y_i) \mid x_i - y_i \rangle + \nu_i \|x_i - y_i\|^2 \end{aligned}$$

for all $g_i(x_i) \in \partial f_i(x_i)$ and all $g_i(y_i) \in \partial f_i(y_i)$. Adding these inequalities yields

$$\langle g_i(x_i) - g_i(y_i) \mid x_i - y_i \rangle \;\geq\; 2\nu_i \|x_i - y_i\|^2. \tag{4.10}$$

Now let us take an element $v^k \in T_{Opt}(w^k) = T_{Opt}(x^k, \mu^k)$ such that $\|v^k - 0\| \leq \operatorname{dist}(0, T_{Opt}(w^k)) + 1/k$ for all $k \in \mathbb{N}$, which is always possible by definition of

the distance function. Recalling the definition of the operator T_{Opt} and using the separability of the function f, we see that this v^k has a representation of the form

$$v^k =: \begin{pmatrix} g_1(x_1^k) \\ \vdots \\ g_i(x_i^k) \\ \vdots \\ g_N(x_N^k) \\ 0 \end{pmatrix} + \begin{pmatrix} A_1^*\mu^k \\ \vdots \\ A_i^*\mu^k \\ \vdots \\ A_N^*\mu^k \\ b - Ax^k \end{pmatrix} + \begin{pmatrix} s_1(x_1^k) \\ \vdots \\ s_i(x_i^k) \\ \vdots \\ s_N(x_N^k) \\ 0 \end{pmatrix}$$

for certain elements $s_i(x_i^k) \in N_{\mathcal{X}_i}(x_i^k)$ and $g_i(x_i^k) \in \partial f_i(x_i^k)$. In view of assertion (a), we also have $0 \in T_{Opt}(w^\infty) = T_{Opt}(x^\infty, \mu^\infty)$. Then we obtain from the monotonicity of the normal cone operators together with (4.10) that

$$\begin{aligned}
&\left\langle v^k - 0 \mid \begin{pmatrix} x^k - x^\infty \\ \mu^k - \mu^\infty \end{pmatrix} \right\rangle \\
&= \sum_{l=1}^N \langle g_l(x_l^k) - g_l(x_l^\infty) \mid x_l^k - x_l^\infty \rangle + \sum_{l=1}^N \langle A_l^*\mu^k - A_l^*\mu^\infty \mid x_l^k - x_l^\infty \rangle \\
&\quad + \sum_{l=1}^N \langle s_l(x_l^k) - s_l(x_l^\infty) \mid x_l^k - x_l^\infty \rangle + \langle (b - Ax^k) - (b - Ax^\infty) \mid \mu^k - \mu^\infty \rangle \\
&\geq \langle g_i(x_i^k) - g_i(x_i^\infty) \mid x_i^k - x_i^\infty \rangle + \langle Ax^\infty - Ax^k \mid \mu^k - \mu^\infty \rangle \\
&\quad - \langle x^\infty - x^k \mid A^*\mu^k - A^*\mu^\infty \rangle \\
&\geq 2\nu_i \|x_i^k - x_i^\infty\|^2.
\end{aligned}$$

Since $\{v^k\}$ converges strongly to zero in view of (d), and $\{w^k\}$ is weakly convergent, the previous chain of inequalities shows that $x_i^k \to x_i^\infty$ (strongly). □

Remark 4.8. (a) As already seen in Proposition 4.6, the operator $(I + \beta Q^{-1}T)^{-1}$ is firmly non-expansive in a suitable Hilbert space. By the Krasnoselsky-Mann iteration for firmly non-expansive operators, see Section 2.3.3, we see that many statements of Theorem 4.7 remain true if we consider the more general iterative procedure

$$\begin{aligned}
x^{k+1} &:= (1 - \rho^k)x^k + \rho^k \hat{x}^k, \\
\mu^{k+1} &:= (1 - \rho^k)\mu^k + \rho^k \hat{\mu}^k,
\end{aligned}$$

where $\hat{x}^k$ and $\hat{\mu}^k$ denote the outcome of one iteration of Algorithm 4.1 and $\rho^k \in [0, 2]$ satisfies $\sum_{k=1}^\infty \rho^k(2 - \rho^k) = \infty$. The choice $\rho^k = 1$ corresponds to our algorithm, whereas $\rho^k < 1$ and $\rho^k > 1$ are often called under- and overrelaxation,

respectively. In view of our limited numerical experience, however, we do not obtain any benefit by taking $\rho^k \neq 1$.

(b) There exist several inexact versions of the proximal-point method in the literature, see for example [41,48,72,94]. The previous analysis clearly shows that it is also possible to exploit these inexact proximal-point methods in order to obtain inexact counterparts of Algorithm 4.1. The corresponding details are left to the reader.

(c) It requires just a minor change in the above proofs to use a regularization with a strongly monotone linear operator R_i, i.e. a regularization of the form $\|x_i - x_i^k\|_{R_i}^2$, as in [28,60,107], instead of the regularization $\gamma\|x_i - x_i^k\|^2$ in (4.3). However, these are technical details that do not provide any additional mathematical insight. $\diamond$

It is not difficult to see that the convergence theory in this section remains valid as long as the bounded operator Q is self-adjoint and strongly monotone. More specifically, if Q as defined in (4.6) is derived from another splitting-type scheme, we obtain the same interpretation as a proximal-point method, and therefore the method inherits its convergence properties. Unfortunately, Q being self-adjoint plays a central role here, as shown below in Example 4.9. This means that our convergence theory cannot be applied to the case where Algorithm 4.1 is replaced by a corresponding regularized Gauss-Seidel-type ADMM-method because the resulting counterpart of the matrix M (hence also Q) as given in (4.5) is, in general, not self-adjoint. But Q not being self-adjoint ruins all desired convergence properties. This is illustrated by the following example in the finite-dimensional setting.

Example 4.9. In order to show that the proximal-point algorithm can only be applied if the operator Q is self-adjoint, let us define

$$T := \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

First we notice that T is maximally monotone, cf. Proposition 2.40. Furthermore, an easy calculation shows that

$$d^T Q d = \frac{1}{2}(d_1 + d_2)^2 + \frac{1}{2}(d_2 + d_3)^2 + \frac{1}{2}(d_1 + d_3)^2 > 0$$

for all $d \neq 0$; hence Q is positive definite and therefore yields a strongly monotone operator. However, the matrix Q is not symmetric, so we do not have a self-adjoint operator here. The proximal-point method is given by

$$x^{k+1} = \left(\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x^k = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} x^k.$$

Further we see that

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

hence $x^{4k} = x^0$ for every $x^0 \in \mathbb{R}^3$ and all $k \in \mathbb{N}$. Thus, the method does not converge. ◇

Convergence in the Inconsistent Case

In the above convergence theory we assumed the existence of a KKT pair and therefore in particular of a solution. Now we want to investigate if it is possible to tell from the behavior of the iterates generated by Algorithm 4.1 whether or not there exists a solution of the optimization problem (Opt). Further we want to investigate the behavior in the case where a solution to the problem exists, but it does not satisfy the KKT conditions. This is often the case in the infinite-dimensional setting, because the desired Lagrange multiplier lies in a weaker space $Y^* \supset \mathcal{K}$ than the augmentation space $\mathcal{K}$. First, we want to see from the convergence behavior of Algorithm 4.1, if there is no solution. Thereafter, using additional assumptions, we deduce convergence results in the case there is a solution which is not a KKT point.

Even though it is well known that a part of the iterates generated by the classical ADMM-method or the Douglas-Rachford splitting diverges in the absence of a solution, see [42], the literature containing a more differentiated analysis of the behavior of ADMM-type methods in the absence of KKT pairs and solutions is quite limited and often assumes a quadratic objective function, see [5, 98] and references therein. To the best of our knowledge, the approach that we present in this section, using asymptotic KKT sequences from Theorem 3.5, is new. Further, it seems that convergence properties in the embedding context for ADMM-type methods were not analyzed before.

In Theorem 3.5 we have shown that if the problem (Opt) admits a minimizer in $\mathcal{H}$, then there is a sequence in $\mathcal{H} \times \mathcal{K}$ approximating the KKT conditions.

Recall that $(x^*, \mu^*) \in \mathcal{H} \times \mathcal{K}$ being a KKT pair is equivalent to (x^*, μ^*) being a fixed point of the operator $(I + \beta Q^{-1} T_{Opt})^{-1}$. The next theorem shows that the operator $(I + \beta Q^{-1} T_{Opt})^{-1}$ almost admits a fixed point, provided that there is a solution of the problem (Opt). Notice that we only need a solution but not necessarily a KKT point.

Theorem 4.10. *Let the standing assumptions for the problem* (Opt) *hold, let* T_{Opt} *be defined as in* (3.5)*, and* $Q \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$ *be defined as in* (4.6)*. Suppose that* Q *is*

self-adjoint and strongly monotone. If the problem (Opt) *admits a solution, then*

$$\inf_{w\in\mathcal{X}\times\mathcal{K}} \left\|w - (I+\beta Q^{-1}T_{Opt})^{-1}w\right\| = 0.$$

Proof. Setting $w^k := (x^k, \lambda^k)$ and using Theorem 3.5, we find $\varepsilon^k \in T_{Opt}(w^k)$ with $\varepsilon^k \to 0$; thus

$$\begin{aligned}\varepsilon^k \in T_{Opt}(w^k) &\iff w^k + \beta Q^{-1}\varepsilon^k \in (I+\beta Q^{-1}T_{Opt})w^k \\ &\iff w^k \in (I+\beta Q^{-1}T_{Opt})^{-1}(w^k + \beta Q^{-1}\varepsilon^k).\end{aligned}$$

Since the resolvent $(I+\beta Q^{-1}T_{Opt})^{-1}$ of the maximally monotone operator $Q^{-1}T_{Opt}$ is uniquely defined, we even have

$$w^k = (I+\beta Q^{-1}T_{Opt})^{-1}(w^k + \beta Q^{-1}\varepsilon^k).$$

Thus we obtain

$$\begin{aligned}&\|w^k - (I+\beta Q^{-1}T_{Opt})^{-1}w^k\|_Q \\ &= \|(I+\beta Q^{-1}T_{Opt})^{-1}(w^k + \beta Q^{-1}\varepsilon^k) - (I+\beta Q^{-1}T_{Opt})^{-1}w^k\|_Q \\ &\le \|(w^k + \beta Q^{-1}\varepsilon^k) - w^k\|_Q \\ &= \|\beta Q^{-1}\varepsilon^k\|_Q.\end{aligned}$$

In the inequality, we used the firm non-expansiveness of $(I+\beta Q^{-1}T_{Opt})^{-1}$, see Proposition 4.6. Since the standard norm and the norm induced by Q are equivalent (see Lemma 2.9), we obtain from $\varepsilon^k \to 0$ that $\|x^k - (I+\beta Q^{-1}T_{Opt})^{-1}x^k\| \to 0$. This shows the claim. □

Theorem 3.5 shows that if we have a solution, we have a sequence (x^k, μ^k) approximating the KKT conditions. From this we have deduced in Theorem 4.10 that the operator $(I+\beta Q^{-1}T_{Opt})^{-1}$ almost admits a fixed point. Now Theorem 2.34 shows that we can expect certain properties of the classical fixed-point iteration. This together yields the next result about Algorithm 4.1.

Theorem 4.11. *Let the standing assumptions for the problem* (Opt) *hold, and let the sequence $(x^k, \mu^k) \in \mathcal{H}\times\mathcal{K}$ be generated by Algorithm 4.1, with $\gamma > 0$ chosen such that Q, defined in* (4.6)*, is strongly monotone and self-adjoint. Then the following statements hold:*

(a) *If there is a $c > 0$ and a subsequence $\mathcal{I} \subset \mathbb{N}$ such that $\|x^{k+1} - x^k\|_{\mathcal{H}}^2 + \|\mu^{k+1} - \mu^k\|_{\mathcal{K}}^2 \ge c$ for all $k \in \mathcal{I}$, then there is no solution of* (Opt).

(b) *If there is a solution of* (Opt)*, then $x^{k+1} - x^k \to 0$ in $\mathcal{H}$ and $\frac{1}{\beta}(\mu^{k+1} - \mu^k) = Ax^{k+1} - b \to 0$ in $\mathcal{K}$. Further, every weak accumulation point of x^{k+1} is feasible.*

Proof. (a) Set $F := (I + \beta Q^{-1} T_{Opt})^{-1}$ in Theorem 2.34, now Proposition 4.6 shows that

$$\begin{aligned} \|x^{k+1} - x^k\|_{\mathcal{X}}^2 + \|\mu^{k+1} - \mu^k\|_{\mathcal{K}}^2 &= \|(x^{k+1}, \mu^{k+1}) - (x^k, \mu^k)\|_{\mathcal{H}\times\mathcal{K}}^2 \\ &= \|w^{k+1} - w^k\|_{\mathcal{H}\times\mathcal{K}}^2 \\ &= \|F(w^k) - w^k\|_{\mathcal{H}\times\mathcal{K}}^2 \\ &\to \inf_{w\in\mathcal{X}\times K} \|(I-F)w\|_{\mathcal{H}\times\mathcal{K}}^2. \end{aligned} \tag{4.11}$$

If $\|x^{k+1} - x^k\|^2 + \|\mu^{k+1} - \mu^k\|^2 \geq c > 0$ on a subsequence, it follows from the above convergence that $\inf_{w\in\mathcal{X}\times K} \|(I-F)w\|_{\mathcal{H}\times\mathcal{K}}^2 > 0$. Theorem 4.10 shows that if $\inf_{w\in\mathcal{X}\times K} \|(I-F)w\|^2 > 0$, there is no solution. This implies the first claim.

(b) Further, Theorem 4.10 shows that if there is a solution, then $\inf_{w\in\mathcal{X}\times K} \|(I-F)w\|^2 = 0$. By (4.11) we see that $x^{k+1} - x^k \to 0$ in $\mathcal{H}$ and $\mu^{k+1} - \mu^k \to 0$ in $\mathcal{K}$. Since $\frac{1}{\beta}(\mu^{k+1} - \mu^k) = Ax^{k+1} - b$, we see $Ax^{k+1} - b \to 0$ in $\mathcal{K}$. Suppose now that $x^k \rightharpoonup_{\mathcal{I}} \bar{x}$ on a subsequence $\mathcal{I} \subset \mathbb{N}$. First notice that $\bar{x} \in \mathcal{X}$ because $\mathcal{X}$ is closed and convex, thus weakly sequentially closed. By the linearity of A, we get $Ax^k - b \to A\bar{x} - b$ in $\mathcal{K}$, and thus $A\bar{x} - b = 0$; thus $\bar{x}$ is feasible. □

Let us stress that in Theorem 4.11 (a) we obtained that there is no solution of (Opt) if $\|x^{k+1} - x^k\|_{\mathcal{H}}^2 + \|\mu^{k+1} - \mu^k\|_{\mathcal{K}}^2 \geq c$ on a subsequence, this is a much stronger result than the non-existence of a KKT pair.

Now we want to investigate the convergence behavior of Algorithm 4.1 if there is a solution which admits no Lagrange multiplier. As mentioned in the introduction of this section in many examples from optimal control the desired Lagrange multiplier lies in a weaker space $Y^* \supset \mathcal{K}$ than the augmentation space $\mathcal{K}$. This is now illustrated by two examples.

Example 4.12. In almost every example from optimal control, the constraint is a mapping from some vector space into the Sobolev space $H_0^1(\Omega)$ (or a subspace of this space, for example $C(\overline{\Omega})$). Therefore, our above setting requires an augmentation of the constraint in $H_0^1(\Omega)$. However, it is often more convenient to augment the constraint in the Lebesgue space $L^2(\Omega)$, because the norm and the projections can be computed faster. For example, the standard tracking-type problem

$$\begin{aligned} \min_{\substack{u\in L^2(\Omega)\\ y\in H_0^1(\Omega)}} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 \\ \text{s.t. } -\triangle y &= u \text{ in } \Omega \\ y &= 0 \text{ in } \partial\Omega \end{aligned}$$

can be reformulated as

$$\min_{\substack{u\in L^2(\Omega)\\ y\in H_0^1(\Omega)}} \|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$
$$\text{s.t. } y = Su,$$

where S denotes the solution operator of the Laplace equation, cf. Section 2.1.3. This is a constraint in the space $H_0^1(\Omega)$; nevertheless, it would be convenient to augment the constraint in the space $L^2(\Omega)$, which contains the space $H_0^1(\Omega)$. ◇

In the last example, the space $H_0^1(\Omega)$ was still a Hilbert space and therefore the augmentation in $L^2(\Omega)$ was convenient but not necessary. The next example demonstrates that there might not be any Hilbert space that is suitable for the augmentation. Moreover, there is no reason to expect that a Lagrange multiplier exists. Hence, any augmented Lagrangian approach needs to make use of embeddings.

Example 4.13. This time, we consider the tracking type problem with state constraints

$$\min_{\substack{u\in L^2(\Omega)\\ y\in H_0^1(\Omega)}} \|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$
$$\begin{aligned}\text{s.t. } -\triangle y &= u \text{ in } \Omega\\ y &= 0 \text{ in } \partial\Omega,\\ y &\le g,\end{aligned}$$

where $g \in C(\overline{\Omega})$. It is well known that, using the solution operator of the Laplace equation, cf. Section 2.1.3, this problem can be reformulated as

$$\min_{u\in L^2(\Omega)} \|Su-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$
$$\text{s.t. } Su \le g.$$

Further, it is known that this constraint, if interpreted as an $L^2(\Omega)$ constraint, possesses no interior points, and thus the existence of a Lagrange multiplier cannot be expected. As usual in optimal control, the constraint is therefore interpreted as a constraint in the Banach space $C(\overline{\Omega})$, which is not a Hilbert space but a subspace of $L^2(\Omega)$. Under a Slater condition, the existence of a Lagrange multiplier is then shown in the dual space $C(\overline{\Omega})^*$, which is the space of the signed Borel measures and not a Hilbert space, cf. [104]. Hence, in principle, a Hilbert space method is not applicable. Nevertheless, in practice, the constraint is augmented in $L^2(\Omega)$ while "hoping" for convergence. As we see later, this hope is legitimate in some cases. If $g \in C^1(\overline{\Omega})$ then a analogous reasoning can be carried out using the spaces $H_0^1(\Omega)$ and $C^1(\overline{\Omega})$ instead of $L^2(\Omega)$ and $C(\overline{\Omega})$. ◇

Let us now proceed from the last two examples to a more general framework for our considerations, which is described in the following assumption.

Assumption 4.14. In addition to the standing assumptions for the problem (Opt), we assume the following:

(a) Y is a reflexive Banach space.

(b) The embedding $Y \hookrightarrow \mathcal{K} \simeq \mathcal{K}^* \hookrightarrow Y^*$ is continuous and dense.

(c) $f : \mathcal{H} \to \mathbb{R}$ is proper, convex, and lower semi-continuous; in particular, $\operatorname{dom} f = \mathcal{H}$.

(d) The A_ν are linear, continuous operators from $\mathcal{H}_\nu$ to Y, and therefore they are also mappings to $\mathcal{K}$. We denote again $Ax = \sum_{\nu=1}^{N} A_\nu x_\nu$.

(e) The vector b is an element of Y.

Even though it might seem strange to assume that the linear operators A_ν map into the more restrictive space Y at first glance, this is actually the right assumption, since the constraint originally lies in Y and is just interpreted as a Hilbert space constraint in $\mathcal{K}$. Since we can interpret the mapping A as a map from $\mathcal{H}$ to Y and, by dint of the embedding, also as a map from $\mathcal{H}$ to $\mathcal{K}$, we obtain $A^* = A^\star \circ I_{\mathcal{K}\to Y^*}$ for the Hilbert space adjoint $A^* : \mathcal{K} \to \mathcal{H}$ and the Banach space adjoint $A^\star : Y^* \to \mathcal{H}$. For notational convenience, we omit the embeddings $I_{Y\to\mathcal{K}} : Y \to \mathcal{K}$ and $I_{\mathcal{K}\to Y^*} : \mathcal{K} \to Y^*$; thus, both $\mu^{k+1} \in \mathcal{K}$ generated by Algorithm 4.1 and its embedding $I_{\mathcal{K}\to Y^*}\mu^{k+1} \in Y^*$ are denoted by μ^{k+1}. By virtue of this embedding, the Hilbert and Banach space adjoint of A can therefore be applied to μ^{k+1}.

Using the Theorem 4.11, we can now deduce some convergence properties in the case where no Lagrange multiplier exists in $\mathcal{K}$.

Theorem 4.15. *Suppose that* (Opt) *admits a solution, and that Assumption 4.14 holds true. Let the sequence* $(x^k, \mu^k) \in \mathcal{H} \times \mathcal{K}$ *be generated by Algorithm 4.1, with* $\gamma > 0$ *chosen such that* Q*, defined in* (4.6)*, is strongly monotone and self-adjoint. Then the following statements hold:*

(a) *Assume that* $A : \mathcal{H} \to Y$ *is a compact, bounded, linear operator, and that there is a subsequence* $\mathcal{I} \subset \mathbb{N}$ *such that* $x^{k+1} \rightharpoonup_{\mathcal{I}} \bar{x}$ *and* $\mu^{k+1} \rightharpoonup^*_{\mathcal{I}} \bar{\mu}$ *in* Y^**. Then* $(\bar{x}, \bar{\mu})$ *is a KKT pair in* $(\mathcal{H}, Y^*)$*.*

(b) *Assume that* f *is differentiable,* ∇f *is weakly sequentially continuous, i.e. it maps weakly convergent sequences to weakly convergent sequences. Further assume that* $A : \mathcal{H} \to Y$ *is surjective,* $\mathcal{X} = \mathcal{H}$*, and that* $x^{k+1} \rightharpoonup_{\mathcal{I}} \bar{x}$ *converges weakly on a subsequence* $\mathcal{I} \subset \mathbb{N}$*. Then* $\mu^{k+1} \rightharpoonup_{\mathcal{I}} \bar{\mu}$ *in* Y^**, where* $\bar{\mu}$ *is the unique multiplier such that* $(\bar{x}, \bar{\mu})$ *is a KKT pair.*

Proof. (a) First notice that $\mu^{k+1} \rightharpoonup_{\mathcal{I}} \bar{\mu}$ since Y is reflexive and, therefore, weak and weak-* convergence coincide. By Theorem 4.11 (b), the accumulation point $\bar{x}$ is feasible. From the compactness of A we get the strong convergence $Ax^{k+1} - b \to_{\mathcal{I}} 0$ in Y. The compactness of $A : \mathcal{H} \to Y^*$ implies the compactness of the Banach space adjoint $A^\star : Y^* \to \mathcal{H}$, now $\mu^{k+1} \rightharpoonup_{\mathcal{I}} \bar{\mu}$ in Y^* implies $A^\star\mu^{k+1} \to_{\mathcal{I}} A^\star\bar{\mu}$. As in the proof of Lemma 4.5, we obtain from the optimality condition of (4.1) that

$$-A^\star\mu^{k+1} - \beta(\gamma I - M)(x^{k+1} - x^k) \in \partial f(x^{k+1}) + N_{\mathcal{X}}(x^{k+1}), \tag{4.12}$$

where $\mu^{k+1} \in Y^*$ through the embedding stated in Assumption 4.14. The left-hand side of this inclusion converges (strongly) to $-A^\star\bar{\mu}$ on the subsequence $\mathcal{I}$ since $x^{k+1} - x^k \to 0$ and $A^\star\mu^{k+1} \to_{\mathcal{I}} A^\star\bar{\mu}$. The operators ∂f and $N_{\mathcal{X}}$ on the right-hand side of this inclusion are maximally monotone. Also, we have $\mathrm{dom}(\partial f) = \mathcal{H}$ by Proposition 2.19, and $\mathrm{dom}(f) = \mathcal{H}$. Thus, Proposition 2.42 shows that their sum $\partial f + N_{\mathcal{X}}$ is maximally monotone as well. Therefore, the graph of $\partial f + N_{\mathcal{X}}$ is weakly-strongly sequentially closed by Proposition 2.41, and we get

$$0 \in \partial f(\bar{x}) + N_{\mathcal{X}}(\bar{x}) + A^\star\bar{\mu}.$$

Together with $A\bar{x} - b = 0$, these are the KKT conditions in $\mathcal{H} \times Y^*$.

(b) Since $\mathcal{X} = \mathcal{H}$, we have $N_{\mathcal{X}}(\bar{x}) = \{0\}$. The weak convergence of x^{k+1} on the subsequence $\mathcal{I}$ yields $\nabla f(x^{k+1}) \rightharpoonup_{\mathcal{I}} \nabla f(\bar{x})$. The optimality condition (4.12) of (4.1), which simplifies to

$$-A^\star\mu^{k+1} - \beta(\gamma I - M)(x^{k+1} - x^k) = \nabla f(x^{k+1}),$$

the convergence $x^{k+1} - x^k \to 0$ resulting from Theorem 4.11 (b), and the weak convergence of $\nabla f(x^{k+1})$ on the subsequence $\mathcal{I}$ imply the weak convergence of $A^\star\mu^{k+1}$ on the subsequence $\mathcal{I}$. Now let $y \in Y$ be arbitrary. By the surjectivity of A, there is an $x \in \mathcal{H}$ such that $y = Ax$ and we see

$$\langle \mu^{k+1} \mid y \rangle_{Y^*,Y} \;=\; \langle \mu^{k+1} \mid Ax \rangle_{Y^*,Y} \;=\; \langle A^\star\mu^{k+1} \mid x \rangle_{\mathcal{H}}.$$

The right-hand side of this equation converges because of the weak convergence of $A^\star\mu^{k+1}$ (on the subsequence $\mathcal{I}$); hence μ^{k+1} is weak-* convergent in Y^* on the subsequence $\mathcal{I}$ to a limit $\bar{\mu} \in Y$. The reflexivity of Y shows $\mu^{k+1} \rightharpoonup_{\mathcal{I}} \bar{\mu}$. Therefore, we have $\nabla f(\bar{x}) + A^\star\bar{\mu} = 0$, and further we see as above that $A\bar{x} = b$. These are the KKT conditions. □

4.3 A Strongly Convergent Algorithm

As shown in the previous Chapter 3, the operator T_{Opt} from (3.5) is maximally monotone. Hence our Jacobi-type ADMM-method stated in Algorithm 4.1 yields

weak convergence of the corresponding sequence $\{w^k\}$ since this method can be interpreted as a proximal-point method in a suitable Hilbert space. We also proved that the sequence x_i^k generated by Algorithm 4.1 converges strongly if f_i is strongly convex. However, most functions are not strongly convex, and it is often appreciated to have a strongly convergent algorithm at hand since it possesses better approximation properties. In Section 2.3.4, we provided a detailed description of a way to obtain strong convergence for a fixed-point iteration of a non-expansive operator, which was originally discovered by Halpern, see [61].

Having in mind the proximal-point interpretation of our regularized Jacobi-type ADMM-method from Algorithm 4.1, it is not surprising that the following algorithm can be understood as a version of Halpern's method.

Algorithm 4.16. (Halpern-Regularized Jacobi-type ADMM-Method)

(S.0) Choose $(x^0, \mu^0), (x, \mu) \in \mathcal{X} \times \mathcal{K}$, parameters $\beta, \gamma > 0$, set $k := 0$, and choose a sequence $\{\rho^k\}_{k\in\mathbb{N}}$ satisfying

$$\rho^k \to 0, \quad \sum_{k=1}^{\infty} \rho^k = +\infty, \quad \sum_{k=1}^{\infty} |\rho^{k+1} - \rho^k| < \infty.$$

(S.1) If a suitable termination criterion is satisfied: STOP.

(S.2) For $i = 1, \ldots, N$, compute

$$\tilde{x}_i^k := \operatorname*{arg\,min}_{x_i \in \mathcal{X}_i} \Big\{ f_i(x_i) + \langle \mu^k \mid A_i x_i \rangle + \frac{\beta}{2} \big(\|A_i x_i + \sum_{l \neq i} A_l x_l^k - b\|^2 + \gamma \|x_i - x_i^k\|^2 \big) \Big\}. \tag{4.13}$$

(S.3) Define

$$\tilde{\mu}^k := \mu^k + \beta \Big(\sum_{l=1}^{N} A_l x_l^{k+1} - b \Big). \tag{4.14}$$

(S.4) Set

$$\begin{aligned} x^{k+1} &:= \rho^k x + (1 - \rho^k) \tilde{x}^k, \\ \mu^{k+1} &:= \rho^k \mu + (1 - \rho^k) \tilde{\mu}^k. \end{aligned}$$

(S.5) Set $k \leftarrow k + 1$, and go to (S.1).

The global and strong convergence properties of this method follow immediately from known results about Halpern's modification of the standard proximal-point method and are summarized, for the sake of completeness, in the following result.

Theorem 4.17. *Let the standing assumptions for the problem* (Opt) *hold. Suppose that Q, defined in* (4.6)*, is self-adjoint and strongly monotone and that there is at least one KKT point of the optimization problem* (Opt)*. Then the sequence $\{w^k\}_{k\in\mathbb{N}} = \{(x^k, \mu^k)\}_{k\in\mathbb{N}}$ generated by Algorithm 4.16 converges strongly to a KKT point $w^\infty = (x^\infty, \mu^\infty)$ of* (Opt)*.*

Proof. The operator $F := (I + \beta Q^{-1}T_{Opt})^{-1}$ is non-expansive by Proposition 4.6, as it is the resolvent of a maximally monotone map in the Hilbert space $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_Q$. It was shown in Proposition 4.6 that the iterates $\tilde{w}^k = (\tilde{x}^k, \tilde{\mu}^k)$ generated by (4.13) and (4.14) are equal to the iterates generated by $\tilde{w}^k := (I + \beta Q^{-1}T_{Opt})^{-1}w^k$, where T_{Opt} is the maximally monotone operator defined in (3.5). Thus, the iterates $w^{k+1} = (x^{k+1}, \mu^{k+1})$ generated by Algorithm 4.16 are equal to the iterates generated by

$$w^{k+1} = \rho^k w + (1-\rho^k)\tilde{w}^k \;=\; \rho^k w + (1-\rho^k)(I + \beta Q^{-1}T_{Opt})^{-1}w^k,$$

where w is a fixed vector and ρ^k has the properties given in Algorithm 4.16. But this is the standard Halpern-type iteration for the non-expansive operator $(I + \beta Q^{-1}T_{Opt})^{-1}$. The assertion now follows from the convergence properties of Halpern's iteration, cf. Theorem 2.35. □

The strong convergence of the iterates w^k to a KKT point of the optimization problem (Opt) immediately implies that all the other statements known from Theorem 4.7, if not superfluous, automatically also hold for Algorithm 4.16 as well.

Two other strongly convergent ADMM-type algorithms can be derived using the two alternatives to Halpern's method already mentioned in Section 2.3.4. More precisely, these are the Haugazeau's scheme, see e.g. [9, 62], and the method described in [15]. Using the firmly non-expansive fixed-point operator $F = (I + \beta Q^{-1}T_{Opt})^{-1}$, they can be obtained in a similar way as Algorithm 4.16.

Chapter 5

Regularized Jacobi-type ADMM-Methods for Generalized Nash Equilibrium Problems

In this chapter, we want to introduce a Jacobi-type ADMM-method and modifications thereof for the problem (GNEP), which was already discussed in Section 3.2. The convergence analysis presented in this section is based on [20]. We start by recalling the generalized Nash equilibrium problem (GNEP) with N players ν. Recall that the optimization problem of player ν is given by

$$\min_{x_\nu \in \mathcal{H}_\nu} \theta_\nu(x_\nu, x_{-\nu}) + \varphi_\nu(x_\nu) \quad \text{s.t.} \quad \sum_{\nu=1}^{N} A_\nu x_\nu = b, \quad x_\nu \in \mathcal{X}_\nu \tag{GNEP}$$

for all $\nu = 1, \ldots, N$, where the symbols are defined as in Section 3.2, i.e. $\mathcal{H}_\nu$ and $\mathcal{K}$ are given Hilbert spaces, $\varphi_\nu : \mathcal{H}_\nu \to \mathbb{R}$ are proper, convex, and lower semi-continuous functions, $\theta_\nu : \mathcal{H}_1 \times \cdots \times \mathcal{H}_N \to \mathbb{R}$ are continuously Fréchet-differentiable with $\theta_\nu(\cdot, x_{-\nu})$ being convex for any fixed $x_{-\nu}$, $\mathcal{X}_\nu \subset \mathcal{H}_\nu$ are non-empty, closed, and convex sets, $A_\nu \in \mathcal{L}(\mathcal{H}_\nu, \mathcal{K})$, and $b \in \mathcal{K}$. As in Section 3.2, we write, following standard notation in Nash games, $x = (x_\nu, x_{-\nu})$ and $(y_\nu, x_{-\nu}) = (x_1, \ldots, x_{\nu-1}, y_\nu, x_{\nu+1}, \ldots, x_N)$. Further, we again use the canonical abbreviations stated in (3.8).

We assume that the generalized Nash equilibrium problem (GNEP) has a non-empty feasible set. Since we have explicit constraints $\mathcal{X}_\nu$, there is essentially no loss of generality in assuming θ_ν and φ_ν are real-valued for all $\nu = 1, \ldots, N$. The assumption that the φ_ν are real-valued can be circumvented by using a technical condition. Moreover, we do not require the operators A_ν to be injective or surjective, which is a condition that is often used for ADMM-type methods in the finite-dimensional context, where the matrices A_ν are assumed to have full rank.

This chapter is organized as follows. Our basic parallel ADMM-type method is stated and analyzed in Section 5.1. First, in Section 5.1.1, the convergence analysis is carried out by applying the convergence theory of the forward-backward method. Then an alternative, self-contained convergence theory is presented in Section 5.1.2. Thereafter, in Section 5.1.3, the generalization of the algorithm to the more general conically constrained problem (GNEP_{conic}) is explained. In addition, two modifications of the basic approach are investigated. The first modification, presented in Section 5.2, generates strongly convergent iterates under a cocoercivity assumption. The second modification, which we discuss in Section 5.3, requires only a Lipschitz and a monotonicity assumption in order to guarantee convergence.

