LEVELING THE LOBBYING PROCESS: EFFORT MAXIMIZATION IN ASYMMETRIC N-PERSON CONTESTS[1]

Jörg Franke[2], Christian Kanzow[3], Wolfgang Leininger[2], and Alexandra Schwartz[3]

Preprint 295 April 2010

[2]University of Dortmund (TU)
Department of Economics
Vogelpothsweg 87
44227 Dortmund
Germany

e-mail: Joerg.Franke@tu-dortmund.de
Wolfgang.Leininger@tu-dortmund.de

[3]University of Würzburg
Institute of Mathematics
Am Hubland
97074 Würzburg
Germany

e-mail: kanzow@mathematik.uni-wuerzburg.de
schwartz@mathematik.uni-wuerzburg.de

April 01, 2010

[1]This research was partially supported by a grant from the international doctorate program "Identification, Optimization, and Control with Applications in Modern Technologies" within the Elite-Network of Bavaria.

Abstract

We model the lobbying process between a politician and lobbyists as a contest game where the politician can encourage or discourage specific lobbyists from participating in the process. This realistic extension of the seminal lobbying set-up has important implications: (i) The exclusion principle, as established in Baye et al. (AER 1993), does not hold, instead there is (endogeneously induced) inclusion of lobbyists; (ii) the politician will optimally level the playing field by encouraging weak lobbyists to participate; (iii) at least three lobbyists will be active in the leveled lobbying process. Results are established using techniques from mathematical programming with equilibrium constraints.

Key Words: Lobbying games, exclusion principle, level playing field, contest theory, bilevel programming.

JEL classification: C72; D72

1 Introduction

Lobbying is a wide-spread policy instrument to influence political decision making on all levels of governmental activity. In the US, for instance, the total lobbying spending in 2008 amounts to 3.3 Billion US Dollar, which is channeled through 14,808 registered lobbyists.[4] Those figures reflect the significant political and economic impact that this type of influencing activities has on contemporary policy making. The analysis of the lobbying process based on game-theoretical models should contribute to gain insights into the underlying mechanisms of lobbying.

The lobbying process can be characterized as being informal and, at the same time, public: It is public in the sense that lobbyists have to register and disclose their lobbying spending. Additionally, the political positions and arguments of specific lobbying groups are usually publicly known, at least by their rivals. However, the lobbying process itself is informal in the sense that influencing activities do usually not follow institutionalized rules. This also implies that no lobbying group can be sure of the effectiveness of its lobbing efforts on the respective decision maker. This uncertainty is aggravated by the fact that the decision maker might not be neutral with respect to the lobbying effort exerted by specific lobby groups, for instance, she might favor or encourage specific lobby groups relatively to others. Frequently observed statements emphasizing that the opinion of each affected group will be considered in the respective political process can be taken as anecdotal evidence for such attempts to encourage specific groups in the lobbying process.[5]

To capture the mentioned characteristics of the lobbying process, i.e. public information, uncertain outcomes, and different lobbying effectiveness of lobbyists, we model this process as an asymmetric lottery contest game under complete information in the style of Tullock (1980) [26], cf. [5]. Here, a politician grants a prize, for instance a monopoly right or production permission, to a lobbying group depending on its relative lobbying effort. The outcome of the lobbying contest is probabilistic which reflects the uncertainty of the lobbyists with respect to the effectiveness of their respective lobbying effort. Additionally, the contest rule, or contest success function, might be biased due to the discretionary power of the politician. This feature of the model captures the informal structure of the lobbying process where a lobby group may know ex-ante that its lobbying effort counts less than those of others. From the perspective of the politician that seeks to maximize total lobbying effort (interpreted as implicit payments to the politician)

[4]Those figures, that do not include campaign contributions, are collected by the Center of Responsive Policy and are based on lobbying disclosure reports filed with the Secretary of the Senate's Office of Public Records.

[5]It is also evident that the influence of specific lobby groups on specific politicians might be very low (for instance if they belong to hostile political fractions) irrespectively of their respective lobbying effort.

the question is then how the lobbying process should be designed to achieve its objective. Should the politician deviate from a neutral contest rule by encouraging less effective lobbyists to engage in the lobbying process? Should she bias the contest rule to discourage highly effective lobby groups from participation in order to increase competitive pressure between remaining lobbyists? Our model is intended to address those types of questions related with the optimal leveling of the lobbying process out of the perspective of the respective decision maker.

Therefore, the lobbying game considered in our approach has a two-stage structure: In the first stage the politician discretionary decides about the specific importance that is granted to the lobbying effort of specific lobby groups by specifying individual weights in order to obtain maximal implicit payments (lobbying effort). In the second stage the lobby groups exert lobbying effort taken as given the bias of the politician and the lobbying efforts of their opponents.

From a mathematical point of view, this two-stage game is a bilevel program or, more precisely, a mathematical program with equilibrium constraints. The latter is of the general form

$$\max_{x,\alpha} f(x,\alpha) \quad \text{subject to} \quad \alpha \in A,\ x \in \mathcal{S}(\alpha),$$

where x, α denote the variables, A is the feasible set for α (usually defined by simple constraints), and $\mathcal{S}(\alpha)$ is the solution set of another optimization or (Nash) equilibrium problem, typically called the lower-level problem, cf. [16, 22, 9] and references therein for an extensive discussion. Note that this lower level problem depends on α. In our case, the lower level problem is the contest game and has a unique solution $x(\alpha)$ (depending on α, the individual weights as specified by the contest organizer) in pure strategies for which an analytic expression is known. This allows us to follow the so-called implicit programming approach from [21, 22] and to replace the unknown x in the objective function f by the unique solution $x(\alpha)$ of the lower level problem. We then obtain the (typically nonsmooth) optimization problem

$$\max_{\alpha} \tilde{f}(\alpha) := f(x(\alpha),\alpha) \quad \text{subject to} \quad \alpha \in A$$

depending on α only. Standard solvers for the solution of such a kind of (usually nonconcave) optimization problem find local maxima or certain stationary points of the objective function, but not necessarily a global maximum. Here we exploit the special structure of our effort maximization problem in order to derive an explicit formula for the global maximum, which represents the optimal bias of the contest rule.

Our results provide unambiguous answers to the questions raised above: We show that the optimal policy for the politician is to bias the contest rule in favor of less effective lobbyists to some extent. Hence, more effective lobbyists are relatively discouraged by the politician while weaker active lobby groups are encouraged. This implies that the politician can endogenously induce a more leveled playing field which spurs competition by itself and results in higher total lobbying effort, i.e. implicit payments to the politician. The resulting playing field is more leveled than under a neutral policy which also implies that there are at least three lobbyists active in the lobbying process, irrespective of the underlying heterogeneity in lobbying effectiveness.[6]

Our framework allows to explicitly characterize the active set of participating lobbyists under the optimally biased contest rule: Although the playing field is more balanced the active set of lobbying groups consists of the most effective lobbyists (up to a threshold). Hence, in our set-up it is never optimal for the politician to discourage highly effective lobbyists from participating.

This result is in some contrast to the so called *exclusion principle*, established in [4]. There a politician has the power to exclude lobbyists from the lobbying process which is modeled as an all-pay auction under complete information, i.e. the lobby group that exerts the highest lobbying effort wins the prize with certainty. The exclusion principle states that excluding *highly effective* lobbyists from the lobbying process might be beneficial for the politician because it results in higher competitive pressure between the remaining (more homogeneous) lobbyists which would induce higher total implicit payments to the politician. However, if the lottery contest framework is used to describe the lobbying process the exclusion principle does not hold. This result has been established in [12] for a symmetric lottery framework where the discretionary power of the politician is restricted to the pure exclusion of specific lobbyists. Our generalization confirms first the non-viability of the exclusion principle in an extended lottery framework. More importantly, it allows us to compare directly the active set of lobbyists under the neutral and the optimally biased contest rule. Instead of exclusion we find additional (endogenously induced) entry of lobbyists under the optimally biased contest rule. This suggests that there is rather an *inclusion principle* at work due to the partially leveled lobbying process that is the result of the optimally biased contest rule. Note that our analysis also implies that a symmetric lottery, in which all contestants get the same weight, is optimal if and only if all contestants are identical, i.e. the well analyzed symmetric lottery model is never optimal for a heterogeneous group of contestants.

Our contest game models the lobbying process in a rather stylized way. Hence, besides the lobbying context the derived results might also be relevant for other competitive situations where weak players are favored, for instance, by handicapping in promotion tournaments, cf. [15], bidding preferences in public procurement, cf. [18, 17], lobbying caps, cf. [6], as well as affirmative

[6]However, we will also show that it is not optimal for the politician to level the playing field to the full extent (with the exception of the two player-case). Hence, even under the optimally biased contest rule full participation might not be achieved, i.e. some weak lobbyists may remain inactive also under the biased rule.

action, cf. [14, 13]. An alternative theroretical approach to model the lobbying process is the mentioned all-pay auction framework with complete information, see [23] for a recent example without natural ordering of players, and [1], where a general class of symmetric contest success functions is used to clarify the relation between all-pay auction and contest models. A lobbying game with incomplete information, where lobbyists are financially constrained and lobbying is informative is analyzed in [11]. From a theoretical perspective our model is, besides [12], closely related to the 'simple contest' presented in [10]. However, in that paper the weighting parameters are interpreted as population weights which results in a different objective function. Additionally, the specification of the cost functions is different. While there the assumption of convex costs has the advantage of making the participation issue obsolete, it also impedes the derivation of closed form solutions as it is possible in our set up.

The remainder of the paper is structured as follows. In the next section we set up the lobbying process as a n-player contest game between heterogeneous players based on an asymmetric contest success function. We derive some properties of the equilibrium in the underlying contest game that are helpful for the subsequent analysis in Section 2. In Section 3 we prove that there exists a vector of weighting factors that yields a global maximum for aggregated equilibrium effort. In Section 4 we derive explicit formulae for the unique optimal set of active contestants and the optimal set of weighting factors. Section 5 is dedicated to some examples and the discussion of the optimal weighting scheme while Section 6 concludes.

Notation: $x^\nu \in \mathbb{R}$ denotes the variable of player ν, $x := (x^1, \ldots, x^n)$ is the vector of all decision variables. In order to emphasize the role of player ν within this vector, we sometimes write $x = (x^\nu, x^{-\nu})$, where the symbol $x^{-\nu}$ subsumes the variables of all other players. We further denote by $B(x;\delta)$ the open Euclidian ball of radius $\delta > 0$ around a given point $x \in \mathbb{R}^n$.

2 The Underlying Contest Game

The lobbying process is modeled as a contest game between heterogeneous lobbyists, from now on called players. Hence, the contest game to be considered here is a special instance of a Nash equilibrium problem and defined as follows: Let $N := \{1, \ldots, n\}$ be the set of players. Furthermore, let

$$\theta_\nu(x^\nu, x^{-\nu}) := \begin{cases} \frac{\alpha_\nu x^\nu}{\sum_{\mu=1}^n \alpha_\mu x^\mu} - \beta_\nu x^\nu, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

be the expected payoff or utility of player $\nu \in N$, where α_ν, β_ν are positive constants for all $\nu \in N$ that represent the individual weights, i.e. the bias of the contest rule, and respectively, the heterogeneity between players affecting marginal costs. In this section these parameters are assumed to be fixed.

Each player $\nu \in N$ chooses a strategy $x^{*,\nu}$ from his strategy space $X_\nu := [0, +\infty)$ in such a way that

$$\theta_\nu(x^{*,\nu}, x^{*,-\nu}) \geq \theta_\nu(x^\nu, x^{*,-\nu}) \quad \forall x^\nu \in X_\nu$$

holds for all $\nu \in N$, i.e. player ν tries to maximize his utility function θ_ν with respect to his own strategy x^ν, whereas the strategies of all other players are fixed (at their optimal values).

The utility functions θ_ν can be interpreted in the following way: All players take part in a lottery, where a price with the value $V = 1$ can be won. Every player ν can increase his probability of winning, which is given by the contest success function $\frac{\alpha_\nu x^\nu}{\sum_{\mu=1}^n \alpha_\mu x^\mu}$ as axiomatized in [2], by increasing his effort x^ν which, however, also increases his costs $\beta_\nu x^\nu$. We could extend our model to the case where different players have different valuations $V_\nu > 0$ of the price. This would lead to the following utility function

$$\tilde{\theta}_\nu(x^\nu, x^{-\nu}) := \begin{cases} \frac{\alpha_\nu x^\nu}{\sum_{\mu=1}^n \alpha_\mu x^\mu} V_\nu - \tilde{\beta}_\nu x^\nu, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

In this case, we can obtain utility functions of the form θ_ν by multiplying the functions $\tilde{\theta}_\nu$ with the factor $\frac{1}{V_\nu}$ and defining $\beta_\nu := \frac{\tilde{\beta}_\nu}{V_\nu}$. Note that re-scaling the function $\tilde{\theta}_\nu(\cdot, x^{*,-\nu})$ with a positive multiplier does not change the location of its maximum, i.e. it does not change the optimal effort $x^{*,\nu}$. Hence, the case of heterogeneous valuations of the price is included in our model.