5.1 Regularized Jacobi-type ADMM-Method

In the following, a regularized Jacobi-type method for the solution of (GNEP) is investigated. Its basic idea is to augment the joint constraints in order to obtain a separable structure in the remaining constraints. We then use the ADMM-idea and interpret the resulting optimization problems of each player as minimization problems of the variables x_ν alone. Note that we also use a linearization of the smooth part θ_ν in our subproblems, which might simplify the solution of the resulting subproblems significantly (whereas the possibly nonsmooth term φ_ν remains unchanged). Finally, a proximal term is added to improve the convergence properties and ensure that the overall method is well-defined.

Algorithm 5.1. (Regularized linearized Jacobi-type ADMM-Method)

(S.0) Choose a starting point $(x^0, \mu^0) \in \mathcal{X} \times \mathcal{K}$, parameters $\beta, \gamma > 0$, and set $k := 0$.

(S.1) If a suitable termination criterion is satisfied: STOP.

(S.2) For $\nu = 1, \ldots, N$, compute

$$x_\nu^{k+1} := \underset{x_\nu \in \mathcal{X}_\nu}{\arg\min} \Big\{ \varphi_\nu(x_\nu) + \langle \nabla_{x_\nu} \theta_\nu(x_\nu^k, x_{-\nu}^k) \mid x_\nu - x_\nu^k \rangle_{\mathcal{H}_\nu} + \langle \mu^k \mid A_\nu x_\nu \rangle_{\mathcal{K}} + \tfrac{\beta}{2} \big(\| A_\nu x_\nu + \textstyle\sum_{i \neq \nu} A_i x_i^k - b \|_{\mathcal{K}}^2 + \gamma \| x_\nu - x_\nu^k \|_{\mathcal{H}_\nu}^2 \big) \Big\}. \tag{5.1}$$

(S.3) Define

$$\mu^{k+1} := \mu^k + \beta \Big(\sum_{\mu=1}^{N} A_\mu x_\mu^{k+1} - b \Big). \tag{5.2}$$

(S.4) Set $k \leftarrow k + 1$, and go to (S.1).

Throughout our convergence analysis, we implicitly assume that Algorithm 5.1 generates an infinite number of iterates. We further note that all subproblems (5.1) are strongly convex for all $\nu = 1, \ldots, N$ and $k \in \mathbb{N}$. Hence $x^{k+1} := \left(x_1^{k+1}, \ldots, x_N^{k+1}\right)$ is uniquely defined. This is due to the quadratic regularization term, which does not occur in standard ADMM methods for two or more components.

The main computational overhead in Algorithm 5.1 results from the solution of the optimization subproblems in (S.2). However, in contrast to augmented Lagrangian-type methods [73, 74], these subproblems are themselves only optimization problems and not Nash equilibrium problems. Moreover, the subproblems occuring in (S.2) can typically be solved in an efficient way, sometimes even analytically.

5.1.1 Convergence Analysis Based on the Forward-Backward Method

We next investigate the convergence properties of Algorithm 5.1. The main idea of our analysis here is to interpret Algorithm 5.1, after a simple linear transformation, as a forward-backward splitting method applied to a suitable inclusion problem in an appropriate Hilbert space. To this end, let us introduce the linear operator $M \in \mathcal{L}(\mathcal{H})$ by

$$Mx := \Big(\sum_{\substack{i=1 \\ i \neq \nu}}^{N} A_\nu^* A_i x_i \Big)_{\nu=1}^{N} = \begin{pmatrix} \sum_{i=2}^{N} A_1^* A_i x_i \\ \vdots \\ \sum_{i=1}^{N-1} A_N^* A_i x_i \end{pmatrix}. \tag{5.3}$$

It is not difficult to see that M is self-adjoint. Furthermore, we define $Q_{\beta,\gamma} \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$ by

$$Q_{\beta,\gamma} \begin{pmatrix} x \\ \mu \end{pmatrix} := \begin{pmatrix} \beta^2(\gamma x - Mx) \\ \mu \end{pmatrix}, \tag{5.4}$$

where β and γ denote the constants from Algorithm 5.1.

The definitions of M and $Q_{\beta,\gamma}$ coincide with the definitions of M and Q from Chapter 4, see (4.5) and (4.6). We use β and γ as a subscript in order to emphasize the dependency of these two parameters, as this dependency is more important in this chapter. Further notice that Remark 4.3 and Lemma 4.4 are still valid. The next remark, which plays a critical role in our subsequent convergence analysis, recapitulates these results.

Remark 5.2. Since M from (5.3) is self-adjoint, it follows that $Q_{\beta,\gamma}$ from (5.4) is also self-adjoint. Moreover, for all $\gamma > 0$ sufficiently large (say $\gamma > \|M\|$), $Q_{\beta,\gamma}$

is strongly monotone. This implies that, in this case, $Q_{\beta,\gamma}$ is both injective and surjective. Hence the inverse of $Q_{\beta,\gamma} \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$ exists and is a linear, continuous, strongly monotone, and self-adjoint operator.
Further we have the estimate $\|M\| \leq (N-1) \max_{\nu=1,\dots,N}\{\|A_\nu\|^2\}$. ◇

A suitable reformulation of the optimality conditions for the subproblems (5.1) and (5.2) is presented in the next result.

Lemma 5.3. *Let the standing assumptions for the problem* (GNEP) *hold. The vector* $w^{k+1} = \big(x^{k+1}, \mu^{k+1}\big)$ *computed in* (5.1) *and* (5.2) *is characterized by the equivalent inclusions*

$$\big(I - \beta Q_{\beta,\gamma}^{-1} P_\theta\big) w^k \in \big(I + \beta Q_{\beta,\gamma}^{-1}(\partial\psi + G + N_{\mathcal{W}})\big) w^{k+1} \tag{5.5}$$

and

$$0 \in \partial\psi(w^{k+1}) + P_\theta(w^k) + G(w^{k+1}) + \frac{1}{\beta} Q_{\beta,\gamma}(w^{k+1} - w^k) + N_{\mathcal{W}}(w^{k+1}). \tag{5.6}$$

Moreover, it is characterized by the following variational inequality: Find $g^{k+1} \in \partial\psi(w^{k+1})$ *such that*

$$\Big\langle g^{k+1} + P_\theta(w^k) + G(w^{k+1}) + \frac{1}{\beta} Q_{\beta,\gamma}(w^{k+1} - w^k) \mid w - w^{k+1} \Big\rangle \geq 0 \tag{5.7}$$

for all $w \in \mathcal{W}$. *Here we use* $\partial\psi$, P_θ, *and* G *defined in* (3.13), (3.14), *and* (3.15), *respectively.*

Proof. Using the optimality conditions for the programs (5.1), it follows that x_ν^{k+1} solves these programs if and only if $x_\nu := x_\nu^{k+1}$ satisfies the optimality conditions

$$\begin{aligned} 0 \in \partial_{x_\nu} \Big(& \varphi_\nu(x_\nu) + \big\langle \nabla_{x_\nu} \theta_\nu(x_\nu^k, x_{-\nu}^k), x_\nu - x_\nu^k \big\rangle_{\mathcal{H}_\nu} + \big\langle \mu^k \mid A_\nu x_\nu \big\rangle_{\mathcal{K}} \\ & + \tfrac{\beta}{2} \big(\|A_\nu x_\nu + \textstyle\sum_{i \neq \nu} A_i x_i^k - b\|_{\mathcal{K}}^2 + \gamma \|x_\nu - x_\nu^k\|_{\mathcal{H}_\nu}^2 \big) \Big) + N_{\mathcal{X}_\nu}(x_\nu) \end{aligned}$$

for all $\nu = 1, \dots, N$. This is equivalent to saying that there exist elements $g_\nu \in \partial\varphi_\nu(x_\nu^{k+1})$ such that the vector

$$-\Big(g_\nu + \nabla_{x_\nu} \theta_\nu(x_\nu^k, x_{-\nu}^k) + A_\nu^* \mu^k + \beta A_\nu^* (A_\nu x_\nu^{k+1} + \sum_{i \neq \nu} A_i x_i^k - b) + \beta\gamma (x_\nu^{k+1} - x_\nu^k) \Big)$$

belongs to the normal cone $N_{\mathcal{X}_\nu}(x_\nu^{k+1})$ for all $\nu = 1, \dots, N$. By definition of the normal cone, this can be rewritten as

$$\begin{aligned} & \Big\langle g_\nu + \nabla_{x_\nu} \theta_\nu(x_\nu^k, x_{-\nu}^k) + A_\nu^* \mu^k \mid x_\nu - x_\nu^{k+1} \Big\rangle_{\mathcal{H}_\nu} \\ & + \Big\langle \beta A_\nu^* \big(A_\nu x_\nu^{k+1} + \textstyle\sum_{i \neq \nu} A_i x_i^k - b \big) + \beta\gamma (x_\nu^{k+1} - x_\nu^k) \mid x_\nu - x_\nu^{k+1} \Big\rangle_{\mathcal{H}_\nu} \geq 0 \end{aligned}$$

for all $x_\nu \in \mathcal{X}_\nu$ and all $\nu = 1, \dots, N$. Using $\mu^{k+1} = \mu^k + \beta\big(\sum_{i=1}^N A_i x_i^{k+1} - b\big)$, cf. (5.2), the last inequality is equivalent to

$$\begin{aligned}&\Big\langle g_\nu + \nabla_{x_\nu}\theta_\nu(x_\nu^k, x_{-\nu}^k) \mid x_\nu - x_\nu^{k+1}\Big\rangle_{\mathcal{H}_\nu}\\ &+\Big\langle A_\nu^*\mu^{k+1} + \beta A_\nu^*\big(\textstyle\sum_{i\neq\nu} A_i(x_i^k - x_i^{k+1})\big) + \beta\gamma(x_\nu^{k+1} - x_\nu^k) \mid x_\nu - x_\nu^{k+1}\Big\rangle_{\mathcal{H}_\nu} \geq 0\end{aligned}$$

for all $x_\nu \in \mathcal{X}_\nu$ and all $\nu = 1, \dots, N$. Exploiting the definition of M in (5.3), the Cartesian product structure of the set $\mathcal{X}$, recalling from (3.13) that $\widehat{P}_\theta(x^k) = (\nabla_{x_1}\theta_1(x^k), \dots, \nabla_{x_N}\theta_N(x^k))$, and setting $\tilde{g} = (g_1, \dots, g_N)$, this can be rewritten more compactly as

$$\Big\langle \tilde{g} + \widehat{P}_\theta(x^k) + A^*\mu^{k+1} + \beta M(x^k - x^{k+1}) + \beta\gamma(x^{k+1} - x^k) \mid x - x^{k+1}\Big\rangle_{\mathcal{H}} \geq 0$$

for all $x \in \mathcal{X}$. Since

$$\Big\langle \frac{1}{\beta}(\mu^{k+1} - \mu^k) + \big(b - \sum_{\nu=1}^N A_\nu x_\nu^{k+1}\big) \mid \mu - \mu^k\Big\rangle_{\mathcal{K}} = 0 \qquad \forall \mu \in \mathcal{K}$$

in view of (5.2), and by setting $g := (\tilde{g}, 0)$, the previous two formulas are equivalent to

$$\Big\langle g + P_\theta(w^k) + G(w^{k+1}) + \frac{1}{\beta}Q_{\beta,\gamma}(w^{k+1} - w^k) \mid w - w^{k+1}\Big\rangle \geq 0$$

for all $w \in \mathcal{W}$, which shows the characterization (5.7). Using the definition of the normal cone $N_{\mathcal{W}}$, we can express this as

$$0 \in g + P_\theta(w^k) + G(w^{k+1}) + \frac{1}{\beta}Q_{\beta,\gamma}(w^{k+1} - w^k) + N_{\mathcal{W}}(w^{k+1}).$$

By definition, we have $g_\nu \in \partial\varphi_\nu(x_\nu^{k+1})$ for all $\nu = 1, \dots, N$; hence the last equation is equivalent to

$$\begin{aligned}&0 \in \partial\psi(w^{k+1}) + P_\theta(w^k) + G(w^{k+1}) + \frac{1}{\beta}Q_{\beta,\gamma}(w^{k+1} - w^k) + N_{\mathcal{W}}(w^{k+1})\\ &\iff \frac{1}{\beta}Q_{\beta,\gamma}w^k - P_\theta(w^k) \in \partial\psi(w^{k+1}) + G(w^{k+1}) + \frac{1}{\beta}Q_{\beta,\gamma}w^{k+1} + N_{\mathcal{W}}(w^{k+1}),\end{aligned}$$

which shows (5.6) and can be rewritten as

$$\big(I - \beta Q_{\beta,\gamma}^{-1}P_\theta\big)(w^k) \in \big(I + \beta Q_{\beta,\gamma}^{-1}(\partial\psi + G + N_{\mathcal{W}})\big)(w^{k+1}).$$

This shows (5.5) and completes the proof. □

Based on the previous results and assuming that $Q_{\beta,\gamma}$ is strongly monotone, cf. Remark 5.2, we obtain the following alternative procedure for the computation of w^{k+1} from Algorithm 5.1: Using the operators T_1 and T_2 from (3.16), we can rewrite formula (5.5) as

$$w^{k+1} \in \big(I + \beta Q_{\beta,\gamma}^{-1} T_2\big)^{-1} \big(I - \beta Q_{\beta,\gamma}^{-1} T_1\big) w^k,$$

which almost looks like a forward-backward method. In fact, if the operator $\mathcal{A} := Q_{\beta,\gamma}^{-1} T_2$ would be maximally monotone, we could rewrite this inclusion as an equation and would obtain the forward-backward iteration

$$w^{k+1} = \big(I + \beta Q_{\beta,\gamma}^{-1} T_2\big)^{-1} \big(I - \beta Q_{\beta,\gamma}^{-1} T_1\big) w^k, \tag{5.8}$$

which is known to converge provided that the second operator $\mathcal{B} := Q_{\beta,\gamma}^{-1} T_1$ has a suitable cocoercivity property. Unfortunately, though the operators T_1 and T_2 themselves are maximally monotone in view of Proposition 3.12, this property does not hold, in general, for the operators $\mathcal{A} = Q_{\beta,\gamma}^{-1} T_2$ and $\mathcal{B} = Q_{\beta,\gamma}^{-1} T_1$. However, this problem can be solved easily by using a suitable weighted scalar product.

Proposition 5.4. *Let the standing assumptions for the problem* (GNEP) *hold. Assume that $Q_{\beta,\gamma}$ from* (5.4) *is self-adjoint and strongly monotone. Let T_1 and T_2 be the operators defined in* (3.16)*. Furthermore, consider the Hilbert space $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle w \mid z \rangle_{Q_{\beta,\gamma}} := \langle Q_{\beta,\gamma} w \mid z \rangle$. Then the following statements hold for this scalar product:*

(a) *The operator $\mathcal{B} := Q_{\beta,\gamma}^{-1} T_1$ is maximally monotone and single-valued.*

(b) *The operator $\mathcal{A} := Q_{\beta,\gamma}^{-1} T_2$ is maximally monotone.*

(c) *The operator $\mathcal{A} + \mathcal{B}$ is maximally monotone.*

Proof. Once we verify statement (b), part (a) follows in the same way. To this end, recall that T_2 is maximally monotone (with respect to the scalar product $\langle \cdot \mid \cdot \rangle$) in view of Proposition 3.12. Hence βT_2 is also maximally monotone. Since $Q_{\beta,\gamma}^{-1}$ is self-adjoint and strongly monotone, it follows from Proposition 2.44 that $\beta Q_{\beta,\gamma}^{-1} T_2$ is maximally monotone in the Hilbert space $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$. Statement (c) follows directly from Proposition 2.42 since $\operatorname{dom} \mathcal{B} = \mathcal{H} \times \mathcal{K}$. □

Proposition 5.4 implies that the sequence $\{w^k\}_{k \in \mathbb{N}}$ generated by Algorithm 5.1 can be equivalently represented by the forward-backward scheme from (5.8), which is known to yield weak convergence under suitable assumptions. However, since we have maximally monotone operators only with respect to the weighted scalar product introduced in Proposition 5.4, we obtain weak and strong convergence

with respect to this scalar product and its induced norm only. But this scalar product is introduced here just for theoretical reasons; one is typically interested in corresponding convergence results in terms of the given Hilbert space endowed with the original scalar product. Lemma 2.9 shows that weak and strong convergence in $\mathcal{H}$ with respect to the original scalar product and the one induced by $Q_{\beta,\gamma}$ coincide. In order to verify convergence, it remains to show that the (forward) operator $\mathcal{B} = Q_{\beta,\gamma}^{-1} T_1$ is cocoercive for some modulus $\alpha > 0$. Then, in principle, the known convergence properties of the forward-backward splitting method can be applied for any choice of the step size β from the interval $(0, 2\alpha)$. However, in our case, the operator $\mathcal{B} = Q_{\beta,\gamma}^{-1} T_1$ itself depends on β via the linear operator $Q_{\beta,\gamma}$. This is actually the reason for not simply denoting this operator by Q, since this dependence is crucial, which therefore causes some additional problems. The following result discusses the cocoercivity of the operator $\mathcal{B}$, which obviously depends on corresponding properties of the mapping P_θ.

Lemma 5.5. *Let the standing assumptions for the problem* (GNEP) *hold. Suppose that P_θ, defined in* (3.14)*, is α-cocoercive in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle$ and $Q_{\beta,\gamma} \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$ from* (5.4) *is a strongly monotone, self-adjoint operator. Then $Q_{\beta,\gamma}^{-1} P_\theta$ is $\alpha / \|Q_{\beta,\gamma}^{-1}\|$-cocoercive in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$.*

Proof. First recall that, for any continuous linear operator Q, the inequality $\langle Qw \mid w \rangle \leq \|Q\| \|w\|^2$ holds for all w. We therefore obtain

$$\begin{aligned}
&\|Q_{\beta,\gamma}^{-1} P_\theta(w) - Q_{\beta,\gamma}^{-1} P_\theta(v)\|_{Q_{\beta,\gamma}}^2 \\
&\quad = \left\langle Q_{\beta,\gamma}\big(Q_{\beta,\gamma}^{-1} P_\theta(w) - Q_{\beta,\gamma}^{-1} P_\theta(v)\big) \mid Q_{\beta,\gamma}^{-1} P_\theta(w) - Q_{\beta,\gamma}^{-1} P_\theta(v) \right\rangle \\
&\quad = \left\langle P_\theta(w) - P_\theta(v) \mid Q_{\beta,\gamma}^{-1}\big(P_\theta(w) - P_\theta(v)\big) \right\rangle \;\leq\; \|Q_{\beta,\gamma}^{-1}\| \|P_\theta(w) - P_\theta(v)\|^2.
\end{aligned}$$

Hence, the assumed α-cocoercivity of P_θ yields

$$\begin{aligned}
\langle Q_{\beta,\gamma}^{-1} P_\theta(w) - Q_{\beta,\gamma}^{-1} P_\theta(v) \mid w - v \rangle_{Q_{\beta,\gamma}} &= \langle P_\theta(w) - P_\theta(v) \mid w - v \rangle \\
\geq\; \alpha \|P_\theta(w) - P_\theta(v)\|^2 &\geq \frac{\alpha}{\|Q_{\beta,\gamma}^{-1}\|} \|Q_{\beta,\gamma}^{-1} P_\theta(w) - Q_{\beta,\gamma}^{-1} P_\theta(v)\|_{Q_{\beta,\gamma}}^2,
\end{aligned}$$

and this completes the proof. □

Note that the cocoercivity assumption on the operator P_θ is very natural and follows immediately from the corresponding cocoercivity of the mapping $\widehat{P}_\theta$. In particular, this assumption holds if $\widehat{P}_\theta$ is strongly monotone and Lipschitz continuous. On the other hand, it is important to note that the latter condition does not imply the strong monotonicity of P_θ. In fact, the operator P_θ is never strongly monotone due to its last component being zero.

We finally need to address the problem that the operator $\mathcal{B}$ depends on β, which causes some issues regarding the choice of this step size. However, these are resolved in the proof of the following main convergence result for Algorithm 5.1.

Theorem 5.6. *Let the standing assumptions for the problem* (GNEP) *hold. Suppose that the operator $\widehat{P}_\theta$ from* (3.9) *is α-cocoercive in $\mathcal{H}$ endowed with the usual scalar product, and that there is at least one variational KKT point of the Nash equilibrium problem* (GNEP). *Assume that the parameters β and γ are chosen such that $\beta \in (0, 2\alpha)$ and $\gamma \geq \frac{1}{\beta^2} + \|M\|$, where M is defined as in* (5.3). *Furthermore, let the sequence $\{w^k\}_{k\in\mathbb{N}} = \{(x^k, \mu^k)\}_{k\in\mathbb{N}}$ be generated by Algorithm 5.1. Then the following statements hold:*

(a) *The operator $Q_{\beta,\gamma}$ from* (5.4) *is self-adjoint and strongly monotone with $\|Q_{\beta,\gamma}^{-1}\| \leq 1$.*

(b) *The operator $\big(I + \beta Q_{\beta,\gamma}^{-1} T_2\big)^{-1}\big(I - \beta Q_{\beta,\gamma}^{-1} T_1\big)$ is $\tilde{\alpha}$-averaged in $\mathcal{H}\times\mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$, where $\tilde{\alpha} = 2\alpha/(4\alpha - \beta)$.*

(c) *The sequence $\{w^k\}_{k\in\mathbb{N}} = \{(x^k, \mu^k)\}_{k\in\mathbb{N}}$ converges weakly to a variational KKT pair $w^* = (x^*, \mu^*)$.*

(d) *The sequence $\{\widehat{P}_\theta(x^k)\}_{k\in\mathbb{N}}$ converges strongly to the unique value $\widehat{P}_\theta(x^*)$.*

(e) *If $\widehat{P}_\theta$ is strongly monotone, the sequence $\{x^k\}_{k\in\mathbb{N}}$ converges strongly to x^*.*

Proof. (a) Since $\gamma > \|M\|$, the strong monotonicity and self-adjointedness follow from Remark 5.2. Due to $\gamma \geq \frac{1}{\beta^2} + \|M\|$, we have

$$\|\beta^2(\gamma I - M)x\|_{\mathcal{H}} \geq \beta^2\big(\|\gamma x\|_{\mathcal{H}} - \|Mx\|_{\mathcal{H}}\big) \geq \beta^2\big(\gamma - \|M\|\big)\|x\|_{\mathcal{H}} \geq \|x\|_{\mathcal{H}}$$

for all $x \in \mathcal{H}$. This yields

$$\begin{aligned}
\frac{1}{\|Q_{\beta,\gamma}^{-1}\|^2} &= \frac{1}{\sup_{w\in(\mathcal{H}\times\mathcal{K})\setminus\{0\}} \frac{\|Q_{\beta,\gamma}^{-1} w\|^2}{\|w\|^2}} \\
&= \inf_{w\in(\mathcal{H}\times\mathcal{K})\setminus\{0\}} \frac{\|w\|^2}{\|Q_{\beta,\gamma}^{-1} w\|^2} \\
&= \inf_{v\in(\mathcal{H}\times\mathcal{K})\setminus\{0\}} \frac{\|Q_{\beta,\gamma} v\|^2}{\|v\|^2} \\
&= \inf_{(x,\mu)\in(\mathcal{H}\times\mathcal{K})\setminus\{0\}} \frac{\|\beta^2(\gamma I - M)x\|_{\mathcal{H}}^2 + \|\mu\|_{\mathcal{K}}^2}{\|x\|_{\mathcal{H}}^2 + \|\mu\|_{\mathcal{K}}^2} \\
&\geq \inf_{(x,\mu)\in(\mathcal{H}\times\mathcal{K})\setminus\{0\}} \frac{\|x\|_{\mathcal{H}}^2 + \|\mu\|_{\mathcal{K}}^2}{\|x\|_{\mathcal{H}}^2 + \|\mu\|_{\mathcal{K}}^2} = 1,
\end{aligned} \tag{5.9}$$

where we substituted $v = Q_{\beta,\gamma}^{-1} w$. Hence, the choice of γ guarantees that $\|Q_{\beta,\gamma}^{-1}\| \leq 1$.

(b) We define $F := \big(I + \beta Q_{\beta,\gamma}^{-1} T_2\big)^{-1} \big(I - \beta Q_{\beta,\gamma}^{-1} T_1\big)$ and now prove that this operator is $\tilde{\alpha}$-averaged with $\tilde{\alpha} := 2\alpha/(4\alpha - \beta)$ in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$. By Proposition 5.4, the operators

$$\begin{aligned} \mathcal{A} &:= Q_{\beta,\gamma}^{-1} T_2 \\ \mathcal{B} &:= Q_{\beta,\gamma}^{-1} T_1 \;=\; Q_{\beta,\gamma}^{-1} P_\theta \end{aligned}$$

are maximally monotone in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$, and F is therefore single-valued, cf. Section 2.5.3. Lemma 5.5 combined with (a) shows that $\mathcal{B}$ is α-averaged in $\mathcal{H} \times \mathcal{K}$ endowed with $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$. Since $F := \big(I + \beta\mathcal{A}\big)^{-1} \big(I - \beta\mathcal{B}\big)$, Proposition 2.54 shows that F is $\tilde{\alpha}$-averaged in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$.

(c) First notice that $P_\theta : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \times \mathcal{K}$ as defined in (3.14) is α-cocoercive if $\widehat{P}_\theta : \mathcal{H} \to \mathcal{H}$ as defined in (3.9) is α-cocoercive. Since $\|Q_{\beta,\gamma}^{-1}\| \leq 1$, it follows from Lemma 5.5 that $Q_{\beta,\gamma}^{-1} P_\theta$ is α-cocoercive in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$. Statement (c) therefore follows from standard convergence properties of the forward-backward splitting method, cf. Theorem 2.55, together with Lemma 2.9.

(d) Since $P_\theta = (\widehat{P}_\theta, 0)$, this statement is a consequence of standard results, cf. Theorem 2.55.

(e) This statement follows directly from (c) and the observation that $\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|_{\mathcal{H}} \geq \rho \|x^k - x^*\|_{\mathcal{H}}$ for some $\rho > 0$, which is a consequence of the strong monotonicity and the Cauchy-Schwarz inequality. □

In practice, the constant $\gamma \geq \frac{1}{\beta^2} + \|M\|$ from Theorem 5.6 might be large, but there also exist examples where this constant is just of moderate size, see Chapter 7 for an instance.

5.1.2 Self-Contained Convergence Analysis

In the above section we have seen that Algorithm 5.1 can be interpreted as a forward-backward splitting method, which is exploited in more detail in the sequel. Here we present an alternative convergence theorem and its proof, using a technique that was discovered in the later work [19] of the authors and therefore was not presented in the underlying manuscript [20]. Since the choice of $\beta > 0$ is arbitrary in this alternative convergence theorem, it is an actual improvement of

Theorem 5.6. The reason why the convergence theory leading to Theorem 5.6 is still presented is that it provides an interesting insight into the method. Moreover, we use this theory to obtain an algorithm that only needs Lipschitz continuity rather than cocoercivity and another algorithm that is strongly convergent under a cocoercivity assumption.

Theorem 5.7. *Let the standing assumption for the problem* (GNEP) *hold. Assume that* (GNEP) *admits a variational KKT point and suppose that the operator* $\widehat{P}_\theta$ *from* (3.9) *is* α*-cocoercive. Moreover, let* $\gamma > \frac{1}{2\alpha\beta} + \|M\|$*, where* M *is the operator introduced in* (5.3)*. Then the iterates* $\{w^{k+1}\}_{k\in\mathbb{N}} = \{(x^{k+1}, \mu^{k+1})\}_{k\in\mathbb{N}}$ *generated by Algorithm 5.1 converge weakly to a variational KKT pair of* (GNEP)*. Further it holds that*

$$x^{k+1} - x^k \to 0, \qquad \mu^{k+1} - \mu^k \to 0, \qquad \text{and} \qquad \widehat{P}_\theta(x^k) \to \widehat{P}_\theta(x^*).$$

If $\widehat{P}_\theta$ *is strongly monotone, then additionally to the above assertions,* $\{x^{k+1}\}_{k\in\mathbb{N}}$ *converges strongly.*

Proof. For $w = (x, \mu) \in \mathcal{W} = \mathcal{H} \times \mathcal{K}$, let us define

$$Rw \;:=\; \frac{1}{\beta} Q_{\beta,\gamma} \;=\; \begin{pmatrix} \gamma\beta x - \beta M x \\ \frac{1}{\beta}\mu \end{pmatrix} \;=\; \begin{pmatrix} (\gamma\beta I - \beta M)x \\ \frac{1}{\beta}\mu \end{pmatrix},$$

which can be easily seen to be self-adjoint and strongly monotone under the given assumptions on γ, and therefore induces a scalar product and norm, which we denote by $\langle \cdot \mid \cdot \rangle_R$ and $\|\cdot\|_R$, respectively. On the other hand, we denote by $\langle \cdot \mid \cdot \rangle$ and $\|\cdot\|$ the original scalar product and its induced norm in the spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{H} \times \mathcal{K}$, respectively. Using the optimality conditions of (5.1) and (5.2), we get from (5.7) stated in Lemma 5.3 that there is a $g^{k+1} \in \partial\psi(w^{k+1})$ such that

$$\left\langle g^{k+1} + P_\theta(w^k) + G(w^{k+1}) + \frac{1}{\beta} Q_{\beta,\gamma}(w^{k+1} - w^k) \mid w - w^{k+1} \right\rangle \geq 0$$

for all $w \in \mathcal{W} = \mathcal{H} \times \mathcal{K}$. Using the above definition of R, this last inequality is equivalent to

$$\left\langle g^{k+1} + P_\theta(w^k) + G(w^{k+1}) + R(w^{k+1} - w^k) \mid w - w^{k+1} \right\rangle \geq 0 \tag{5.10}$$

for all $w \in \mathcal{W}$. Further, (5.6) from Lemma 5.3 shows that

$$0 \in \partial\psi(w^{k+1}) + P_\theta(w^k) + G(w^{k+1}) + R(w^{k+1} - w^k) + N_{\mathcal{W}}(w^{k+1}). \tag{5.11}$$

Let $w^* = (x^*, \mu^*)$ be an arbitrary KKT pair. Setting $w = w^*$ in the inequality (5.10) yields

$$\begin{aligned}
0 \quad &\leq \quad \langle g^{k+1} + P_\theta(w^k) + G(w^{k+1}) + R(w^{k+1} - w^k) \mid w^* - w^{k+1}\rangle \\
&\overset{\text{Lem. 3.11}}{\leq} \quad \langle w^{k+1} - w^k \mid w^* - w^{k+1}\rangle_R \\
&\qquad +\langle g^{k+1} + P_\theta(w^k) + G(w^{k+1}) - (g^* + P_\theta(w^*) + G(w^*)) \mid w^* - w^{k+1}\rangle \\
&\overset{\partial\psi, G \text{ mon.}}{\leq} \quad \langle w^{k+1} - w^k \mid w^* - w^{k+1}\rangle_R + \langle P_\theta(w^k) - P_\theta(w^*) \mid w^* - w^{k+1}\rangle \\
&\overset{P_\theta = (\widehat{P}_\theta, 0)}{=} \quad \langle w^{k+1} - w^k \mid w^* - w^{k+1}\rangle_R + \langle \widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*) \mid x^* - x^k\rangle \\
&\qquad +\langle \widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*) \mid x^k - x^{k+1}\rangle \\
&\overset{\widehat{P}_\theta \text{ cocoercive}}{\leq} \quad \langle w^{k+1} - w^k \mid w^* - w^{k+1}\rangle_R - \alpha\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|^2 \\
&\qquad +\langle \widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*) \mid x^k - x^{k+1}\rangle \\
&\overset{CSI}{\leq} \quad \langle w^{k+1} - w^k \mid w^* - w^{k+1}\rangle_R - \alpha\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|^2 \\
&\qquad +\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\| \cdot \|x^{k+1} - x^k\| \\
&\overset{Young}{\leq} \quad \langle w^{k+1} - w^k \mid w^* - w^{k+1}\rangle_R - \alpha\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|^2 \\
&\qquad +\frac{\varepsilon}{2}\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|^2 + \frac{1}{2\varepsilon}\|x^{k+1} - x^k\|^2 \\
&= \quad \langle w^{k+1} - w^k \mid w^* - w^{k+1}\rangle_R - (\alpha - \frac{\varepsilon}{2})\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|^2 \\
&\qquad +\frac{1}{2\varepsilon}\|x^{k+1} - x^k\|^2 \\
&\overset{\text{Lem. 2.60}}{=} \quad \frac{1}{2}\|w^k - w^*\|_R^2 - \frac{1}{2}\|w^{k+1} - w^*\|_R^2 - \frac{1}{2}\|w^{k+1} - w^k\|_R^2 \\
&\qquad -(\alpha - \frac{\varepsilon}{2})\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|^2 + \frac{1}{2\varepsilon}\|x^{k+1} - x^k\|^2.
\end{aligned}$$

This can be equivalently written as

$$\begin{aligned}
0 \ \leq\ & \|x^k - x^*\|^2_{\gamma\beta I - \beta M} + \frac{1}{\beta}\|\mu^k - \mu^*\|^2 \\
& - \|x^{k+1} - x^*\|^2_{\gamma\beta I - \beta M} - \frac{1}{\beta}\|\mu^{k+1} - \mu^*\|^2 \\
& - \|x^{k+1} - x^k\|^2_{(\gamma\beta - 1/\varepsilon) I - \beta M} - \frac{1}{\beta}\|\mu^{k+1} - \mu^k\|^2 \\
& - (2\alpha - \varepsilon)\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|^2.
\end{aligned} \tag{5.12}$$

Since $\gamma > \frac{1}{2\alpha\beta} + \|M\|$, we can find $\varepsilon > 0$ such that

$$\alpha \geq \frac{\varepsilon}{2}$$

and

$$\gamma\beta - \frac{1}{\varepsilon} - \beta\|M\| > 0.$$

With this ε, the linear operators

$$(\gamma\beta - 1/\varepsilon)I - \beta M, \qquad \gamma\beta I - \beta M, \qquad \text{and} \qquad R$$

can easily be seen to be self-adjoint and strongly monotone, and therefore they induce norms that are equivalent to the original norm, cf. Lemma 2.4. Further, the factor $\alpha - \varepsilon/2$ is positive. Therefore, (5.12) shows

$$\begin{aligned} \|w^{k+1} - w^*\|_R^2 &= \|x^{k+1} - x^*\|_{\gamma\beta I - \beta M}^2 + \frac{1}{\beta}\|\mu^{k+1} - \mu^k\|^2 \\ &\leq \|x^k - x^*\|_{\gamma\beta I - \beta M}^2 + \frac{1}{\beta}\|\mu^k - \mu^k\|^2 \\ &= \|w^k - w^*\|_R^2. \end{aligned} \tag{5.13}$$

Hence the sequence $\{w^{k+1}\}_{k\in\mathbb{N}} := \{(x^{k+1}, \mu^{k+1})\}_{k\in\mathbb{N}}$ is Fejér-monotone and bounded in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_R := \langle R\cdot \mid \cdot \rangle$. Thus, $\{w^{k+1}\}_{k\in\mathbb{N}}$ possesses a weakly convergent subsequence $w^{k+1} \rightharpoonup_{\mathcal{I}} \bar{w} = (\bar{x}, \bar{\mu})$ in $\mathcal{H} \times \mathcal{K}$ with $\langle \cdot \mid \cdot \rangle_R$, and by Lemma 2.9 we get $w^{k+1} \rightharpoonup_{\mathcal{I}} \bar{w}$ in $\mathcal{H} \times \mathcal{K}$ endowed with the original scalar product. Since $\mathcal{W} = \mathcal{X} \times \mathcal{K}$ is closed and convex as a product of two closed, convex sets, it is weakly sequentially closed; therefore $\bar{w} \in \mathcal{X} \times \mathcal{K}$. Summing (5.12) yields

$$\begin{aligned} &\sum_{k=0}^{\ell} \Big(\|x^{k+1} - x^k\|_{(\gamma\beta - 1/\varepsilon)I - \beta M}^2 + \frac{1}{\beta}\|\mu^{k+1} - \mu^k\|^2 + (\alpha - \frac{\varepsilon}{2})\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|^2 \Big) \\ &\quad \leq \|x^0 - x^*\|_{\gamma\beta I - \beta M}^2 + \frac{1}{\beta}\|\mu^0 - \mu^*\|^2 - \|x^{l+1} - x^*\|_{\gamma\beta I - \beta M}^2 - \frac{1}{\beta}\|\mu^{l+1} - \mu^*\|^2 \\ &\quad \leq \|x^0 - x^*\|_{\gamma\beta I - \beta M}^2 + \frac{1}{\beta}\|\mu^0 - \mu^*\|^2, \end{aligned}$$

where the right-hand side of the equation is constant; therefore passing $\ell \to \infty$ shows

$$\sum_{k=0}^{\infty} \Big(\|x^{k+1} - x^*\|_{(\gamma\beta - 1/\varepsilon)I - \beta M}^2 + \frac{1}{\beta}\|\mu^{k+1} - \mu^k\|^2 + (\alpha - \frac{\varepsilon}{2})\|\widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*)\|^2 \Big) < \infty.$$

By $\alpha - \varepsilon/2 > 0$, $\beta > 0$, and $(\gamma\beta - 1/\varepsilon)I - \beta M$ strongly monotone, it holds that

$$x^{k+1} - x^k \to 0, \qquad \mu^{k+1} - \mu^k \to 0, \qquad \text{and} \qquad \widehat{P}_\theta(x^k) \to \widehat{P}_\theta(x^*).$$

in $\mathcal{H}$ or $\mathcal{K}$, respectively, endowed with the original scalar product by the equivalence of norms, cf. Lemma 2.4. Notice that the value of $\widehat{P}_\theta(x^*)$ is independent of the solution point x^* due to the cocoercivity assumption. By (5.11), we have

$$\begin{aligned} -R(w^{k+1}-w^k) - P_\theta(w^k) &\in \partial\psi(w^{k+1}) + G(w^{k+1}) + N_{\mathcal{W}}(w^{k+1}) \\ &= (\partial\psi + G + N_{\mathcal{W}})w^{k+1}. \end{aligned}$$

Our previous discussion shows that the left-hand side converges strongly to the unique value $P_\theta(w^*) = (\widehat{P}_\theta(x^*), 0)$, whereas $w^{k+1} \rightharpoonup_{\mathcal{I}} \bar{w}$. Further, the operator $\partial\psi + G + N_{\mathcal{W}} = T_2$ is maximally monotone, cf. Proposition 3.12. Thus, by the strong-weak-sequential closedness of the graph of a maximally monotone operator, we obtain $P_\theta(\bar{w}) \in (\partial\psi + G + N_{\mathcal{W}})\bar{w}$. Note that

$$\begin{aligned} & P_\theta(\bar{w}) \in (\partial\psi + G + N_{\mathcal{W}})\bar{w} \\ \iff\ & 0 \in (\partial\psi + P_\theta + G + N_{\mathcal{W}})\bar{w} \ = \ T_{GNEP}(\bar{w}), \end{aligned}$$

where T_{GNEP} was defined in (3.16). Lemma 3.10 and the fact that $\bar{w} \in \mathcal{W} = \mathcal{X} \times \mathcal{K}$ therefore imply that $\bar{w}$ is a variational KKT point of (GNEP). Thus every weak cluster point of w^{k+1} is a zero of T_{GNEP}. Since (5.13) eventually implies that the sequence $\{w^k\}_{k\in\mathbb{N}}$ is Fejér-monotone with respect to the solution set $S := \{w \in \mathcal{W} \mid 0 \in T_{GNEP}(w)\}$ and that every weak cluster point is a variational KKT point, it follows from Proposition 2.29 that $w^{k+1} \rightharpoonup \bar{w}$ in $\mathcal{H} \times \mathcal{K}$ endowed with $\langle \cdot \mid \cdot \rangle_R$. By Lemma 2.9, it holds that $w^{k+1} \rightharpoonup \bar{w}$ in $\mathcal{H} \times \mathcal{K}$ endowed with the original scalar product.

If the operator $\widehat{P}_\theta(x^k)$ is strongly monotone, strong convergence of $\{x^k\}_{k\in\mathbb{N}}$ follows straight from $\widehat{P}_\theta(x^k) \to \widehat{P}_\theta(x^*)$ and

$$0 \leftarrow \langle \widehat{P}_\theta(x^k) - \widehat{P}_\theta(x^*) \mid x^{k+1} - x^* \rangle \geq \rho \|x^{k+1} - x^*\|^2,$$

where $\rho > 0$ is the strong monotonicity constant of $\widehat{P}_\theta(x^*)$. □

Some further comments on this technique of proof can be found at the end of Chapter 6.

5.1.3 Application to Conic Constraints

In Section 3.2.2 we have shown that the conically constrained generalized Nash problem (GNEP_{conic}) can be reformulated as an equality constrained problem of the form (GNEP), which was done by introducing slack variables s_ν or s. The resulting optimization problems that need to be solved in each iteration of Algorithm 5.1 are then minimization problems in the variables (x_ν, s_ν) or (x_N, s), respectively. Following the idea of Rockafellar [93], however, it turns out that the

minimization with respect to s_ν (or s) can be carried out exactly so that we have to solve, once again, minimization problems in x_ν alone.

We first illustrate this for the subproblems arising from the formulation in (3.18). The resulting optimization problems consists of computing, for each player $\nu = 1, \ldots, N$, the solution pair

$$\begin{aligned}
&(x_\nu^{k+1}, s_\nu^{k+1}) \\
&\quad := \quad \arg\min_{\substack{x_\nu \in \mathcal{X}_\nu \\ s_\nu \in \mathcal{C}}} \Big\{ \varphi_\nu(x_\nu) + \langle \nabla_{x_\nu} \theta_\nu(x_\nu^k, x_{-\nu}^k) \mid x_\nu - x_\nu^k \rangle_{\mathcal{H}_\nu} \\
&\qquad\quad + \langle \lambda^k \mid B_\nu x_\nu - s_\nu \rangle_{\mathcal{K}} + \tfrac{\beta}{2} \| B_\nu x_\nu + \textstyle\sum_{i \neq \nu} B_i x_i^k - b - s_\nu - \sum_{i \neq \nu} s_i^k \|_{\mathcal{K}}^2 \\
&\qquad\quad + \tfrac{\beta\gamma}{2} \| x_\nu - x_\nu^k \|_{\mathcal{H}_\nu}^2 + \tfrac{\beta\gamma}{2} \| s_\nu - s_\nu^k \|_{\mathcal{K}}^2 \Big\} \\
&\quad = \quad \arg\min_{\substack{x_\nu \in \mathcal{X}_\nu \\ s_\nu \in \mathcal{C}}} \Big\{ \varphi_\nu(x_\nu) + \langle \nabla_{x_\nu} \theta_\nu(x_\nu^k, x_{-\nu}^k) \mid x_\nu - x_\nu^k \rangle_{\mathcal{H}_\nu} + \tfrac{\beta\gamma}{2} \| x_\nu - x_\nu^k \|_{\mathcal{H}_\nu}^2 \\
&\qquad\quad + \tfrac{\beta\gamma}{2(\gamma+1)} \| B_\nu x_\nu + \textstyle\sum_{i \neq \nu} B_i x_i^k - b - \sum_{i=1}^N s_i^k + \frac{\lambda^k}{\beta} \|_{\mathcal{K}}^2 \\
&\qquad\quad + \tfrac{\beta}{2(\gamma+1)} \| (\gamma+1) s_\nu - (B_\nu x_\nu + \textstyle\sum_{i \neq \nu} B_i x_i^k - b - \sum_{i \neq \nu} s_i^k + \frac{\lambda^k}{\beta} + \gamma s_\nu^k) \|_{\mathcal{K}}^2 \Big\},
\end{aligned}$$

where the second equality follows after some elementary (though lengthy) algebraic calculations.

This shows that

$$s_\nu^{k+1} := s_\nu^{k+1}(x_\nu) = \mathrm{Proj}_{\mathcal{C}} \left(\frac{1}{\gamma+1} \Big(\frac{\lambda^k}{\beta} + B_\nu x_\nu + \sum_{i \neq \nu} B_i x_i^k - b - \sum_{i \neq \nu} s_i^k + \gamma s_\nu^k \Big) \right).$$

Using the squared distance function, we therefore obtain x_ν^{k+1} from

$$\begin{aligned}
x_\nu^{k+1} \quad = \quad & \arg\min_{x_\nu \in \mathcal{X}_\nu} \Big\{ \varphi_\nu(x_\nu) + \langle \nabla_{x_\nu} \theta_\nu(x_\nu^k, x_{-\nu}^k) \mid x_\nu - x_\nu^k \rangle_{\mathcal{H}_\nu} \\
& + \tfrac{\beta\gamma}{2(\gamma+1)} \| B_\nu x_\nu + \textstyle\sum_{i \neq \nu} B_i x_i^k - b - \sum_{i=1}^N s_i^k + \frac{\lambda^k}{\beta} \|_{\mathcal{K}}^2 + \tfrac{\beta\gamma}{2} \| x_\nu - x_\nu^k \|_{\mathcal{H}_\nu}^2 \\
& + \tfrac{\beta(\gamma+1)}{2} \operatorname{dist}_{\mathcal{C}}^2 \Big(\tfrac{1}{\gamma+1} (B_\nu x_\nu + \textstyle\sum_{i \neq \nu} B_i x_i^k - b - \sum_{i \neq \nu} s_i^k + \frac{\lambda^k}{\beta} + \gamma s_\nu^k) \Big) \Big\}
\end{aligned}$$

for all $\nu = 1, \ldots, N$. The corresponding multiplier update becomes

$$\lambda^{k+1} = \lambda^k + \beta \Big(\sum_{\nu=1}^N B_\nu x_\nu^{k+1} - b - \sum_{\nu=1}^N s_\nu^{k+1} \Big).$$

For the reformulation with one slack variable in (3.19), the resulting x_ν-subproblems for $\nu = 1, \ldots, N-1$ stay essentially the same as in Algorithm 5.1. The only difference arises for the last player, who has to solve the subproblem

$$\begin{aligned}
&(x_N^{k+1}, s^{k+1}) \\
&\quad := \quad \arg\min_{\substack{x_N \in \mathcal{X}_N \\ s \in \mathcal{C}}} \Big\{ \varphi_N(x_N) + \langle \nabla_{x_N} \theta_N(x_N^k, x_{-N}^k) \mid x_N - x_N^k \rangle_{\mathcal{H}_\nu} \\
&\qquad\quad + \langle \lambda^k \mid B_N x_N - s \rangle_{\mathcal{K}} + \tfrac{\beta\gamma}{2} \| x_N - x_N^k \|_{\mathcal{H}_\nu}^2 + \tfrac{\beta\gamma}{2} \| s - s^k \|_{\mathcal{K}}^2 \\
&\qquad\quad + \tfrac{\beta}{2} \| B_N x_N + \textstyle\sum_{i \neq N} B_i x_i^k - b - s \|_{\mathcal{K}}^2 \Big\}.
\end{aligned}$$

Applying the same technique as before yields the following updating formulas:

$$\begin{aligned}
x_N^{k+1} &:= \arg\min_{x_N \in \mathcal{X}_N} \Big\{ \varphi_N(x_N) + \langle \nabla_{x_N} \theta_N(x_N^k, x_{-N}^k) \mid x_N - x_N^k \rangle_{\mathcal{H}_\nu} \\
&\quad + \tfrac{\beta\gamma}{2(\gamma+1)} \| B_N x_N + \textstyle\sum_{i \neq N} B_i x_i^k - b - s^k + \frac{\lambda^k}{\beta} \|_{\mathcal{K}}^2 + \frac{\beta\gamma}{2} \| x_N - x_N^k \|_{\mathcal{H}_\nu}^2 \\
&\quad + \tfrac{\beta(\gamma+1)}{2} \operatorname{dist}_{\mathcal{C}}^2 \Big(\tfrac{1}{\gamma+1} (B_N x_N + \textstyle\sum_{i \neq N} B_i x_i^k - b + \frac{\lambda^k}{\beta} + \gamma s^k) \Big) \Big\}, \\
s^{k+1} &:= \operatorname{Proj}_{\mathcal{C}} \left(\tfrac{1}{\gamma+1} \big(\tfrac{\lambda^k}{\beta} + (B_N x_N^{k+1} + \textstyle\sum_{i \neq N} B_i x_i^k - b) + \gamma s^k \big) \right), \\
\lambda^{k+1} &:= \lambda^k + \beta (\textstyle\sum_{\nu=1}^N B_\nu x_\nu^{k+1} - b - s^{k+1}).
\end{aligned}$$

Note that the projections onto $\mathcal{C}$ are often easy to compute; e.g., in finite dimensions, $\mathcal{C}$ often equals a Cartesian product of intervals like $(-\infty, 0]$, where the projection is simply given by $\operatorname{Proj}_{\mathcal{C}}(y) = \min\{0, y\}$. A similar observation holds for the distance function $\operatorname{dist}_{\mathcal{C}}$ since a corresponding computation of the projection onto $\mathcal{C}$ immediately yields the distance.

Remark 5.8. In order to obtain convergence of Algorithm 5.1, we need to estimate the constant $\gamma > 1/\beta^2 + \|M\|$ or $\gamma > 1/(2\alpha\beta) + \|M\|$, where M is defined in (5.3). Please be aware that in the last section we changed the operators B_ν from problem (GNEP$_{conic}$) to either an operator $A_\nu(x_\nu, s_\nu) = B_\nu x_\nu - s_\nu$ in the case where we introduced N slack variables, or $A_\nu x_\nu = B_\nu x_\nu$ for $\nu = 1, \ldots, N-1$ and $A_\nu(x_\nu, s) = B_\nu x_\nu - s$ in the case where we introduced one slack variable. Recall that the operator M is defined for the equality constrained problem; hence we need to use the modified operators A_ν to compute the operator norm $\|M\|$. To do so, we recall from Remark 3.15 that $\|A_\nu\| \leq \|B_\nu\| + 1$ in the first case, or $\|A_\nu\| = \|B_\nu\|$ for $\nu = 1, \ldots, N-1$ and $\|A_N\| \leq \|B_N\| + 1$ in the second case. Together with Lemma 4.4, we obtain $\|M\| \leq (N-1) \max_{\nu=1,\ldots,N} \{(\|B_\nu\|^2 + 1)^2\}$. ◇

5.2 Strongly Convergent Jacobi-type ADMM-Method

Motivated by the fact that Algorithm 5.1 turned out to be a forward-backward splitting method in a suitable Hilbert space, and that the forward-backward splitting can be interpreted as a Krasnoselsky-Mann iteration, as outlined in Section 2.5.3, it is natural to apply the strongly convergent Halpern method to our setting in order to obtain strong convergence of the iterates (x^{k+1}, μ^{k+1}). This means that we have to correct the outcome of Algorithm 5.1 in a suitable way. The details are given in the following algorithm.

Algorithm 5.9. (Strongly Convergent Regularized Jacobi-type ADMM-Method)

(S.0) Choose a starting point $(x^0, \mu^0), (x, \mu) \in \mathcal{X} \times \mathcal{K}$, parameters $\beta, \gamma > 0$, set $k := 0$, and choose a sequence $\{\rho^k\}_{k\in\mathbb{N}}$ satisfying

$$\rho^k \to 0, \quad \sum_{k=1}^{\infty} \rho^k = +\infty, \quad \sum_{k=1}^{\infty} |\rho^{k+1} - \rho^k| < \infty.$$

(S.1) If a suitable termination criterion is satisfied: STOP.

(S.2) For $\nu = 1, \ldots, N$, compute

$$\begin{aligned}\hat{x}_\nu^k := \underset{x_\nu \in \mathcal{X}_\nu}{\arg\min} \Big\{ & \varphi_\nu(x_\nu) + \langle \nabla_{x_\nu} \theta_\nu(x_\nu^k, x_{-\nu}^k) \mid x_\nu - x_\nu^k \rangle_{\mathcal{H}_\nu} + \langle \mu^k \mid A_\nu x_\nu \rangle_{\mathcal{K}} \\ & + \tfrac{\beta}{2} \big(\|A_\nu x_\nu + \textstyle\sum_{i\neq\nu} A_i x_i^k - b\|_{\mathcal{K}}^2 + \gamma \|x_\nu - x_\nu^k\|_{\mathcal{H}_\nu}^2 \big) \Big\}. \end{aligned} \tag{5.14}$$

(S.3) Define

$$\hat{\mu}^{k+1} := \mu^k + \beta \Big(\sum_{\nu=1}^{N} A_\nu \hat{x}_\nu^k - b \Big). \tag{5.15}$$

(S.4) Let $\hat{x}^k := \big(\hat{x}_1^k, \ldots, \hat{x}_N^k\big)$, and compute

$$\begin{aligned} x^{k+1} &:= \rho^k x + (1 - \rho^k) \hat{x}^k, \\ \mu^{k+1} &:= \rho^k \mu + (1 - \rho^k) \hat{\mu}^k. \end{aligned}$$

(S.5) Set $k \leftarrow k + 1$, and go to (S.1).

The global and strong convergence properties of this method follow immediately from known results about Halpern's modification and Theorem 5.6. They are summarized in the following result.

Theorem 5.10. *Let the standing assumptions for the problem* (GNEP) *hold. Suppose that the operator* $\widehat{P}_\theta$ *from* (3.9) *is* α*-cocoercive in* $\mathcal{H}$ *endowed with the usual scalar product, and that there is at least one variational KKT point of the Nash equilibrium problem* (GNEP). *Furthermore, assume that the parameters* β *and* γ *are chosen such that* $\beta \in (0, 2\alpha)$ *and* $\gamma \geq \frac{1}{\beta^2} + \|M\|$. *Then the following statements hold:*

(a) *The sequences* $\{w^k\}_{k\in\mathbb{N}} = \{(x^k, \mu^k)\}_{k\in\mathbb{N}}$ *and* $\{\hat{w}^k\}_{k\in\mathbb{N}} = \{(\hat{x}^k, \hat{\mu}^k)\}_{k\in\mathbb{N}}$ *generated by Algorithm 5.9 converge strongly to a variational KKT pair* $w^* = (x^*, \mu^*)$.

(b) *The sequence* $\{\widehat{P}_\theta(x^k)\}_{k\in\mathbb{N}}$ *converges strongly to the unique value* $\widehat{P}_\theta(x^*)$.

Proof. (a) By Theorem 5.6 (a), the operator $Q_{\beta,\gamma}$ from (5.4) is self-adjoint and strongly monotone with $\|Q_{\beta,\gamma}^{-1}\| \leq 1$. In Lemma 5.3 and (5.8) we have seen that the iterates $\hat{w}^{k+1} = (\hat{x}^{k+1}, \hat{\mu}^{k+1})$ and $w^{k+1} = (x^{k+1}, \mu^{k+1})$ generated by Algorithm 5.9 can be expressed as

$$\begin{aligned} \hat{w}^k &= \big(I + \beta Q_{\beta,\gamma}^{-1} T_2\big)^{-1} \big(I - \beta Q_{\beta,\gamma}^{-1} T_1\big) w^k, \\ w^{k+1} &= \rho^k w + (1-\rho^k)\hat{w}^k, \end{aligned}$$

where $w = (x, \mu)$ is chosen in (S.0) of the algorithm. Let us define $F := \big(I + \beta Q_{\beta,\gamma}^{-1} T_2\big)^{-1} \big(I - \beta Q_{\beta,\gamma}^{-1} T_1\big)$. Then Theorem 5.6 (b) shows that F is $\tilde{\alpha}$-averaged with $\tilde{\alpha} := 2\alpha/(4\alpha - \beta)$ in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$. In particular, by our assumptions on β and γ, the operator F is non-expansive in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$. The iterates can now be expressed as

$$w^{k+1} = \rho^k w + (1-\rho^k) F w^k.$$

Theorem 2.35 shows the strong convergence of the sequence $\{w^{k+1}\}_{k\in\mathbb{N}} = \{(x^{k+1}, \mu^{k+1})\}_{k\in\mathbb{N}}$ to a fixed point of F in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$. The strong convergence in the original norm follows from the equivalence of the norms, cf. Lemma 2.9. Since $\rho^k \to 0$, we see that $w^{k+1} - \hat{w}^k \to 0$; therefore $\{\hat{w}^k\}_{k\in\mathbb{N}}$ converges strongly to the same limit point. By

$$\begin{aligned} 0 \in T_1 x^* + T_2 x^* &\iff -T_1 x^* \in T_2 x^* \\ &\iff (I - \beta Q_{\beta,\gamma}^{-1} T_1) x^* \in (I + \beta Q_{\beta,\gamma}^{-1} T_2) x^* \\ &\iff x^* \in (I + \beta Q_{\beta,\gamma}^{-1} T_2)^{-1} (I - \beta Q_{\beta,\gamma}^{-1} T_1) x^* \\ &\iff x^* = (I + \beta Q_{\beta,\gamma}^{-1} T_2)^{-1} (I - \beta Q_{\beta,\gamma}^{-1} T_1) x^*, \end{aligned}$$

these fixed points turn out to be zeros of $T_1 + T_2 = T_{GNEP}$, and Lemma 3.10 shows that these zeros are variational KKT pairs of (GNEP).

(b) Follows straight from the continuity of $\widehat{P}_\theta$ and (a). □

5.3 Modified Regularized Jacobi-type ADMM-Methods

Another approach, which again is motivated by the fact that Algorithm 5.1 is a certain type of forward-backward splitting method, consists of applying Tseng's method, as outlined in Section 2.5.4, to our setting in order to weaken the cocoercivity assumption. This means that we have to adjust the outcome of Algorithm 5.1 accordingly. The details are given in the following algorithm.

Algorithm 5.11. (Modified Regularized Linearized Jacobi-type ADMM-Method)

(S.0) Choose a starting point $(x^0, \mu^0) \in \mathcal{X} \times \mathcal{K}$, parameters $\beta, \gamma > 0$, and set $k := 0$.

(S.1) If a suitable termination criterion is satisfied: STOP.

(S.2) For $\nu = 1, \ldots, N$, compute

$$\hat{x}_\nu^k := \underset{x_\nu \in \mathcal{X}_\nu}{\arg\min} \Big\{ \varphi_\nu(x_\nu) + \langle \nabla_{x_\nu} \theta_\nu(x_\nu^k, x_{-\nu}^k) \mid x_\nu - x_\nu^k \rangle_{\mathcal{H}_\nu} + \langle \mu^k \mid A_\nu x_\nu \rangle_{\mathcal{K}} + \tfrac{\beta}{2} \big(\|A_\nu x_\nu + \textstyle\sum_{i \neq \nu} A_i x_i^k - b\|_{\mathcal{K}}^2 + \gamma \|x_\nu - x_\nu^k\|_{\mathcal{H}_\nu}^2 \big) \Big\}. \tag{5.16}$$

(S.3) Define

$$\mu^{k+1} := \mu^k + \beta \Big(\sum_{\nu=1}^N A_\nu \hat{x}_\nu^k - b \Big). \tag{5.17}$$

(S.4) Let $\hat{x}^k := \big(\hat{x}_1^k, \ldots, \hat{x}_N^k\big)$, and compute

$$x^{k+1} := \hat{x}^k + (\beta\gamma I - \beta M)^{-1} \big(\widehat{P}_\theta(x^k) - \widehat{P}_\theta(\hat{x}^k) \big). \tag{5.18}$$

(S.5) Set $k \leftarrow k + 1$, and go to (S.1).

In the following, we show that this algorithm has essentially the same convergence properties as Algorithm 5.1, requiring a Lipschitz continuity assumption, which is weaker than the above required cocoercivity assumption. The price we have to pay is the solution of the linear operator equation (5.18) (with a constant operator, independent of k). In the finite-dimensional case, this means that we have to solve a linear system of equations, but, since the matrix is the same for all iterations, only one (single) matrix decomposition has to be computed, which can then be reused for all further iterations.

Let us define the two vectors

$$\widehat{w}^{k+1} := \begin{pmatrix} \hat{x}^k \\ \mu^{k+1} \end{pmatrix} \quad \text{and} \quad w^{k+1} := \begin{pmatrix} x^{k+1} \\ \mu^{k+1} \end{pmatrix}.$$

Then we see that $\widehat{w}^{k+1}$ corresponds exactly to one iteration of Algorithm 5.1 and can therefore be viewed as a forward-backward method in a Hilbert space with a suitably modified scalar product. In order to verify convergence of Algorithm 5.11, we show that this method may be understood as a Tseng-type splitting method in the same (modified) Hilbert space. To this end, note that we can indeed compute w^{k+1} by the formula

$$w^{k+1} := \widehat{w}^{k+1} + \beta \big(Q_{\beta,\gamma}^{-1} P_\theta(w^k) - Q_{\beta,\gamma}^{-1} P_\theta(\widehat{w}^{k+1}) \big). \tag{5.19}$$

To use the general convergence theory for Tseng's method, we apply his result to the modified Hilbert space. In particular, this requires Lipschitz continuity. This task is taken care of in the following result.

Lemma 5.12. *Suppose that P_θ from* (3.14) *is $1/\alpha$-Lipschitz continuous in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle$, and that $Q_{\beta,\gamma} \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$ is a strongly monotone, self-adjoint operator. Then $Q_{\beta,\gamma}^{-1} P_\theta$ is $\|Q_{\beta,\gamma}^{-1}\|/\alpha$-Lipschitz continuous in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$.*

Proof. Applying Lemma 2.4 (d) with $Q := Q_{\beta,\gamma}$ implies

$$\begin{aligned}
&\|Q_{\beta,\gamma}^{-1} P_\theta(w) - Q_{\beta,\gamma}^{-1} P_\theta(v)\|_{Q_{\beta,\gamma}}^2 \\
&= \Big\langle Q_{\beta,\gamma}\big(Q_{\beta,\gamma}^{-1} P_\theta(w) - Q_{\beta,\gamma}^{-1} P_\theta(v)\big) \mid Q_{\beta,\gamma}^{-1} P_\theta(w) - Q_{\beta,\gamma}^{-1} P_\theta(v) \Big\rangle \\
&= \Big\langle P_\theta(w) - P_\theta(v) \mid Q_{\beta,\gamma}^{-1}\big(P_\theta(w) - P_\theta(v)\big) \Big\rangle \\
&\leq \|Q_{\beta,\gamma}^{-1}\| \|P_\theta(w) - P_\theta(v)\|^2 \\
&\leq \frac{\|Q_{\beta,\gamma}^{-1}\|}{\alpha^2} \|w - v\|^2 \leq \frac{\|Q_{\beta,\gamma}^{-1}\|^2}{\alpha^2} \|w - v\|_{Q_{\beta,\gamma}}^2.
\end{aligned}$$

This completes the proof. □

As in the case where P_θ needs to be cocoercive, we again have the problem that the operator $\mathcal{B} = Q_{\beta,\gamma}^{-1} \widehat{P}_\theta$ itself depends on β. However, as already discussed in Section 5.1, we can resolve this problem by selecting γ accordingly.

Theorem 5.13. *Let $Q_{\beta,\gamma} \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$ be as defined in* (5.4). *Suppose that $\widehat{P}_\theta$ from* (3.9) *is $1/\alpha$-Lipschitz continuous and monotone in $\mathcal{H}$ endowed with the usual scalar product, and that there is at least one variational KKT point of the Nash equilibrium problem* (GNEP). *Choose $\beta \in (0, \alpha)$ and take $\gamma \geq \frac{1}{\beta^2} + \|M\|$. Then the following statements hold:*

(a) *$Q_{\beta,\gamma}$ is self-adjoint and strongly monotone.*

(b) *The sequences $\{w^k\}_k \in \mathbb{N} = \{(x^k, \mu^k)\}_{k \in \mathbb{N}}$ and $\{\hat{w}^k\}_{k \in \mathbb{N}} = \{(\hat{x}^k, \mu^{k+1})\}_{k \in \mathbb{N}}$ generated by Algorithm 5.11 converge weakly to a variational KKT pair (x^*, μ^*).*

Proof. Since $\gamma > \|M\|$, assertion (a) follows from Remark 5.2. To verify statement (b), first note that $P_\theta : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \times \mathcal{K}$ (from (3.14)) is also $1/\alpha$-Lipschitz continuous since $\widehat{P}_\theta : \mathcal{H} \to \mathcal{H}$ is $1/\alpha$-Lipschitz continuous (with respect to the given norm). Using $\gamma \geq \frac{1}{\beta^2} + \|M\|$, we notice that Theorem 5.6 (a) is valid; thus we have $\|Q_{\beta,\gamma}^{-1}\| \leq 1$. Lemma 5.12 therefore yields that $Q_{\beta,\gamma}^{-1} P_\theta$ is $1/\alpha$-Lipschitz continuous in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$. From Proposition 5.4,

we see that $Q_{\beta,\gamma}^{-1}T_{GNEP}$, $Q_{\beta,\gamma}^{-1}T_1$, $Q_{\beta,\gamma}^{-1}T_2$, where T_{GNEP}, T_1, T_2 are defined as in (3.16), are maximally monotone in $(\mathcal{H} \times \mathcal{K}, \langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}})$. Further, (5.19) together with Lemma 5.3 and (5.8) shows that (5.16) and (5.17) from Algorithm 5.11 can jointly be interpreted as a forward-backward-forward splitting in $\mathcal{H} \times \mathcal{K}$ endowed with the scalar product $\langle \cdot \mid \cdot \rangle_{Q_{\beta,\gamma}}$. Thus, the standard convergence result for Tseng's splitting method (from, e.g., Theorem 2.56) together with Lemma 2.9 yields statement (b). □

Chapter 6

Regularized Gauss-Seidel-type ADMM-Methods for Generalized Nash Equilibrium Problems

In this chapter, we introduce two Gauss-Seidel-type ADMM-methods for the problem (GNEP) discussed in Section 3.2. The first one uses a fixed penalty parameter, whereas the second one increases the penalty parameter if necessary and therefore it can be expected to converge faster than the first method. The convergence analysis presented here is based on [19]. We start again by recalling the generalized Nash equilibrium problem (GNEP) with N players ν. Recall that the optimization problem of player ν is given by

$$\min_{x_\nu \in \mathcal{H}_\nu} \theta_\nu(x_\nu, x_{-\nu}) + \varphi_\nu(x_\nu) \quad \text{s.t.} \quad \sum_{\nu=1}^{N} A_\nu x_\nu = b, \quad x_\nu \in \mathcal{X}_\nu \tag{GNEP}$$

for all $\nu = 1, \ldots, N$, where the symbols are defined as in Section 3.2, i.e. $\mathcal{H}_\nu$ and $\mathcal{K}$ are given Hilbert spaces, $\varphi_\nu : \mathcal{H}_\nu \to \mathbb{R}$ are proper, convex, and lower semi-continuous functions, $\theta_\nu : \mathcal{H}_1 \times \cdots \times \mathcal{H}_N \to \mathbb{R}$ are continuously Fréchet-differentiable with $\theta_\nu(\cdot, x_{-\nu})$ being convex for any fixed $x_{-\nu}$, $\mathcal{X}_\nu \subset \mathcal{H}_\nu$ are non-empty, closed, and convex sets, $A_\nu \in \mathcal{L}(\mathcal{H}_\nu, \mathcal{K})$, and $b \in \mathcal{K}$. As in Section 3.2, we write, following standard notation in Nash games, $x = (x_\nu, x_{-\nu})$ and $(y_\nu, x_{-\nu}) = (x_1, \ldots, x_{\nu-1}, y_\nu, x_{\nu+1}, \ldots, x_N)$. Further, we again use the canonical abbreviations defined in (3.8).

We assume again that the generalized Nash equilibrium problem (GNEP) has a non-empty feasible set. Since we have explicit constraints $\mathcal{X}_\nu$, there is essentially no loss of generality in assuming that θ_ν and φ_ν are real-valued for all $\nu = 1, \ldots, N$. The assumption that the φ_ν are real-valued can be circumvented by using a technical condition. Again, we do not require the operators A_ν to be injective or surjective.

This chapter is organized as follows. Section 6.1 states the assumptions under which global convergence results are shown in the subsequent sections. Our first ADMM-type method with fixed regularization parameters is introduced in Section 6.2, which also contains the main convergence result as well as a class of examples showing that regularization is necessary already for $N = 2$ players, and that the regularization parameters have to be sufficiently large (for any $N \geq 2$). A modified ADMM-type method using an updating technique for the regularization parameter together with the corresponding convergence analysis is presented in Section 6.3. We close this chapter with some comments concerning other related methods in Section 6.4.