We now summarize a number of properties of this Nash equilibrium problem. The following existence and uniqueness result for the above problem was established in [7, 25, 24].

Theorem 2.1 *The above Nash equilibrium problem has a unique solution x^*.*

Note that the previous result holds for any fixed parameters α_ν and β_ν, but that, of course, the solution depends on the exact values of these parameters. More precisely, we have the following representation, see [7, 24] for a proof of these statements.

Theorem 2.2 *Let x^* be the unique solution of the above equilibrium problem, let $K := \{\nu \in N \mid x^{*,\nu} > 0\}$ be the corresponding set of* active *players, and let $k := |K|$ the number of active players. Then:*

(a) K consists of at least two elements.

(b) The active players can be characterized as follows:

$$\nu \in K \Longleftrightarrow (k-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu \in K} \frac{\beta_\mu}{\alpha_\mu}. \tag{1}$$

(c) The components $x^{*,\nu}$ of the solution satisfy the relation

$$x^{*,\nu} = \max\left\{0, \frac{1}{\alpha_\nu}\left(1 - \frac{\beta_\nu}{\alpha_\nu}\frac{k-1}{\sum_{\mu \in K}\frac{\beta_\mu}{\alpha_\mu}}\right)\frac{k-1}{\sum_{\mu \in K}\frac{\beta_\mu}{\alpha_\mu}}\right\}$$

for all $\nu \in N$.

Theorem 2.2 characterizes the equilibrium based on an implicit description of the set K of active players in (1). It should be noted that also the expression for the unique solution x^* of the Nash equilibrium problem is implicit because it depends on the set K of active players.

From expression (1) it is clear that the set K of active players consists of those players with the lowest combined parameters $\frac{\beta_\nu}{\alpha_\nu}$, which we might term the effective cost parameters with respect to a 'homogenized' contest success function of the form $\frac{x^\nu}{\sum_{\mu=1}^n \alpha_\mu x^\mu}$. A homogenized contest can be obtained from the original contest game by multiplying the functions θ_ν with the factor $\frac{1}{\alpha_\nu}$ for $\nu = 1, \ldots, n$. Again, this re-scaling does not change the location of its maximum. Note, however, that the homogenized contest success function still treats players with different α_ν's differently.

The following result due to [24] allows an explicit calculation of the set K consisting of the most effective players. Together with Theorem 2.2, we are then in a position to compute the unique solution of our Nash equilibrium problem.

Theorem 2.3 *Assume without loss of generality that the players ν are ordered in such a way that*

$$\frac{\beta_1}{\alpha_1} \leq \frac{\beta_2}{\alpha_2} \leq \ldots \leq \frac{\beta_n}{\alpha_n}. \tag{2}$$

Furthermore, let x^* be the unique solution of the Nash equilibrium problem. Then the corresponding set K of active players is given by

$$K = \left\{\nu \in N \;\middle|\; (\nu-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu=1}^{\nu} \frac{\beta_\mu}{\alpha_\mu}\right\}.$$

A simple consequence of the previous result is the following corollary that will be used later in the proof of Lemma A.1. It provides an upper bound on the effective cost parameter of an active player.

Corollary 2.4 *Assume without loss of generality that the players* ν *are ordered as in* (2)*. Let* x^* *be the unique Nash equilibrium and* K *be the corresponding index set of active players. Then*

$$K \subseteq \left\{ \nu \in N \,\middle|\, \frac{\beta_\nu}{\alpha_\nu} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right\}.$$

Proof. Assumption (2) implies that $K = \{1, \ldots, k\}$. Now, the inequality $\frac{\beta_\nu}{\alpha_\nu} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}$ obviously holds for $\nu = 1, 2$. For $\nu = 3, \ldots, k$, this inequality follows inductively taking into account the inequality

$$(\nu - 1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu=1}^{\nu} \frac{\beta_\mu}{\alpha_\mu} \quad \text{for} \quad \nu = 3, \ldots, k,$$

from Theorem 2.3. □

Note that the inclusion in this corollary can be an equality but, in general, will be a strict inclusion as the following example illustrates.

Example 2.5 Consider a game with four players and $\alpha = (1, 1, 1, 1)^T$.

(a) If $\beta = (2, 3, 3.5, 4)^T$, we have

$$K = \{1, 2, 3, 4\} = \left\{ \nu \in N \,\middle|\, \frac{\beta_\nu}{\alpha_\nu} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} = 5 \right\}.$$

(b) If, however, $\beta = (2, 3, 3.5, 4.5)^T$, we have

$$K = \{1, 2, 3\} \subsetneq \{1, 2, 3, 4\} = \left\{ \nu \in N \,\middle|\, \frac{\beta_\nu}{\alpha_\nu} < \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} = 5 \right\}.$$

◇

We next provide a characterization of the unique Nash equilibrium in terms of the subsets $K \subseteq N$. Hence, not only equilibrium effort x^* but also the corresponding set of active players K is unique. This characterization will turn out to be useful for our analysis in the subsequent sections.

Theorem 2.6 *Let $L, M \subseteq N$ be subsets with $l := |L| \geq 2$, $m := |M| \geq 2$, and the property that*

$$\left((l-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu \in L} \frac{\beta_\mu}{\alpha_\mu} \quad \Longleftrightarrow \quad \nu \in L \right) \quad \text{and} \quad \left((m-1)\frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu \in M} \frac{\beta_\mu}{\alpha_\mu} \quad \Longleftrightarrow \quad \nu \in M \right).$$

Then $L = M$, hence the index set of active players corresponding to the unique Nash equilibrium is the only subset of N with the properties mentioned above.

Proof. Let x^* be the unique Nash equilibrium. Then we know from Theorem 2.2 (a), (b) that the index set of active players

$$K := \{\nu \in N \mid x^{*,\nu} > 0\}$$

has the postulated properties. Hence we only have to verify that $L = M$ follows for all sets $L, M \subseteq N$ with these properties. Assume now that there are two such sets with $L \neq M$. If we assume without loss of generality that the players are ordered according to (2), this implies

$$L = \{1, \ldots, l\} \quad \text{and} \quad M = \{1, \ldots, m\}.$$

Since $L \neq M$, we can assume without loss of generality that $l > m$. Then $l \notin M$, and together with (2) it follows that

$$(l-1)\frac{\beta_l}{\alpha_l} = (l-m)\frac{\beta_l}{\alpha_l} + (m-1)\frac{\beta_l}{\alpha_l} \geq \sum_{\mu=m+1}^{l} \frac{\beta_\mu}{\alpha_\mu} + \sum_{\mu=1}^{m} \frac{\beta_\mu}{\alpha_\mu} = \sum_{\mu=1}^{l} \frac{\beta_\mu}{\alpha_\mu},$$

a contradiction to $l \in L$. Consequently, we have $L = M$. □

We summarize the previous results in the following note which, basically, says that we have a Nash equilibrium if and only if we are able to find a set K satisfying the requirements (a) and (b) from Theorem 2.2.

Remark 2.7 The following statements hold:

(a) If x^* is the unique Nash equilibrium and K the corresponding set of active players, then K has at least two elements and satisfies the conditions from (1).

(b) Conversely, if we have a subset $K \subseteq N$ with at least two elements such that (1) holds, then K is the set of active players corresponding to the unique Nash equilibrium of our game.

3 Effort Maximization: Existence of Solution

We now consider the problem of the contest organizer who can specify positive individual weights α_ν for all players $\nu = 1, \dots, n$ (which determine the bias of the contest success function) in order to maximize total equilibrium effort from the lower-level contest game. As the individual cost parameters for producing lobbying effort are inherently given by the contestants fixed abilities, they remain being fixed parameters in the maximization problem of the contest organizer.

Since the unique solution x^* of the Nash equilibrium problem from Section 2 depends on α_ν (and β_ν which, however, are fixed), we now write $x(\alpha)$ for this solution as well as $K(\alpha)$ for the corresponding set of active players. Moreover, let $k(\alpha) := |K(\alpha)|$ be the number of active players. In view of Theorem 2.2, the components $x^\nu(\alpha)$ satisfy

$$x^\nu(\alpha) = \max \left\{ 0, \frac{1}{\alpha_\nu} \left(1 - \frac{\beta_\nu}{\alpha_\nu} \frac{k(\alpha) - 1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \right) \frac{k(\alpha) - 1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \right\} \quad \forall \nu \in N,$$

whereas the characterization

$$\nu \in K(\alpha) \quad \Longleftrightarrow \quad (k(\alpha) - 1) \frac{\beta_\nu}{\alpha_\nu} < \sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu} \tag{3}$$

holds for the set $K(\alpha)$.

The problem that we deal with in this and the next section is the following one:

$$\max \sum_{\nu=1}^{n} x^\nu(\alpha) \quad \text{s.t.} \quad \alpha \in (0, \infty)^n. \tag{4}$$

Recall that $x^\nu(\alpha)$ is the (Nash) *equilibrium* effort of player ν, $\nu = 1, .., n$, if the contest success function uses the vector of weights α. A contest administrator - or more general, mechanism designer - can now mediate the contest by *choice* of the weights α in order to elicit maximal total effort from the n potential contestants (some of whom may choose to stay inactive). Hence, it is the contest organizer's problem that is described by (4). Note also that the β-parameters describe personal characteristics of the contestants, which consequently cannot be altered neither by the contestants themselves nor the contest organizer.

Taking into account the previous representation of $x^\nu(\alpha)$, the objective function of (4) can be written in the following form:

$$f(\alpha) := \sum_{\nu=1}^{n} x^\nu(\alpha) = \sum_{\nu \in K(\alpha)} x^\nu(\alpha) = \frac{k(\alpha)-1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \left(\sum_{\mu \in K(\alpha)} \frac{1}{\alpha_\mu} - \frac{k(\alpha)-1}{\sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu}} \sum_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu^2} \right). \tag{5}$$

The aim of this section is to show that the maximization problem (4) has a solution. This is not clear a priori since the feasible set for α is both unbounded and open. The unboundedness turns out to be less serious (and will be dealt with in Lemma 3.1), the really crucial part is the fact that the objective function f is not defined as soon as $\alpha_\nu = 0$ for one player ν. This, however, refers directly to the fact that player ν would effectively be excluded from the contest, since not exerting effort, i.e. $x^\nu = 0$, would become a dominant strategy for player ν if $\alpha_\nu = 0$.

We begin with some results to show that both $f(\alpha)$ and $K(\alpha)$ remain unchanged under certain manipulations of α. A first result of this kind is the following lemma which shows that both $f(\alpha)$ and $K(\alpha)$ are homogeneous of degree zero in α. This is not surprising since the contest success function and therefore also the utility functions θ_ν themselves are homogeneous of degree zero in α, but the lemma can also be proven directly using the definitions of $f(\alpha)$ and $K(\alpha)$.

Lemma 3.1 *For all $\alpha \in (0, \infty)^n$ and all $c > 0$, we have*

$$K(c\alpha) = K(\alpha) \quad \textit{and} \quad f(c\alpha) = f(\alpha).$$

Proof. First note that $c\alpha$ is feasible (i.e., belongs to $(0, \infty)^n$) for all feasible α. The characterization (3) together with the uniqueness of the set $K(\alpha)$ by Remark 2.7 then immediately implies $K(c\alpha) = K(\alpha)$. Taking this into account and calculating $f(c\alpha)$ gives precisely the same value as $f(\alpha)$ since the factor c can be cancelled. □

Another manipulation of α which leaves the function value $f(\alpha)$ unchanged is presented in the following result which basically says that, given a fixed parameter α^*, we can replace the components α^*_ν of α^* corresponding to the inactive players by arbitrary small numbers α_ν and still have $K(\alpha^*) = K(\alpha)$ and $f(\alpha^*) = f(\alpha)$. Hence, players that decided not to participate given the weights α^* will not alter their decision if they face an alternative weighting scheme α as defined below.

Lemma 3.2 *Let $\alpha^* \in (0, \infty)^n$ be arbitrarily given. Then $K(\alpha^*) = K(\alpha)$ and $f(\alpha^*) = f(\alpha)$ hold for all $\alpha \in (0, \infty)^n$ satisfying the following properties:*

(a) For all $\nu \in K(\alpha^*)$, we have

$$\alpha_\nu = \alpha_\nu^*.$$

(b) For all $\nu \notin K(\alpha^*)$, we have

$$\alpha_\nu \in \left(0, \frac{(k(\alpha^*) - 1)\beta_\nu}{\sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*}}\right].$$

Proof. Choose $\alpha \in (0, \infty)^n$ in such a way that the two properties (a) and (b) hold. Then Remark 2.7 shows that the corresponding index set $K(\alpha)$ is uniquely defined. Using property (a), we obtain for all $\nu \in K(\alpha^*)$

$$(k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha_\nu} = (k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha_\nu^*} < \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*} = \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu}.$$

On the other hand, property (b) implies for all $\nu \notin K(\alpha^*)$

$$(k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha_\nu} \geq \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*} = \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu}.$$

The uniqueness of $K(\alpha)$ therefore gives $K(\alpha) = K(\alpha^*)$. Together with property (a) we then obtain $f(\alpha) = f(\alpha^*)$. □

So far, we are not able to prove the existence of a solution for the maximization problem (4). However, Lemmas 3.1 and 3.2 show that such a solution (if it exists) is certainly not unique. The last Lemma says in particular that lowering the weight of an inactive player in the contest success function leaves him inactive. In order to verify the existence of a solution, we first verify the continuity of the function f on $(0, \infty)^n$. Note that continuity is not obvious since the index set $K(\alpha^*)$ may change in points arbitrarily close to α^*.