6.1 Assumptions

In this section, we state and discuss the assumptions of the current chapter. In addition, we compare them with those usually made in splitting-type methods. Our aim is to directly extend the well-known alternating direction method of multipliers for optimization problems to generalized Nash equilibrium problems with shared constraints. As we want to deal with more than two players, we cannot expect to deal with weaker conditions than those known for the multi-block ADMM, i.e. the ADMM with more than two functions, for finite-dimensional optimization problems. In [32], it is shown that the multi-block ADMM is not necessarily convergent when there are more than two (only convex) functions involved. On the other hand, the recent paper [103] shows that the multi-block ADMM is convergent if all except two functions are strongly convex.

In our generalized Nash equilibrium setting, this tells us that it is not enough to require $\partial\varphi + \widehat{P}_\theta$ from (3.14) to be monotone, but that we need $\partial\varphi + \widehat{P}_\theta$ to be strongly monotone with respect to a certain part of the variable $x = (x_1, \ldots, x_N)$. In fact, we require $\partial\varphi + \widehat{P}_\theta$ to be strongly monotone in all except one variable, i.e. in one variable more than in [103]. Taking into account that our class of problems is far more general and difficult to deal with, we believe that this condition is sufficiently weak.

In contrast to the multi-block ADMM, the objective functions of our players also depend on the strategies of the other players. In order to have more control over the consecutive iterates, we also need a certain Lipschitz continuity condition on the gradient of the players' objective functions. The Lipschitz condition applied here is weaker than the Lipschitz assumption that is typically used in the convergence analysis of first-order methods since we assume the Lipschitz continuity to hold only with respect to certain input variables. Hence both our strong monotonicity condition and our Lipschitz condition are weaker than the standard assumptions used in the context of splitting methods.

A precise statement of our assumptions follows.

Assumption 6.1. (U) For all $\nu = 1, \dots, N$, let $A_\nu \in \mathcal{L}(\mathcal{H}_\nu, \mathcal{K})$, $b \in \mathcal{K}$, $\varphi_\nu : \mathcal{H}_\nu \to \mathbb{R}$ be proper, lower semi-continuous, convex, $\operatorname{dom} \varphi_\nu = \mathcal{H}_\nu$, $\theta_\nu(\cdot, x_{-\nu})$ convex, and continuously Fréchet-differentiable with the derivative being continuous with respect to all variables $x := (x_1, \dots, x_N)$. Let $\mathcal{X}_\nu$ be non-empty, closed, and convex, and let $\mathcal{F} := \{x \in \mathcal{X} \mid Ax = b\}$ be non-empty.

(S) Let $\widehat{P}_\theta$ and $\partial\varphi$ be defined as in (3.9) and (3.10), respectively. Suppose that $\partial\varphi + \widehat{P}_\theta$ is monotone in x and strongly monotone in $x_{-N} := (x_1, \dots, x_{N-1})$, i.e. there exists $\rho > 0$ such that for all $x, y \in \mathcal{H}$ and all $g_x \in \partial\varphi(x)$, $g_y \in \partial\varphi(y)$ we have

$$\begin{aligned} \big\langle \big(g_x + \widehat{P}_\theta(x)\big) - \big(g_y + \widehat{P}_\theta(y)\big) \mid x - y \big\rangle &\geq \rho \|x_{-N} - y_{-N}\|^2 \\ &= \rho \sum_{\nu=1}^{N-1} \|x_\nu - y_\nu\|^2. \end{aligned}$$

(L) Assume that $\nabla_{x_\nu} \theta_\nu$ is Lipschitz continuous in the last $N - \nu$ components, i.e. there is an $L_\nu > 0$ such that

$$\begin{aligned} &\|\nabla_{x_\nu} \theta_\nu(x_1, \dots, x_\nu, y_{\nu+1}, \dots, y_N) - \nabla_{x_\nu} \theta_\nu(x_1, \dots, x_\nu, x_{\nu+1}, \dots, x_N)\|^2 \\ &\qquad \leq L_\nu^2 \sum_{i=\nu+1}^{N} \|x_i - y_i\|^2 \end{aligned} \tag{6.1}$$

for all $\nu = 1, \dots, N-1$.

Assumption 6.1 (S) gives us the opportunity to treat problems that are strongly monotone in $N - 1$ components only. This allows us to deal with conic and inequality constraints by means of the approach described in Section 3.2.2.

For notational convenience, let us define

$$C_\nu := \max_{i=\nu+1,\dots,N} \|A_\nu^* A_i\|^2. \tag{6.2}$$

6.2 ADMM-Method with Fixed Regularization

This section is devoted to a regularized ADMM-method for solving the problem (GNEP). The method presented here uses a fixed regularization parameter. A modification that includes an update of the regularization parameter is discussed

in the subsequent section. The precise statement of the algorithm is given in Section 6.2.1, whereas its global convergence is analyzed in Section 6.2.2. Finally, Section 6.2.3 contains a brief discussion regarding the necessity of the regularization. In particular, we show that the regularization is (in general) necessary even in the case of just two players.

6.2.1 Statement of the Algorithm

The following regularized alternating direction method of multipliers for the solution of (GNEP) is investigated in this section. Its basic idea is to augment the joint constraints in order to obtain a separable structure in the remaining constraints ($x_\nu \in \mathcal{X}_\nu$). We then use the direct extension of the ADMM and view the resulting optimization problems of each player as minimization problems of the variables x_ν alone. A proximal term is added to improve the convergence properties.

Algorithm 6.2. (ADMM-Method with Fixed Regularization)

(S.0) Choose a starting point $(x^0, \mu^0) \in \mathcal{X} \times \mathcal{K}$, parameters $\beta > 0$, $\gamma_\nu > 0$ for all $\nu = 1, \ldots, N$, and set $k := 0$.

(S.1) If a suitable termination criterion is satisfied: STOP.

(S.2) For $\nu = 1, \ldots, N$, compute

$$\begin{aligned} x_\nu^{k+1} := \underset{x_\nu \in \mathcal{X}_\nu}{\arg\min} \Big\{ & \varphi_\nu(x_\nu) + \theta_\nu(x_1^{k+1}, \ldots, x_{\nu-1}^{k+1}, x_\nu, x_{\nu+1}^k, \ldots, x_N^k) \\ & + \langle \mu^k \mid A_\nu x_\nu \rangle_{\mathcal{K}} + \tfrac{\gamma_\nu}{2} \|x_\nu - x_\nu^k\|_{\mathcal{H}_\nu}^2 \\ & + \tfrac{\beta}{2} \|A_\nu x_\nu + \textstyle\sum_{i=1}^{\nu-1} A_i x_i^{k+1} + \sum_{i=\nu+1}^{N} A_i x_i^k - b\|_{\mathcal{K}}^2 \Big\}. \end{aligned} \tag{6.3a}$$

(S.3) Define

$$\mu^{k+1} := \mu^k + \beta \left(\sum_{\nu=1}^{N} A_\nu x_\nu^{k+1} - b \right). \tag{6.3b}$$

(S.4) Set $k \leftarrow k + 1$, and go to (S.1).

We note that every subproblem (6.3a) is strongly convex for all ν and all iterations k. Hence, all iterates $x^{k+1} := \left(x_1^{k+1}, \ldots, x_N^{k+1}\right)$ are well-defined and uniquely determined. Throughout our convergence analysis, we implicitly assume that Algorithm 6.2 generates an infinite number of iterates.

The main computational overhead in Algorithm 6.2 stems from solving the optimization subproblems in (S.2). However, in contrast to, for example, augmented Lagrangian-type methods [73, 74], these subproblems are only optimization problems and not Nash equilibrium problems. Moreover, the subproblems occuring in (S.2) can typically be solved in an efficient way, sometimes even analytically.

6.2.2 Convergence

In this section, we analyze the convergence of Algorithm 6.2 under Assumption 6.1. To keep the notation reasonably simple, we make extensive use of the abbreviations introduced in (3.12)-(3.15) (recall that $w = (x, \mu)$). Moreover, it is convenient to define the auxiliary vectors

$$\hat{x}^{\nu,k} := (x_1^{k+1}, \ldots, x_\nu^{k+1}, x_{\nu+1}^k, \ldots, x_N^k) \qquad \text{for } \nu = 1, \ldots N; \tag{6.4}$$

in particular, it holds that $\hat{x}^{N,k} = x^{k+1}$.

We begin our analysis with two simple results.

Lemma 6.3. *Suppose that Assumption 6.1 (U) and (S) hold, and let $w^* = (x^*, \mu^*)$ be a variational KKT point of* (GNEP). *Then*

$$\begin{aligned} &\langle g + P_\theta(w) + G(w) + s \mid w - w^* \rangle \\ &\quad \geq \langle g + P_\theta(w) + G(w) \mid w - w^* \rangle \\ &\quad \geq \rho \|x_{-N} - x_{-N}^*\|^2 \end{aligned}$$

holds for all $x \in \mathcal{X}$, $\mu \in \mathcal{K}$, $g \in \partial\psi(w)$, and $s \in N_{\mathcal{W}}(w)$, where, again, $w = (x, \mu)$ and ψ is defined in (3.13).

Proof. The first inequality follows from $s \in N_{\mathcal{W}}(w)$, which implies $\langle s \mid w - w^* \rangle \geq 0$. In order to verify the second inequality, first note that the assumed strong monotonicity of $\partial\varphi + \hat{P}_\theta$ with respect to x_{-N} together with Lemma 3.9 yields

$$\big\langle g + P_\theta(w) + G(w) - \big(g^* + P_\theta(w^*) + G(w^*)\big) \mid w - w^* \big\rangle \;\geq\; \rho \|x_{-N} - x_{-N}^*\|^2$$

for all $g^* \in \partial\psi(w^*)$ and $g \in \partial\psi(w)$. Now, in view of Lemma 3.11, there exists a particular element $g^* \in \partial\psi(w^*)$ corresponding to the variational KKT pair w^* such that $\langle g^* + P_\theta(w^*) + G(w^*) \mid w - w^* \rangle \geq 0$ holds for all $w \in \mathcal{W}$. By combining this with the previous inequality for this particular g^*, the statement follows. □

The second preliminary result is concerned with the partial uniqueness of a variational KKT pair.

Lemma 6.4. *Suppose that Assumption 6.1 (U) and (S) hold, and let $w^* = (x^*, \mu^*)$ and $\bar{w} = (\bar{x}, \bar{\mu})$ be two variational KKT points of* (GNEP). *Then $x_{-N}^* = \bar{x}_{-N}$.*

Proof. By Definition 3.7 of a variational KKT point, we see that there exist $\bar{g} \in \partial\psi(\bar{w})$ and $\bar{s} \in N_{\mathcal{W}}(\bar{w})$ such that $0 = \bar{g} + P_\theta(\bar{w}) + G(\bar{w}) + \bar{s}$. Using Lemma 6.3 and the fact that w^* is a variational KKT pair, we therefore obtain

$$0 \;=\; \langle 0 \mid \bar{w} - w^* \rangle \;=\; \langle \bar{g} + P_\theta(\bar{w}) + G(\bar{w}) + \bar{s} \mid \bar{w} - w^* \rangle \geq \rho \|\bar{x}_{-N} - x_{-N}^*\|^2,$$

hence $\bar{x}_{-N} = x_{-N}^*$. □

In order to formulate the optimality conditions of (6.3) as a suitable inclusion, we introduce two linear operators $Q_N, R_N \in \mathcal{L}(\mathcal{H} \times \mathcal{K})$. The first operator

$$Q_N(w) \;:=\; Q_N(x,\mu) \;:=\; \begin{pmatrix} \gamma_1 x_1 \\ \vdots \\ \gamma_N x_N \\ \frac{1}{\beta}\mu \end{pmatrix} \tag{6.5}$$

represents the regularization. It is easy to see that Q_N is self-adjoint and strongly monotone for all $\gamma_\nu > 0$ and $\beta > 0$. The second operator is given by

$$R_N(w) := \begin{pmatrix} 0 & -\beta A_1^* A_2 & \dots & \dots & -\beta A_1^* A_N & 0 \\ 0 & 0 & -\beta A_2^* A_3 & \dots & -\beta A_2^* A_N & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & & & & -\beta A_{N-1}^* A_N & 0 \\ 0 & 0 & \dots & & 0 & 0 \\ 0 & 0 & \dots & & 0 & 0 \end{pmatrix} w. \tag{6.6}$$

These two operators allow the following formulation of the subproblems from (6.3) as a generalized equation.

Lemma 6.5. *Under Assumption 6.1 (U), the iteration* (6.3) *is equivalent to finding* $w^{k+1} \in \mathcal{H} \times \mathcal{K}$ *such that*

$$\begin{aligned} 0 \in T_{GNEP}(w^{k+1}) + Q_N(w^{k+1} - w^k) + R_N(w^{k+1} - w^k) \\ + \begin{pmatrix} \nabla_{x_1}\theta_1(\hat{x}^{1,k}) - \nabla_{x_1}\theta_1(x^{k+1}) \\ \vdots \\ \nabla_{x_{N-1}}\theta_{N-1}(\hat{x}^{N-1,k}) - \nabla_{x_{N-1}}\theta_{N-1}(x^{k+1}) \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{6.7}$$

where T_{GNEP} *is the set-valued operator defined in* (3.16).

Proof. First note that, due to the Cartesian structure of $\mathcal{W}$, we have $N_{\mathcal{W}}(w^{k+1}) = N_{\mathcal{X}_1}(x_1^{k+1}) \times \dots \times N_{\mathcal{X}_N}(x_N^{k+1}) \times N_{\mathcal{K}}(\mu^{k+1})$. Furthermore, using the sum rule for the convex subdifferential, cf. Proposition 2.20 (which can be applied because of Assumption 6.1 (U)), as well as the definitions of $\hat{x}^{\nu,k}$ and μ^{k+1}, the optimality

conditions of the convex optimization problem (6.3a) can be written as

$$\begin{aligned}
0 \;\in\; & \partial_{x_\nu}\varphi_\nu(x_\nu^{k+1}) + \nabla_{x_\nu}\theta_\nu(\hat{x}^{\nu,k}) + A_\nu^*\mu^k + \beta A_\nu^*(A\hat{x}^{\nu,k} - b) \\
& + \gamma_\nu(x_\nu^{k+1} - x_\nu^k) + N_{\mathcal{X}_\nu}(x_\nu^{k+1}) \\
= \; & \partial_{x_\nu}\varphi_\nu(x_\nu^{k+1}) + \nabla_{x_\nu}\theta_\nu(\hat{x}^{\nu,k}) + A_\nu^*\mu^{k+1} - \beta \sum_{i=\nu+1}^{N} A_\nu^* A_i(x_i^{k+1} - x_i^k) \\
& + \gamma_\nu(x_\nu^{k+1} - x_\nu^k) + N_{\mathcal{X}_\nu}(x_\nu^{k+1}) \\
= \; & \Big[\partial_{x_\nu}\varphi_\nu(x_\nu^{k+1}) + \nabla_{x_\nu}\theta_\nu(x^{k+1}) + A_\nu^*\mu^{k+1} + N_{\mathcal{X}_\nu}(x_\nu^{k+1})\Big] + \Big[\gamma_\nu(x_\nu^{k+1} - x_\nu^k)\Big] \\
& + \Big[- \beta \sum_{i=\nu+1}^{N} A_\nu^* A_i(x_i^{k+1} - x_i^k)\Big] + \Big[\nabla_{x_\nu}\theta_\nu(\hat{x}^{\nu,k}) - \nabla_{x_\nu}\theta_\nu(x^{k+1})\Big]
\end{aligned}$$

for all $\nu = 1, \ldots, N$. It is easy to see that the terms in brackets correspond to the ν-th row of the operator on the right-hand side of (6.7) (for $\nu = N$, we exploited the fact that $\hat{x}^{N,k} = x^{k+1}$).

The last row of (6.7) results from the updating rule (6.3b), which can be rewritten as

$$0 \;=\; \frac{1}{\beta}(\mu^{k+1} - \mu^k) - \left(\sum_{\nu=1}^{N} A_\nu x_\nu^{k+1} - b\right),$$

which, in turn, is equivalent to

$$0 \;\in\; \Big[- Ax^{k+1} + b + N_{\mathcal{K}}(\mu^{k+1})\Big] + \Big[\frac{1}{\beta}\big(\mu^{k+1} - \mu^k\big)\Big] + \Big[0\Big] + \Big[0\Big],$$

since $N_{\mathcal{K}}(\mu^{k+1}) = \{0\}$. This completes the proof. □

Using the definitions of the operator T_{GNEP} and the normal cone, Lemma 6.5 immediately yields the following result.

Lemma 6.6. *Under the Assumption 6.1 (U), the iteration* (6.3) *is equivalent to finding* $w^{k+1} := (x_1^{k+1}, \ldots, x_N^{k+1}, \mu^{k+1})$ *and* $g^{k+1} \in \partial\psi(w^{k+1})$*, with* ψ *from* (3.13)*, such that*

$$\begin{aligned}
\big\langle g^{k+1} + P_\theta(w^{k+1}) + G(w^{k+1}) + Q_N(w^{k+1} - w^k) \mid w - w^{k+1}\big\rangle & \\
+ \sum_{\nu=1}^{N-1} \langle \nabla_{x_\nu}\theta_\nu(\hat{x}^{\nu,k}) - \nabla_{x_\nu}\theta_\nu(x^{k+1}) \mid x_\nu - x_\nu^{k+1}\rangle & \\
- \beta \sum_{\nu=1}^{N-1}\sum_{i=\nu+1}^{N} \langle A_\nu^* A_i(x_i^{k+1} - x_i^k) \mid x_\nu - x_\nu^{k+1}\rangle \geq 0 \qquad \forall w \in \mathcal{W}. &
\end{aligned} \tag{6.8}$$

The previous results enable us to prove our main global convergence theorem.

Theorem 6.7. *Assume that problem* (GNEP) *admits a variational KKT point, and that Assumption 6.1 holds. Further suppose that* $\gamma_1 > 0$ *and*

$$\gamma_\nu > \frac{1}{\rho}\sum_{i=1}^{\nu-1} L_i^2 + \frac{1}{\rho}\beta^2 \sum_{i=1}^{\nu-1} C_i(N-i) \quad \text{for all } \nu = 2, \ldots, N, \tag{6.9}$$

where L_ν *and* C_ν *are defined in* (6.1) *and* (6.2)*, respectively. Then the iterates* $\{w^{k+1}\}_{k\in\mathbb{N}}$ *generated by Algorithm 6.2 converge weakly to a variational KKT pair* $(x_1^*, \ldots, x_N^*, \mu^*)$ *of* (GNEP)*. Furthermore,* $\{x_{-N}^{k+1}\}_{k\in\mathbb{N}}$ *converges strongly to* x_{-N}^**.*

Before proving this theorem, let us add some comments. The first regularization parameter γ_1 can be chosen as an arbitrary positive constant. On the other hand, the remaining regularization parameters have to satisfy condition (6.9), which implies, in particular, that $\gamma_2 < \gamma_3 < \ldots < \gamma_N$, i.e. the lower bounds get more and more restrictive. Large values of these regularization parameters typically slow down the convergence. We therefore present a modification of Algorithm 6.2 with an adaptive update rule for these regularization parameters to avoid unnecessarily large values of these parameters in the next section. Since the convergence theory for this adapted version depends on the convergence result for Algorithm 6.2, we have to investigate the properties of Algorithm 6.2 first.

Let us briefly discuss the special case of $N = 2$ players in Theorem 6.7. Assumption 6.1 then requires $\partial\varphi + \widehat{P}_\theta$ to be strongly monotone in x_1 only, and $\nabla_{x_1}\theta_1$ to be Lipschitz continuous in the second component x_2. Both conditions are significantly weaker than the usual assumption of $\widehat{P}_\theta$ being strongly monotone and Lipschitz continuous. The conditions regarding the regularization parameters are $\gamma_1 > 0$ and $\gamma_2 > \frac{1}{\rho}\big(L_1^2 + \beta^2\|A_1^* A_2\|\big)$. Then we obtain that $\{(x_1^{k+1}, x_2^{k+1}, \mu^{k+1})\}_{k\in\mathbb{N}}$ converges weakly to a solution and that $\{x_1^{k+1}\}_{k\in\mathbb{N}}$ even converges strongly.

Since the Algorithms 5.1 and 6.8 are closely related and the proofs of Theorem 5.7 and Theorem 6.7 use a similar technique, we have to be aware of the fact that the regularization factor in Algorithm 5.1 is $\beta\gamma$, whereas, we use the factor γ in Algorithm 6.8. This is just a minor change made for notational convenience. However, it must be taken into consideration when comparing the conditions stated in the Theorems 5.6, 5.7, and 6.7.

Proof of Theorem 6.7. First recall that the linear operator Q_N from (6.5) is self-adjoint and strongly monotone. Hence it induces a scalar product $\langle x \mid y\rangle_{Q_N} := \langle x \mid Q_N y\rangle$ in the Hilbert space $\mathcal{H} \times \mathcal{K}$.

Throughout this proof, the norm induced by this scalar product is denoted by $\|\cdot\|_{Q_N}$ while we are omitting the index for the original scalar product and the original norm.

Using Lemma 6.3, we obtain

$$\langle g^{k+1} + P_\theta(w^{k+1}) + G(w^{k+1}) \mid w^{k+1} - w^* \rangle \geq \rho \|x_{-N}^{k+1} - x_{-N}^*\|^2 \tag{6.10}$$

for an arbitrary $g^{k+1} \in \partial\psi(w^{k+1})$. Setting $w = w^*$ in (6.8), applying the Cauchy-Schwarz inequality (CSI), and using Young's inequality twice, the first time with some $\varepsilon > 0$ and the second time with some $\delta > 0$ (at this stage of the proof, $\delta, \varepsilon > 0$ are arbitrary; however, later these values will be specified in a suitable way), we obtain

$$\begin{aligned}
0 \overset{\text{Lem. 6.6}}{\leq} & \left\langle g^{k+1} + P_\theta(w^{k+1}) + G(w^{k+1}) + Q_N(w^{k+1} - w^k) \mid w^* - w^{k+1} \right\rangle \\
& + \sum_{\nu=1}^{N-1} \langle \nabla_{x_\nu} \theta_\nu(\hat{x}^{\nu,k}) - \nabla_{x_\nu} \theta_\nu(x^{k+1}) \mid x_\nu^* - x_\nu^{k+1} \rangle \\
& - \beta \sum_{\nu=1}^{N-1} \sum_{i=\nu+1}^{N} \langle A_\nu^* A_i (x_i^{k+1} - x_i^k) \mid x_\nu^* - x_\nu^{k+1} \rangle \\
\overset{(6.10)}{\leq} & \langle w^{k+1} - w^k \mid w^* - w^{k+1} \rangle_{Q_N} - \rho \|x_{-N}^{k+1} - x_{-N}^*\|^2 \\
& + \sum_{\nu=1}^{N-1} \langle \nabla_{x_\nu} \theta_\nu(\hat{x}^{\nu,k}) - \nabla_{x_\nu} \theta_\nu(x^{k+1}) \mid x_\nu^* - x_\nu^{k+1} \rangle \\
& - \beta \sum_{\nu=1}^{N-1} \left\langle \sum_{i=\nu+1}^{N} A_\nu^* A_i (x_i^{k+1} - x_i^k) \mid x_\nu^* - x_\nu^{k+1} \right\rangle \\
\overset{CSI}{\leq} & \langle w^{k+1} - w^k \mid w^* - w^{k+1} \rangle_{Q_N} - \rho \|x_{-N}^{k+1} - x_{-N}^*\|^2 \\
& + \sum_{\nu=1}^{N-1} \left\| \nabla_{x_\nu} \theta_\nu(\hat{x}^{\nu,k}) - \nabla_{x_\nu} \theta_\nu(x^{k+1}) \right\| \cdot \|x_\nu^{k+1} - x_\nu^*\| \\
& + \beta \sum_{\nu=1}^{N-1} \left\| \sum_{i=\nu+1}^{N} A_\nu^* A_i (x_i^{k+1} - x_i^k) \right\| \cdot \|x_\nu^{k+1} - x_\nu^*\| \\
\overset{Young}{\leq} & \langle w^{k+1} - w^k \mid w^* - w^{k+1} \rangle_{Q_N} - \rho \|x_{-N}^{k+1} - x_{-N}^*\|^2 \\
& + \sum_{\nu=1}^{N-1} \left(\frac{\varepsilon}{2} \|\nabla_{x_\nu} \theta_\nu(\hat{x}^{\nu,k}) - \nabla_{x_\nu} \theta_\nu(x^{k+1})\|^2 + \frac{1}{2\varepsilon} \|x_\nu^{k+1} - x_\nu^*\|^2 \right) \\
& + \sum_{\nu=1}^{N-1} \left(\frac{\beta^2 \delta}{2} \left\| \sum_{i=\nu+1}^{N} A_\nu^* A_i (x_i^{k+1} - x_i^k) \right\|^2 + \frac{1}{2\delta} \|x_\nu^{k+1} - x_\nu^*\|^2 \right)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(6.1)}{\leq} && \langle w^{k+1}-w^k \mid w^*-w^{k+1}\rangle_{Q_N} - \rho\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&&& + \sum_{\nu=1}^{N-1}\frac{\varepsilon}{2}L_\nu^2 \sum_{i=\nu+1}^{N}\|x_i^{k+1}-x_i^k\|^2 + \frac{1}{2\varepsilon}\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&&& + \sum_{\nu=1}^{N-1}\frac{\beta^2\delta}{2}\Big\|\sum_{i=\nu+1}^{N}A_\nu^*A_i(x_i^{k+1}-x_i^k)\Big\|^2 + \frac{1}{2\delta}\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&\overset{\text{Lem. 2.59}}{\leq} && \langle w^{k+1}-w^k \mid w^*-w^{k+1}\rangle_{Q_N} - \rho\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&&& + \sum_{\nu=1}^{N-1}\frac{\varepsilon}{2}L_\nu^2 \sum_{i=\nu+1}^{N}\|x_i^{k+1}-x_i^k\|^2 + \frac{1}{2\varepsilon}\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&&& + \sum_{\nu=1}^{N-1}\frac{\beta^2\delta(N-\nu)}{2}\sum_{i=\nu+1}^{N}\big\|A_\nu^*A_i\big\|^2\big\|x_i^{k+1}-x_i^k\big\|^2 + \frac{1}{2\delta}\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&= && \langle w^{k+1}-w^k \mid w^*-w^{k+1}\rangle_{Q_N} - \big(\rho-\frac{1}{2\varepsilon}-\frac{1}{2\delta}\big)\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&&& + \sum_{\nu=1}^{N-1}\sum_{i=\nu+1}^{N}\Big(\frac{\varepsilon}{2}L_\nu^2 + \frac{\beta^2\delta(N-\nu)}{2}\big\|A_\nu^*A_i\big\|^2\Big)\|x_i^{k+1}-x_i^k\|^2 \\
&\overset{(6.2)}{\leq} && \langle w^{k+1}-w^k \mid w^*-w^{k+1}\rangle_{Q_N} - \big(\rho-\frac{1}{2\varepsilon}-\frac{1}{2\delta}\big)\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&&& + \frac{1}{2}\sum_{\nu=1}^{N-1}\sum_{i=\nu+1}^{N}\Big(\varepsilon L_\nu^2 + \beta^2\delta C_\nu(N-\nu)\Big)\|x_i^{k+1}-x_i^k\|^2 \\
&= && -\langle w^{k+1}-w^k \mid w^{k+1}-w^*\rangle_{Q_N} - \big(\rho-\frac{1}{2\varepsilon}-\frac{1}{2\delta}\big)\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&&& + \frac{1}{2}\sum_{\nu=2}^{N}\|x_\nu^{k+1}-x_\nu^k\|^2\cdot\sum_{i=1}^{\nu-1}\Big(\varepsilon L_i^2 + \beta^2\delta C_i(N-i)\Big) \\
&\overset{\text{Lem. 2.60}}{=} && \frac{1}{2}\|w^k-w^*\|_{Q_N}^2 - \frac{1}{2}\|w^{k+1}-w^k\|_{Q_N}^2 - \frac{1}{2}\|w^{k+1}-w^*\|_{Q_N}^2 \\
&&& - \big(\rho-\frac{1}{2\varepsilon}-\frac{1}{2\delta}\big)\|x_{-N}^{k+1}-x_{-N}^*\|^2 \\
&&& + \frac{1}{2}\sum_{\nu=2}^{N}\|x_\nu^{k+1}-x_\nu^k\|^2\cdot\sum_{i=1}^{\nu-1}\Big(\varepsilon L_i^2 + \beta^2\delta C_i(N-i)\Big),
\end{aligned}
$$

where the penultimate equality follows from an elementary reordering of the corresponding terms.

Multiplying this inequality by two, using the definition of Q_N, and exploiting the block structure of $w = (x, \mu) = (x_1, x_{-1}, \mu)$, we therefore obtain

$$\|w^{k+1} - w^*\|_{Q_N}^2 + \big(2\rho - \frac{1}{\varepsilon} - \frac{1}{\delta}\big)\|x_{-N}^{k+1} - x_{-N}^*\|^2 + \frac{1}{\beta}\|\mu^{k+1} - \mu^k\|^2 + \gamma_1\|x_1^{k+1} - x_1^k\|^2 + \sum_{\nu=2}^{N} \|x_\nu^{k+1} - x_\nu^k\|^2 \cdot \left(\gamma_\nu - \varepsilon \sum_{i=1}^{\nu-1} L_i^2 - \delta\beta^2 \sum_{i=1}^{\nu-1} C_i(N-i)\right) \;\leq\; \|w^k - w^*\|_{Q_N}^2. \tag{6.11}$$

Assumption (6.9) now guarantees that we can find suitable $\varepsilon, \delta \in (\frac{1}{\rho}, \infty)$ such that

$$\hat{\gamma}_\nu := \left(\gamma_\nu - \varepsilon \sum_{i=1}^{\nu-1} L_i^2 - \delta\beta^2 \sum_{i=1}^{\nu-1} C_i(N-i)\right) > 0 \quad \text{for all } \nu = 2, \dots, N.$$

For the sake of completeness, let us also define $\hat{\gamma}_1 := \gamma_1 > 0$. Note that the choice of δ and ε implies

$$\hat{\rho} := 2\rho - \frac{1}{\varepsilon} - \frac{1}{\delta} > 0.$$

With these positive scalars, we can reformulate (6.11) as

$$\|w^{k+1} - w^*\|_{Q_N}^2 + \hat{\rho}\|x_{-N}^{k+1} - x_{-N}^*\|^2 + \frac{1}{\beta}\|\mu^{k+1} - \mu^k\|^2 + \sum_{\nu=1}^{N} \hat{\gamma}_\nu \|x_\nu^{k+1} - x_\nu^k\|^2 \;\leq\; \|w^k - w^*\|_{Q_N}^2. \tag{6.12}$$

Summation of (6.12) for $k = 0, 1, \dots, \ell$ yields

$$\sum_{k=0}^{\ell} \left(\hat{\rho}\|x_{-N}^{k+1} - x_{-N}^*\|^2 + \frac{1}{\beta}\|\mu^{k+1} - \mu^k\|^2 + \sum_{\nu=1}^{N} \hat{\gamma}_\nu \|x_\nu^{k+1} - x_\nu^k\|^2\right) \;\leq\; \|w^0 - w^*\|_{Q_N}^2.$$

Taking the limit $\ell \to \infty$, we therefore obtain $\|\mu^{k+1} - \mu^k\| \to 0$, $\|x_\nu^{k+1} - x_\nu^k\| \to 0$ for all $\nu = 1, \dots, N$, and $\|x_{-N}^{k+1} - x_{-N}^*\| \to 0$. Hence $w^{k+1} - w^k \to 0$ and $x_{-N}^{k+1} \to x_{-N}^*$, which is the second part of our claim. It therefore remains to be shown that $\{x_N^k\}_{k\in\mathbb{N}}$ and $\{\mu^k\}_{k\in\mathbb{N}}$ are weakly convergent, and that the weak limit is a solution.

In view of (6.12), the sequence $\{w^{k+1}\}_{k\in\mathbb{N}}$ is bounded; thus it has a weakly convergent subsequence $w^{k+1} \rightharpoonup_{\mathcal{I}} \bar{w}$. The fact that $w^{k+1} - w^k \to 0$ then also implies $w^k \rightharpoonup_{\mathcal{I}} \bar{w}$. Since $\mathcal{W} = \mathcal{X} \times \mathcal{K}$ is closed and convex as a product of two

closed convex sets it is weak sequentially closed, and therefore $\bar{w} \in \mathcal{X} \times \mathcal{K}$. By the assumed Lipschitz continuity of $\nabla_{x_\nu}\theta_\nu$ in the last $N-\nu$ components, we obtain

$$\|\nabla_{x_\nu}\theta_\nu(\hat{x}^{\nu,k}) - \nabla_{x_\nu}\theta_\nu(x^{k+1})\|^2 \;\leq\; L_\nu^2 \sum_{i=\nu+1}^{N} \|x_i^{k+1} - x_i^k\|^2 \to 0. \tag{6.13}$$

Let us define

$$\Delta^k \;:=\; Q_N(w^{k+1}-w^k) + R_N(w^{k+1}-w^k) + \begin{pmatrix} \nabla_{x_1}\theta_1(\hat{x}^{1,k}) - \nabla_{x_1}\theta_1(x^{k+1}) \\ \vdots \\ \nabla_{x_{N-1}}\theta_{N-1}(\hat{x}^{N-1,k}) - \nabla_{x_{N-1}}\theta_{N-1}(x^{k+1}) \\ 0 \\ 0 \end{pmatrix},$$

which converges to zero as a consequence of $w^{k+1} - w^k \to 0$ and (6.13). Recall from Lemma 6.5 that the optimality conditions of the subproblems from (6.3) can now be rewritten as

$$-\Delta^k \;\in\; T_{GNEP}(w^{k+1}),$$

where T_{GNEP} denotes the operator from (3.16). Our previous discussion shows that the left-hand side converges strongly to zero, whereas $w^{k+1} \rightharpoonup_{\mathcal{I}} \bar{w}$. Thus, by the strong-weak-sequential closedness of the graph of a maximally monotone operator and the maximal monotonicity of T_{GNEP}, cf. Proposition 3.12, we obtain $0 \in T_{GNEP}(\bar{w})$. Lemma 3.10, (3.17), and $\bar{w} \in \mathcal{W} = \mathcal{X} \times \mathcal{K}$ therefore imply that $\bar{w}$ is a variational KKT point of (GNEP). Since (6.12) implies that the sequence $\{w^k\}_{k\in\mathbb{N}}$ is Fejér-monotone with respect to the solution set and that every cluster point is a variational KKT point, it follows from Proposition 2.29 that $w^{k+1} \rightharpoonup \bar{w}$. □

One natural question that now naturally arises is how the above proof differs from the proof of standard ADMM-methods. All proofs of all kinds of ADMM-methods and related proximal methods make use of the optimality conditions of the x_ν-subproblems. In these optimality conditions, the (generalized) derivative of the objective function only depends on the new iterate x^{k+1}, which is due to the separable structure of the objective function. In our method, the (generalized) derivative of the ν-th player's objective function $\varphi_\nu + \theta_\nu$ depends in parts on the current iterate x^{k+1} and other parts still depend on the old iterate x^k, i.e. the generalized derivative in the optimality condition is $\nabla_{x_\nu}\theta_\nu(x_1^{k+1}, \ldots, x_{\nu-1}^{k+1}, x_\nu^{k+1}, x_{\nu-1}^k, \ldots, x_n^k) + \partial_{x_\nu}\varphi_\nu(x_\nu^{k+1})$, see the proof of Lemma 6.5. This situation requires special care,

which is provided in the above proof. Comparing proofs, our method seems to be more closely related to the forward-backward method than the standard ADMM, see [31] and Chapter 5. The manuscript [31] uses a related technique to prove convergence of a certain kind of forward-backward method. Furthermore, the method presented in Chapter 5 was shown to be equivalent to a forward-backward method in a weighted scalar product, and a stronger convergence theorem was proved in Section 5.1.2 using a techniques that is related to the one applied above. The above considerations are supported by the fact that we solve a non-symmetric variational inequality, i.e. in the finite-dimensional case the derivative of $\widehat{P}_\theta$ is a non-symmetric matrix. Therefore, there exists no associated optimization problem.

6.2.3 Necessity of Regularization

Taking into account the situation of the standard ADMM-method for optimization problems, one might expect that (a) no regularization is necessary for GNEPs with $N = 2$ players, and (b) arbitrary (possibly small) regularization parameters $\gamma_\nu > 0$ are sufficient for the global convergence of GNEPs with $N \geq 3$ players. The subsequent discussion shows that none of these statements hold. Hence a regularization is also necessary for two players, and the corresponding regularization parameters have to be sufficiently large. This shows that the GNEP is a significantly more difficult class of problems than optimization problems. Altogether, this justifies that one cannot expect to prove a much stronger convergence result than the one given in Theorem 6.7.

In order to verify the above statements, let us consider the finite-dimensional GNEP

$$\begin{aligned} &\min_{x_1 \in \mathbb{R}^{n_1}} \ \frac{1}{2} x_1^T U_{11} x_1 + x_1^T U_{12} x_2 \\ &\min_{x_2 \in \mathbb{R}^{n_2}} \ \frac{1}{2} x_2^T U_{22} x_2 + x_2^T U_{21} x_1 \end{aligned} \qquad \text{s.t. } A_1 x_1 + A_2 x_2 = 0$$

with $N = 2$ players. Since both objective functions are quadratic, the corresponding subproblems (6.3) are simple quadratic programs, and hence their optimality conditions result in a linear system of equations. In fact, an elementary calculation shows that the corresponding updating scheme from Lemma 6.5 boils down to the matrix iteration

$$\begin{pmatrix} x_1^{k+1} \\ x_2^{k+1} \\ \mu^{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} U_{11} + \gamma_1 I & -\beta A_1^* A_2 & A_1^* \\ U_{21} & U_{22} + \gamma_2 I & A_2^* \\ -A_1 & -A_2 & \frac{1}{\beta} I \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 I & -U_{12} - \beta A_1^* A_2 & 0 \\ 0 & \gamma_2 I & 0 \\ 0 & 0 & \frac{1}{\beta} I \end{pmatrix}}_{=:M} \begin{pmatrix} x_1^k \\ x_2^k \\ \mu^k \end{pmatrix}.$$

Standard results on splitting methods guarantee that this iteration converges, for all starting points, to a solution if and only if $\rho(M) < 1$, where $\rho(M)$ denotes the

spectral radius of M. In other words, for $\rho(M) \geq 1$, there exist starting points such that the above matrix iteration does not converge to a solution. Hence, in the following, we only have to compute the spectral radius for a particular instance of the above GNEP (note that a similar reasoning was also used in [32] in order to verify that the standard ADMM-method is not necessarily convergent for finite-dimensional optimization problems with more than two blocks).

To be more specific, let us take $U_{11} = 1$, $U_{12} = -10$, $U_{21} = 10$, and $U_{22} = 1$. Then it is easy to see that $\widehat{P}_\theta$ is strongly monotone with modulus $\rho = 1$ and Lipschitz continuous with constant $L = 11$. In particular, this implies that Assumption 6.1 holds. Let us further choose $A_1 = (1, 0)^T$, $A_2 = (0, 1)^T$, $\beta = 1$, and $\gamma_1 = 0.01$. Then the spectral radius of M depends on the parameter γ_2. Using a simple program, we can compute this spectral radius for different values of γ_2. The corresponding result is shown in Figure 6.1 which shows that $\rho(M) < 1$ holds for values of γ_2 larger than (approximately) 32.1, whereas for all values of γ_2 less than 32.1, the spectral radius is larger than one. This example clearly shows that a regularization is necessary for two-player games, and convergence cannot be expected, in general, for arbitrary $\gamma_\nu > 0$.

We close this section by noting that our theoretical lower bound for γ_2 from (6.9) yields $\gamma_2 > 100$ since $L_1 = 10$ and $A_1^* A_2 = 0$ in our case. Hence our theoretical lower bound is not sharp, though also not a too rough overestimate.

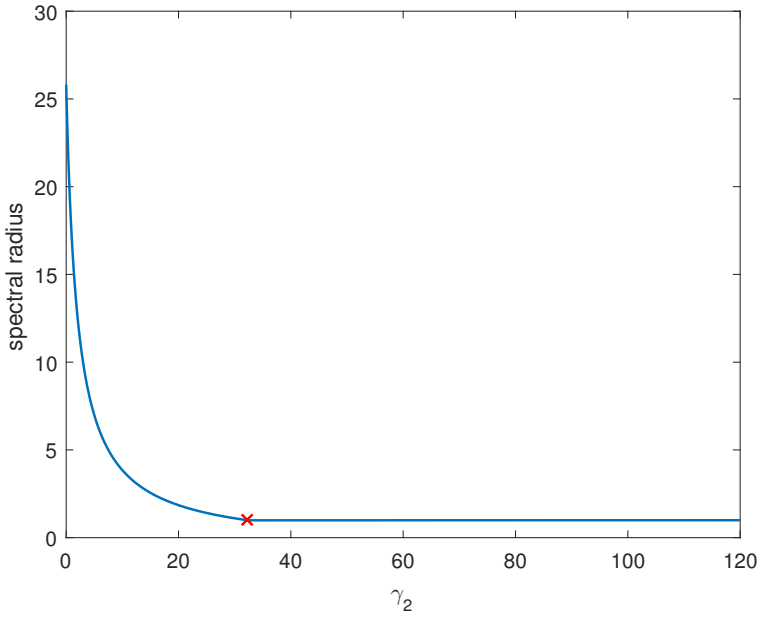

Figure 6.1: The spectral radius of the iteration matrix plotted against γ_2. The red cross indicates where the value drops below one.

6.3 ADMM-Method with Adaptive Regularization

Theorem 6.7 proves convergence of Algorithm 6.2 for all sufficiently large regularization parameters γ_ν. As the counterexample from Section 6.2.3 shows, one cannot expect convergence if these regularization parameters are not large enough. On the other hand, the same counterexample also indicates that the theoretical bounds from (6.9) are not sharp. Moreover, numerical tests indicate that Algorithm 6.2 often converges for significantly smaller regularization parameters. Since smaller regularization parameters typically lead to faster convergence, this topic plays a crucial role for the practical solution of GNEPs by ADMM-type methods.

In Section 6.3.1, we therefore present a modification of Algorithm 6.2 that uses an adaptive update rule for the regularization parameter. Its convergence, which is heavily based on the results from the previous section, is discussed in Section 6.3.2.

6.3.1 Statement of the Algorithm

The following method is a modification of Algorithm 6.2 that replaces the fixed regularization parameters by an adaptive updating procedure. In addition, it includes a second acceptance criterion for the new iterate that might also be satisfied for some small values of the regularization parameter. For a concise formulation of this modified algorithm, let us introduce the abbreviation

$$r^k := \sum_{\nu=1}^{N-1} \left\| g_\nu^k + \nabla_{x_\nu} \theta_\nu(x^k) + A_\nu^* \mu^k + s_\nu^k \right\|^2 + \|\gamma^{k-1}(x_N^k - x_N^{k-1})\|^2 + \left\| \sum_{\nu=1}^{N} A_\nu x_\nu^k - b \right\|^2, \tag{6.14}$$

for every k, with arbitrary elements $g_\nu^k \in \partial_{x_\nu} \varphi_\nu(x^k)$ and $s_\nu^k \in N_{\mathcal{X}_\nu}(x^k)$, see below for a discussion. To keep the notation simple, we also assume that all players take the same regularization parameter at each step, i.e. $\gamma_1^k = \cdots = \gamma_N^k =: \gamma^k$.

Algorithm 6.8. (ADMM-Method with Adaptive Regularization)

(S.0) Choose a starting point $(x^0, \mu^0) \in \mathcal{X} \times \mathcal{K}$, parameters $\alpha \in (0,1)$, $\beta > 0$, $\gamma^0 > 0$, $\tau > 0$, $\kappa \in \mathbb{N} \cup \{0\}$, $\Upsilon > 0$ satisfying (6.9), and set $k := 0$.

(S.1) If a suitable termination criterion is satisfied: STOP.

(S.2a) For $\nu = 1, \ldots, N$, compute

$$\begin{aligned} x_\nu^{k+1} := \operatorname*{arg\,min}_{x_\nu \in \mathcal{X}_\nu} \Big\{ & \varphi_\nu(x_\nu) + \theta_\nu(x_1^{k+1}, \ldots, x_{\nu-1}^{k+1}, x_\nu, x_{\nu+1}^k, \ldots, x_N^k) \\ & + \langle \mu^k \mid A_\nu x_\nu \rangle_{\mathcal{K}} + \tfrac{\gamma^k}{2} \|x_\nu - x_\nu^k\|_{\mathcal{H}_\nu}^2 \\ & + \tfrac{\beta}{2} \|A_\nu x_\nu + \textstyle\sum_{i=1}^{\nu-1} A_i x_i^{k+1} + \sum_{i=\nu+1}^{N} A_i x_i^k - b\|_{\mathcal{K}}^2 \Big\}. \end{aligned} \tag{6.15a}$$

(S.2b) For all $\nu = 1, \ldots, N-1$, compute $g_\nu^{k+1} \in \partial_{x_\nu} \varphi_\nu(x^{k+1})$ and $s_\nu^{k+1} \in N_{\mathcal{X}_\nu}(x^{k+1})$.

(S.3) Define

$$\mu^{k+1} := \mu^k + \beta\Big(\sum_{\nu=1}^{N} A_\nu x_\nu^{k+1} - b\Big). \tag{6.15b}$$

(S.4) If $\gamma^k \geq \Upsilon$, or γ^k was increased at least once during the last κ iterations, or if

$$r^{k+1} \leq \alpha r^k \qquad \text{(with } r^k \text{ defined in (6.14))} \tag{6.16}$$

set $\gamma^{k+1} := \gamma^k$

else

set $\gamma^{k+1} := \gamma^k + \tau$.

(S.5) Set $k \leftarrow k+1$, and go to (S.1).

Notice that the main steps (6.15a) and (6.15b) of this algorithm are the same as (6.3a) and (6.3b) of Algorithm 6.2. Thus Lemma 6.3, Lemma 6.5, and Lemma 6.6 still hold. The idea of the method is to start the iteration with a small γ^0 and to increase this regularization parameter only if this is really necessary. Note that an estimate Υ for the lower bounds stated in (6.9) is still required, but not in such a crucial way as in Algorithm 6.2 where the corresponding lower bound directly influences the size of the regularization parameters, whereas here we can also choose a rough overestimate for Υ and still deal with relatively small regularization parameters.

This is due to the third acceptance criterion used in (S.4), which is a sufficient decrease condition that implies that there exists a sequence $\{\varepsilon^{k+1}\}_{k\in\mathbb{N}} \subset \mathcal{H} \times \mathcal{K}$ such that $\varepsilon^{k+1} \to 0$ and $\varepsilon^{k+1} \in T_{GNEP}(w^{k+1})$. Essentially, (6.16) is a linear convergence condition whose satisfaction can reasonably be expected since our problem satisfies a strong convexity condition, and splitting-type algorithms are often linearly convergent under a strong convexity/monotonicity assumption.

The second acceptance condition states that the regularization parameter γ^k is not increased if γ^k was increased within the last κ iterations. This is motivated by our numerical experience, which shows that after the regularization parameter γ^k was increased, the method needs a couple of iterations to get back to its normal convergence behavior. Something similar can be observed in standard ADMM-methods, i.e. in the beginning of the iteration, the method often fails to decrease the KKT conditions that are related to the quantity r^k from (6.14); this relation will now be specified.

In order to understand the definition of r^k used in the sufficient decrease condition (6.16), note that r^{k+1} represents the squared norm of the variational KKT conditions of the problem (GNEP), except the player N, see Definition 3.7. The only term that needs some explanation is the difference $\gamma^k(x_N^{k+1} - x_N^k)$ occurring

in r^{k+1}. We claim that this term represents the first-order optimality condition of (6.15a) for player $\nu = N$ (this is why the first term in the definition of r^{k+1} is a sum from $\nu = 1$ to $N-1$ only). In order to see this, note that the optimality condition is given by

$$0 = g_N^{k+1} + \nabla_{x_N}\theta_N(x^{k+1}) + A_N^*\mu^k + \gamma^k(x_N^{k+1} - x_N^k) + \beta A_N^*(Ax^{k+1} - b) + s_N^{k+1}$$

for suitable elements $g_N^{k+1} \in \partial_{x_N}\varphi_N(x^{k+1})$ and $s_N^{k+1} \in N_{\mathcal{X}_N}(x^{k+1})$. Using the definition (6.15b) of μ^{k+1}, we therefore obtain

$$\begin{aligned}\gamma^k(x_N^k - x_N^{k+1}) &= g_N^{k+1} + \nabla_{x_N}\theta_N(x^{k+1}) + A_N^*\mu^k + \beta A_N^*(Ax^{k+1} - b) + s_N^{k+1} \\ &= g_N^{k+1} + \nabla_{x_N}\theta_N(x^{k+1}) + A_N^*\mu^{k+1} + s_N^{k+1}, \end{aligned} \tag{6.17}$$

which explains our definition of r^k.

Note that, for all other players $\nu = 1, \ldots, N-1$, Algorithm 6.8 requires the computation of arbitrary elements $g_\nu^{k+1} \in \partial_{x_\nu}\varphi_\nu(x^{k+1})$ and $s_\nu^{k+1} \in N_{\mathcal{X}_\nu}(x^{k+1})$. Apart from the fact that φ_ν might be differentiable or the set $\mathcal{X}_\nu$ might be the entire space, so that these elements are uniquely defined, these elements do not need to be those particular ones that occur in the optimality conditions of the corresponding optimization problem of player ν. These subgradients are used in our method only in the sufficient decrease condition, and there we can work with arbitrary elements. From our numerical experience, however, good choices of g_ν^{k+1} and s_ν^{k+1} can be obtained through the subproblem solver of (6.15a).

Finally, let us note that we decided to use the update $\gamma^{k+1} := \gamma^k + \tau$ for our algorithm. Alternatively, we could have chosen the update $\gamma^{k+1} := \tau\gamma^k$ for some $\tau > 1$, but our choice keeps γ^k smaller and seems to work quite well in numerical experiments.

6.3.2 Convergence

The global convergence of Algorithm 6.8 is heavily based on the known global convergence of Algorithm 6.2. Basically, we just have to verify that the additional decrease condition (6.16) does not destroy the global convergence properties stated in Theorem 6.7. To this end, let us take a closer look at Algorithm 6.2. The sequence $\{\gamma^k\}$ is monotonically increasing. Moreover, the update $\gamma^{k+1} = \gamma^k + \tau$ can occur only finitely many times due to the test $\gamma^k \geq \Upsilon$ in (S.4). Hence it is clear that $\{\gamma^k\}_{k\in\mathbb{N}}$ eventually stays fixed. For the convergence analysis, we can therefore assume that γ^k is a constant sequence, say $\gamma^k = \gamma > 0$ for all k sufficiently large.

Taking this into account, there are two possible situations: Either $\gamma \geq \Upsilon$ or $\gamma < \Upsilon$. In the first case, we are precisely in the situation of the previous section, and convergence follows from Theorem 6.7. Hence it remains to consider the second case, i.e. $\gamma < \Upsilon$, which means that the sufficient decrease condition (6.16)

is satisfied for all sufficiently large k. This situation is discussed separately in the following result.

Proposition 6.9. *Suppose that* (GNEP) *admits a variational KKT point, that the sequence* $\{w^{k+1}\}_{k\in\mathbb{N}} = \{(x^{k+1}, \mu^{k+1})\}_{k\in\mathbb{N}}$ *is generated by Algorithm 6.8, that Assumption 6.1 holds, and that there is a* $k_0 \in \mathbb{N}$ *such that* (6.16) *holds for all* $k \geq k_0$. *Then the sequence* $\{w^{k+1}\}_{k\in\mathbb{N}}$ *converges strongly to a variational KKT point of* (GNEP).

Proof. The proof is divided into four parts: In the first two parts, we prove that $\{\mu^{k+1}\}_{k\in\mathbb{N}}$ and $\{x_N^{k+1}\}_{k\in\mathbb{N}}$ are Cauchy sequences and, therefore, (strongly) convergent. We use this in part three to conclude that the sequence $\{x_{-N}^{k+1}\}_{k\in\mathbb{N}}$ is strongly convergent. Finally, in part four we show that every limit point of the sequence $\{w^{k+1}\}_{k\in\mathbb{N}}$ is a variational equilibrium of (GNEP). Throughout this proof, we denote by $w^* = (x^*, \mu^*)$ an arbitrary variational KKT pair, whose x_{-N}^*-part is unique due to Lemma 6.4.

Part 1: First notice that $\sqrt{\alpha} \in (0, 1)$. Taking the square root of (6.16), we inductively obtain

$$\begin{aligned}\Big(\sum_{\nu=1}^{N-1} &\|g_\nu^{k+1} + \nabla_{x_\nu}\theta_\nu(x^{k+1}) + A_\nu^*\mu^{k+1} + s_\nu^{k+1}\|^2 \\ &+\|\gamma^k(x_N^{k+1} - x_N^k)\|^2 + \|Ax^{k+1} - b\|^2\Big)^{1/2} \\ \leq\ &\ldots \leq \alpha^{k/2} r \end{aligned} \tag{6.18}$$

where $r := \sqrt{r^1}$ with r^1 defined in (6.14); note that r is essentially a constant (just depending on x^1 and x^0). Summation yields

$$\begin{aligned}&\sum_{k=1}^{\infty}\|Ax^{k+1} - b\| \\ &\quad\leq \sum_{k=1}^{\infty}\Big(\sum_{\nu=1}^{N-1}\|g_\nu^{k+1} + \nabla_{x_\nu}\theta_\nu(x^{k+1}) + A_\nu^*\mu^{k+1} + s_\nu^{k+1}\|^2 \\ &\qquad\qquad + \|\gamma^k(x_N^{k+1} - x_N^k)\|^2 + \|Ax^{k+1} - b\|^2\Big)^{1/2} \\ &\quad\leq r\sum_{k=1}^{\infty}\sqrt{\alpha}^k = \frac{r}{1-\sqrt{\alpha}}. \end{aligned} \tag{6.19}$$

Hence (6.15b) implies

$$\sum_{k=1}^{\infty}\|\mu^{k+1} - \mu^k\| = \beta\sum_{k=1}^{\infty}\|Ax^{k+1} - b\| \leq \beta\frac{r}{1-\sqrt{\alpha}}.$$

Consequently, the triangle inequality shows that $\{\mu^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence and, therefore, convergent to some element $\bar{\mu}$.

Part 2: Similar to the derivation of (6.19), we obtain from (6.18) that

$$\sum_{k=1}^{\infty} \gamma^k \|x_N^{k+1} - x_N^k\| \leq \cdots \leq \frac{r}{1-\sqrt{\alpha}}.$$

Since $\{\gamma^k\}_{k\in\mathbb{N}}$ is eventually constant, this implies that $\{x_N^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, thus convergent to some element $\bar{x}_N$.

Part 3: Note that (6.16) or (6.18) together with (6.17) and the previous parts imply

$$g^{k+1} + P_\theta(w^{k+1}) + G(w^{k+1}) + s^{k+1} \to 0 \tag{6.20}$$

where $g^{k+1} := (g_1^{k+1}, \ldots, g_N^{k+1}, 0) \in \partial\psi(w^{k+1})$ and $s^{k+1} := (s_1^{k+1}, \ldots, s_N^{k+1}, 0) \in N_{\mathcal{W}}(w^{k+1})$ are chosen as in (S.2b) or, for $\nu = N$, given by (6.17). Assume now that there is a subsequence $\mathcal{I}$ and some $c > 0$ such that $\|x_{-N}^{k+1} - x_{-N}^*\| \geq c > 0$ for all $k \in \mathcal{I}$. Then (6.20), Lemma 6.3, and the boundedness of $\{x_N^{k+1}\}_{k\in\mathbb{N}}$ and $\{\mu^{k+1}\}_{k\in\mathbb{N}}$ imply

$$\rho\|x_{-N}^{k+1} - x_{-N}^*\| \;\leq\; \frac{\langle g^{k+1} + P_\theta(w^{k+1}) + G(w^{k+1}) + s^{k+1} \mid w^{k+1} - w^*\rangle}{\|x_{-N}^{k+1} - x_{-N}^*\|} \to_{\mathcal{I}} 0.$$

Therefore $\|x_{-N}^{k+1} - x_{-N}^*\| \to_{\mathcal{I}} 0$, which contradicts the assumption. Hence $x_{-N}^{k+1} \to x_{-N}^*$ (recall that the x_{-N}^*-part is unique for all variational KKT points).

Part 4: We have already shown that $x_{-N}^{k+1} \to x_{-N}^*$, $x_N^{k+1} \to \bar{x}_N$, and $\mu^{k+1} \to \bar{\mu}$. It remains to verify that $\bar{w} := ((x_{-N}^*, \bar{x}_N), \bar{\mu})$ is a variational KKT pair. By the strong convergence of the block components, we have $w^{k+1} - w^k \to 0$. Therefore Assumption 6.1 (L) yields

$$\|\nabla_{x_\nu}\theta_\nu(\hat{x}^{\nu,k}) - \nabla_{x_\nu}\theta_\nu(x^{k+1})\|^2 \;\leq\; L_\nu^2 \sum_{i=\nu+1}^{N} \|x_i^{k+1} - x_i^k\|^2 \to 0.$$

Since $\{\gamma^k\}_{k\in\mathbb{N}}$ is eventually constant, we can use the linear operators defined in (6.5) and (6.6) to obtain that

$$\Delta^k \;:=\; Q_N(w^{k+1} - w^k) + R_N(w^{k+1} - w^k) + \begin{pmatrix} \nabla_{x_1}\theta_1(\hat{x}^{1,k}) - \nabla_{x_1}\theta_1(x^{k+1}) \\ \vdots \\ \nabla_{x_{N-1}}\theta_{N-1}(\hat{x}^{N-1,k}) - \nabla_{x_{N-1}}\theta_{N-1}(x^{k+1}) \\ 0 \\ 0 \end{pmatrix}$$

converges to zero. In view of Lemma 6.5, we have $-\Delta^k \in T_{GNEP}(w^{k+1})$. Using the strong-weak sequential closedness (thus also strong-strong sequential closedness) of the graph of a maximally monotone operator and exploiting Proposition 3.12, we obtain $0 \in T_{GNEP}(\bar{w})$. Hence $\bar{w} = ((x^*_{-N}, \bar{x}_N), \bar{\mu})$ is a variational KKT pair, see (3.17) or Lemma 3.10. □

Combining Theorem 6.7 and Proposition 6.9, we obtain from the discussion at the beginning of this section the following convergence result for Algorithm 6.8.

Theorem 6.10. *Assume that problem* (GNEP) *admits a variational KKT point and that Assumption 6.1 holds. Further suppose that*

$$\Upsilon > \frac{1}{\rho}\sum_{i=1}^{N-1} L_i^2 + \frac{1}{\rho}\beta^2 \sum_{i=1}^{N-1} C_i(N-i), \tag{6.21}$$

where L_ν and C_ν are defined in (6.1) *and* (6.2)*, respectively. Then the iterates $\{w^{k+1}\}_{k\in\mathbb{N}}$ generated by Algorithm 6.8 converge weakly to a variational KKT point w^* of* (GNEP)*. Furthermore, $\{x^{k+1}_{-N}\}_{k\in\mathbb{N}}$ converges strongly to x^*_{-N}.*

6.4 Comments

In this chapter and the previous Chapter 5 we presented methods for solving linearly constrained generalized Nash equilibrium problems. These methods were motivated by the very popular alternating direction method of multipliers (ADMM) for the solution of optimization problems that possess a certain structure. The convergence rate of ADMM methods is usually slow, but they can often be applied to large-scale problems since they typically have to solve only small-dimensional optimization problems at each iteration.

There are several possible modifications of Algorithms 5.1, 6.2, and 6.8 that might be of interest of their own. For example, the following changes of steps (5.1), (6.3a), or (6.15a) might be investigated in a way similar to the analysis given in this chapter and in Section 5.1.2:

- A linearized Gauss-Seidel-ADMM-type modification

$$\begin{aligned} x_\nu^{k+1} := \operatorname*{arg\,min}_{x_\nu \in \mathcal{X}_\nu} \Big\{ & \langle \nabla_{x_\nu}\theta_\nu(x_1^{k+1}, \dots, x_{\nu-1}^{k+1}, x_\nu^k, x_{\nu+1}^k, \dots, x_N^k) \mid x_\nu - x_\nu^k \rangle \\ & + \varphi_\nu(x_\nu) + \langle \mu^k \mid A_\nu x_\nu \rangle_{\mathcal{K}} + \tfrac{\gamma_\nu}{2}\|x_\nu - x_\nu^k\|^2_{\mathcal{H}_\nu} \\ & + \tfrac{\beta}{2}\|A_\nu x_\nu + \textstyle\sum_{i=1}^{\nu-1} A_i x_i^{k+1} + \sum_{i=\nu+1}^{N} A_i x_i^k - b\|^2_{\mathcal{K}} \Big\}. \end{aligned}$$

  This method differs from (6.3a) by replacing the function θ_ν by its (Gauss-Seidel-) linearization; hence the subproblems to be solved at each iteration become even simpler.

- A linearized Gauss-Seidel-Jacobi-ADMM-type modification

$$x_\nu^{k+1} := \operatorname*{arg\,min}_{x_\nu \in \mathcal{X}_\nu} \Big\{ \varphi_\nu(x_\nu) + \langle \nabla_{x_\nu} \theta_\nu(x^k) \mid x_\nu - x_\nu^k \rangle \\ + \langle \mu^k \mid A_\nu x_\nu \rangle_{\mathcal{K}} + \tfrac{\gamma_\nu}{2} \|x_\nu - x_\nu^k\|_{\mathcal{H}_\nu}^2 \\ + \tfrac{\beta}{2} \|A_\nu x_\nu + \textstyle\sum_{i=1}^{\nu-1} A_i x_i^{k+1} + \sum_{i=\nu+1}^{N} A_i x_i^k - b\|_{\mathcal{K}}^2 \Big\}.$$

  This method differs from the previous one by using the old iterate x^k in the linearization of θ_ν, whereas the penalty term includes the new information x_i^{k+1} for all $i = 1, \ldots, \nu - 1$.

- The Jacobi-ADMM-type modification

$$x_\nu^{k+1} := \operatorname*{arg\,min}_{x_\nu \in \mathcal{X}_\nu} \Big\{ \varphi_\nu(x_\nu) + \theta_\nu(x_\nu, x_{-N}^k) + \langle \mu^k \mid A_\nu x_\nu \rangle_{\mathcal{K}} \\ + \tfrac{\beta}{2} \|A_\nu x_\nu + \sum_{\substack{i=1 \\ i \neq \nu}}^{N} A_i x_i^k - b\|_{\mathcal{K}}^2 + \tfrac{\gamma_\nu}{2} \|x_\nu - x_\nu^k\|_{\mathcal{H}_\nu}^2 \Big\}.$$

  This update rule corresponds to (6.3a) except that x_i^{k+1} is replaced everywhere by x_i^k for $i = 1, \ldots, \nu - 1$, leading to a method that is fully parallel.

We believe that it is possible to extend our convergence theory to the above modifications, but leave the details for our future research.

Chapter 7

Applications

This chapter presents some applications and numerical results where the algorithms stated in Chapters 4, 5, and 6 are applicable. Since we can expect the numerical behavior of the modified methods described in Sections 4.3, 5.2, and 5.3 to be equal to the one of the initially introduced Algorithms 4.1 and 5.1, and because Algorithm 6.8 converges usually much faster than Algorithm 6.2, we concentrate on Algorithms 4.1, 5.1, and 6.8. The aim is to demonstrate the convergence behavior of the above-mentioned methods and see that the ADMM-type approach sometimes terminates after a suprisingly small number of iterations despite the fact that the global convergence rate of ADMM-like schemes is usually not that fast.

Since our examples include (linear) inequality constraints, let us first note that we can also handle problems like (GNEP$_{conic}$). In fact, using a slack variable, it was already noted in Chapter 3 that it is possible to reformulate these problems using slack variables so that they fit into our framework described in (GNEP). This allows the application of our methods also to problems of the form (GNEP$_{conic}$).

If not stated differently, we stop the applied algorithms as soon as the value of

$$r^k := \sum_{\nu=1}^{N} \left\| g_\nu^k + \nabla_{x_\nu} \theta_\nu(x^k) + A_\nu^* \mu^k + s_\nu^k \right\|^2 + \left\| \sum_{\nu=1}^{N} A_\nu x_\nu^k - b \right\|^2,$$

with arbitrary elements $g_\nu^k \in \partial_{x_\nu} \varphi_\nu(x^k)$ and $s_\nu \in N_{\mathcal{X}_\nu}(x_\nu^k)$, is less than 10^{-8}, which corresponds to the variational KKT conditions being satisfied up to an accuracy of 10^{-4} in the $\mathcal{H} \times \mathcal{K}$-norm. For optimization problems the same criterion is used by choosing $\theta_\nu \equiv 0$ and $g_\nu \in \partial_{x_\nu} f_\nu(x_\nu^k)$ for all $\nu = 1, \ldots, N$. Notice that, in general, this termination criterion is not guaranteed to work due to the fact that the subgradients s^k and g^k, which occur in the definition of r^k, are taken arbitrarily and might not be the correct ones that imply that r^k tends to zero. On the other hand, if r^k converges to zero, this is clearly an indication that we are close to a

(variational) KKT point. In our numerical experiments, we did not observe any difficulties with this termination criterion.

If not stated differently, the ADMM-subproblems (4.1), (5.1), and (6.15a) were solved using the MATLAB® function `fmincon`, with the gradient and Hessian user supplied. The `OptimalityTolerance` and `ConstraintTolerance` were left on the default values 10^{-6}.

In order to choose the penalty parameter β required for Algorithms 4.1 and 5.1, and 6.8, there are some considerations to be made. First, we need to recognize that there are two quantities included in the condition $r^k \leq \varepsilon$. The first one is basically the first order optimality condition of player ν

$$\|g_\nu^k + \nabla_{x_\nu}\theta_\nu(x^k) + A_\nu^*\mu^k + s_\nu^k\|^2,$$

and the second one is the feasibility of the iterates

$$\Big\|\sum_{\nu=1}^{N} Ax^{k+1} - b\Big\|^2;$$

both should be zero in a KKT pair (x^*, μ^*). If we choose β large, we see in the numerical behavior of all presented methods that the feasibility decreases fast, but the first order optimality condition stays quite long high. This occurs because the update of the Lagrange multiplier μ^{k+1} is very small as a consequence of the small constraint. Also, the large penalty parameter β increases the suitable regularization in Algorithms 4.1 and 5.1. This, however, can lead to slower convergence of the primal optimality condition because the step size in the primal variable x^{k+1} is penalized, which can slow down the convergence speed further. On the other hand, if β is chosen small, the first order optimality condition decreases faster and the constraint stays large for a relatively long time. Hence, it is necessary to strike a balance between these three effects, and we see by dint of noting the connection of larger β and larger regularization that, in general, a good choice of β for Algorithms 4.1 and 5.1 might be smaller than for Algorithm 6.8.

Further, it is well known that some constraints are harder to satisfy than others. Thus, it is reasonable to choose the penalty parameter β larger for difficult constraints and smaller for easier ones. If not stated differently, in the case that the augmented constraints only consist of identities, we therefore choose $\beta = 1$ in all the algorithms. If the augmented constraint is more difficult than summing parts of the x_ν, we choose $\beta = 10$ in Algorithm 4.1, $\beta = 10$ in Algorithm 5.1, and $\beta = 1000$ in Algorithm 6.8. In Algorithm 6.8, the parameters $\gamma^0 := 0.1$, $\alpha := 0.99999$, $\tau := 1$, and $\kappa = 10$ are chosen. Finally, we initialize the algorithms using the starting point $(x^0, \mu^0) := (0, 0)$.