Theorem 3.3 *The objective function f is continuous on $(0, \infty)^n$. Moreover, this function is continuously differentiable in an open neighbourhood of any vector $\alpha^* \in (0, \infty)^n$ having the following property:*

$$\nu \notin K(\alpha^*) \Longrightarrow (k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha_\nu^*} > \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*}. \tag{6}$$

Proof. The statement regarding the continuous differentiability is clear since condition (6) guarantees that, locally, the index set $K(\alpha^*)$ is constant, hence $K(\alpha) = K(\alpha^*)$ for all α from a sufficiently small neighbourhood of α^*. In particular, f is continuous in these points.

In order to verify the continuity of f on the whole set $(0,\infty)^n$, it therefore remains to consider a point $\alpha^* \in (0,\infty)^n$ such that the index set

$$L := \left\{ \nu \in N \;\middle|\; (k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha^*_\nu} = \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha^*_\mu} \right\}$$

is nonempty. Now, it is not difficult to see that there is a neighbourhood $U \subseteq (0,\infty)^n$ of α^* such that

$$K \subseteq K(\alpha) \subseteq K \cup L \quad \forall \alpha \in U,$$

where, for simplicity of notation, we use the abbreviation $K := K(\alpha^*)$. Let us further write $k := |K|$ und $l := |L|$. Then, for each $\alpha \in U$, we have $K(\alpha) = M$ for one of the 2^l sets M satisfying $K \subseteq M \subseteq K \cup L$. Setting $m := |M|$ and using

$$\frac{\beta_\nu}{\alpha^*_\nu} = \frac{\sum_{\mu\in K} \frac{\beta_\mu}{\alpha^*_\mu}}{k-1}$$

for all $\nu \in M\backslash K$, we obtain for all these index sets M

$$\begin{aligned}
f_M(\alpha^*) &:= \frac{m-1}{\sum_{\mu\in M}\frac{\beta_\mu}{\alpha^*_\mu}}\left(\sum_{\mu\in M}\frac{1}{\alpha^*_\mu} - \frac{m-1}{\sum_{\mu\in M}\frac{\beta_\mu}{\alpha^*_\mu}}\sum_{\mu\in M}\frac{\beta_\mu}{(\alpha^*_\mu)^2}\right)\\
&= \frac{m-1}{\sum_{\mu\in K}\frac{\beta_\mu}{\alpha^*_\mu}\cdot\left(1+\frac{m-k}{k-1}\right)}\left[\sum_{\mu\in K}\frac{1}{\alpha^*_\mu} + \sum_{\mu\in M\backslash K}\frac{\sum_{\lambda\in K}\frac{\beta_\lambda}{\alpha^*_\lambda}}{(k-1)\beta_\mu}\right.\\
&\quad \left. - \frac{m-1}{\sum_{\mu\in K}\frac{\beta_\mu}{\alpha^*_\mu}\cdot\left(1+\frac{m-k}{k-1}\right)}\left(\sum_{\mu\in K}\frac{\beta_\mu}{(\alpha^*_\mu)^2} + \sum_{\mu\in M\backslash K}\frac{\left(\sum_{\lambda\in K}\frac{\beta_\lambda}{\alpha^*_\lambda}\right)^2}{(k-1)^2\beta_\mu}\right)\right]\\
&= \frac{k-1}{\sum_{\mu\in K}\frac{\beta_\mu}{\alpha^*_\mu}}\left[\sum_{\mu\in K}\frac{1}{\alpha^*_\mu} + \frac{\sum_{\mu\in K}\frac{\beta_\mu}{\alpha^*_\mu}}{(k-1)}\sum_{\mu\in M\backslash K}\frac{1}{\beta_\mu}\right.\\
&\quad \left. - \frac{k-1}{\sum_{\mu\in K}\frac{\beta_\mu}{\alpha^*_\mu}}\left(\sum_{\mu\in K}\frac{\beta_\mu}{(\alpha^*_\mu)^2} + \left(\frac{\sum_{\mu\in K}\frac{\beta_\mu}{\alpha^*_\mu}}{k-1}\right)^2\sum_{\mu\in M\backslash K}\frac{1}{\beta_\mu}\right)\right]\\
&= f(\alpha^*).
\end{aligned}$$

Since the 2^l functions f_M are continuous in α^*, we obtain for an arbitrary $\varepsilon > 0$ and all M a suitable $\delta_M > 0$ such that $|f_M(\alpha) - f_M(\alpha^*)| < \varepsilon$ for all $\alpha \in B(\alpha^*; \delta_M)$. Define $\delta := \min\{\delta_M \mid K \subseteq$

$M \subseteq K \cup L\}$. Then we obtain for all $\alpha \in B(\alpha^*;\delta)$ that $K(\alpha) = M$ for one of the above index sets M and, therefore, $\left|f(\alpha) - f(\alpha^*)\right| = \left|f_M(\alpha) - f_M(\alpha^*)\right| < \varepsilon$. This proves continuity of f in α^*. □

So far, we know that f is continuous on $(0,\infty)^n$. However, this set is unbounded and open. Based on the following argument the problem of unboundedness becomes irrelevant in our context: Consider an arbitrary $\alpha \in (0,\infty)^n$. Then Lemma 3.1 implies

$$f(\alpha) = f\left(\frac{1}{\sum_{\mu=1}^n \alpha_\mu}\alpha\right).$$

Therefore, defining the set

$$A := \left\{\alpha \in (0,\infty)^n \;\middle|\; \sum_{\mu=1}^n \alpha_\mu = 1\right\}, \tag{7}$$

we obtain $f((0,\infty)^n) = f(A)$, and the function f attains a global maximum on $(0,\infty)^n$ if and only if it has a maximizer on the bounded set A. But this set A is not closed, hence not compact. However, Theorem 3.4 below shows that the function f can be extended continuously onto the closure $\bar{A}$ of A. This continuous extension of f then attains a maximum on $\bar{A}$ by a standard compactness argument. We then show that this, in turn, implies the existence of a maximizer of f on its original domain $(0,\infty)^n$.

In order to simplify our notation, let us define the index set

$$J(\alpha) := \{\nu \in N \mid \alpha_\nu = 0\}$$

corresponding to a given $\alpha \in [0,\infty)^n$. Then the following existence result holds.

Theorem 3.4 *The function $f : A \to \mathbb{R}$ has a continuous extension onto the closure $\bar{A}$ of A and, therefore, attains a global maximum on $\bar{A}$. Moreover, no vector $\alpha \in \bar{A}$ with $|J(\alpha)| = n-1$ is a global maximum.*

Proof. The fact that f can be continuously extended from A onto $\bar{A}$ follows from Lemmas A.1 and A.2 in the appendix, where, in particular, it is shown that this extension satisfies $f(\alpha) = 0$ for all $\alpha \in \bar{A}$ with $|J(\alpha)| = n-1$, hence none of these vectors is a global maximum of f since f attains positive values on A. The existence of a global maximum then follows immediately from the fact that $\bar{A}$ is a compact set. □

The global maximizer from Theorem 3.4 might belong to the set $\bar{A}\backslash A$. However, the feasible set of our original maximization problem is the set A or (without scaling) the set $(0, \infty)^n$. Using a suitable scaling together with a small perturbation, we now obtain the existence of a maximizer for our original problem from (4). Note that the following result shows that we can choose the maximizer in such a way that it also has some additional differentiability properties that will be exploited in Section 4.

Corollary 3.5 *The function f attains a global maximum in $(0, \infty)^n$. Moreover, this global maximum can be chosen in such a way that condition (6) from Theorem 3.3 holds.*

Proof. By Theorem 3.4, the function f attains a global maximum in $\bar{A}$, and this maximum necessarily belongs to the set

$$\{\alpha \in \bar{A} \mid |J(\alpha)| \in \{0, \dots, n-2\}\}.$$

However, as a consequence of Lemma 3.1, we have $f(c\alpha) = f(\alpha)$ for all α from this set and for all scalars $c > 0$. Consequently, the function f attains a global maximum α^* on the set

$$\{\alpha \in [0, \infty)^n \mid |J(\alpha)| \in \{0, \dots, n-2\}\}.$$

If, for this maximum, we have $|J(\alpha^*)| \in \{1, \dots, n-2\}$, i.e. $\alpha^* \notin (0, \infty)^n$, Lemma 3.2 shows that there is a point $\alpha^{**} \in (0, \infty)^n$ with the same function value so that α^{**} is also a global maximizer. Consequently, f has a global maximum in the set $(0, \infty)^n$, too. If this maximum does not satisfy condition (6) from Theorem 3.3, we can apply Lemma 3.2 once more and get another point in $(0, \infty)^n$ with the same function value (which, therefore, is also a maximum) such that (6) holds. $\square$

4 Effort Maximization: Computation of Solution

In this section we are going to provide an explicit analytical solution of the optimization problem of the contest organizer. Corollary 3.5 shows that there exists at least one global maximum α^* of the optimization problem from (4). Moreover, this result tells us that the maximum can be chosen in such a way that the objective function f is differentiable in a neighbourhood of this maximum. Since the feasible set $(0, \infty)^n$ is open, it follows that each such maximizer satisfies $\nabla f(\alpha^*) = 0$.

Basically, the aim of this section is to compute a global maximum by looking for all possible solutions of the nonlinear system of equations $\nabla f(\alpha) = 0$ at those points x where the derivative of f exists. The previous discussion shows that this technique will eventually provide us a global maximum of (4). Moreover, our analysis will prove that the global maximum ist unique in some sense and give an explicit formula.

Unfortunately, computing the zeros of $\nabla f(\alpha) = 0$ is not an easy task, especially since the derivative with respect to α leads to complicated formulas. In order to simplify our calculations we continue our analysis in terms of the effective cost parameters. Note, that those can be influenced by the contest organizer. We therefore use the transformation $\gamma : (0, \infty)^n \to (0, \infty)^n$ defined by

$$\gamma_\nu(\alpha) := \frac{\beta_\nu}{\alpha_\nu} \tag{8}$$

for all $\nu \in N$. Since $\beta \in (0, \infty)^n$, the mapping γ is a diffeomorphism from $(0, \infty)^n$ onto $(0, \infty)^n$. We further write $\gamma = \frac{\beta}{\alpha}$ for the vector whose components are given by $\frac{\beta_\nu}{\alpha_\nu}$. For some $\gamma \in (0, \infty)^n$, we also write

$$K(\gamma) := \left\{ \nu \in N \;\middle|\; (k(\gamma) - 1)\gamma_\nu < \sum_{\mu \in K(\gamma)} \gamma_\mu \right\}$$

with $k(\gamma) := |K(\gamma)|$. Using Theorem 2.6, it follows that for each $\gamma \in (0, \infty)^n$, there is precisely one such set $K(\gamma)$. Based on the set $K(\gamma)$, we now define the function $g : (0, \infty)^n \to \mathbb{R}$ by

$$g(\gamma) := \frac{k(\gamma) - 1}{\sum_{\mu \in K(\gamma)} \gamma_\mu} \left(\sum_{\mu \in K(\gamma)} \frac{\gamma_\mu}{\beta_\mu} - \frac{k(\gamma) - 1}{\sum_{\mu \in K(\gamma)} \gamma_\mu} \sum_{\mu \in K(\gamma)} \frac{\gamma_\mu^2}{\beta_\mu} \right).$$

Since

$$K(\gamma(\alpha)) = K(\alpha)$$

for all $\alpha \in (0, \infty)^n$, we have $g = f \circ \gamma^{-1}$. Hence, for all global maxima α^* of the function f satisfying condition (6) of Theorem 3.3, the function g has a global maximum in $\gamma^* := \frac{\beta}{\alpha^*}$ and is continuously differentiable in a neighbourhood of γ^*, since

$$(k(\gamma^*) - 1)\gamma_\nu^* = (k(\alpha^*) - 1)\frac{\beta_\nu}{\alpha_\nu^*} > \sum_{\mu \in K(\alpha^*)} \frac{\beta_\mu}{\alpha_\mu^*} = \sum_{\mu \in K(\gamma^*)} \gamma_\mu^* \quad \forall \nu \notin K(\gamma^*).$$

Conversely, if γ^* denotes a global maximum of g with the property

$$\nu \notin K(\gamma^*) \quad \Longrightarrow \quad (k(\gamma^*) - 1)\gamma_\nu^* > \sum_{\mu \in K(\gamma^*)} \gamma_\mu^*, \tag{9}$$

then $\alpha^* := \frac{\beta}{\gamma^*}$ is a global maximum of f such that condition (6) of Theorem 3.3 holds. Hence we have the following result.