This Chapter is organized as follows. We first present all infinite-dimensional examples, and in the end we also investigate some finite-dimensional ones. We

begin, in Section 7.1, with an infinite-dimensional optimization problem of splitting a partial differential equation into smaller and more easily manageable ones by a domain decomposition approach. Algorithm 4.1 is applied to this problem. Thereafter, in Sections 7.2 and 7.3, we discuss two infinite-dimensional elliptic control GNEPs that are actually potential games and therefore can be reduced to an optimization problem. To these problems, all the presented algorithms can be applied. Then, in Section 7.4, Algorithm 4.1 is applied to an elliptic control Nash equilibrium problem that is a potential game but does not possess a shared constraint and therefore is not a GNEP. Another infinite-dimensional GNEP is discussed in Section 7.5, and Algorithms 5.1 and 6.8 are applied. We close with some finite-dimensional examples in Section 7.6. First, in Section 7.6.1, some finite-dimensional GNEPs and their numerical results for Algorithms 5.1 and 6.8 are presented. Thereafter, in Section 7.6.2, Algorithm 4.1 is compared to other Jacobi-type ADMM-methods using the l_1-minimization problem.

7.1 Application to Domain Decomposition

Domain decomposition is a technique for the solution of boundary value problems that splits the given domain into smaller ones. Prominent examples are the methods by Schwarz, which, for certain problems, were shown to be equivalent to an augmented Lagrangian method applied to the corresponding optimization problem, see [57, 80]. Here, we follow a similar idea and show how our regularized Jacobi-type ADMM-methods from Chapter 4 can be used to obtain suitable domain decomposition methods. The central idea is described in Section 7.1.1. An explicit realization of our method and its application to a particular instance are discussed in Section 7.1.2, whereas we compute a suitable lower bound for the choice of our regularization parameter γ from Algorithm 4.1 in Section 7.1.3.

7.1.1 Non-Overlapping Domain Decomposition

In this subsection, we follow [2] to decompose the domain of a partial differential equation (PDE). For example, Figure 7.1 shows how the unit square can be decomposed into four different squares. Although we describe the idea of the domain decomposition method in a general context, we often refer to this particular example for its explicit realization. This way, we are able to avoid some technical notation; moreover, the central idea can be explained much better for this special setting.

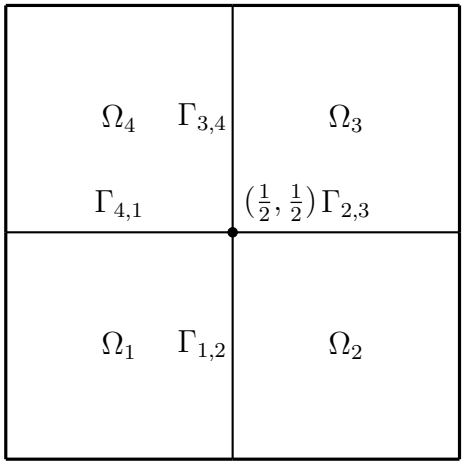

Figure 7.1: The unit square Ω decomposed into four squares Ω_i.

Let us consider the problem of solving the Laplace equation

$$-\Delta y(x) = u(x) \quad \forall x \in \Omega, \qquad y(x) = 0 \quad \forall x \in \partial\Omega,$$

where Ω is a bounded, convex Lipschitz domain in $\mathbb{R}^d$ and $u \in L^2(\Omega)$. Then the associated weak formulation is

$$\langle \nabla y \mid \nabla v \rangle_{L^2(\Omega)} = \langle u \mid v \rangle_{L^2(\Omega)} \qquad \forall v \in H_0^1(\Omega). \tag{7.1}$$

The existence of a weak solution follows from the Lax-Milgram theorem, cf. Section 2.1.3. Moreover, it is well known that solving (7.1) is equivalent to finding a solution of the optimization problem

$$\min_{y \in H_0^1(\Omega)} \Big\{ \frac{1}{2} \langle \nabla y \mid \nabla y \rangle_{L^2(\Omega)} - \langle u \mid y \rangle_{L^2(\Omega)} \Big\}. \tag{7.2}$$

Now let us decompose our domain Ω into N disjoint, convex Lipschitz subdomains Ω_i such that $\Omega = \Omega_1 \,\dot\cup \ldots \dot\cup\, \Omega_N$. We equip H_0^1 with the standard H^1-norm, which implies that there exist unique solutions of the subproblems even on subdomains Ω_i that are in the interior of Ω. Note that this is not true for the methods from [2,80] where the subdomains are required to have a non-empty intersection with the boundary of Ω, since the underlying PDE's are Robin boundary value problems. This difficulty can also be circumvented for the method from [2] by using the H^1-norm instead of the H_0^1-norm. Nevertheless, even in this case, this method is not (at least not directly) capable of handling more than two subdomains (with guaranteed convergence). Furthermore, we stress that the main idea of domain decomposition is to deal with smaller subproblems in a parallel way. This is totally reflected by our approach, whereas [2] is a sequential method.

Unfortunately, describing the main idea in a concise way, requires a technical overhead; in particular, we have to define some appropriate index sets. To this end, let J denote the set of all pairs (i,j) such that $\partial\Omega_i \cap \partial\Omega_j$ does not have vanishing $d-1$-dimensional measure; thus, in Figure 7.1, we have $J = \{(1,2),(2,3),(3,4),(4,1),(2,1),(3,2),(4,2),(1,4)\}$, whereas, e.g., the pairs $(1,3)$ and $(2,4)$ do not belong to J. Further, we denote by J^o a subset of J that does not contain permutations where the small index "o" stands for ordered; thus, we choose $J^o = \{(1,2),(2,3),(3,4),(4,1)\}$. Now let us define $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$ for all $(i,j) \in J$ and $\Gamma_i = \partial\Omega \cap \partial\Omega_i$. We denote by $H^1_{\Gamma_i}(\Omega_i)$ the $H^1(\Omega_i)$ functions with trace equal to zero on Γ_i.

In the following, we distinguish between the trace operators $\mathrm{trace}_{\Gamma_{i,j}}$ and $\mathrm{trace}_{\Gamma_{j,i}}$. The former is defined on the subdomain Ω_i with assigned values on the boundary $\Gamma_{i,j}$, whereas the latter denotes the trace operator defined on the subdomain Ω_j with assigned values on the (same) boundary $\Gamma_{j,i} = \Gamma_{i,j}$.

Remark 7.1. By partial integration, it can be seen easily that

$$\begin{aligned} & y \in H^1_0(\Omega) \\ \iff\ & \big(y_i \in H^1_{\Gamma_i}(\Omega_i)\ \forall i = 1,\dots,N \text{ and } \mathrm{trace}_{\Gamma_{i,j}}(y_i) = \mathrm{trace}_{\Gamma_{j,i}}(y_j)\ \forall (i,j) \in J^o\big); \end{aligned}$$

furthermore, it holds that $u \in L^2(\Omega)$ if and only if $u_i \in L^2(\Omega_i)$ for all $i = 1,\dots,N$, where u_i denotes the restriction of the given mapping u on Ω_i. ◇

Hence (7.2) can be written as

$$\begin{aligned} \min_{\substack{y_i \in H^1_{\Gamma_i}(\Omega_i) \\ i=1,\dots,N}} & \Big\{ \sum_{i=1}^{N} \big(\frac{1}{2} \langle \nabla y_i \mid \nabla y_i \rangle_{L^2(\Omega_i)} - \langle u_i \mid y_i \rangle_{L^2(\Omega_i)} \big) \Big\} \\ \text{s.t. } & \mathrm{trace}_{\Gamma_{i,j}}(y_i) = \mathrm{trace}_{\Gamma_{j,i}}(y_j) \qquad \forall (i,j) \in J^o. \end{aligned} \tag{7.3}$$

Since the trace operator is linear and continuous, this optimization problem is exactly of the form (Opt); therefore, we can apply Algorithm 4.1 or 4.16 to (7.3). Using the notation introduced in (Opt), we have for $i = 1,\dots,N$ that

- $\mathcal{X}_i = \mathcal{H}_i = H^1_{\Gamma_i}(\Omega_i)$,
- $\mathcal{K} = \prod_{(i,j)\in J^o} L^2(\Gamma_{i,j})$,
- $f_i : \mathcal{X}_i \to \mathbb{R}$ where $f_i(y_i) := \frac{1}{2}\langle \nabla y_i \mid \nabla y_i \rangle_{L^2(\Omega_i)} - \langle u_i \mid y_i \rangle_{L^2(\Omega_i)}$,
- $A_i : \mathcal{X}_i \to \mathcal{K}$ corresponds to the linear trace constraints stated in (7.3).

The latter is somewhat technical to describe in general, but for the particular domain from Figure 7.1, it is easy to see that the corresponding A_i's are given by

$$A_1 = \begin{pmatrix} \operatorname{trace}_{\Gamma_{1,2}} \\ 0 \\ 0 \\ -\operatorname{trace}_{\Gamma_{1,4}} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -\operatorname{trace}_{\Gamma_{2,1}} \\ \operatorname{trace}_{\Gamma_{2,3}} \\ 0 \\ 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 \\ -\operatorname{trace}_{\Gamma_{3,2}} \\ \operatorname{trace}_{\Gamma_{3,4}} \\ 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 \\ 0 \\ -\operatorname{trace}_{\Gamma_{4,3}} \\ \operatorname{trace}_{\Gamma_{4,1}} \end{pmatrix}. \tag{7.4}$$

We will later need these definitions to estimate the proximal constant γ. Further, it is easy to see that the subproblems in Algorithms 4.1 and 4.16 consist of solving a PDE on the subdomains Ω_i in every step, or, in a finite element context, of solving N lower-dimensional linear systems of equations. The corresponding details for the particular domain illustrated in Figure 7.1 are given in the next section.

Remark 7.2. If $\partial\Omega_i \cap \partial\Omega$ has $(d-1)$-dimensional measure larger than zero for all i, then the Poincaré-inequality implies that all functions f_i are strongly convex. Consequently, the corresponding iterates generated by the application of Algorithm 4.1 to the domain decomposition converge strongly in view of Theorem 4.7 (g). ◇

7.1.2 Application of the Optimization Algorithm

Now we want to apply Algorithm 4.1 to problem (7.3) for the domain displayed in Figure 7.1. Similar considerations hold for the application of Algorithm 4.16; the corresponding details are left to the reader.

The constraints in (7.3) are equivalent to $\operatorname{trace}_{\Gamma_{i,j}} y_i - \operatorname{trace}_{\Gamma_{j,i}} y_j = 0$. Following standard notation in the field of applied analysis, we omit the trace operator and identify y_i on the boundary with its trace; thus, the corresponding constraint is $y_i - y_j = 0$ on $\Gamma_{i,j}$. Consequently, the subproblems resulting from the four domains are:

$$\begin{aligned} y_1^{k+1} = \operatorname*{arg\,min}_{y_1 \in H^1_{\Gamma_1}(\Omega_1)} \Bigg\{ & \left(\frac{1}{2} \langle \nabla y_1 \mid \nabla y_1 \rangle_{L^2(\Omega_1)} - \langle u_1 \mid y_1 \rangle_{L^2(\Omega_1)} \right) \\ & + \langle \mu_{1,2}^k \mid y_1 \rangle_{L^2(\Gamma_{1,2})} - \langle \mu_{4,1}^k \mid y_1 \rangle_{L^2(\Gamma_{4,1})} \\ & + \frac{\beta}{2} \big(\|y_1 - y_2^k\|^2_{L^2(\Gamma_{1,2})} + \|y_1 - y_4^k\|^2_{L^2(\Gamma_{4,1})} \big) + \frac{\beta\gamma}{2} \|y_1 - y_1^k\|^2_{H^1_{\Gamma_1}(\Omega_1)} \Bigg\}, \end{aligned}$$

$$y_2^{k+1} = \underset{y_2 \in H^1_{\Gamma_2}(\Omega_2)}{\arg\min} \left\{ \left(\frac{1}{2} \langle \nabla y_2 \mid \nabla y_2 \rangle_{L^2(\Omega_2)} - \langle u_2 \mid y_2 \rangle_{L^2(\Omega_2)} \right) \right.$$
$$+ \langle \mu^k_{2,3} \mid y_2 \rangle_{L^2(\Gamma_{2,3})} - \langle \mu^k_{1,2} \mid y_2 \rangle_{L^2(\Gamma_{1,2})}$$
$$\left. + \frac{\beta}{2} \big(\|y_2 - y_1^k\|^2_{L^2(\Gamma_{1,2})} + \|y_2 - y_3^k\|^2_{L^2(\Gamma_{2,3})} \big) + \frac{\beta\gamma}{2} \|y_3 - y_3^k\|^2_{H^1_{\Gamma_2}(\Omega_2)} \right\},$$

$$y_3^{k+1} = \underset{y_3 \in H^1_{\Gamma_3}(\Omega_3)}{\arg\min} \left\{ \left(\frac{1}{2} \langle \nabla y_3 \mid \nabla y_3 \rangle_{L^2(\Omega_3)} - \langle u_3 \mid y_3 \rangle_{L^2(\Omega_3)} \right) \right.$$
$$+ \langle \mu^k_{3,4} \mid y_3 \rangle_{L^2(\Gamma_{3,4})} - \langle \mu^k_{2,3} \mid y_3 \rangle_{L^2(\Gamma_{2,3})}$$
$$\left. + \frac{\beta}{2} \big(\|y_3 - y_2^k\|^2_{L^2(\Gamma_{2,3})} + \|y_3 - y_4^k\|^2_{L^2(\Gamma_{3,4})} \big) + \frac{\beta\gamma}{2} \|y_3 - y_3^k\|^2_{H^1_{\Gamma_3}(\Omega_3)} \right\},$$

$$y_4^{k+1} = \underset{y_4 \in H^1_{\Gamma_4}(\Omega_4)}{\arg\min} \left\{ \left(\frac{1}{2} \langle \nabla y_4 \mid \nabla y_4 \rangle_{L^2(\Omega_4)} - \langle u_4 \mid y_4 \rangle_{L^2(\Omega_4)} \right) \right.$$
$$+ \langle \mu^k_{4,1} \mid y_4 \rangle_{L^2(\Gamma_{4,1})} - \langle \mu^k_{3,4} \mid y_4 \rangle_{L^2(\Gamma_{3,4})}$$
$$\left. + \frac{\beta}{2} \big(\|y_4 - y_3^k\|^2_{L^2(\Gamma_{3,4})} + \|y_4 - y_1^k\|^2_{L^2(\Gamma_{4,1})} \big) + \frac{\beta\gamma}{2} \|y_3 - y_3^k\|^2_{H^1_{\Gamma_4}(\Omega_4)} \right\},$$

and the multiplier update is

$$\mu^{k+1}_{1,2} = \mu^k_{1,2} + \beta(y_1^{k+1} - y_2^{k+1}), \qquad \mu^{k+1}_{2,3} = \mu^k_{2,3} + \beta(y_2^{k+1} - y_3^{k+1}),$$
$$\mu^{k+1}_{3,4} = \mu^k_{3,4} + \beta(y_3^{k+1} - y_4^{k+1}), \qquad \mu^{k+1}_{4,1} = \mu^k_{4,1} + \beta(y_4^{k+1} - y_1^{k+1}),$$

where the equalities hold in the boundary spaces $L^2(\Gamma_{i,j})$ for $(i,j) \in J^o$.

Using the sign vector defined by

$$\alpha_{i,j} = \begin{cases} +1 & \text{if } (i,j) \in J^o = \{(1,2),(2,3),(3,4),(4,1)\} \\ -1 & \text{if } (i,j) \in J \setminus J^o = \{(2,1),(3,2),(4,3),(1,4)\} \end{cases},$$

we can rewrite these subproblems more compactly as

$$y_i^{k+1} = \underset{y_i \in H^1_{\Gamma_i}(\Omega_i)}{\arg\min} \left\{ \left(\frac{1}{2} \langle \nabla y_i \mid \nabla y_i \rangle_{L^2(\Omega_i)} - \langle u_i \mid y_i \rangle_{L^2(\Omega_i)} \right) \right.$$
$$+ \sum_{j:(i,j)\in J} \alpha_{i,j} \langle \mu^k_{i,j} \mid y_i \rangle_{L^2(\Gamma_{i,j})}$$
$$\left. + \frac{\beta}{2} \sum_{j:(i,j)\in J} \|y_i - y_j^k\|^2_{L^2(\Gamma_{i,j})} + \frac{\beta\gamma}{2} \|y_i - y_i^k\|^2_{H^1_{\Gamma_i}(\Omega_i)} \right\} \qquad (7.5)$$

for all $i = 1, \dots, 4$. The associated multiplier update is

$$\mu_{i,j}^{k+1} = \mu_{i,j}^{k} + \beta(\alpha_{i,j} y_i^{k+1} + \alpha_{j,i} y_i^{k+1}) \qquad \forall (i,j) \in J^o \qquad \mu_{i,j}^{k+1} \;=\; \mu_{j,i}^{k+1} \tag{7.6}$$

in $L^2(\Gamma_{i,j})$.

The optimality conditions of (7.5) are necessary and sufficient since the subproblems are strongly convex. These optimality conditions are given by

$$\begin{aligned} \langle \nabla y_i^{k+1}, \nabla v_i \rangle_{L^2(\Omega_i)} + \sum_{j:(i,j)\in J} \alpha_{i,j} \langle \mu_{i,j}^k \mid v_i \rangle_{L^2(\Gamma_{i,j})} + \beta \sum_{j:(i,j)\in J} \langle y_i^{k+1} - y_j^k \mid v_i \rangle_{L^2(\Gamma_{i,j})} \\ + \beta\gamma \langle y_i^{k+1} - y_i^k \mid v_i \rangle_{L^2(\Omega_i)} + \beta\gamma \langle \nabla y_i^{k+1} - \nabla y_i^k \mid \nabla v_i \rangle_{L^2(\Omega_i)} = \langle u_i \mid v_i \rangle_{L^2(\Omega_i)} \end{aligned} \tag{7.7}$$

for all $v_i \in H^1_{\Gamma_i}(\Omega_i)$, $i = 1, \dots, 4$. This is the weak formulation of the PDE

$$\begin{aligned} -(1+\beta\gamma)\triangle y_i + \beta\gamma y_i &= u + \beta\gamma y_i^k - \beta\gamma \triangle y_i^k && \text{in } \Omega_i \\ \beta y_i + (1+\beta\gamma)\tfrac{\partial y_i}{n_i} &= \beta y_j^k + \beta\gamma \tfrac{\partial y_i^k}{n_i} - \alpha_{i,j}\mu_{i,j}^k && \text{in } \Gamma_{i,j}\ \forall j : (i,j) \in J \\ y_i &= 0 && \text{in } \partial\Omega \cap \partial\Omega_i, \end{aligned} \tag{7.8}$$

where n_i denotes the outer normal of Ω_i, $i = 1, \dots, 4$. The strong convexity of the subproblems shows that these partial differential equations admit unique solutions even if the partial domain Ω_i does not have any common boundary with the full domain Ω. Thus, Algorithms 4.1 and 4.16 basically consist of solving the (uniquely determined) PDEs (7.7) or (7.8), respectively, and thereafter updating the Lagrange multiplier μ as stated in (7.6).

7.1.3 Estimating the Proximal Constant γ

Now we want to figure out how to choose the constant γ in Algorithms 4.1 and 4.16 for the problem (7.3) on the domain displayed in Figure 7.1. Thus, we have to estimate the operator norm of M, defined in (4.5). To this end, we first state a lemma that estimates the operator norm of the trace operator that maps to a certain boundary part.

Lemma 7.3. *Suppose $\Omega_i \subset \mathbb{R}^2$ is a rectangle with side lengths L_1 and L_2, i.e. only through rotation and translation, it can be taken to a form $(0, L_1) \times (0, L_2)$. Suppose that Γ is a side of Ω_i with length L_2. Then $\| \operatorname{trace}_\Gamma \|^2_{H^1(\Omega)\to L^2(\Gamma)} \leq \big(L_1 + \frac{1}{L_1}\big)$.*

Proof. To prove this lemma, we follow [78, Thm. A.4]. Now first suppose that $v \in C^1([0, L])$; with $L > 0$, hence $v(x) = v(y) + \int_y^x v'(s)ds$, and therefore

$$|v(x)| \leq |v(y)| + \int_0^L |v'(s)|ds \leq |v(y)| + L^{\frac{1}{2}} \|v'\|_{L^2(0,L)} \quad \forall x, y \in [0, L].$$

Squaring both sides, integrating with respect to y, and using Young's inequality with $\varepsilon = L^2$, we obtain

$$Lv(x)^2 \leq (L^2+1)\|v\|^2_{L^2(0,L)} + (L^2+1)\|v'\|^2_{L^2(0,L)}$$

or, equivalently,

$$v(x)^2 \leq \big(L + \frac{1}{L}\big)\|v\|^2_{L^2(0,L)} + \big(L+\frac{1}{L}\big)\|v'\|^2_{L^2(0,L)}. \tag{7.9}$$

Now suppose that $y \in C^1(\overline{\Omega}_i)$, and w.l.o.g. let $\Gamma = \{0\} \times (0, L_2) \subset \partial\Omega_i$ be one boundary of Ω_i. We obtain by (7.9) that

$$y(0,x_2)^2 \leq \big(L_1 + \frac{1}{L_1}\big)\int_0^{L_1} y(x_1,x_2)^2 dx_1 + \big(L_1+\frac{1}{L_1}\big)\int_0^{L_1} \partial_1 y(x_1,x_2)^2 dx_1.$$

Integrating this equation with respect to x_2, we obtain

$$\begin{aligned}
\|\operatorname{trace}_\Gamma y\|^2_{L^2(\Gamma)} = \|y\|^2_{L^2(\Gamma)} &= \int_0^{L_2} y(0,x_2)^2 dx_2 \\
&\leq \big(L_1 + \frac{1}{L_1}\big)\|y\|^2_{L^2(\Omega_i)} + \big(L_1+\frac{1}{L_1}\big)\|\nabla y\|^2_{L^2(\Omega_i)} \\
&= \big(1 + \frac{1}{L_1}\big)\|y\|^2_{H^1(\Omega_i)}.
\end{aligned}$$

The claim follows from the density of $C^1(\overline{\Omega}_i)$ in $H^1(\Omega_i)$ and from the fact that rotation and translation do not change the operator norm. □

Lemma 7.4. *For our example domain displayed in Figure 7.1 we obtain* $\|M\| < 5.7$*, where* M *is defined as in* (4.5).

Proof. We see from (7.4) and $\langle A_i^* A_j x_j \mid y_i\rangle = \langle A_j x_j \mid A_i y_i\rangle$ that

$$\begin{aligned}
A_1^*A_2 &= -\operatorname{trace}^*_{\Gamma_{1,2}} \operatorname{trace}_{\Gamma_{2,1}}, \quad A_2^*A_1 = -\operatorname{trace}^*_{\Gamma_{2,1}} \operatorname{trace}_{\Gamma_{1,2}}, \\
A_1^*A_4 &= -\operatorname{trace}^*_{\Gamma_{1,4}} \operatorname{trace}_{\Gamma_{4,1}}, \quad A_4^*A_1 = -\operatorname{trace}^*_{\Gamma_{4,1}} \operatorname{trace}_{\Gamma_{1,4}}, \\
A_2^*A_3 &= -\operatorname{trace}^*_{\Gamma_{2,3}} \operatorname{trace}_{\Gamma_{3,2}}, \quad A_3^*A_2 = -\operatorname{trace}^*_{\Gamma_{3,2}} \operatorname{trace}_{\Gamma_{2,3}}, \\
A_3^*A_4 &= -\operatorname{trace}^*_{\Gamma_{3,4}} \operatorname{trace}_{\Gamma_{4,3}}, \quad A_3^*A_4 = -\operatorname{trace}^*_{\Gamma_{3,4}} \operatorname{trace}_{\Gamma_{4,3}}, \\
&A_1^*A_3 = 0, \quad A_3^*A_1 = 0, \quad A_2^*A_4 = 0, \quad A_4^*A_2 = 0.
\end{aligned}$$

With this we further notice that

$$\begin{aligned}
\|Mx\|^2 &= \left\| \left(\sum_{\substack{l=1 \\ l\neq i}}^4 A_i^* A_l x_l \right)_{i=1}^4 \right\|^2_{\mathcal{H}} \\
&\leq \|A_2^*A_1x_1\|^2 + \|A_4^*A_1x_1\|^2 + \|A_1^*A_2x_2\|^2 + \|A_3^*A_2x_2\|^2 \\
&\quad + \|A_2^*A_3x_3\|^2 + \|A_4^*A_3x_3\|^2 + \|A_1^*A_4x_4\|^2 + \|A_3^*A_4x_4\|^2 \\
&\leq 8 \cdot \max_{(i,j)\in J^o} \{\|\operatorname{trace}_{\Gamma_{i,j}}\|^2\}\|x\|^2;
\end{aligned}$$

hence, with the last lemma and $L = 0.5$, we obtain $\|M\| \leq \sqrt{8 \cdot 5/2} = \sqrt{20} < 5.7$. □

The previous result gives us an estimate of the constant γ appearing in Algorithms 4.1 and 4.16 since it is required that $\gamma > \|M\|$.

7.1.4 Numerical Results of the Domain Decomposition

We implemented the domain decomposition algorithm described in Section 7.1.1 with Python and the FEniCS program package, version 2017.1, see `https://fenicsproject.org/`. We used the test example

$$-\Delta y = -6 \quad \text{in } \Omega, \qquad y(x) = 1 + x_1^2 + 2x_2^2 \quad \forall x \in \partial\Omega, \tag{7.10}$$

whose exact solution is given by $y(x) = 1 + x_1^2 + 2x_2^2$, cf. [77]. The theory from Section 7.1.1 applies with standard arguments also to arbitrary Dirichlet conditions in $H^{1/2}(\partial\Omega)$; thus, (7.10) is covered by our theory. Motivated by the discussion in Section 7.1.3, we chose $\gamma = 5.7$ and $\beta = 1$ as the parameters in our explicit implementation of Algorithm 4.1. As a termination criterion, we used $\|y_i^{k+1} - y_i^k\|^2_{L^2(\Omega_i)} \leq \varepsilon$ and $\|y_i^{k+1} - y_j^{k+1}\|_{L^2(\Gamma_{i,j})} \leq \varepsilon$, where the L^2-norm is the approximate L^2-norm provided by FEniCS. We are aware that in the first criterion the H^1-norm would be better, but this norm is quite difficult to compute in FEniCS.

We made some experiments with different mesh sizes and different ε; the corresponding results are summarized in Table 7.1. The results indicate that the number of iterations is (almost) independent of the mesh size. Moreover, taking into account the dimension of the discretized problem, the number of iterations is relatively small for all test problem instances. Finally, the last column in Table 7.1 shows that the exact error, which is not used as a termination criterion in our implementation since usually the exact solution is unknown, is surprisingly small for a method whose local rate of convergence is (in general) sublinear.

To visualize the solution process, we also present some images of approximate solutions generated by the regularized Jacobi-type ADMM-method from Algorithm 4.1, see Figures 7.2–7.4. These figures correspond to three different choices of ε and show the computed solution for a mesh size whose biggest edge length is always the same and around 0.013.

Table 7.1: Some results of the regularized Jacobi-type ADMM-method from Algorithm 4.1 with different choices of the termination parameter ε and different mesh sizes for each fixed ε.

ε	largest edge in mesh	number of iterations =: k	$\|y^k - y^{exact}\|_{L^2(\Omega)}$
0.01	0.042	28	$2.5 \cdot 10^{-3}$
0.01	0.025	28	$2.3 \cdot 10^{-3}$
0.01	0.013	28	$2.3 \cdot 10^{-3}$
0.01	0.0042	28	$2.3 \cdot 10^{-3}$
0.01	0.0013	28	$2.3 \cdot 10^{-3}$
0.001	0.042	63	$4.2 \cdot 10^{-4}$
0.001	0.025	62	$1.8 \cdot 10^{-4}$
0.001	0.013	63	$1.1 \cdot 10^{-4}$
0.001	0.0042	64	$9.4 \cdot 10^{-5}$
0.001	0.0013	64	$9.3 \cdot 10^{-5}$
0.0001	0.042	289	$3.9 \cdot 10^{-4}$
0.0001	0.025	250	$1.2 \cdot 10^{-4}$
0.0001	0.013	284	$3.2 \cdot 10^{-5}$
0.0001	0.0042	302	$6.6 \cdot 10^{-6}$
0.0001	0.0013	308	$3.9 \cdot 10^{-6}$

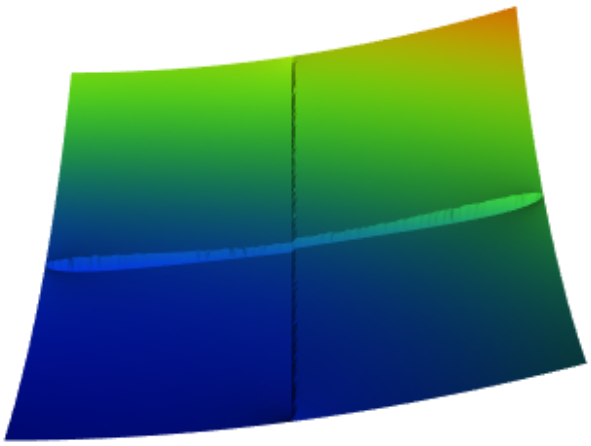

Figure 7.2: Plot of solution with $\varepsilon = 0.1$, number of iterations 5, $\|y^5 - y^{exact}\|_{L^2(\Omega)} = 0.047$. There are strong edges between the solutions on the subdomains.

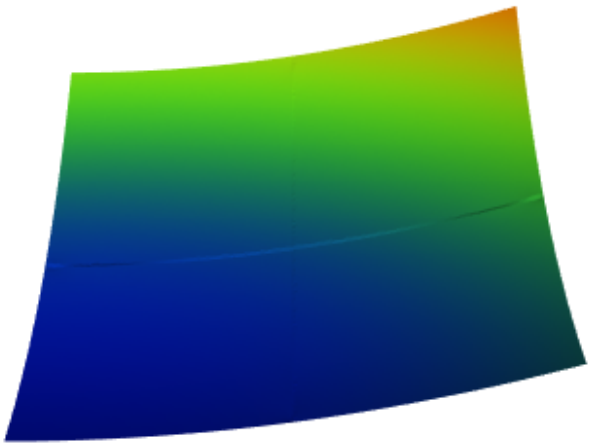

Figure 7.3: Plot of solution with $\varepsilon = 0.01$, number of iterations 28, $\|y^{28} - y^{exact}\|_{L^2(\Omega)} = 0.0023$. The edges between the solutions on the subdomains are still clearly visible.

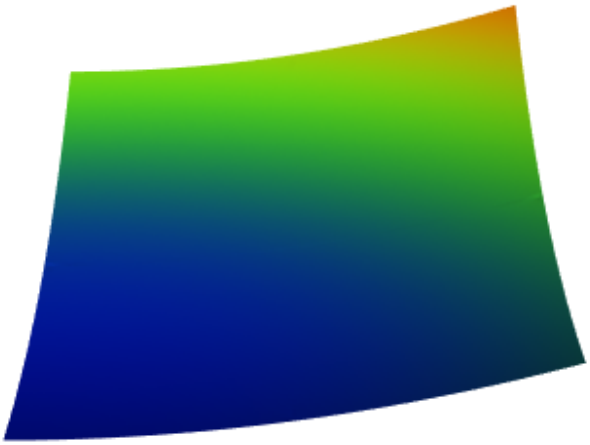

Figure 7.4: Plot of solution with $\varepsilon = 0.001$, number of iterations 63, $\|y^{63} - y^{exact}\|_{L^2(\Omega)} = 1.1 \cdot 10^{-4}$. If we zoom in, we can still see a small inaccuracy in the middle of the right edge. This inaccuracy is still there for a finer grid, but it disappears when smaller values are chosen for ε.

7.2 Elliptic Optimal Control GNEPs with Accumulated Control Bound

As an example, we now discuss a class of GNEPs that is closely related to those previously used in [67, 73, 74], where different solution methods are considered. Our aim is to verify that this example satisfies the requirements that guarantee convergence of our solution methods introduced in Chapters 5 and 6. Further we show that this particular GNEP class is equivalent to an optimization problem and that the algorithms from Chapter 4 can be applied.

In order to describe the example, we slightly change our notation in this section to be consistent with the standard notation used in the optimal control setting. The players' strategies $x_\nu \in \mathcal{X}_\nu$ are now called the controls and denoted by $u_\nu \in L^2(\Omega)$, Ω being a suitable domain in $\mathbb{R}^d$. The so-called state variable $y \in H_0^1(\Omega)$ is the solution of an elliptic partial differential equation that depends on the players' strategies $u = (u_1, \ldots, u_N) \in L^2(\Omega)^N$. We then consider the optimal control generalized Nash problem

$$\min_{\substack{u_\nu \in L^2(\Omega) \\ y \in H_0^1(\Omega)}} \left\{ \frac{1}{2} \|y(u_\nu, u_{-\nu}) - y_\nu^d\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2} \|u_\nu\|_{L^2(\Omega)}^2 \right\} \tag{7.11a}$$

$$\begin{aligned} \text{s.t.} \qquad -\triangle y &= \textstyle\sum_{\nu=1}^N u_\nu && \text{in } \Omega, \\ y &= 0 && \text{in } \partial\Omega, \\ u_\nu(x) &\in [u_\nu^a(x), u_\nu^b(x)] && \text{f.a.a. } x \in \Omega, \\ \textstyle\sum_{\nu=1}^N u_\nu(x) &\leq \psi(x) && \text{f.a.a. } x \in \Omega \end{aligned} \tag{7.11b}$$

with a sufficiently smooth domain Ω. Hence, we have a tracking-type objective function for each player $\nu = 1, \ldots, N$, pointwise lower and upper bounds on the controls u_ν, and an additional upper bound on the sum of controls $\sum_{\nu=1}^N u_\nu$.

7.2.1 Theoretical Considerations

It is desired to apply the algorithms from Chapters 4, 5, and 6 to the problem (7.11). Before we can do this, we have to investigate the problem in more detail. For the algorithms from Chapter 4, the problem (7.11) has to be transformed into the form of (Opt), and for the algorithms from Chapters 5 and 6 it is necessary to check the assumptions made in Theorems 5.6, 5.7, and 6.10. We start with the reformulation as an optimization problem (Opt).