Lemma 4.1 *$\alpha^* \in (0,\infty)^n$ is a global maximum of f satisfying property* (6) *of Theorem 3.3 if and only if $\gamma^* = \frac{\beta}{\alpha^*}$ is a global maximum of g satisfying condition* (9)*.*

Consequently, if we find all global maxima of g satisfying (9), expressed as effective cost parameters, then a simple re-transformation gives us all the global maxima of f satisfying condition (6) from Theorem 3.3, expressed as weighting factors of the original contest success function.

If a global maximum γ^* satisfies (9), then there is a neighbourhood of γ^* such that $K(\gamma) \equiv K(\gamma^*)$ for all γ from this neighbourhood and, hence, g is continuously differentiable in this neighbourhood. Since γ^* is a maximum, we therefore have $\nabla g(\gamma^*) = 0$. Consequently, we have to compute the zeros of ∇g. To this end, we first state two simple properties of the function g whose analogues were already shown for the function f.

Lemma 4.2 *(a) For all $\gamma \in (0,\infty)^n$ and all $c > 0$, we have $K(\gamma) = K(c\gamma)$ and $g(\gamma) = g(c\gamma)$.*

(b) Let $\gamma^* \in (0,\infty)^n$ be arbitrary. Then $K(\gamma^*) = K(\gamma)$ and $g(\gamma^*) = g(\gamma)$ hold for all $\gamma \in (0,\infty)^n$ satisfying

$$\gamma_\nu = \gamma^*_\nu \;\; \forall \nu \in K(\gamma^*) \quad \textit{and} \quad \gamma_\nu \geq \frac{1}{k(\gamma^*)-1} \sum_{\mu \in K(\gamma^*)} \gamma^*_\mu \;\; \forall \nu \notin K(\gamma^*).$$

Lemma 4.2 (a) shows that it suffices to compute maxima γ^* of g such that $\sum_{\mu \in K(\gamma^*)} \gamma^*_\mu = 1$. The inequality in part (b) implies that increasing the effective cost parameter of an inactive player leaves him inactive. The following result summarizes some properties of global maxima satisfying this additional condition.

Theorem 4.3 *Let $\gamma^* \in (0,\infty)^n$ be a global maximum of the function g satisfying $\sum_{\mu \in K(\gamma^*)} \gamma^*_\mu = 1$ and* (9)*. Then the following statements hold:*

(a) For all active players $\nu \in K(\gamma^*)$, we have

$$\gamma^*_\nu = \frac{1}{2(k(\gamma^*)-1)} \left[1 + (k(\gamma^*)-2) \frac{\beta_\nu}{\sum_{\mu \in K(\gamma^*)} \beta_\mu} \right].$$

(b) For all inactive players $\nu \notin K(\gamma^*)$, we have

$$\gamma^*_\nu > \frac{1}{k(\gamma^*)-1}.$$

(c) *For all active players $\nu \in K(\gamma^*)$, we have*

$$(k(\gamma^*) - 2)\beta_\nu < \sum_{\mu \in K(\gamma^*)} \beta_\mu.$$

(d) *Total equilibrium effort is given by:*

$$g(\gamma^*) = \frac{1}{4}\left[\sum_{\mu \in K(\gamma^*)} \frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu \in K(\gamma^*)} \beta_\mu}\right].$$

Proof. Since the maximum γ^* satisfies condition (9), there is a neighbourhood U of γ^* with

$$K(\gamma) = K(\gamma^*) =: K \quad \text{and} \quad k(\gamma) = k(\gamma^*) =: k.$$

Hence g is continuously differentiable in this neighbourhood of γ^* and, therefore, being an (essentially unconstrained) global maximum, we have $\nabla g(\gamma^*) = 0$.

The only statement we obtain for the components γ_ν with $\nu \notin K$ follows from (9):

$$\gamma_\nu^* > \frac{1}{k-1} \sum_{\mu \in K} \gamma_\mu^* = \frac{1}{k-1}.$$

This shows that statement (b) holds.

Moreover, for all $\nu \in K$, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial \gamma_\nu} g(\gamma^*) \\ &= -\frac{k-1}{\left(\sum_{\mu \in K} \gamma_\mu^*\right)^2} \sum_{\mu \in K} \frac{\gamma_\mu^*}{\beta_\mu} + \frac{k-1}{\sum_{\mu \in K} \gamma_\mu^*} \frac{1}{\beta_\nu} + \frac{2(k-1)^2}{\left(\sum_{\mu \in K} \gamma_\mu^*\right)^3} \sum_{\mu \in K} \frac{(\gamma_\mu^*)^2}{\beta_\mu} - \frac{2(k-1)^2}{\left(\sum_{\mu \in K} \gamma_\mu^*\right)^2} \frac{\gamma_\nu^*}{\beta_\nu} \\ &= -(k-1) \sum_{\mu \in K} \frac{\gamma_\mu^*}{\beta_\mu} + (k-1)\frac{1}{\beta_\nu} + 2(k-1)^2 \sum_{\mu \in K} \frac{(\gamma_\mu^*)^2}{\beta_\mu} - 2(k-1)^2 \frac{\gamma_\nu^*}{\beta_\nu}. \end{aligned} \tag{10}$$

Summing up equation (10) over all $\nu \in K$, we get

$$-(k-1) \sum_{\mu \in K} \frac{\gamma_\mu^*}{\beta_\mu} + 2(k-1)^2 \sum_{\mu \in K} \frac{(\gamma_\mu^*)^2}{\beta_\mu} = \frac{1}{k}\left(2(k-1)^2 \sum_{\mu \in K} \frac{\gamma_\mu^*}{\beta_\mu} - (k-1) \sum_{\mu \in K} \frac{1}{\beta_\mu}\right).$$

Inserting this into (10) and cancelling the factor $k-1$, we obtain for all $\nu \in K$:

$$\begin{aligned} \frac{2(k-1)}{k}\sum_{\mu\in K}\frac{\gamma_\mu^*}{\beta_\mu} - \frac{1}{k}\sum_{\mu\in K}\frac{1}{\beta_\mu} + \frac{1}{\beta_\nu} - 2(k-1)\frac{\gamma_\nu^*}{\beta_\nu} &= 0 \\ \iff \gamma_\nu^* - \frac{1}{k}\sum_{\mu\in K}\frac{\beta_\nu}{\beta_\mu}\gamma_\mu^* &= \frac{1}{2(k-1)}\left[1 - \frac{\beta_\nu}{k}\sum_{\mu\in K}\frac{1}{\beta_\mu}\right]. \end{aligned}$$

Consequently, the vector $\gamma_K^* := (\gamma_\nu^*)_{\nu\in K}$ is a solution of the linear system of equations

$$\left(I_{k\times k} - \frac{1}{k}\left(\frac{\beta_\nu}{\beta_\mu}\right)_{\nu,\mu\in K}\right)\left(\gamma_\mu\right)_{\mu\in K} = \frac{1}{2(k-1)}\left(1 - \frac{\beta_\nu}{k}\sum_{\lambda\in K}\frac{1}{\beta_\lambda}\right)_{\nu\in K}.$$

Using the abbreviations $\beta_K := (\beta_\nu)_{\nu\in K}$ and $\beta_K^{-1} := (\frac{1}{\beta_\nu})_{\nu\in K}$, the matrix of this linear system can be written as

$$I_{k\times k} - \frac{1}{k}\left(\frac{\beta_\nu}{\beta_\mu}\right)_{\nu,\mu\in K} = I_{k\times k} - \frac{1}{k}\beta_K(\beta_K^{-1})^T =: M.$$

This matrix M is singular, more precisely, it has rank $k-1$ and its null space (kernel) is given by $\ker(M) = \operatorname{span}\{\beta_K\}$ (this singularity reflects the fact that the function value $g(\gamma)$ is independent of the scaling of γ, cf. Lemma 4.2, hence M cannot be expected to be nonsingular at an arbitrary stationary point). Now it is easy to see that one particular solution of the above linear system of equations is the vector from the right-hand side:

$$\tilde{\gamma}_\nu = \frac{1}{2(k-1)}\left(1 - \frac{\beta_\nu}{k}\sum_{\lambda\in K}\frac{1}{\beta_\lambda}\right) \quad \forall \nu \in K.$$

Therefore, the vector γ_K^* is of the form $\gamma_K^* = \tilde{\gamma}_K + c\beta_K$, where $c \in \mathbb{R}$ has to be chosen in such a way that $\sum_{\mu\in K}\gamma_\mu^* = 1$. It follows that

$$c = \frac{1}{2(k-1)}\left[\frac{k-2}{\sum_{\mu\in K}\beta_\mu} + \frac{1}{k}\sum_{\mu\in K}\frac{1}{\beta_\mu}\right]$$

and, therefore,

$$\gamma_\nu^* = \frac{1}{2(k-1)}\left[1 + (k-2)\frac{\beta_\nu}{\sum_{\mu\in K}\beta_\mu}\right] \ (> 0)$$

for all $\nu \in K$. Hence statement (a) holds.

By the definition of $K = K(\gamma^*)$, we have for all $\nu \in K$:

$$(k-1)\gamma_\nu^* \ < \ \sum_{\mu \in K} \gamma_\mu^* \ = \ 1 \quad \Longleftrightarrow \quad (k-2)\beta_\nu \ < \ \sum_{\mu \in K} \beta_\mu.$$

This verifies statement (c). Inserting the representation of γ_K^* gives the desired representation of $g(\gamma^*)$ from assertion (d). $\square$

Note that Theorem 4.3 (c) does not say that the inequality

$$(k(\gamma^*) - 2)\beta_\nu < \sum_{\mu \in K(\gamma^*)} \beta_\mu$$

is violated for all $\nu \notin K(\gamma^*)$.

The next lemma shows that, in some sense, the converse of Theorem 4.3 also holds.

Lemma 4.4 *Let $K \subseteq N$ be arbitrarily given, let $k := |K| \geq 2$ and suppose that*

$$(k-2)\beta_\nu < \sum_{\mu \in K} \beta_\mu \quad \forall \nu \in K. \tag{11}$$

Define the vector $\gamma^* \in (0, \infty)^n$ in such a way that $\gamma_\nu^* > \frac{1}{k-1}$ is arbitrary for all $\nu \notin K$ and

$$\gamma_\nu^* = \frac{1}{2(k-1)} \left[1 + (k-2) \frac{\beta_\nu}{\sum_{\mu \in K} \beta_\mu} \right] \quad \forall \nu \in K.$$

Then the following statements hold:

(a) $\sum_{\mu \in K} \gamma_\mu^* = 1$.

(b) $K(\gamma^*) = K$ *and* γ^* *satisfies condition* (9).

(c) The function g is continuously differentiable in a neighbourhood of γ^* with $\nabla g(\gamma^*) = 0$.

(d) $g(\gamma^*) = \frac{1}{4} \left[\sum_{\mu \in K} \frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu \in K} \beta_\mu} \right]$.

Proof. Statement (a) can be verified easily using the definition of γ_μ^* for $\mu \in K$. Assertions (c) and (d), on the other hand, follow in essentially the same way as in the proof of Theorem 4.3 since our *definition* of γ_ν^* is exactly the same as the *representation* of γ_ν^* obtained in Theorem 4.3 for γ_ν^* ($\nu \in K$). To see that statement (b) holds, we verify that

$$(k-1)\gamma_\nu^* < \sum_{\mu \in K} \gamma_\mu^* = 1 \Longleftrightarrow \nu \in K. \tag{12}$$

The definition of the index set $K(\gamma^*)$ together with the uniqueness of this index set then shows $K = K(\gamma^*)$. Now, for $\nu \in K$, we obtain from the definition of γ^*_ν together with (11) that

$$(k-1)\gamma^*_\nu = \frac{1}{2}\left[1 + (k-2)\frac{\beta_\nu}{\sum_{\mu\in K}\beta_\mu}\right] < \frac{1}{2}[1+1] = 1,$$

hence the implication "$\Longleftarrow$" holds in (12). On the other hand, for $\nu \notin K$, we have $(k-1)\gamma^*_\nu > 1$ which, by contraposition, shows that also the implication "$\Longrightarrow$" holds in (12). □

Lemma 4.4 and Theorem 4.3 are the foundation of the following idea how to find all global maxima which satisfy the additional conditions from Theorem 4.3. All other global maxima can be derived from those satisfying the conditions from Theorem 4.3 using the variations from Lemma 4.2.

Theorem 4.3 allows the following interpretation: If γ^* is a global maximum of g satisfying $\sum_{\mu\in K(\gamma^*)}\gamma^*_\mu = 1$ and (9), then we necessarily have

$$|K(\gamma^*)| \geq 2 \quad \text{and} \quad (k(\gamma^*)-2)\beta_\nu < \sum_{\mu\in K(\gamma^*)}\beta_\mu \quad \forall \nu \in K(\gamma^*)$$

by statement (c). (Assertions (a) and (b) only give the structure of the maximizer γ^*, whereas statement (d) calculates the corresponding function value $g(\gamma^*)$.) Now, Lemma 4.4 takes an arbitrary index set $K \subseteq N$ with

$$k := |K| \geq 2 \quad \text{and} \quad (k-2)\beta_\nu < \sum_{\mu\in K}\beta_\mu \quad \forall \nu \in K, \tag{13}$$

defines corresponding values for γ^*_ν ($\nu \in N$) and then states that, in particular, we have $K = K(\gamma^*)$ and that γ^*_ν satisfies $\sum_{\mu\in K(\gamma^*)}\gamma^*_\mu = 1$ as well as condition (9) and $\nabla g(\gamma^*) = 0$. (Consequently, the vector γ^* corresponding to K is a candidate for a global maximum.) Hence, we can compute all global maxima satisfying the additional conditions from Theorem 4.3 by searching for those index sets $K \subseteq N$ with (13) that yield the maximal value $g(\gamma^*)$. Remember that there are only finitely many index sets $K \subseteq N$.

To make this idea more precise, we have to introduce some notation first. Using the formula for $g(\gamma^*)$ given in Lemma 4.4 (d), we define the function

$$h(K) := \frac{1}{4}\left[\sum_{\mu\in K}\frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu\in K}\beta_\mu}\right]$$

for all $K \subseteq N$ with $k := |K| \geq 2$ and $(k-2)\beta_\nu < \sum_{\mu \in K} \beta_\mu$ for all $\nu \in K$. Furthermore, we want to introduce the following terminology that will simplify our subsequent discussion to some extent.

Definition 4.5 *A set $K \subseteq N$ with $k := |K|$ is called*

(a) feasible*, if $k \geq 2$ and $(k-2)\beta_\nu < \sum_{\mu \in K} \beta_\mu$ for all $\nu \in K$.*

(b) maximal*, if K is feasible and there is no feasible superset $\tilde{K} \subseteq N$ of K.*

(c) optimal*, if K is feasible and $h(K) \geq h(\tilde{K})$ for all feasible sets $\tilde{K}$.*

We stress that a feasible set K still allows the existence of players $\nu \notin K$ such that the inequality

$$(k-2)\beta_\nu < \sum_{\mu \in K} \beta_\mu$$

holds. A feasible set is maximal if it is not strictly contained in another feasible set. Furthermore, an optimal set K is a feasible set such that the expression $h(K)$ is maximal among all feasible sets. The existence of such a set is clear since the number of feasible sets is finite (though typically exponentially large).

With this terminology, we state our idea from above more formally: According to Lemma 4.4 and Theorem 4.3, γ^* is a global maximum of g satisfying the conditions of Theorem 4.3 if and only if $K(\gamma^*)$ is optimal, i.e. $K(\gamma^*)$ is a solution of

$$\max \; h(K) \quad \text{s.t.} \quad K \text{ is feasible.} \tag{14}$$

The reason, why we are interested in maximal sets, is the following proposition.

Proposition 4.6 *Let $K, M \subseteq N$ be feasible subsets such that $M \subsetneq K$. Then we have $h(M) < h(K)$.*

Proof. Using the well-known inequality between the arithmetic and harmonic mean together with some elementary calculations, we obtain

$$\begin{aligned}
h(K) - h(M) &= \frac{1}{4}\left[\sum_{\mu \in K \setminus M} \frac{1}{\beta_\mu} - \frac{(k-2)^2}{\sum_{\mu \in K} \beta_\mu} + \frac{(m-2)^2}{\sum_{\mu \in M} \beta_\mu}\right] \\
&\geq \frac{1}{4}\left[\frac{(k-m)^2}{\sum_{\mu \in K \setminus M} \beta_\mu} - \frac{(k-m)^2 + 2(k-m)(m-2) + (m-2)^2}{\sum_{\mu \in K \setminus M} \beta_\mu + \sum_{\mu \in M} \beta_\mu} + \frac{(m-2)^2}{\sum_{\mu \in M} \beta_\mu}\right] \\
&= \frac{1}{4}\left[\frac{\left((k-m)\sum_{\mu \in M} \beta_\mu - (m-2)\sum_{\mu \in K \setminus M} \beta_\mu\right)^2}{\sum_{\mu \in K \setminus M} \beta_\mu \sum_{\mu \in K} \beta_\mu \sum_{\mu \in M} \beta_\mu}\right]
\end{aligned}$$

$$\geq \quad 0,$$

and equality $h(K) = h(M)$ holds if and only if

$$\sum_{\mu \in K \setminus M} \frac{1}{\beta_\mu} = \frac{(k-m)^2}{\sum_{\mu \in K \setminus M} \beta_\mu} \quad \text{and} \quad \sum_{\mu \in M} \beta_\mu = (m-2) \frac{\sum_{\mu \in K \setminus M} \beta_\mu}{k-m}.$$

Since all β_μ ($\mu \in K \setminus M$) are positive, the harmonic mean and the arithmetic mean coincide if and only if all β_μ ($\mu \in K \setminus M$) coincide, i.e. if $\beta_\mu = \beta$ for all $\mu \in K \setminus M$ and a suitable $\beta > 0$. Hence, the second equation implies that, for all $\mu \in K \setminus M$, we have

$$(k-2)\beta_\mu \;=\; (k-m)\beta + (m-2)\beta \;=\; \sum_{\mu \in K \setminus M} \beta_\mu + \sum_{\mu \in M} \beta_\mu \;=\; \sum_{\mu \in K} \beta_\mu,$$

which, however, is a contradiction to the feasibility of K. □

Proposition 4.6 says that strict subsets of feasible sets cannot be optimal. Thus, all solutions of (14) have to be maximal subsets of N. In other words, excluding a player from a feasible set will always decrease total maximal effort. The necessity of maximality for a feasible set to be optimal can hence be interpreted as a weak form of an inclusion principle.

Now, consider the case of $N \geq 3$ players. Further note that every subset $M \subseteq N$ consisting of two players is feasible. Then, take an arbitrary element $\nu \in N \setminus M$ and define $K := M \cup \{\nu\}$. This set K consists of three players containing M as a strict subset and is feasible, too. In view of Proposition 4.6, it follows that M cannot be an optimal set. Hence we obtain the following result.

Theorem 4.7 *Consider the effort maximization problem* (4) *with* $|N| \geq 3$. *Then there are at least three active players in every global maximum.*

Theorem 4.7 is remarkable: It not only improves on previous knowledge as summarized in Theorem 2.2 (a); it is also in marked contrast to well-established results from contests which are modeled as all-pay auctions, i.e. the contest success function is such that the highest effort wins with certainty (in case of m highest bids each wins with probability $\frac{1}{m}$). Then the equilibrium of the n-player complete information contest is generically unique and exhibits *precisely* two active players. Moreover, in the non-generic case with multiple equilibria, total effort in equilibrium is highest in the equilibrium with only two active players (see Baye et al. [3]). Hence allowing for free entry into the contest cannot improve the competitiveness of the contest as the equilibrium strategies of the two active players do not depend on the number and identity of inactive players.

This is not true in our model: a third player can always improve on the effort levels obtained in a two-player contest from the contest organizer's point of view.

As we will prove later, one of the maximal subsets mentioned above is

$$K^* := \left\{ \nu \in N \,\middle|\, (|K^*| - 2)\beta_\nu < \sum_{\mu \in K^*} \beta_\mu \right\}, \tag{15}$$

Our aim ist to prove that K^* is the unique optimal set. To this end, first note that the definition of K^* is given in an implicit form since K^* also occurs in the expression within the parenthesis. Therefore, it is neither clear whether this object is well-defined and unique nor whether it is a useful expression for the explicit calculation of the set K^*. The following result gives an alternative (and explicit) expression for K^* (provided that, without loss of generality, the coefficients β_μ are ordered in such a way that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$). This expression also implies that K^* exists. In fact, it turns out that there is precisely one set K^* satisfying the definition (15).

Lemma 4.8 *(a) If the coefficients β_μ are ordered in such a way that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, then the definition of K^* is equivalent to the following definition:*

$$K^* := \left\{ \nu \in N \,\middle|\, (\nu - 2)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu \right\}.$$

(b) The set K^* exists and is unique.

Proof. We verify statements (a) and (b) simultaneously, first under the assumption that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$. To this end, let us denote the set mentioned above by

$$M := \left\{ \nu \in N \,\middle|\, (\nu - 2)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu \right\},$$

and define $m := |M|$. Obviously, M exists and is unique. Furthermore, one can use the ordering of β_μ to verify the implications

$$\nu \in M \implies \nu - 1 \in M \quad \text{or, equivalently,} \quad \nu \notin M \implies \nu + 1 \notin M.$$

Hence, the set M is of the form $M = \{1, \ldots, m\}$.

Our first step is to show that the set M has the same properties as K^*. To this end, choose an arbitrary $\nu \in M$. Then the ordering of β_μ implies

$$(m-2)\beta_\nu = (\nu-2)\beta_\nu + (m-\nu)\beta_\nu < \sum_{\mu=1}^{\nu}\beta_\mu + \sum_{\mu=\nu+1}^{m}\beta_\mu = \sum_{\mu\in M}\beta_\mu.$$

On the other hand, we can use the definition of M to obtain

$$((m+1)-2)\beta_{m+1} \geq \sum_{\mu=1}^{m+1}\beta_\mu \quad \Longleftrightarrow \quad (m-2)\beta_{m+1} \geq \sum_{\mu\in M}\beta_\mu.$$

This, however, implies that the following is true for all $\nu \notin M$:

$$(m-2)\beta_\nu \geq (m-2)\beta_{m+1} \geq \sum_{\mu\in M}\beta_\mu.$$

Consequently, M satisfies the conditions imposed on K^*.

So far, we have shown that there is at least one set which suffices the definition of K^*, namely the set M. Furthermore, the ordering of β_μ implies that every set K^* has to be of the form $K^* = \{1,\ldots,k^*\}$. Now, it remains to prove that every set K^* has the same properties as M. To this end, we choose an arbitrary $\nu \notin K^*$. Then the ordering of β_μ implies

$$(\nu-2)\beta_\nu = (k^*-2)\beta_\nu + (\nu-k^*)\beta_\nu \geq \sum_{\mu\in K^*}\beta_\mu + \sum_{\mu=k^*+1}^{\nu}\beta_\mu = \sum_{\mu=1}^{\nu}\beta_\mu.$$

On the other hand, we know

$$(k^*-2)\beta_{k^*} < \sum_{\mu\in K^*}\beta_\mu \quad \Longleftrightarrow \quad ((k^*-1)-2)\beta_{k^*} < \sum_{\mu=1}^{k^*-1}\beta_\mu.$$

Using this reformulation and the ordering of the β_μ, we obtain

$$(\nu-2)\beta_\nu \leq (\nu-2)\beta_{\nu+1} < \sum_{\mu=1}^{\nu}\beta_\mu$$

inductively for all $\nu = k^*-1,\ldots,2$ (the case $\nu = 1$ is trivial). Altogether, we have shown that every set satisfying the definition of K^* also satisfies the definition of M.

Therefore, we have shown that the definitions of K^* and M coincide whenever the β_μ are ordered in an increasing way. Note that this implies existence and uniqueness of K^* in the ordered case.

Now, let us consider the case where β_ν are not necessarily sorted in increasing order. The definition of K^* is obviously independent of the numbering of the coefficients β_μ. Hence, we can use a permutation $\pi : N \to N$ to obtain an increasing ordering of the form $\beta_{\pi(1)} \leq \beta_{\pi(2)} \leq \ldots \leq \beta_{\pi(n)}$. To shorten the notation, we define $\tilde{\beta}_\mu := \beta_{\pi(\mu)}$ and

$$\tilde{K} := \left\{ \nu \in N \;\middle|\; (\nu - 2)\tilde{\beta}_\nu < \sum_{\mu \in \tilde{K}} \tilde{\beta}_\mu \right\}.$$

We are now in a position to apply the first part of our proof and obtain existence and uniqueness of $\tilde{K}$. On the other hand, we have $\nu \in K^*$ if and only if $\pi^{-1}(\nu) \in \tilde{K}$, i.e. $K^* = \pi(\tilde{K})$, where the permutation is meant to be applied elementwise on the set $\tilde{K}$. By combining these two facts, we can derive existence and uniqueness of the set K^* as well. □

Note that the assumption $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$ can be stated without loss of generality. Then Lemma 4.8 shows that K^* consists precisely of the $k^* := |K^*|$ smallest elements of the coefficients β_μ, i.e. $K^* = \{1, 2, \ldots, k^*\}$ (see also the proof of Lemma 4.8). This is an intuitive efficiency property of any solution: only the most able contestants, i.e. those with the lowest cost to provide effort, are chosen to be active by the contest organizer. This expression of K^* is very useful for the actual computation of this set. On the other hand, in our subsequent analysis, we typically exploit the implicit definition of K^* from (15).

Since we want to show that K^* is an optimal (in fact, the optimal) set, we know from Proposition 4.6 that it has to be at least a maximal set. The following result therefore verifies that K^* is indeed a maximal set, so that it remains a candidate for being an optimal set.