Reformulation as an Optimization Problem

In order to apply Algorithm 4.1, we have to show that problem (7.11) is equivalent to an optimization problem of the form (Opt), i.e. that it is a potential game of

a certain form. By rewriting (7.11) as a variational inequality, it can be seen as in [68, Prop. 3.10] that finding a variational equilibrium of (7.11) is equivalent to solving the optimization problem

$$\min_{\substack{u_1,\dots,u_N\in L^2(\Omega)\\ y\in H_0^1(\Omega)}} \left\{ \frac{1}{2}\|y\|^2_{L^2(\Omega)} + \sum_{\nu=1}^{N} \left(\frac{\alpha_\nu}{2}\|u_\nu\|^2_{L^2(\Omega)} - \langle y_\nu^d \mid S_\nu u_\nu\rangle_{L^2(\Omega)} \right) \right\} \tag{7.12a}$$

$$\begin{aligned} \text{s.t.} \qquad -\triangle y &= \textstyle\sum_{\nu=1}^N u_\nu && \text{in } \Omega, \\ y &= 0 && \text{in } \partial\Omega, \\ \textstyle\sum_{\nu=1}^N u_\nu &\le \psi(x) && \text{f.a.a. } x\in\Omega, \\ u_\nu(x) &\in [u_\nu^a(x), u_\nu^b(x)] && \text{f.a.a. } x\in\Omega, \end{aligned} \tag{7.12b}$$

where $S_\nu u_\nu$ denotes the weak solution of

$$\begin{aligned} -\triangle z &= u_\nu && \text{in } \Omega, \\ z &= 0 && \text{in } \partial\Omega. \end{aligned}$$

Thus, $S_\nu \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega))$ is the solution operator of the Laplace equation from Section 2.1.3. Taking $u_1,\dots,u_N$ and y as variables, this problem has a separable structure as needed for Algorithm 4.1, where the linear constraints are

$$\begin{aligned} \sum_{\nu=1}^{N} u_\nu &\le \psi && \text{in } \Omega, \\ -\triangle y = \sum_{\nu=1}^{N} u_\nu & && \text{in } \Omega, \\ y = 0 & && \text{in } \partial\Omega. \end{aligned}$$

Since the space $\mathcal{H}_i$ from (Opt) is chosen to be $L^2(\Omega)$, we note that the Hilbert-triple $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \cong L^2(\Omega)^* \hookrightarrow H^{-1}(\Omega)$ is used implicitly. Taking into account the Hilbert-triple, we interpret the operators S_ν as maps from $L^2(\Omega)$ to $H_0^1(\Omega)$.

Let us first note that we have already discussed that the operator S_ν and $-\triangle$ are isometric isomorphisms between the spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$; thus, $\|-\triangle\|_{H_0^1(\Omega)\to H^{-1}(\Omega)} = 1 = \|S\|_{H^{-1}(\Omega)\to H_0^1(\Omega)}$, but the norm of S_ν as a map from $L^2(\Omega)$ to $H_0^1(\Omega)$ or $L^2(\Omega)$ to $L^2(\Omega)$, is not necessarily equal to one.

In view of the theory presented in Chapters 4, 5, and 6, in particular the definition of the operator M stated in (4.5) or (5.3), we have to find upper estimates of $\|S_\nu\|_{L^2(\Omega)\to L^2(\Omega)}$ and $\|S_\nu\|_{L^2(\Omega)\to H_0^1(\Omega)}$.

Proposition 7.5. *Suppose that the domain $\Omega \subset \mathbb{R}^d$ is contained in a cube with edges of length $c > 0$. Then the solution operator $S_\nu : L^2(\Omega) \to H_0^1(\Omega)$ of*

$$\langle \nabla y_\nu \mid \nabla v_\nu \rangle_{L^2(\Omega)} \;=\; \langle u_\nu \mid v_\nu \rangle \qquad \forall v_\nu \in H_0^1(\Omega) \tag{7.13}$$

satisfies $\|S_\nu\|_{L^2(\Omega)\to L^2(\Omega)} \le c^2$ and $\|S_\nu\|_{L^2(\Omega)\to H_0^1(\Omega)} \le c$.

Proof. We omit the index ν in this proof. The definition of c together with the Poincaré inequality from, e.g., Theorem 2.10 implies that $\|u\|_{L^2(\Omega)} \le c\|\nabla u\|_{L^2(\Omega)}$ for all $u \in H_0^1(\Omega)$. We therefore obtain

$$\begin{aligned}\frac{1}{c^2}\|Su\|_{L^2(\Omega)}^2 \;&\le\; \|\nabla Su\|_{L^2(\Omega)}^2 \;=\; \langle \nabla Su \mid \nabla Su\rangle_{L^2(\Omega)} \;=\; \langle u \mid Su\rangle_{L^2(\Omega)} \\ &\le\; \|u\|_{L^2(\Omega)}\|Su\|_{L^2(\Omega)},\end{aligned}$$

from which we get

$$\|S\|_{L^2(\Omega)\to L^2(\Omega)} \;=\; \sup_{u\neq 0} \frac{\|Su\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} \;\le\; c^2,$$

which is the first of our assertions. Further, we obtain again with the Poincaré inequality

$$\begin{aligned}\|Su\|_{H_0^1(\Omega)}^2 \;&=\; \|\nabla Su\|_{L^2(\Omega)}^2 \;=\; \langle \nabla Su \mid \nabla Su\rangle_{L^2(\Omega)} \;=\; \langle u \mid Su\rangle_{L^2(\Omega)} \\ &\le\; \|u\|_{L^2(\Omega)}\|Su\|_{L^2(\Omega)} \;\le\; c\|u\|_{L^2(\Omega)}\|\nabla Su\|_{L^2(\Omega)} \\ &=\; c\|u\|_{L^2(\Omega)}\|Su\|_{H_0^1(\Omega)},\end{aligned}$$

from which we get

$$\|S\|_{L^2(\Omega)\to H_0^1(\Omega)} \;=\; \sup_{u\neq 0} \frac{\|Su\|_{H_0^1(\Omega)}}{\|u\|_{L^2(\Omega)}} \;\le\; c,$$

which is the second of our assertions. □

Verification of Cocoercivity and Strong Monotonicity

Now we put the problem (7.11) in the form (GNEP$_{conic}$) and check the assumptions that are needed for the convergence of the algorithms from Chapters 5 and 6. Using the control-to-state-map

$$S : L^2(\Omega)^N \to H_0^1(\Omega) \cap C(\overline{\Omega}), \quad u \mapsto y, \quad Su = \sum_{\nu=1}^{N} S_\nu u_\nu,$$

where the last expression uses the linearity of the solution mapping S, we can rewrite (7.11) as

$$\begin{aligned} \min_{u_\nu \in L^2(\Omega)} & \Big\{ \frac{1}{2} \Big\| \sum_{\nu=1}^N S_\nu u_\nu - y_\nu^d \Big\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2} \|u_\nu\|_{L^2(\Omega)}^2 \Big\} \\ \text{s.t.} \quad & u_\nu \in [u_\nu^a, u_\nu^b], \\ & \sum_{\nu=1}^N u_\nu \leq \psi, \end{aligned} \tag{7.14}$$

and therefore obtain a GNEP of the form (GNEP$_{conic}$) by taking

$$\begin{aligned} \theta_\nu(u) &= \frac{1}{2} \Big\| \sum_{\nu=1}^N S_\nu u_\nu - y_\nu^d \Big\|_{L^2(\Omega)}^2, \qquad \varphi_\nu(u_\nu) = \frac{\alpha_\nu}{2} \|u_\nu\|_{L^2(\Omega)}^2, \\ \mathcal{X}_\nu &= \big\{ u_\nu \in L^2(\Omega) \mid u_\nu(x) \in [u_\nu^a(x), u_\nu^b(x)] \quad \text{f.a.a. } x \in \Omega \big\}, \\ B_\nu &= Id, \qquad b = \psi. \end{aligned}$$

Our aim is to show that the resulting operator $\widehat{P}_\theta$ is α-cocoercive for a suitable $\alpha > 0$. To this end, note that the subsequent analysis makes use of the Gelfand triple $H_0^1(\Omega) \subset L^2(\Omega) \cong L^2(\Omega)^* \subset H^{-1}(\Omega)$, where we identify the Hilbert space $L^2(\Omega)$ with its dual. Moreover, recall that Su and $S_\nu u_\nu$ are elements of $H_0^1(\Omega)$, but we often view them as elements of the space $L^2(\Omega)$. Formally, this means that we often consider the mappings $I \circ S$ and $I \circ S_\nu$, where $I := I_{H_0^1 \to L^2}$ denotes the canonical embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$. For notational convenience, we follow the standard convention and omit writing the mapping I everywhere.

Proposition 7.6. *For the problem* (7.11) *the in* (3.9) *defined operator* $\widehat{P}_\theta : L^2(\Omega)^N \to L^2(\Omega)^N$ *is* α*-cocoercive with*

$$\alpha := \frac{1}{N} \min_{\nu=1,\ldots,N} \Big\{ \frac{1}{\|S_\nu\|_{L^2(\Omega) \to L^2(\Omega)}^2} \Big\},$$

where $S_\nu : L^2(\Omega) \to L^2(\Omega)$ *is the solution operator of* (7.13).

Proof. By an elementary calculation, see [73, Lem. 6.2], we obtain

$$\langle \widehat{P}_\theta(u) - \widehat{P}_\theta(v) \mid u - v \rangle_{L^2(\Omega)^N} = \|Su - Sv\|_{L^2(\Omega)}^2.$$

Since $\|S_\nu^* Su\| \le \|S_\nu^*\| \, \|Su\|$, and thus $\|S_\nu^* Su\|/\|S_\nu^*\| \le \|Su\|$, we therefore obtain

$$\begin{aligned}
\langle \widehat{P}_\theta(u) - \widehat{P}_\theta(v) \mid u - v \rangle_{L^2(\Omega)^N} &= \|Su - Sv\|^2_{L^2(\Omega)} \\
&= \frac{1}{N} \sum_{\nu=1}^{N} \|Su - y_\nu^d - (Sv - y_\nu^d)\|^2_{L^2(\Omega)} \\
&\ge \frac{1}{N} \sum_{\nu=1}^{N} \frac{1}{\|S_\nu^*\|^2} \big\| S_\nu^*(Su - y_\nu^d) - S_\nu^*(Sv - y_\nu^d) \big\|^2_{L^2(\Omega)} \\
&= \frac{1}{N} \sum_{\nu=1}^{N} \frac{1}{\|S_\nu\|^2} \big\| \big[\widehat{P}_\theta(u) - \widehat{P}_\theta(v)\big]_\nu \big\|^2_{L^2(\Omega)} \\
&\ge \frac{1}{N} \min_{\nu=1,\dots,N} \left\{ \frac{1}{\|S_\nu\|^2} \right\} \big\| \widehat{P}_\theta(u) - \widehat{P}_\theta(v) \big\|^2_{L^2(\Omega)^N},
\end{aligned}$$

which is precisely the statement. □

Summarizing the previous results, we finally obtain the α-cocoercivity of the operator $\widehat{P}_\theta$ with a computable constant α; thus, Algorithms 5.1 and 5.9 are applicable.

Theorem 7.7. *Suppose that $\Omega \subset \mathbb{R}^d$ is contained in a cube with side length $c > 0$. Then the operator $\widehat{P}_\theta : L^2(\Omega)^N \to L^2(\Omega)^N$ for the problem* (7.11)*, as defined in* (3.9)*, is α-cocoercive with $\alpha := \frac{1}{Nc^4}$.*

Proof. The statement follows immediately from Propositions 7.5 and 7.6 . □

The next theorem shows that Algorithms 6.2 and 6.8 are also applicable.

Theorem 7.8 ([73, Lem. 6.2])**.** *Suppose that $\Omega \subset \mathbb{R}^d$ is contained in a cube with side length $c > 0$. Then the operator $\partial\varphi + \widehat{P}_\theta$ for the problem* (7.11)*, as defined in* (3.9) *and* (3.10)*, is ρ-strongly monotone with $\rho = \min_{\nu=1,\dots,N} \alpha_\nu$, where the α_ν are the Tikhonov parameters from* (7.11)*.*

Proof. This follows straight from Theorem 7.7 and

$$\begin{aligned}
\sum_{\nu=1}^{N} \left\langle \nabla_{u_\nu}\Big(\frac{\alpha_\nu}{2}\|u_\nu\|^2_{L^2(\Omega)}\Big) - \nabla_{v_\nu}\Big(\frac{\alpha_\nu}{2}\|v_\nu\|^2_{L^2(\Omega)}\Big) \mid u_\nu - v_\nu \right\rangle &= \sum_{\nu=1}^{N} \alpha_\nu \|u_\nu - v_\nu\|^2_{L^2(\Omega)} \\
&\ge \rho \|u - v\|^2_{L^2(\Omega)^N},
\end{aligned}$$

for all $u = (u_1, \dots, u_N), v = (v_1, \dots, v_N) \in L^2(\Omega)^N$. □

7.2.2 Numerical Results

Using $N = 4$ and $\Omega = (0,1)^2$ as in our numerical example below, we can choose $c = 1$ in Theorem 7.7 and then obtain the α-cocoercivity of $\widehat{P}_\theta$ with $\alpha = 1/4$ from Theorem 7.7. From this cocoercivity, we obtain the Lipschitz continuity of $\widehat{P}_\theta$. The strong monotonicity of $\partial\varphi + \widehat{P}_\theta$ was derived in Theorem 7.8.

The norm of the operator M, defined in (4.5) and (5.3), is important for the application of Algorithm 4.1 and Algorithm 5.1. We estimated the norm of this operator M for conic constraints in Remark 3.15 and Remark 5.8. Regarding Algorithm 5.1, this operator norm is required for estimating the right choice of the regularization constant γ from Theorem 5.6 or Theorem 5.7. By Remark 5.8, we obtain $\|M\| \leq 4(N-1) = 12$. Thus, we have all constants that are necessary for applying Algorithm 5.1.

As we have seen above, the problem discussed in this section is equivalent to a separable optimization problem in the variables $(y, x_1, \ldots, x_N)$, i.e. we have introduced the additional dummy "player" in the variable y. Both the PDE constraint and the inequality $\sum u_\nu \leq \psi$ will be augmented. Therefore, we obtain $\|M\| \leq 4 \cdot N = 16$ for the application of Algorithm 4.1.

As already mentioned, an exact estimate of the value Υ, which corresponds to the maximal γ^k in Algorithm 6.8, is not of such importance because the value of γ^k usually stays way below this maximal Υ. Therefore, we choose $\Upsilon = 1.1 \cdot 3 \cdot (1 + \beta^2 N)/\min_{\nu=1,\ldots,N}(\alpha_\nu)$.

We implemented the elliptic optimal control GNEP presented above in MATLAB®, with $N = 4$, $\Omega = (0,1)^2$, $\alpha_\nu = 1$ for all $\nu = 1, \ldots, 4$,

$$y_\nu^d(x) \;=\; 10^3 \max\left(0, 4\Big(\frac{1}{4} - \max\big(|x_1 - z_\nu^1|, |x_2 - z_\nu^2|\big)\Big)\right),$$

where $z^1 = (0.25, 0.75, 0.25, 0.75)$ and $z^2 = (0.25, 0.25, 0.75, 0.75)$, and the box constraints $[u_\nu^a(x), u_\nu^b(x)] = [-1, 1]$ for all $\nu = 1, \ldots, N$. We present results for the two choices

$$\begin{aligned}
\psi_1(x) &:= \frac{3}{2}\left(\cos\left(5\sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2}\right) + 1\right) \quad \text{and} \\
\psi_2(x) &:= 3\left(\cos\left(5\sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2}\right) + 1\right).
\end{aligned}$$

Recall that, in Section 3.2.2, we presented two different ways to rewrite problem (GNEP_{conic}) as an equality constrained problem either by inserting N slack variables, see (3.18), or by using a single slack variable, see (3.19). We call these two different approaches A. (3.18) and A. (3.19). However, due to the strong monotonicity assumption of Algorithm 6.8, only A. (3.19) is applicable for this

algorithm. In order to keep the presentation simple, A. (3.19) is also used in Algorithm 4.1.

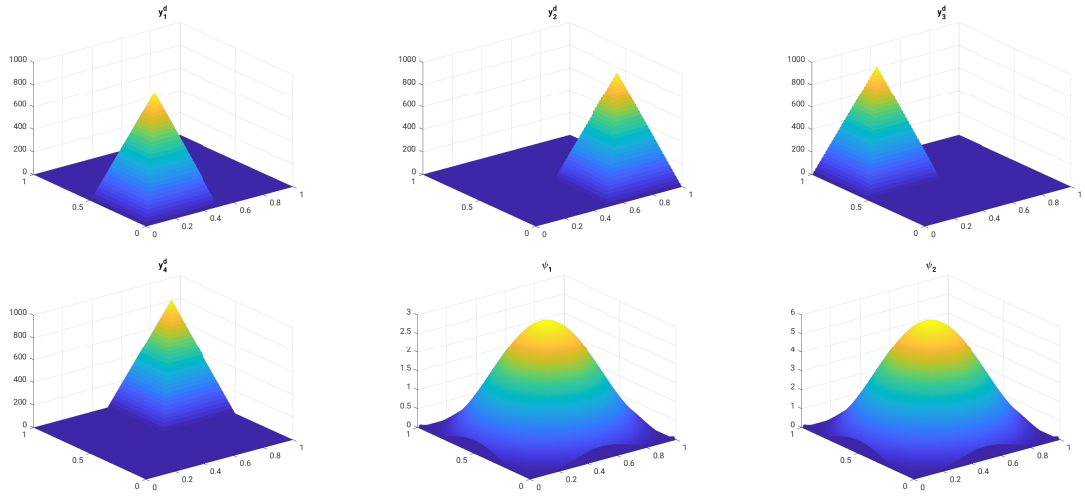

Figure 7.5: Desired states $y_d^1, \ldots, y_d^4$ and bounds ψ_1, ψ_2.

The numerical results for different discretization widths are given in Tables 7.2 and 7.3.

Note that in Algorithm 6.8, the value of γ^k was not increased once for ψ_1. This led to a very small number of iterations until the termination criterion was satisfied. However, for ψ_2, the value of γ^k was increased for the discretization widths 16 and 64; this led to a higher number of iterations. A possible explanation can be seen from the solution, especially by looking at the pictures of the corresponding Lagrange multipliers. Figure 7.6 indicates that the Lagrange multiplier for ψ_1 is in $L^2(\Omega)$, whereas Figure 7.7 allows the interpretation that, for ψ_2, the optimal multiplier seems to belong to a measure space only. Comparing Algorithms 5.1 and 6.8, the iteration count for the Gauss-Seidel-type Algorithm 6.8 is smaller than the iteration number of the Jacobi-type Algorithm 5.1, at least when both algorithms are applied with one slack variable, i.e. A. (3.19) is used. This is most likely due to the fact that the Gauss-Seidel scheme uses the newest information to compute the player's iterate x_ν^{k+1}, as well as the possibility of a smaller regularization, which is due to the increase of the regularization during the iteration.

The iteration numbers for the parallel (GNEP-) Algorithm 5.1 are smaller than the iteration numbers for the parallel optimization Algorithm 4.1. A possible explanation is that in the optimization Algorithm 4.1, there is no other way than augmenting the inequality constraint as well as the PDE constraint; however, in Algorithm 5.1, which is tailored for GNEPs, only the shared inequality constraint has to be augmented. This could lead to a smaller iteration number, since the

constraint is easier to satisfy. Further, the optimization in Algorithm 4.1 is carried out in the variables $(y, x_1, \ldots, x_4)$. This is one variable more than used in Algorithms 5.1 and 6.8. This could also lead to a higher iteration number.

In Chapter 3, two approaches of handling conic constraints were presented. The first, A. (3.18), used N slack variables, whereas the second, A. (3.19), only used one. Even if the approach with more slack variables can sometimes keep the number of iterations lower, the approach from (3.19) is superior to (3.18) taking into account that the subproblems are harder to solve.

Table 7.2: Number of iterations for the elliptic optimal control GNEP (7.11) with ψ_1 until the KKT conditions are satisfied with accuracy 10^{-4}.

Discretization	16	32	64
Iterations of Alg. 4.1, $\beta = 10$, $\gamma = 17.6$	1604	1610	1619
Iterations of Alg. 5.1 with A. (3.18), $\beta = 1$, $\gamma = 15.4$	146	146	147
Iterations of Alg. 5.1 with A. (3.19), $\beta = 1$, $\gamma = 15.4$	146	146	147
Iterations of Alg. 6.8, $\beta = 1$	16	16	16
Final γ in Alg. 6.8	0.1	0.1	0.1

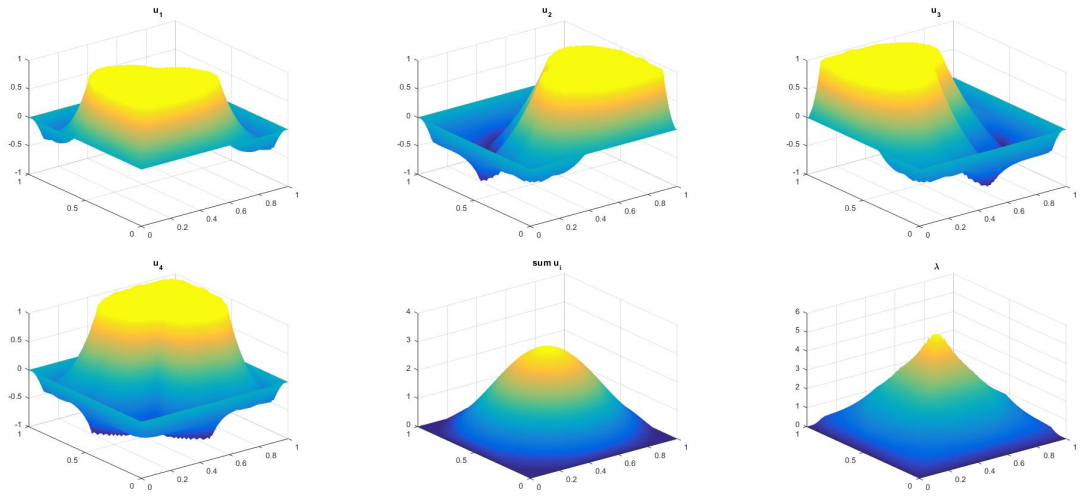

Figure 7.6: The controls $u_1, \ldots, u_4$, their sum, and the Lagrange multiplier for the problem with ψ_1.

Table 7.3: Number of iterations for the elliptic optimal control GNEP (7.11) with ψ_2 until the KKT conditions are satisfied with accuracy 10^{-4}.

Discretization	16	32	64
Iterations of Alg. 4.1, $\beta = 10$, $\gamma = 17.6$	1529	1535	1550
Iterations of Alg. 5.1 with A. (3.18), $\beta = 1$, $\gamma = 15.4$	142	142	182
Iterations of Alg. 5.1 with A. (3.19), $\beta = 1$, $\gamma = 15.4$	275	276	287
Iterations of Alg. 6.8, $\beta = 1$	55	43	215
Final γ in Alg. 6.8	3.1	0.1	13.1

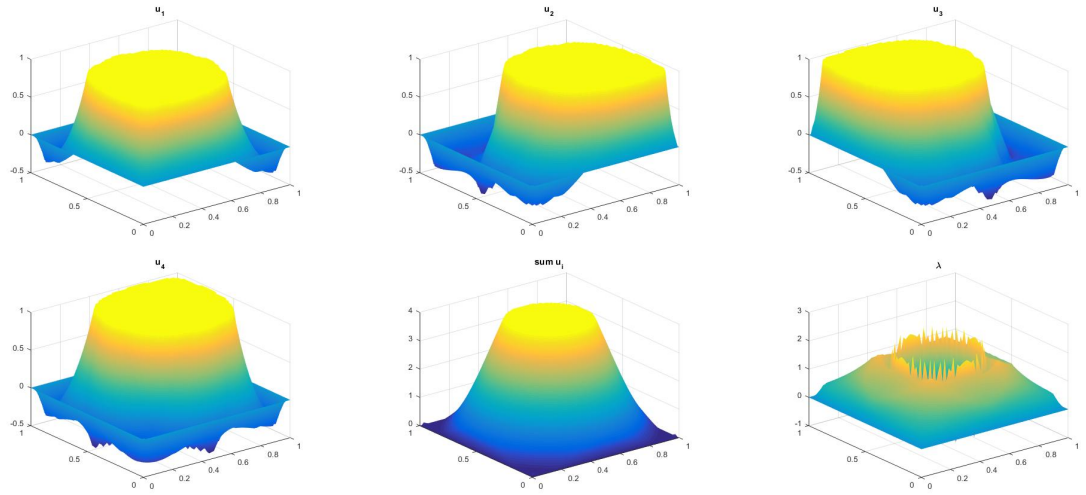

Figure 7.7: The controls $u_1, \ldots, u_4$, their sum, and the Lagrange multiplier for the problem with ψ_2.

7.3 Elliptic Optimal Control GNEPs with State Bound

Here, we discuss a class of examples that is closely related to the one in Section 7.2. We replace the constraint $\sum u_\nu \leq \psi$ on the controls by a more involving constraint on the state $y \geq \psi$; this class of examples was previously used and discussed in [67, 73, 74], where different solution methods were considered.

As in Section 7.2, to be consistent with the standard notation used in the optimal control setting, we slightly change our notation in this section in order to describe the example. Again, the players' strategies $x_\nu \in \mathcal{X}_\nu$ are now called the controls and denoted by $u_\nu \in L^2(\Omega)$, Ω being a suitable and sufficiently smooth domain in $\mathbb{R}^d$. The so-called state variable $y \in H_0^1(\Omega)$ is the solution

of an elliptic partial differential equation that depends on the players' strategies $u = (u_1, \dots, u_N) \in L^2(\Omega)^N$. We then consider the optimal control generalized Nash problem

$$\min_{u_\nu \in L^2(\Omega)} \Big\{ \frac{1}{2} \|y(u_\nu, u_{-\nu}) - y_\nu^d\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2} \|u_\nu\|_{L^2(\Omega)}^2 \Big\} \tag{7.15a}$$

$$\begin{aligned} \text{s.t.} \quad -\triangle y &= \textstyle\sum_{\nu=1}^N u_\nu && \text{in } \Omega, \\ y &= 0 && \text{in } \partial\Omega, \\ u_\nu(x) &\in [u_\nu^a(x), u_\nu^b(x)] && \text{f.a.a. } x \in \Omega, \\ y(x) &\geq \psi(x) && \text{f.a.a. } x \in \Omega. \end{aligned} \tag{7.15b}$$

Hence, we have a tracking-type objective function for each player $\nu = 1, \dots, N$, pointwise lower and upper bounds on the controls u_ν, and an additional upper bound on the state.

Again, let $S_\nu u_\nu$ denote the solution of

$$\begin{aligned} -\triangle z &= u_\nu && \text{in } \Omega, \\ z &= 0 && \text{in } \partial\Omega, \end{aligned}$$

and let S denote the all-player-control-to-state-map

$$S : L^2(\Omega)^N \to H_0^1(\Omega) \cap C(\overline{\Omega}), \quad u \mapsto y, \quad Su = \sum_{\nu=1}^N S_\nu u_\nu.$$

The linearity of the solution mapping of the Laplace equation shows that Su is the solution of $-\triangle y = u = \sum u_\nu$ in Ω and $y = 0$ in $\partial\Omega$. Thus, we can rewrite (7.15) as

$$\min_{u_\nu \in L^2(\Omega)} \Big\{ \frac{1}{2} \Big\| \sum_{\nu=1}^N S_\nu u_\nu - y_\nu^d \Big\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2} \|u_\nu\|_{L^2(\Omega)}^2 \Big\} \tag{7.16a}$$

$$\begin{aligned} \text{s.t.} \quad u_\nu(x) &\in [u_\nu^a(x), u_\nu^b(x)] && \text{f.a.a. } x \in \Omega, \\ \textstyle\sum_{\nu=1}^N S_\nu u_\nu &\geq \psi(x) && \text{f.a.a. } x \in \Omega, \end{aligned} \tag{7.16b}$$

and therefore obtain a GNEP of the form (GNEP_{conic}) by taking

$$\begin{aligned} \theta_\nu(u) &= \frac{1}{2} \Big\| \sum_{\nu=1}^N S_\nu u_\nu - y_\nu^d \Big\|_{L^2(\Omega)}^2, \qquad \varphi_\nu(u_\nu) = \frac{\alpha_\nu}{2} \|u_\nu\|_{L^2(\Omega)}^2, \\ \mathcal{X}_\nu &= \big\{ u_\nu \in L^2(\Omega) \mid u_\nu(x) \in [u_\nu^a(x), u_\nu^b(x)] \quad \text{f.a.a. } x \in \Omega \big\}, \\ B_\nu &= S_\nu, \qquad b = \psi. \end{aligned}$$

Note, that φ_ν, θ_ν and $\widehat{P}_\theta$ are the same as in Section 7.2, and therefore the above discussed results Proposition 7.5 to Theorem 7.8 still hold true, i.e. the operators

$\widehat{P}_\theta$ and $\partial\varphi + \widehat{P}_\theta$ are cocoercive and strongly monotone, respectively. If Ω is a cube with edges of length c, then, in Proposition 7.5, the operator-norm of S_ν was estimated. By Remark 5.8, we therefore obtain $\|M\| \leq 4(N-1) = 12$ if $\Omega = (0,1)^2$ for the application of Algorithm 5.1.

For Algorithm 6.8 we choose $\Upsilon = 1.1((N-1) + 3\beta^2 N)/\min(\alpha_\nu)$.

As in Section 7.2, we also see that the GNEP (7.15) is equivalent to the optimization problem

$$\min_{\substack{u_1,\dots,u_N \in L^2(\Omega) \\ y \in H_0^1(\Omega)}} \left\{ \frac{1}{2}\|y\|^2_{L^2(\Omega)} + \sum_{\nu=1}^{N} \left\{ \frac{\alpha_\nu}{2}\|u_\nu\|^2_{L^2(\Omega)} - \langle y_\nu^d \mid S_\nu u_\nu \rangle_{L^2(\Omega)} \right\} \right\} \tag{7.17a}$$

$$\begin{array}{rrll} \text{s.t.} & -\triangle y & = \sum_{\nu=1}^N u_\nu & \text{in } \Omega, \\ & y & = 0 & \text{in } \partial\Omega, \\ & y & \geq \psi(x) & \text{f.a.a. } x \in \Omega, \\ & u_\nu(x) & \in [u_\nu^a(x), u_\nu^b(x)] & \text{f.a.a. } x \in \Omega. \end{array} \tag{7.17b}$$

Here, the constraint $y \geq \psi$ can be implemented as a box constraint for the variable y. Thus, the augmented constraint is just $-\triangle y = \sum u_\nu$ or $y = \sum S_\nu u_\nu$ in the corresponding norm. By Lemma 4.4 and Proposition 7.5, we therefore obtain $\|M\| \leq N$ if $\Omega = (0,1)^2$ for the application of Algorithm 4.1.

We implemented the elliptic optimal control GNEP presented above with the same functions and parameters as [67,73,74] in MATLAB®, i.e. $N = 4$, $\Omega = (0,1)^2$, $\alpha = (2.8859, 4.3374, 2.5921, 3.9481)$, and $u_\nu^a = -12$, $u_\nu^b = 12$ for all $\nu = 1, \dots, 4$. Further we choose

$$y_1^d = \xi_1 - \xi_4, \qquad y_2^d = \xi_2 - \xi_3, \qquad y_3^d = \xi_3 - \xi_2, \qquad y_4^d = \xi_4 - \xi_1 \tag{7.18}$$

where

$$\xi_i(x) \;=\; 10^3 \max\left(0, 4\Big(\frac{1}{4} - \max\big(|x_1 - z_i^1|, |x_2 - z_i^2|\big)\Big)\right)$$

and $z^1 = (0.25, 0.75, 0.25, 0.75)$, $z^2 = (0.25, 0.25, 0.75, 0.75)$. The constraint function is

$$\psi(x) \;:=\; \left(\cos\left(5\sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2}\right) + 0.1\right).$$

The desired states of the players $\nu = 1, \dots, 4$ are plotted in Figure 7.8.

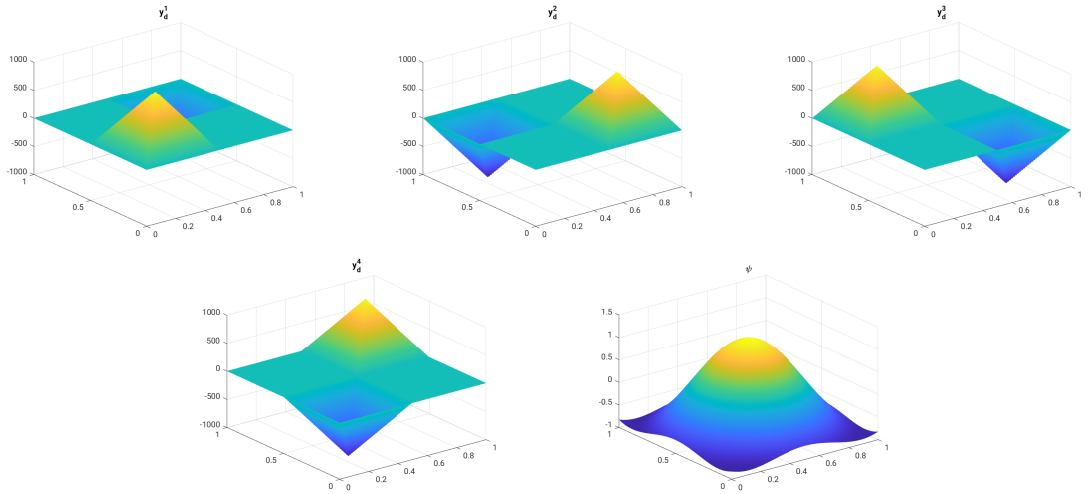

Figure 7.8: Desired states $y_d^1, \ldots, y_d^4$ and state bound ψ.

The numerical results for different algorithms and discretization widths are given in Table 7.4. Here, almost everything already said in Section 7.2.2 about the convergence behavior of Algorithms 4.1, 5.1, and 6.8 and the two approaches A. (3.18) and A. (3.19) could be repeated. Only the convergence behavior of Algorithm 4.1 changes a bit because now the state bound $y \geq \psi$ was implemented as a box constraint, keeping the iteration number lower than the iteration number of Algorithm 5.1.