Lemma 4.9 *The set K^* is maximal in the sense of Definition 4.5.*

Proof. By definition, the set K^* is obviously feasible. In order to prove maximality, we will assume without loss of generality that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$. Then K^* has to be of the form $\{1, 2, \ldots, k^*\}$. Assume that K^* is not maximal. Then there is a set $M \subseteq N \backslash K^*$ such that $m := |M| \neq 0$ and $K^* \cup M$ is feasible. Let $\tilde{\mu}$ be the largest index in $K^* \cup M$. Then $\tilde{\mu} \notin K^*$ and thus the definition of K^* and the increasing order of the β_μ imply

$$((k + m) - 2)\beta_{\tilde{\mu}} = (k - 2)\beta_{\tilde{\mu}} + m\beta_{\tilde{\mu}} \geq \sum_{\mu \in K^*} \beta_\mu + \sum_{\mu \in M} \beta_\mu = \sum_{\mu \in K^* \cup M} \beta_\mu,$$

a contradiction to the feasibility of $K^* \cup M$. □

The next result is, basically, a technical lemma that will be exploited in our subsequent analysis. It is, however, also of some interest on its own by giving a necessary condition on the size of the β_μ that belong to any feasible set K: These β_μ have to be strictly smaller than the sum of the three smallest elements β_ν with $\nu \in K$. A comparison with the respective condition from Section 2, presented in Corollary 2.4, indicates that the active set of players under the optimal weights might be larger than, for instance, under the neutral weighting scheme, see Section 5 for a detailed example of this comparison.

Lemma 4.10 *Let K be feasible with $k := |K| \geq 3$. Then all β_ν ($\nu \in K$) are strictly smaller than the sum of the three smallest β_μ ($\mu \in K$).*

Proof. Assume once again, without loss of generality, that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$ and let $K = \{\mu_1, \ldots, \mu_k\}$ in increasing order, i.e., $\beta_{\mu_1} \leq \beta_{\mu_2} \leq \ldots \leq \beta_{\mu_k}$. Then the three smallest β_μ ($\mu \in K$) are $\beta_{\mu_1}, \beta_{\mu_2}, \beta_{\mu_3}$. The definition of K implies

$$(k-2)\beta_{\mu_k} < \sum_{j=1}^{k} \beta_{\mu_j} \quad \Longleftrightarrow \quad ((k-1)-2)\beta_{\mu_k} < \sum_{j=1}^{k-1} \beta_{\mu_j}.$$

Using this reformulation and the ordering of the β_{μ_j}, we obtain

$$(i-2)\beta_{\mu_i} \leq (i-2)\beta_{\mu_{i+1}} < \sum_{j=1}^{i} \beta_{\mu_j}$$

inductively for all $i = k-1, \ldots, 2$ (there is nothing to prove for $i = 1$). Now, the assertion is obviously true for $\beta_{\mu_1}, \beta_{\mu_2}, \beta_{\mu_3}$. For $\beta_{\mu_4}, \ldots, \beta_{\mu_k}$, the statement can be derived inductively using the formula above. □

In order to state our main result, we need another technical lemma whose proof is quite lengthy. So we state the result here, but give the proof in an appendix.

Lemma 4.11 *Suppose $n \geq 4$. For every feasible set K with $k := |K| \geq 3$ and $K \setminus K^* \neq \emptyset$ the following estimation holds:*

$$-\sum_{\mu \in K \setminus K^*} \frac{1}{\beta_\mu} + \frac{(k-2)^2}{\sum_{\mu \in K} \beta_\mu} \geq -\frac{(k-d)(k^*-2)}{\sum_{\mu \in K^*} \beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2+k^*-d)\sum_{\mu \in K^*} \beta_\mu - (k^*-2)\sum_{\mu \in K^* \setminus K} \beta_\mu},$$

where $d := |K^* \cap K|$.

Based on the previous results, we are now in a position to state our main theorem which gives an analytic solution of the effort maximization problem.

Theorem 4.12 *The set K^* is the unique optimal set (for the given parameters $\beta_\mu, \mu \in N$).*

Proof. For $n = 2$, $K^* = \{1, 2\}$ is the only feasible and thus optimal set. For $n = 3$, the assertion follows from Theorem 4.7. Now suppose $n \geq 4$. We show that for every feasible set $K \neq K^*$, there exists another feasible set $\tilde{K}$ with $h(K) < h(\tilde{K})$. To this end, let $K \neq K^*$ be an arbitrary feasible set. If $k := |K| = 2$ or $K \subsetneq K^*$ we can find such a set $\tilde{K}$ according to Proposition 4.6 (note that, in particular, every subset of N consisting of three players is feasible, so every feasible set K consisting of just two players can be enlarged by a set with three players, and then Proposition 4.6 can be applied also to this case). Hence, the only remaining case is $k \geq 3$ and $K \backslash K^* \neq \emptyset$. Define $k^* := |K^*|$ and $d := |K^* \cap K|$. In this case, we are in the situation where Lemma 4.11 can be applied, and we obtain

$$\begin{aligned}
4(h(K^*) - h(K)) &= \sum_{\mu \in K^* \backslash K} \frac{1}{\beta_\mu} - \frac{(k^*-2)^2}{\sum_{\mu \in K^*} \beta_\mu} - \sum_{\mu \in K \backslash K^*} \frac{1}{\beta_\mu} + \frac{(k-2)^2}{\sum_{\mu \in K} \beta_\mu} \\
&\geq \frac{(k^*-d)^2}{\sum_{\mu \in K^* \backslash K} \beta_\mu} - \frac{(k^*-2)^2}{\sum_{\mu \in K^*} \beta_\mu} - \frac{(k-d)(k^*-2)}{\sum_{\mu \in K^*} \beta_\mu} \\
&\quad + \frac{(k-2)^2(k^*-2)}{(k-d+k^*-2)\sum_{\mu \in K^*} \beta_\mu - (k^*-2)\sum_{\mu \in K^* \backslash K} \beta_\mu} \\
&= \frac{(k^*-2+k-d)\left((k^*-d)\sum_{\mu \in K^*} \beta_\mu - (k^*-2)\sum_{\mu \in K^* \backslash K} \beta_\mu\right)^2}{\left(\sum_{\mu \in K^* \backslash K} \beta_\mu\right)\left(\sum_{\mu \in K^*} \beta_\mu\right)\left((k-d+k^*-2)\sum_{\mu \in K^*} \beta_\mu - (k^*-2)\sum_{\mu \in K^* \backslash K} \beta_\mu\right)} \\
&> 0,
\end{aligned}$$

where the first equation follows from the definition of the function $h(K)$, the first inequality uses both the inequality between the arithmetic and the harmonic mean applied to the first term and Lemma 4.11 applied to the last two terms, the second equation follows by direct computation using a common denominator for all terms, and the final strict inequality exploits the feasibility of K^*, $k^* \geq 3$ and $k > d$ (the latter holds since $K \setminus K^* \neq \emptyset$). This shows $h(K) < h(K^*)$ for the remaining case. Consequently, K^* is the unique optimal set. □

Based on this theorem we know that the unique set of active players in the optimum is given by

$$K^* = \left\{ \nu \in N \,\middle|\, (k^*-2)\beta_\nu < \sum_{\mu \in K^*} \beta_\mu \right\}.$$

By applying Theorem 4.3 (a) and (b), we can obtain one set of optimal weighting parameters α^*, namely

$$\alpha^*_\nu = \frac{2(k^*-1)\beta_\nu}{1+\frac{(k^*-2)\beta_\nu}{\sum_{\mu\in K^*}\beta_\mu}} \quad \text{for } \nu \in K^*, \qquad \alpha^*_\nu < (k^*-1)\beta_\nu \quad \text{for } \nu \notin K^*. \tag{16}$$

Note that the parameters for the inactive players are not uniquely determined. However, this had to be expected due to Lemma 3.1 and 3.2. All other sets of optimal weighting parameters can be derived from α^* using the variations from those Lemmas. All solutions share an equal treatment property; namely, that active players with identical cost parameters get identical optimal weights.

The expression for α^*_ν is clearly increasing in β_ν for active players. This implies that under the optimal weighting scheme players with high costs are favored relatively more than players with low costs. Hence, with optimally specified weights the heterogeneity between active players is reduced to some extent. A closer look at the formula from Theorem 4.3 (a) reveals, however, that the effective cost parameter,

$$\gamma^*_\nu = \frac{\beta_\nu}{\alpha^*_\nu} = \frac{1}{2(k-1)}\left[1+(k-2)\frac{\beta_\nu}{\sum_{\mu\in K(\gamma^*)}\beta_\mu}\right],$$

is still increasing in β_ν, whenever there are more than two players active under the optimal weighting scheme. Hence, although the heterogeneity is reduced to some extent, the cost disadvantage of players with a higher cost factor still remains. We summarize these results in the following corollary and refer for further clarification to the detailed examples in the next section.

Corollary 4.13 *Under the optimal weighting scheme α^* active players with higher cost parameters obtain higher weights; but the optimal bias does not equalize effective cost parameters except if $n = 2$, i.e. the playing field is not leveled to the full extent if $n > 2$ holds.*

Based on the explicit characterization of the active set of players and the corresponding optimal weighting scheme we are now in a position to derive explicit formulae for equilibrium values. The optimal effort of player ν is then given by

$$x^\nu(\alpha^*) = \begin{cases} \frac{1}{4\beta_\nu}\left[1-\left(\frac{(k^*-2)\beta_\nu}{\sum_{\mu\in K^*}\beta_\mu}\right)^2\right] & \text{for } \nu \in K^*, \\ 0 & \text{for } \nu \notin K^*, \end{cases} \tag{17}$$

and, after some elementary calculations, one obtains

$$\theta_\nu(x^\nu(\alpha^*), x^{-\nu}(\alpha^*)) = \begin{cases} \frac{1}{4}\left[1 - \frac{(k^*-2)\beta_\nu}{\sum_{\mu\in K^*}\beta_\mu}\right]^2 & \text{for } \nu \in K^*, \\ 0 & \text{for } \nu \notin K^* \end{cases} \tag{18}$$

as the corresponding payoff for player ν. Note that the expression for equilibrium utility in (18) is never identical for players with different cost parameters, in fact it is decreasing in β_ν. Hence, the playing field is not leveled to the full extent (with the exception of the two-player case as presented in Example 5.1) which confirms the statement from the corollary.

Finally, total equilibrium effort is given by

$$f(\alpha^*) = \frac{1}{4}\left[\sum_{\mu\in K^*}\frac{1}{\beta_\mu} - \frac{(k^*-2)^2}{\sum_{\mu\in K^*}\beta_\mu}\right]. \tag{19}$$

5 Discussion and Examples

The expressions in (16) to (19) provide closed form equilibrium solutions for the problem of the contest organizer. These results are now applied to two special cases that have already been discussed in the literature, cf. [20] and [8].

Example 5.1 In the 2-player case, the set of active players in the global maximum is $K^* = \{1, 2\}$ and Theorem 4.3 implies that the optimal parameters are

$$\gamma_\nu^* = \frac{\beta_\nu}{\alpha_\nu^*} = \frac{1}{2}, \quad \text{hence} \quad \alpha_\nu^* = 2\beta_\nu \quad \forall \nu = 1, 2.$$

Therefore, heterogeneity between the players is completely removed in the optimum. The optimal set of weighting parameters yields the following equilibrium results:

$$\begin{aligned} x^{*,\nu} &= \frac{1}{4\beta_\nu} \quad \forall \nu = 1, 2; \\ f(\alpha^*) &= \frac{\beta_1 + \beta_2}{4\beta_1\beta_2}; \\ \theta_\nu(x^{*,\nu}, x^{*,-\nu}) &= \frac{1}{4} \quad \forall \nu = 1, 2. \end{aligned}$$

The complete removal of heterogeneity is also reflected by the fact that expected payoff in equilibrium is identical for both players. ◇

Example 5.2 In the homogeneous n-player case, where $\beta_\nu = \beta_\mu$ (=: β) for all $\nu, \mu \in N$, all subsets $K \subseteq N$ with $k := |K| \geq 2$ are feasible. Hence, as the optimal set has to be a maximal set, the set of active players in the global maximum is $K^* = N$, and Theorem 4.3 shows that the corresponding optimal parameters are

$$\gamma_\nu^* = \frac{\beta_\nu}{\alpha_\nu^*} = \frac{1}{n}, \quad \text{hence} \quad \alpha_\nu^* = n\beta_\nu \quad \forall \nu \in N.$$

In particular, all players are active in the optimum. Equilibrium results are the following:

$$x^{*,\nu} = \frac{n-1}{n^2\beta}; \quad f(\alpha^*) = \frac{n-1}{n\beta}; \quad \theta_\nu(x^{*,\nu}, x^{*,-\nu}) = \frac{1}{n^2} \quad \forall \nu \in N.$$

◇

In the next examples the optimal weighting scheme is compared with a neutral, i.e. unbiased weighting scheme $\hat{\alpha}$, where $\hat{\alpha}_\nu = \hat{\alpha}$ for $\nu = 1, \ldots, n$. The neutral weighting scheme coincides with the symmetric contest success function that is usually applied in the literature. For simplicity we assume again without loss of generality that the coefficients β_μ are ordered in such a way that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$.

A glance on the characterization of the respective active set of players, i.e. K^* versus $K(\hat{\alpha})$ calculated from Theorem 2.3, immediately implies that the active set of players under the optimal weighting scheme is (weakly) larger than under the neutral weighting scheme:

$$K^* := \left\{ \nu \in N \;\middle|\; (\nu - 2)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu \right\} \text{ vs. } K(\hat{\alpha}) := \left\{ \nu \in N \;\middle|\; (\nu - 1)\beta_\nu < \sum_{\mu=1}^{\nu} \beta_\mu \right\}.$$

The condition on the cost parameter is less strict under the optimal weighting scheme. Hence, with optimal weights there is no exclusion of players that goes beyond exclusion under the neutral weighting scheme. Note that the two conditions are identical for the case of homogeneous players with identical cost parameters. This is also the only case where the neutral weighting scheme is optimal. The following numerical example shows that optimal weights may lead to the inclusion of players over and above those active under neutral weights.

Example 5.3 The following distribution of cost parameters is considered: $\beta = (1, 2, 2, 4)^T$. Using the above mentioned characterization of the active set, it is obvious that the player with $\beta_4 = 4$ is not active with neutral weights but that he is induced to participate under the optimal weighting scheme. Hence, in this case there is inclusion of players due to the optimal weighting scheme. Equilibrium results are presented in the following table where the weighting factors are normalized to facilitate the comparison between the two weighting schemes.

	Neutral Weights $\hat{\alpha}$	Optimal Weights α^*
α	$(0.25, 0.25, 0.25, 0.25)$	$(0.143, 0.243, 0.243, 0.371)$
$K(\alpha)$	$\{1, 2, 3\}$	$\{1, 2, 3, 4\}$
x^*	$(0.24, 0.08, 0.08, 0)$	$(0.238, 0.100, 0.100, 0.013)$
$\frac{\alpha_\nu x^{*,\nu}}{\sum_{\mu=1}^n \alpha_\mu x^{*,\mu}}$	$(0.6, 0.2, 0.2, 0)$	$(0.389, 0.278, 0.278, 0.056)$
$\theta_\nu(x^*)$	$(0.36, 0.04, 0.04, 0)$	$(0.151, 0.077, 0.077, 0.003)$
$f(\alpha)$	0.4	0.451

Table 1: Example 5.3

Under the optimal weighting scheme the last player is induced to participate due to the relative large weights that he obtains in comparison to the other players. Note also that the dispersion in winning probabilities between the other players is reduced in comparison to the neutral weighting scheme which illustrates that the playing field is more leveled. Both effects, i.e. additional inclusion in combination with the balanced competition result in higher total equilibrium effort under the optimal weighting scheme. $\diamond$

The next example illustrates the isolated effect of balancing competition through the optimal weighting scheme where participation of players is not affected by setting optimal weights.

Example 5.4 The distribution of cost parameters is slightly altered for the last player: $\beta = (1, 2, 2, 6)^T$. In this case the last player even remains inactive under the optimal weights, i.e. the set of active players coincides for both weighting schemes.

	Neutral Weights $\hat{\alpha}$	Optimal Weights α^*
α	$(0.25, 0.25, 0.25, 0.25)$	$(0.226, 0.387, 0.387, 0)$
$K(\alpha)$	$\{1, 2, 3\}$	$\{1, 2, 3\}$
x^*	$(0.24, 0.08, 0.08, 0)$	$(0.24, 0.105, 0.105, 0)$
$\frac{\alpha_\nu x^{*,\nu}}{\sum_{\mu=1}^n \alpha_\mu x^{*,\mu}}$	$(0.6, 0.2, 0.2, 0)$	$(0.4, 0.3, 0.3, 0)$
$\theta_\nu(x^*)$	$(0.36, 0.04, 0.04, 0)$	$(0.16, 0.09, 0.09, 0)$
$f(\alpha)$	0.4	0.45

Table 2: Example 5.4

As in the previous example the dispersion in winning probabilities of active players declines, i.e. the heterogeneity among the players is reduced under the optimal weighting scheme. Again, setting optimal weights results in higher total equilibrium. A comparison with the previous example also implies that there is a positive effect from the inclusion of players because total effort under the optimal weighting scheme is slightly higher in Example 5.3 where all players are successfully encouraged to participate. ◇

6 Concluding Remarks

We have used a simple contest model to analyze political lobbying where the politician has some power over the lobbying process in the sense that she can discretionary bias the importance of the lobbying effort in favor or against specific lobbyists. Despite the simplicity of the contest game, the contest organizer's problem of maximization total effort by specifying the optimal vector of weighting parameters results in a complex nonsmooth optimization problem because the number of active contestants contributing to total effort is endogenous and depends on the weighting parameters (given valuation and cost parameters). Nevertheless, we are able to derive closed form solutions for the optimal vector of weighting factors by using techniques from bilevel programming. Moreover, the derived solution is easily computable because we can show that the set of active contestants that induces maximal total effort in equilibrium consists of the most efficient contestants. Once this optimal set of active contestants is deduced, the corresponding weighting factors are automatically determined.

The derived results suggest that it is optimal for the politician to bias the lobbying process in favor of weaker lobbyists to some extent but not completely. Hence, there is an (imperfect) leveling of the playing field which also induces more players to participate in comparison to a neutral policy (same weights to all players) of the politician. This inclusion principle also implies that there will be always at least three players active under the optimal weights scheme.

A tentative policy implication of our analysis would be to restrict the discretionary power of the politician by establishing formalized institutions in which political decision making takes place, e.g. public hearings, etc. In our terminology this would imply that the politician cannot deviate from neutral weighting which would result in less total lobbying effort (illustrated in Examples 5.3 and 5.4). As our results imply that the neutral policy is never optimal for the politician (except for the homogeneous n-player case), the amount of wasted resources invested into lobbying activities would be reduced at least to some extent.

References

[1] J. Alcalde and M. Dahm: *Rent seeking and rent dissipation: A neutrality result.* Journal of Public Economics 94, 2010, pp. 1–7.

[2] D.J. Clark and C. Riis: *Contest success functions: An extension.* Economic Theory 11, 1998, pp. 201–204.

[3] M.R. Baye, D. Kovenock, and C.G. de Vries: *The all-pay auction with complete information.* Economic Theory 8, 1996, pp. 291–305.

[4] M.R. Baye, D. Kovenock, and C.G. de Vries: *Rigging the lobbying process: An application of the all-pay auction.* American Economic Review 83, 1993, pp. 289–294.

[5] R.D. Congleton, A.L. Hillman, and K.A. Konrad (eds.): *40 Years of Research on Rent Seeking (Vol. 1 and Vol. 2).* Springer, 2008.

[6] Y.K. Che and I.L. Gale: *Caps on political lobbying.* American Economic Review 88, 1998, pp. 643–651.

[7] R. Cornes and R. Hartley: *Asymmetric contests with general technologies.* Economic Theory 26, 2005, pp. 923–946.

[8] A. Dasgupta and K.O. Nti: *Designing an optimal contest.* European Journal of Political Economy 14, 1998, pp. 587–603.

[9] S. Dempe: *Foundations of Bilevel Programming.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[10] J.M. Esteban, and D. Ray: *Conflict and Distribution.* Journal of Economic Theory 87, 1999, pp. 379–415.

[11] J.M. Esteban, and D. Ray: *Inequality, Lobbying, and Resource Allocation.* American Economic Review 96, 2006, pp. 257–279.

[12] H. Fang: *Lottery versus all-pay auction models of lobbying.* Public Choice 112, 2002, pp. 351–371.

[13] J. Franke: *Does affirmative action reduce effort incentives? A contest game analysis.* Mimeo, TU Dortmund, 2009.

[14] Q. Fu: *A theory of affirmative action in college admissions.* Economic Inquiry 44, 2006, pp. 420–428.

[15] E.P. Lazear, and S. Rosen: *Rank-Order Tournaments as optimum labor contracts.* Journal of Political Economy 89, 1981, pp. 841–864.

[16] Z.-Q. Luo, J.-S. Pang, and D. Ralph: *Mathematical Programs with Equilibrium Constraints.* Cambridge University Press, Cambridge, England, 1996.

[17] J. Marion: *Are bid preferences benign? The effect of small business subsidies in highway procurement auctions.* Journal of Public Economics 91, 2007, pp. 1591—1624.

[18] P.R. McAfee, and J. McMillan: *Government procurement and international trade.* Journal of International Economics 26, 1989, pp. 291–308.

[19] K.O. Nti: *Rent-seeking with asymmetric valuations.* Public Choice 98, 1999, pp. 415–430.

[20] K.O. Nti: *Maximum efforts in contests with asymmetric valuations.* European Journal of Political Economy 20, 2004, pp. 1059–1066.

[21] J.V. Outrata: *On optimization problems with variational inequality constraints.* SIAM Journal on Optimization 4, 1994, pp. 340–357.

[22] J.V. Outrata, M. Kocvara, and J. Zowe: *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[23] R. Siegel: *All-pay contests.* Econometrica 77(1), 2009, pp. 71–92.

[24] W.E. Stein: *Asymmetric rent-seeking with more than two contestants.* Public Choice 113, 2002, pp. 325–336.

[25] F. Szidarovsky and K. Okuguchi: *On the existence and uniqueness of pure Nash equilibrium in rent-seeking games.* Games and Economic Behavior 18, 1997, pp. 135–140.

[26] G. Tullock: *Efficient rent seeking.* In J. Buchanan, R. Tollison, and G. Tullock (eds.): *Towards a Theory of the Rent-Seeking Society.* A & M University Press, 1980, pp. 97-112.

A Proof of Theorem 3.4

Here we give a proof of Theorem 3.4 which is the central existence result from Section 3. In particular, we have to show that the function $f : A \to \mathbb{R}$ from (5) has a continuous extension from the set A defined in (7) onto its closure

$$\bar{A} = \left\{ \alpha \in [0, \infty)^n \;\middle|\; \sum_{\mu=1}^{n} \alpha_\mu = 1 \right\}.$$

To this end, we first recall the definition of the index set

$$J(\alpha) := \{\nu \in N \mid \alpha_\nu = 0\}$$

for a given $\alpha \in [0, \infty)^n$. We already know from Theorem 3.3 that f is a continuous function on A, i.e. f is continuous at any point $\alpha \in \bar{A}$ such that $|J(\alpha)| = 0$. In a first step, we will show in Lemma A.1 that f has a continuous extension to all $\alpha \in \bar{A}$ with $|J(\alpha)| \leq n - 2$. Then, we will prove in Lemma A.2 that f can also be extended continuously to all points $\alpha \in \bar{A}$ such that $|J(\alpha)| = n - 1$ by defining $f(\alpha) := 0$ in these points. Since the case $|J(\alpha)| = n$ cannot occur for $\alpha \in \bar{A}$, this yields Theorem 3.4.

Here is our first result regarding the extension of f to points α with $|J(\alpha)| \leq n - 2$.

Lemma A.1 *The function f, viewed as a mapping from A to $\mathbb{R}$, can be extended continuously onto the set* $\{\alpha \in \bar{A} \mid |J(\alpha)| \leq n - 2\}$.

Proof. Recall from the proof of Theorem 3.3 that f is continuous on the set

$$A = \{\alpha \in \bar{A} \mid |J(\alpha)| = 0\}$$

(in fact, it is continuous on $(0, \infty)^n$). Now, let $\alpha^* \in \bar{A}$ with $|J(\alpha^*)| \in \{1, \dots, n - 2\}$ be arbitrarily given. Then let us define the set of players $N^* := N \setminus J(\alpha^*)$. Since we have $|N^*| \geq 2$, it follows that the Nash game with the set of players N^* replacing the set of players N has all the properties that were already shown. Consequently, if we let

$$f^*(\alpha) := \sum_{\nu \in N^*} x^\nu(\alpha)$$

be the objective function of this new game, we, in particular, obtain from Theorem 3.3 that f^* is continuous in a sufficiently small neighbourhood of α^* simply since we eliminated the critical players ν with $\alpha^*_\nu = 0$ from the set N. We will show in the next paragraph that, for all α from a sufficiently small neighbourhood U of α^*, we have $K(\alpha) \subseteq N^*$. This then implies $f(\alpha) = f^*(\alpha)$ for all $\alpha \in U$ and, in this way, we obtain the desired continuous extension of f in α^*.

To verify the above claim, we have to find a sufficiently small neighbourhood U of α^* such that $K(\alpha) \subseteq N^*$ for all $\alpha \in U$, i.e., for all $\alpha \in U$ and all indices ν with $\nu \in K(\alpha)$, we necessarily have $\alpha_\nu > 0$. By contraposition, this is equivalent to showing that, for all $\alpha \in U$ and all indices ν with $\alpha_\nu = 0$, we have $\nu \notin K(\alpha)$.