Also note that, for this example, there was no need to increase γ^k at any iteration of Algorithm 6.8, which kept the total number of iterations very small for essentially all levels of discretization.

Table 7.4: Number of iterations for the elliptic optimal control GNEP (7.15) until the KKT conditions are satisfied with accuracy 10^{-4}.

Discretization	16	32	64
Iterations of Alg. 4.1, $\beta = 10$, $\gamma = 4.4$	330	1012	4303
Iterations of Alg. 5.1 with A. (3.18), $\beta = 10$, $\gamma = 13.42$	818	2660	7420
Iterations of Alg. 5.1 with A. (3.19), $\beta = 10$, $\gamma = 13.42$	824	2670	7456
Iterations of Alg. 6.8, $\beta = 1000$	95	95	95
Final γ in Alg. 6.8	0.1	0.1	0.1

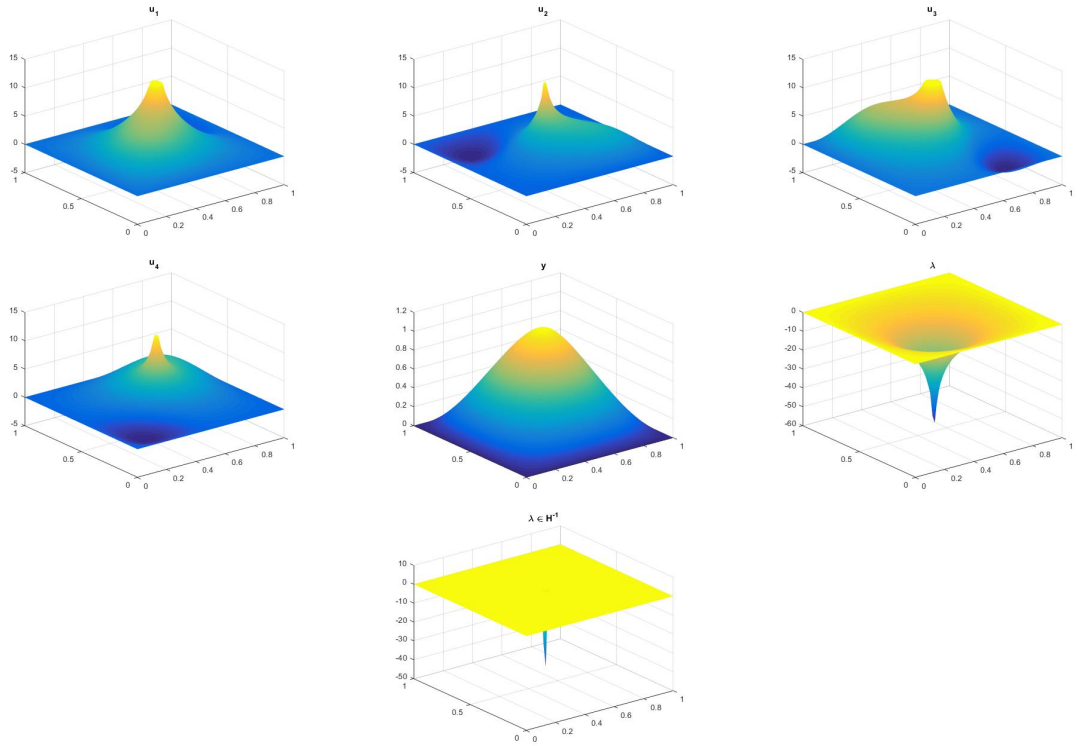

Figure 7.9: Plot of the controls $u_1, \ldots, u_4$ and the state y; the last two figures display the Lagrange multiplier λ, once in the space $H_0^1(\Omega)$, corresponding to our algorithms, and once in (the isomorphic space) $H^{-1}(\Omega)$, which corresponds to the multiplier if Algorithms 5.1 and 6.8 would have been applied to the $H^{-1}(\Omega)$-setting), for the problem with the state constraint $y \geq \psi$.

7.4 Elliptic Optimal Control NEP

The elliptic control GNEPs from Sections 7.2 and 7.3 were reformulated as an optimization problem in order to illustrate the convergence behavior of the optimization algorithm from Chapter 4. Omitting the constraints $\sum u_\nu \leq \psi$ or $y \geq \psi$ in the problems (7.11) or (7.15), respectively, leads to the problem

$$\min_{u_\nu \in L^2(\Omega)} \left\{ \frac{1}{2} \|y(u_\nu, u_{-\nu}) - y_\nu^d\|_{L^2(\Omega)}^2 + \frac{\alpha_\nu}{2} \|u_\nu\|_{L^2(\Omega)}^2 \right\} \tag{7.19a}$$

$$\begin{aligned} \text{s.t.} \quad -\triangle y &= \textstyle\sum_{\nu=1}^N u_\nu && \text{in } \Omega, \\ y &= 0 && \text{in } \partial\Omega, \\ u_\nu(x) &\in [u_\nu^a(x), u_\nu^b(x)] && \text{f.a.a. } x \in \Omega. \end{aligned} \tag{7.19b}$$

However, this is now a Nash problem and not a GNEP; therefore, an application of the algorithms designed for GNEPs is not appropriate. As seen in Section 7.2.1,

this Nash problem can be reformulated into the linearly constrained optimization problem

$$\min_{\substack{u_1,\dots,u_N\in L^2(\Omega)\\ y\in H_0^1(\Omega)}} \left\{ \frac{1}{2}\|y\|^2_{L^2(\Omega)} + \sum_{\nu=1}^{N} \left\{ \frac{\alpha_\nu}{2}\|u_\nu\|^2_{L^2(\Omega)} - \langle y_\nu^d \mid S_\nu u_\nu\rangle_{L^2(\Omega)} \right\} \right\} \tag{7.20a}$$

$$\begin{aligned} \text{s.t.} \quad -\triangle y &= \textstyle\sum_{\nu=1}^{N} u_\nu && \text{in } \Omega, \\ y &= 0 && \text{in } \partial\Omega, \\ u_\nu(x) &\in [u_\nu^a(x), u_\nu^b(x)] && \text{f.a.a. } x \in \Omega. \end{aligned} \tag{7.20b}$$

Therefore, this example is well suited to demonstrate the convergence behavior of the optimization algorithm from Chapter 4.

We use the control constraints $u_\nu^a \equiv -2$ and $u_\nu^b \equiv 2$, and take $\beta = 10$ as our penalty parameter. The other input data stay the same as in Section 7.3. The iteration counts for different discretization widths are displayed in Table 7.5, and the solutions are plotted in Figure 7.10.

Table 7.5: Number of iterations of Algorithm 4.1 for (7.20) until the KKT conditions are satisfied with accuracy 10^{-4}, where $\beta = 10$ and $\gamma = 4.4$. "Discretization" denotes the number of nodes of the discretization mesh in each dimension.

Discretization	16	32	64
Iterations, $\beta = 10$	176	175	175

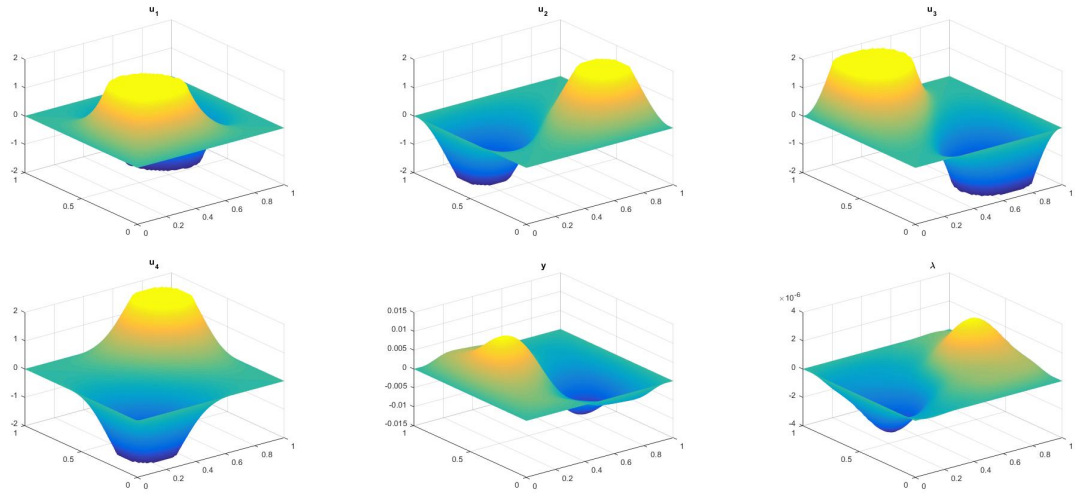

Figure 7.10: Plot of the controls $u_1, \dots, u_4$, the state y, and the Lagrange multiplier λ; the last two figures display the Lagrange multiplier λ.

7.5 Environmental Differential Games

Another infinite-dimensional GNEP results from a class of differential games, which are quite popular in the literature, cf. [52, 53, 90]. We use a modification of an example from [73], where a more detailed discussion can be found. The only modification we made is that we changed some constants. The problem of player ν is given by

$$\min_{u_\nu \in L^2(0,1), y_\nu \in H^1(0,1)} \int_0^1 \Big[q_\nu\big(u_\nu(t), y_\nu(t)\big) + r_\nu\big(y_1(t), \dots, y_N(t)\big) \Big] \, dt$$

$$\text{s.t.} \sum_{\mu=1}^{N} e_\mu y_\mu(t) \le \psi(t), \tag{7.21a}$$

$$u_\nu(t) \in [0, u_\nu^{max}(t)], \tag{7.21b}$$

$$\dot{y}_\nu(t) + b_\nu y_\nu(t) = u_\nu(t), \tag{7.21c}$$

$$y_\nu(0) = y_\nu^0. \tag{7.21d}$$

We choose $N = 2$,

$$q_\nu\big(u_\nu(t), y_\nu(t)\big) = \frac{a_1}{2} y_\nu(t)^2 + \frac{a_2}{2} u_\nu(t)^2, \qquad r_\nu\big(y_1(t), \dots, y_N(t)\big) = \frac{-c \cdot y_\nu(t)}{\sum_{i=1}^N y_i(t) + d},$$

and the constants $a_1 = 0.1$, $a_2 = 0.5$, $b_1 = 0.2$, $b_2 = 0.5$, $c = 1.5$, $e_1 = 2$, $e_2 = 1$, $d = 1$, $y_1^0 = 0$, $y_2^0 = 1$, $u_1^{max}(t) = u_2^{max}(t) = +\infty$, as well as the mapping

$$\psi(t) = \begin{cases} 1.01 & \text{if } t \in [0, 1/3], \\ 1.00 & \text{if } t \in (1/3, 2/3], \\ 0.99 & \text{if } t \in (2/3, 1]. \end{cases}$$

To satisfy the constraints (7.21c) and (7.21d), we use an affine linear solution operator S such that $Su_\nu = y_\nu$, as in Section 7.2. The box constraints (7.21b) represent our set $\mathcal{X}_\nu$, while (7.21a) is enforced through the augmented Lagrangian approach. Some involving but easy computations show that the problem is Lipschitz continuous and strongly monotone in u.

To estimate the constant γ from Algorithm 5.1, we first recognize that $y_2^0 = 1$ and $u_1(t), u_2(t) \ge 0$ imply that we never divide by zero in the definition of r_ν. Regarding the constants α and $\|M\|$, we take $\alpha = a_2/(c \cdot d)$, $\|M\| = 1.8$.

The numerical results for this example, using different discretization levels of this differential game, are summarized in Table 7.6. Recalling that the regularization parameter γ^k generated by Algorithm 6.8 must be divided by β in order to compare it with the regularization parameter γ used in Algorithm 5.1, cf. page 112,

the final value of γ^k generated by Algorithm 6.8 is still quite small. The difference in the iteration count of the application of Algorithm 6.8 with 128 discretization points and the other levels of discretization can be explained by the fact that the final regularization parameter is 2.1 for the 128 discretization width and 1.1 for the others. This demonstrates very well that the approach of increasing the regularization parameter can sustainably reduce the number of iterations. The corresponding solutions are depicted in Figure 7.11.

Table 7.6: Number of iterations for the elliptic optimal control GNEP (7.21) until the KKT conditions are satisfied with accuracy 10^{-4}.

Discretization	128	256	512	1024	2048
Its. of Alg. 5.1, A. (3.18), $\beta = 10$, $\gamma = 1.867$	641	642	642	643	645
Its. of Alg. 5.1, A. (3.19), $\beta = 10$, $\gamma = 1.867$	621	622	622	621	621
Its. of Alg. 6.8, $\beta = 1000$	595	358	358	359	359
Final γ in Alg. 6.8	2.1	1.1	1.1	1.1	1.1

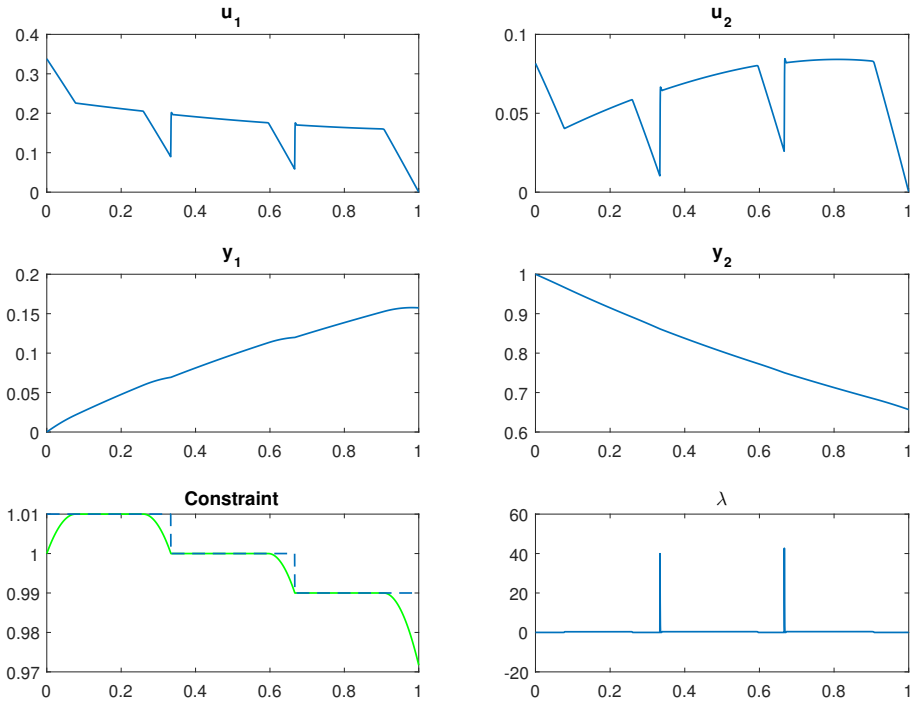

Figure 7.11: Optimal controls u_1, u_2, optimal states y_1, y_2, satisfaction of the constraint, and Lagrange multiplier for the environmental differential game.

7.6 Finite-Dimensional Examples

In this section, we demonstrate the convergence behavior of the presented methods on some finite-dimensional examples. First, we state some finite-dimensional GNEP examples and apply Algorithms 5.1 and 6.8 to them. Thereafter, a comparison of different Jacobi-type methods is conducted on the l_1-minimization problem.

7.6.1 Additional Examples of GNEPs

Here, we consider some finite-dimensional GNEP examples taken from the literature. Since we sometimes modify them slightly to fit within our framework, we restate these examples explicitly. Note that we take $\varphi_\nu \equiv 0$ for all ν in these examples.

Example 7.9. The first example is taken from [45]. There are two players ν, and each is in charge of a single one-dimensional decision variable $x_\nu \in \mathbb{R}$. The problems of these two players are given by

$$\begin{array}{l|l} \min_{x_1 \in \mathbb{R}} (x_1 - 1)^2 & \min_{x_2 \in \mathbb{R}} (x_2 - \frac{1}{2})^2 \\ \text{s.t. } x_1 + x_2 \leq 1 & \text{s.t. } x_1 + x_2 \leq 1. \end{array}$$

While there is only one variational Nash equilibrium $(3/4, 1/4)$, the set of all generalized Nash equilibria for this example is $\{(\alpha, 1-\alpha) \mid \alpha \in [1/2, 1]\}$. For this example, we choose $\beta = 1$ and $\gamma = 5.5$ in Algorithm 5.1, and $\beta = 1$, $\Upsilon = 1000$ in Algorithm 6.8.

$\diamond$

Example 7.10. The second GNEP is taken from [47, 82] and also consists of two players. The first one controls the two-dimensional variable $x_1 = (x_{1,1}, x_{1,2})^T =: (y_1, y_2)^T \in \mathbb{R}^2$, whereas the second player controls a one-dimensional variable $x_2 =: y_3 \in \mathbb{R}$. The optimization problem of the first player is

$$\begin{array}{ll} \min_{y_1, y_2} & y_1^2 + y_1 y_2 + y_2^2 + (y_1 + y_2) y_3 - 25 y_1 - 38 y_2 \\ \text{s.t.} & y_1, y_2 \geq 0, \\ & y_1 + 2y_2 - y_3 \leq 14, \\ & 3y_1 + 2y_2 + y_3 \leq 30, \end{array}$$

and the optimization problem of the second player is

$$\begin{array}{ll} \min_{y_3} & y_3^2 + (y_1 + y_2) y_3 - 25 y_3 \\ \text{s.t.} & y_3 \geq 0, \\ & y_1 + 2y_2 - y_3 \leq 14, \\ & 3y_1 + 2y_2 + y_3 \leq 30. \end{array}$$

The set of all Nash equilibria is given by $\{(\alpha, 11-\alpha, 8-\alpha) \mid \alpha \in [0,2]\}$, where $(0, 11, 8)$ is a normalized Nash equilibrium. For this example, we choose $\beta = 1$ and $\gamma = 30.8$ in Algorithm 5.1, and $\beta = 1$, $\Upsilon = 1000$ in Algorithm 6.8. ◇

Example 7.11. The third example is a modification of the duopoly model from [76]. The $N = 2$ players ν have control of the one-dimensional variable $x_\nu \in \mathbb{R}$, representing their production of a given product. Their objective functions, representing the profit, are given by

$$\theta_\nu(x_\nu, x_{-\nu}) = x_\nu\big(\rho(x_1 + x_2) + \lambda - d\big), \qquad \nu = 1, 2.$$

The production capacity of each player is limited by the individual constraints $x_\nu \in [0, 10]$. Furthermore, both players share a resource constraint given by $x_1 + x_2 \leq r$. We choose $d = 20$, $\lambda = 4$, $\rho = 1$, and $r = 9$. For this example, we choose $\beta = 1$ and $\gamma = 9.43$ in Algorithm 5.1, and $\beta = 1$, $\Upsilon = 1000$ in Algorithm 6.8. ◇

Example 7.12. The fourth example is a modification of the electric demand response management example from [110]. We have $N = 5$ players, each controlling a one-dimensional variable $x_\nu \in \mathbb{R}$, representing their energy consumption. The objective function of player ν is

$$\theta_\nu(x_\nu, x_{-\nu}) = a(x_\nu - c_\nu)^2 + \Big(b\sum_{\mu=1}^{N} x_\mu + p\Big)x_\nu.$$

Because of environmental concerns or supply shortages, the total energy consumption has to be smaller than a certain bound r, i.e., $\sum_{\mu=1}^{N} x_\mu \leq r$. Further, the players are in a certain way stubborn and want their energy consumption to be in an interval around c_ν; for simplicity, we say that we have the box constraint $x_\nu \in [0.8c_\nu, 1.2c_\nu]$. The implementation uses the parameters $a = 1$, $b = 0.04$, $p = 5$, $r = 250$, and $c_\nu = 50 + 5(\nu - 1)$ for $\nu = 1, \ldots, 5$. For this example, we choose $\beta = 1$ and $\gamma = 12.5$ in Algorithm 5.1, and $\beta = 1$, $\Upsilon = 1000$ in Algorithm 6.8. ◇

The numerical results obtained from Algorithm 5.1 and Algorithm 6.8 are summarized in Table 7.7. Note that the algorithms are applied to these problems by taking $\varphi_\nu \equiv 0$ for all examples. To this end, the corresponding cocoercivity modulus α is computed by the maximal and minimal eigenvalues of the symmetric part of the constant (maximally monotone) matrix $(\widehat{P})'$. Then, we choose $\beta = 1$ and $\gamma = 1.1(1/(2\alpha\beta) + \|M\|)$, where $\|M\|$ is approximately the maximal eigenvalue of the symmetric, positive definite matrix M. Since all the constraints are considered easy, $\beta = 1$ was chosen.

Table 7.7: Number of iterations for the elliptic optimal control GNEP (7.11) with ψ_2 until the KKT conditions are satisfied with accuracy 10^{-4}.

Example	7.9	7.10	7.11	7.12
Iterations of Alg. 5.1 with A. (3.18)	25	279	35	75
Iterations of Alg. 5.1 with A. (3.19)	25	279	67	70
Iterations of Alg. 6.8	14	19	23	56
Final γ in Alg. 6.8	0.1	0.1	0.1	3.1

7.6.2 l^1 Minimization

One of the most used test problems for separable, convex algorithms is the class of l^1 minimization problems. Among this class of problems, we chose the basis pursuit problem to compare the different Jacobi-type ADMM-methods outlined in Remark 4.2 with each other. Hence, we consider the optimization problem

$$\min_{x\in\mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad Ax = b,$$

where $A \in \mathbb{R}^{m\times n}$. Thus, we are able to split our problem into n one-dimensional problems whose solutions can be computed analytically, see [63, Section 7.4.1].

We use the technique of performance profiles for benchmarking optimization algorithms as introduced in [36]. Let us explain this technique a bit: We have a set of test problems P and apply different solvers from a set of solvers S to them. Let the number of iterations that the solver $s \in S$ needs for the problem $p \in P$ is denoted by $t_{p,s}$. If the solver $s \in S$ does not solve the problem after a maximal iteration count, set $t_{p,s} = \infty$. Define the *performance ratio* of solver s to the problem p by

$$r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,\tilde{s}} \mid \tilde{s} \in S\}}.$$

The *performance ratio* of the solver $s \in S$ is now

$$\rho_s(\tau) = \frac{1}{|P|} \operatorname{size}\Big\{p \in P \mid r_{p,s} \leq \tau\Big\}.$$

That means $\rho_s(\tau)$ describes the number of test problems that the method s solves with a maximum of $\tau \cdot \min\{t_{p,s} \mid s \in S\}$ iterations.

The test problem set that was used in our numerical test is the SPEAR collection from `http://wwwopt.mathematik.tu-darmstadt.de/spear/` that provides us also with the exact solution x_{exact}. In order to keep the overall computational time of the comparison within acceptable bounds, we only used those test problems whose number of columns was smaller than 3000. As a termination criterion

we took $\|x^k - x_{exact}\|_\infty \leq 10^{-4}$, and the problem was considered not solved if the algorithm required more than one million iterations.

Our comparison includes the following algorithms:

1. The regularized Jacobi-type ADMM-method from Algorithm 4.1 with parameters $\beta = 0.002$ and $\gamma = 1.1 \cdot \|A^T A - \mathrm{diag}(A^T A)\|_\infty$, where $\|A\|_\infty$ denotes the maximum absolute row sum of the current matrix A from the test problem set.
2. The $A_i^T A_i$-norm regularized Jacobi-type ADMM as described in Remark 4.2 (d) using the parameter $\beta = 0.003$.
3. The Jacobi-type ADMM as described in Remark 4.2 (a), with step size $\alpha = 1.999 \cdot (1 - \sqrt{\frac{N}{N+1}})$ as suggested in [63], where N denotes the number of columns of the matrix A. The penalty parameter β was chosen to be $\beta = 0.2$.
4. The twisted ADMM described in Remark 4.2 (e), with penalty parameter $\beta = 0.0001$ and proximal constant $\gamma = 1.1 \cdot (\max\{\mathrm{diag}(A^T A)\} - \min\{\mathrm{diag}(A^T A)\})$, as suggested in [107].
5. The regularized Jacobi-type ADMM-method from [34] that is equal to the one from Algorithm 4.1 except for the choice of the proximal constant γ and a step size τ in the dual variable, as already discussed in Remark 4.2 (c). We choose the parameters $\beta = 0.003$, $\tau = 0.7$, and $\gamma_i = 1.1 \cdot (\frac{N}{2-\tau} - 1) A_i^T A_i$.
6. The Jacobi-type ADMM as described in Remark 4.2 (a), but this time with the step size strategy

$$\alpha = 1 \cdot \frac{\|w^k - \hat{w}^k\|_G^2 + 2(\mu^k - \hat{\mu}^k)^T (A(x^k - \hat{x}^k))}{\|w^k - \hat{w}^k\|_G^2}$$

introduced in [63] and the penalty parameter $\beta = 0.06$.

The above choices of the parameters are either motivated by the corresponding theory or based on some preliminary numerical experiments to get an optimal behavior for each of the algorithms investigated here.

The amount of work per iteration for the first five methods is essentially the same. Hence, the performance profile presented in Figure 7.12 based on the iteration count gives a good idea of the relative performance of each of these methods. The reason for using the iteration count and not the computation time is that we implemented the algorithms in MATLAB® and CPU times provided by MATLAB® seem to be somewhat unreliable.

In Figure 7.12, Algorithm 4.1 has by far the best performance among all Jacobi-type ADMM-methods considered here. The criteria for the choice of γ in

the twisted ADMM-method from [107] and the regularized Jacobi ADMM from [34] seem to be more restrictive and therefore lead to slower convergence of the corresponding algorithms. Furthermore, since all test problems have a relatively high dimension with $N \geq 1024$, it follows that the regularization method involving the $A_i^T A_i$ term yields a very high proximal constant γ, which leads to the poor behavior of this method; this disadvantage may vanish for problems with smaller dimensions. The Jacobi ADMM with constant step size has such a poor numerical behavior, since it has only a very small step size when the number of subproblems is high.

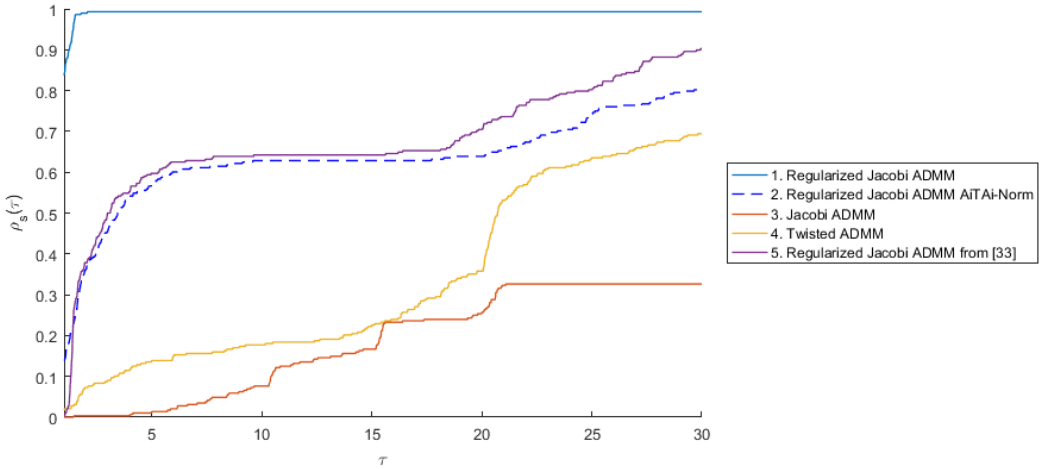

Figure 7.12: Performance profile for the first five Jacobi-type ADMM-methods.

For the second performance profile in Figure 7.13, we also include the sixth method mentioned above. The comparison is again based on the iteration count; however, in this case one should take into account that each iteration of the sixth method, which needs to compute a certain step size at each iteration, is (at least) twice as expensive as all the other methods. Nevertheless, Figure 7.13 indicates that this step size rule makes this method more efficient, even more than Algorithm 4.1. On the other hand, even though Algorithm 4.1 works quite well, it was not our intention to create the fastest method, but to show that certain regularized Jacobi-type ADMM-methods can be interpreted as a proximal-point method.

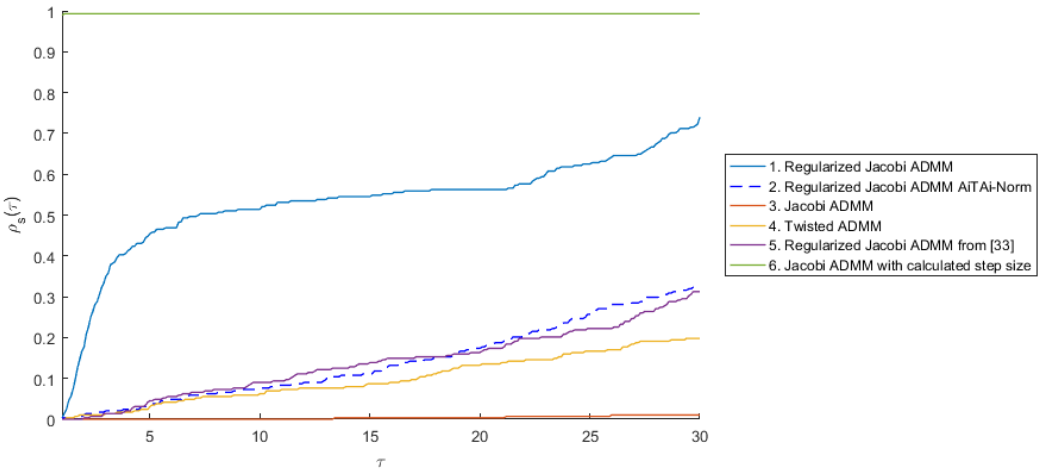

Figure 7.13: Performance profile for all six Jacobi-type ADMM-methods, the sixth method being (at least) twice as expensive per iteration as all other methods.

Chapter 8

Comments and Outlook

The results presented in the preceding chapters provide a fairly comprehensive picture of alternating direction methods of multipliers for generalized Nash equilibrium problems in Hilbert spaces, as well as some insight into ADMM-methods for linearly constrained optimization. By construction, these approaches do not converge fast, but since they have to solve only small-dimensional optimization problems at each iteration, they can usually be applied to large instances. In this chapter, we conclude the thesis by summarizing the main results, highlighting some essential assumptions, and discussing possible topics of future research.

Constrained Optimization

In Chapter 3, we showed that the KKT conditions of a convex, separable, linearly equality constrained optimization problem can be transformed into an equivalent formulation using maximally monotone operators. The main results regarding this reformulation are stated in Lemma 3.3 and Proposition 3.4. This reformulation was then exploited in Chapter 4, where two regularized Jacobi-type ADMM-methods for optimization were presented and analyzed. The two algorithms are, on the one hand, the weakly convergent Algorithm 4.1, which is equivalent to a proximal-point method in a certain scalar product and, on the other hand, the strongly convergent Algorithm 4.16, which can be interpreted as a particular realization of Halpern's method. The main results concerning these algorithms are their respective convergence theorems, i.e. Theorems 4.7 and 4.17. The corresponding proofs of convergence are based on the proximal-point method or its modification introduced by Halpern. Even though we have shown by means of an example that our proximal-point interpretation might not converge for non-symmetric matrices Q, the convergence of the regularized Gauss-Seidel alternating direction method for

$N > 2$ remains an open question. The current technique of proof is not applicable in this setting, so that further research on this issue is necessary.

Generalized Nash Problems

In Chapter 3, we also discussed the solution concept of variational equilibria and variational KKT pairs for generalized Nash equilibrium problems. Inspired by the results for optimization problems, the variational KKT conditions were equivalently rewritten as an inclusion problem with maximally monotone operators, and as a variational inequality. These results are stated in Lemmas 3.10 and 3.11 as well as in Proposition 3.12. Further, it was discussed how to incorporate linear inclusion constraints in the framework of linear equality constraints. These reformulations with maximally monotone operators and variational inequalities were then used to derive ADMM-type algorithms for linearly equality constrained GNEPs in the Chapters 5 and 6.

In Chapter 5, we presented three new methods for solving linearly constrained generalized Nash equilibrium problems, which were inspired by the alternating direction method of multipliers. The first method is the basic regularized Jacobi-type ADMM-method as stated in Algorithm 5.1. The weak convergence of this method under a cocoercivity assumption was shown through a reinterpretation as a forward-backward splitting in Theorem 5.6 and, in addition, a self-contained proof was presented in Section 5.1.2. The second and third method were then based on the forward-backward interpretation of this basic regularized Jacobi-type ADMM-method. The second one is the strongly convergent modification described in Section 5.2, where the convergence theory is based on a reinterpretation as Halpern's method. This modification still requires a cocoercivity assumption, as can be seen in Theorem 5.10, which is the convergence theorem of this method. The third method, Algorithm 5.11, is again weakly convergent; however, it requires only a Lipschitz continuity assumption, which is weaker than cocoercivity. All three methods are fully distributed (Jacobi-like).

An alternative would be to replace the Jacobi-decomposition in Algorithm 5.1 by a Gauss-Seidel approach. This Gauss-Seidel approach typically converges faster, but so far we have not been able to develop a splitting-type interpretation of this approach. However, we gave a convergence theory for this approach in Chapter 6. In this Chapter 6, we presented two more new methods for solving linearly constrained generalized Nash equilibrium problems. Both methods require strong monotonicity and Lipschitz continuity in all but one block-component and regularize the resulting subproblems. It was shown in Section 6.2.3 that this regularization is necessary for guaranteed convergence. The first method, presented in Algorithm 6.2, keeps the regularization parameters constant and above the the-

oretical bound needed for convergence, see Theorem 6.7. The second method, Algorithm 6.8, is based on the fact that, in many cases, a smaller regularization parameter than the theoretical worst case estimate is sufficient for convergence. Therefore, the regularization parameter is increased only if a certain convergence measure does not indicate convergence. Using the strong monotonicity, convergence was also shown in this case, see Theorem 6.10. In the numerical application, this increasing regularization proved very efficient; therefore, more research should be conducted on the application of this technique in other algorithms that use regularization. An application of ADMM-methods to potential games seems promising as well; however, this is left for further research.

Final Comments

It is the author's hope that the theory and practical results presented throughout this thesis will prove useful to other researchers and users. Moreover, the author hopes that his contribution to the research on ADMM- and splitting-type methods facilitates and enhances further research on these methods as well as their application to several optimization-related problems.

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