To see this, we first choose a sufficiently small neighbourhood of α^* such that $|J(\alpha)| \in \{0, 1, \dots, n-2\}$ for all $\alpha \in U$. We then define a function $c(\alpha)$ on U as the sum of the two smallest quotients $\frac{\beta_\mu}{\alpha_\mu}$ ($\mu \in N$). Then $c(\alpha)$ is continuous and finite. Moreover, Corollary 2.4 shows that we always have $K(\alpha) \subseteq \{\nu \in N \,|\, \frac{\beta_\nu}{\alpha_\nu} < c(\alpha)\}$. By taking a possibly smaller neighbourhood U, we may assume by continuity that $c(\alpha) < 2c(\alpha^*)$ for all $\alpha \in U$ and, in addition, that $\frac{\beta_\nu}{\alpha_\nu} > 2c(\alpha^*)$ for all $\nu \in J(\alpha^*)$. This implies the desired claim since, now, we obtain $\frac{\beta_\nu}{\alpha_\nu} > 2c(\alpha^*) > c(\alpha)$ for all $\alpha \in U$ and all $\nu \in J(\alpha^*)$, hence $\nu \notin K(\alpha)$. □

It remains to consider the case $|J(\alpha)| = n - 1$. This is done in the following result.

Lemma A.2 *The function f, viewed as a mapping from $\{\alpha \in \bar{A} \mid |J(\alpha)| \leq n-2\}$ to $\mathbb{R}$, can be extended continuously onto the set $\bar{A}$ by setting $f(\alpha^*) = 0$ for all $\alpha^* \in \bar{A}$ with $|J(\alpha^*)| = n - 1$.*

Proof. We begin with some preliminary comments. In order to verify our claim, we have to show that, given an arbitrary vector $\alpha^* \in \bar{A}$ with $|J(\alpha^*)| = n - 1$ as well as a sequence $\{\alpha\} \to \alpha^*$ with $\alpha \in \bar{A}$ satisfying $|J(\alpha)| \leq n - 2$ for all α, we have $f(\alpha) \to f(\alpha^*)$. Now, for all $\alpha \in A$ (so all components of α are positive), we have the representation

$$f(\alpha) = \sum_{\nu \in K(\alpha)} x^\nu(\alpha) \tag{20}$$

of our objective function, where $K(\alpha)$ is the set of active players, cf. (5). On the other hand, if one or more (at most $n-2$) components of α are equal to zero, we obtained f by a continuous extension in the proof of Lemma A.1, hence the representation (20) does not necessarily hold in this case. However, we showed in the proof of Lemma A.1 that $K(\alpha) \cap J(\alpha) = \emptyset$ so that players ν with $\alpha_\nu = 0$ are certainly not active. This means that for all $\alpha \in \bar{A}$ with $|J(\alpha)| \leq n-2$, the representation (20) is still valid, and we will work with it throughout this proof.

Now, take an arbitrary $\alpha^* \in \bar{A}$ with $|J(\alpha^*)| = n - 1$, i.e. $\alpha^* = e_j$ for some $j \in \{1, \dots, n\}$. Then we obtain for all $\alpha \in \bar{A} \setminus \{\alpha^*\}$ sufficiently close to α^* that, on the one hand, $|J(\alpha)| \in \{0, \dots, n-2\}$

and, on the other hand,

$$\frac{\beta_j}{\alpha_j} = \min_{\mu \in K(\alpha)} \frac{\beta_\mu}{\alpha_\mu},$$

hence $j \in K(\alpha)$. Consider an arbitrary sequence $\{\alpha\} \subset \bar{A}\backslash\{\alpha^*\}$ with $\alpha \to \alpha^*$. We can divide the sequence into finitely many subsequences such that, within each subsequence, the set $K(\alpha)$ is constant. We verify the statement for each of these subsequences which then, obviously, implies that the statement holds for the entire sequence. We now consider one of these subsequences and call it, once again, $\{\alpha\}$. In view of the previous remark, we have $K(\alpha) \equiv K$ and $k(\alpha) \equiv k$ for all α. We now verifiy the limit $f(\alpha) = \sum_{\nu \in K} x^\nu(\alpha) \to 0$ by showing that $x^\nu(\alpha) \to 0$ holds for all $\nu \in K$. For $\nu = j$, this follows immediately from

$$x^j(\alpha) = \left(1 - \frac{\beta_j(k-1)}{\sum_{\mu \in K} \beta_\mu \frac{\alpha_j}{\alpha_\mu}}\right) \frac{(k-1)}{\sum_{\mu \in K} \beta_\mu \frac{\alpha_j}{\alpha_\mu}} \;\to\; (1-0)0 = 0.$$

Moreover, for $k = 2$, the statement also follows easily for $\nu \in K\backslash\{j\}$:

$$x^\nu(\alpha) = \left(1 - \frac{\beta_\nu}{\beta_\nu + \beta_j \frac{\alpha_\nu}{\alpha_j}}\right) \frac{1}{\beta_\nu + \beta_j \frac{\alpha_\nu}{\alpha_j}} \;\to\; (1-1)\frac{1}{\beta_\nu} = 0.$$

It therefore remains to verify $x^\nu(\alpha) \to 0$ for all $\nu \in K\backslash\{j\}$ in the case $k \geq 3$. To this end, we show that, for all $k = 3, 4, \dots$ and all $\nu, \mu \in K\backslash\{j\}$ with $\nu \neq \mu$, we have

$$\lim_{\alpha \to e_j} \frac{\alpha_\nu}{\alpha_\mu} = \frac{\beta_\nu}{\beta_\mu}. \tag{21}$$

Using (21), we then obtain for all $\nu \in K\backslash\{j\}$ and all $k \geq 3$

$$x^\nu(\alpha) = \left(1 - \frac{(k-1)}{\sum_{\mu \in K} \frac{\beta_\mu}{\beta_\nu} \frac{\alpha_\nu}{\alpha_\mu}}\right) \frac{(k-1)}{\beta_\nu \sum_{\mu \in K} \frac{\beta_\mu}{\beta_\nu} \frac{\alpha_\nu}{\alpha_\mu}} \;\to\; (1-1)\frac{1}{\beta_\nu} = 0$$

and therefore the desired statement. To verify (21), it suffices to show that, for all $k = 3, 4, \dots$ and all $\nu, \mu \in K\backslash\{j\}$ with $\nu \neq \mu$, we have

$$\limsup_{\alpha \to e_j} \frac{\alpha_\nu}{\alpha_\mu} \leq \frac{\beta_\nu}{\beta_\mu}. \tag{22}$$

Exchanging the roles of ν and μ then yields (21).

To verify (22), we first consider the case $k = 3$. Therefore, let $\nu, \mu \in K\backslash\{j\}$ be given with

$\nu \neq \mu$. We then obtain for an arbitrary α, exploiting the characteristic property (1) of $\mu \in K$, that

$$\frac{\alpha_\nu}{\alpha_\mu} = \frac{\beta_\nu}{\beta_\mu}\frac{\alpha_\nu}{\beta_\nu}\frac{\beta_\mu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu}\frac{\alpha_\nu}{\beta_\nu}\frac{1}{2}\left(\frac{\beta_j}{\alpha_j} + \frac{\beta_\nu}{\alpha_\nu} + \frac{\beta_\mu}{\alpha_\mu}\right).$$

Rewriting this expression gives

$$\frac{\alpha_\nu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu}\left(\frac{\alpha_\nu\beta_j}{\beta_\nu\alpha_j} + 1\right).$$

Taking into account $\alpha \to e_j$, we obtain (22).

Next, consider the case $k = 4$. To this end, choose arbitrary $\nu, \mu \in K\backslash\{j\}$ with $\nu \neq \mu$, and let $K = \{j, \nu, \mu, \lambda\}$. Using $\lambda \in K$, we have

$$\frac{\beta_\lambda}{\alpha_\lambda} < \frac{1}{3}\sum_{\rho\in K}\frac{\beta_\rho}{\alpha_\rho} \Longleftrightarrow \frac{\beta_\lambda}{\alpha_\lambda} < \frac{1}{2}\left(\frac{\beta_j}{\alpha_j} + \frac{\beta_\nu}{\alpha_\nu} + \frac{\beta_\mu}{\alpha_\mu}\right). \tag{23}$$

Exploiting once again (1), we obtain from $\mu \in K$ the inequality

$$\frac{\alpha_\nu}{\alpha_\mu} = \frac{\beta_\nu}{\beta_\mu}\frac{\alpha_\nu}{\beta_\nu}\frac{\beta_\mu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu}\frac{\alpha_\nu}{\beta_\nu}\frac{1}{3}\left(\frac{\beta_j}{\alpha_j} + \frac{\beta_\nu}{\alpha_\nu} + \frac{\beta_\mu}{\alpha_\mu} + \frac{\beta_\lambda}{\alpha_\lambda}\right).$$

Estimating the right-hand side by using (23) and rearranging the resulting terms, we obtain the same inequality

$$\frac{\alpha_\nu}{\alpha_\mu} < \frac{\beta_\nu}{\beta_\mu}\left(\frac{\alpha_\nu}{\beta_\nu}\frac{\beta_j}{\alpha_j} + 1\right)$$

as above, so that $\alpha \to e_j$ also yields (22) for the case $k = 4$. For $k = 5, 6, \ldots$, the statement can be verified in an analogous way. $\square$

B Proof of Lemma 4.11

Recall from Lemma 4.11 that we are in the situation where we have $n \geq 4$ players, K is a feasible set with at least three elements, $k := |K|$ and $K \setminus K^*$ is nonempty. Our aim is to find a suitable lower bound for the expression

$$-\sum_{\mu\in K\backslash K^*}\frac{1}{\beta_\mu} + \frac{(k-2)^2}{\sum_{\mu\in K}\beta_\mu} = -\sum_{\mu\in K\backslash K^*}\frac{1}{\beta_\mu} + \frac{(k-2)^2}{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\backslash K^*}\beta_\mu},$$

because this lower bound played a very crucial role in the proof of our main result, Theorem 4.12. Standard estimates for this expression did not work in the proof of that result, so we need a very sharp lower bound. To this end, we compute analytically the solution of a related optimization problem. More precisely, we will prove that the lower bound given in Lemma 4.11 is the global minimum of the problem

$$\begin{aligned}
\min_{b_\nu, \nu\in K\backslash K^*} \quad & -\sum_{\mu\in K\backslash K^*} \frac{1}{b_\mu} + \frac{(k-2)^2}{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\backslash K^*} b_\mu} \\
\text{s.t.} \quad & -(k^*-2)b_\nu + \sum_{\mu\in K^*}\beta_\mu \le 0 \quad \forall \nu\in K\backslash K^*, \\
& (k-2)b_\nu - \sum_{\mu\in K\backslash K^*} b_\mu - \sum_{\mu\in K\cap K^*}\beta_\mu \le 0 \quad \forall \nu \in K\backslash K^*,
\end{aligned} \tag{24}$$

where β_ν ($\nu \in K\cap K^*$) are viewed as fixed and b_ν ($\nu\in K\backslash K^*$) are the variables. To do so, we will proceed in two steps: First, we will prove the existence of a global minimum and then we will calculate it explicitly.

Existence of a global minimum:
First note that the feasible set is nonempty since the definitions of the index sets K and K^* immediately imply that the vector $\beta_{K\backslash K^*}$ is feasible.

Since $n \ge 4$, it follows from Theorem 4.7 that $k^* \ge 3$. The maximality of K^* together with $K\backslash K^* \neq \emptyset$ implies $K^*\backslash K \neq \emptyset$, i.e. $k^* - d > 0$, where $d := |K^* \cap K|$. At first, we will deal with the case $d \ge 3$.

We claim that, under this additional assumption, the feasible set of (24) is bounded, hence compact. To this end, let $b_{K\backslash K^*}$ be any feasible vector for this program. This implies that for all $\gamma \in K^*\cap K$ and all $\nu\in K\backslash K^*$

$$0 < \beta_\gamma < \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2} \le b_\nu.$$

Because of $d \ge 3$, this implies that the three smallest elements of $\{b_\nu \mid \nu \in K\backslash K^*\} \cup \{\beta_\nu \mid \nu \in K\cap K^*\}$ belong to indices $\nu \in K^*\cap K$. Define c as the sum over the three smallest β_ν ($\nu\in K^*\cap K$). This constant is independent from $b_{K\backslash K^*}$, and one can prove analogously to Lemma 4.10 that every feasible $b_{K\backslash K^*}$ satisfies $b_\nu \le c$ for all $\nu \in K\backslash K^*$. So the feasible set of problem (24) is not only closed but also bounded, i.e. it is compact. Hence, the continuous objective function attains a global minimum in the feasible set.

Now, we have to deal with the remaining case $d < 3$. Unfortunately, the feasible set is unbounded for $d \in \{0, 1, 2\}$, so we have to use a slightly different argumentation here.

We can find a sequence of feasible vectors $\{b^m_{K\backslash K^*}\}_m$ such that the corresponding values of the objective function converge to the infimum of the function on the feasible set (which could be $-\infty$). If any subsequence of $\{b^m_{K\backslash K^*}\}_m$ converges to a finite limit point, the closedness of the feasible set guarantees that this limit point is feasible and thus a global minimum.

Now let us assume that for every subsequence, at least one component b^m_ν ($\nu \in K\backslash K^*$) is unbounded. Then we can find a subsequence of $\{b^m_{K\backslash K^*}\}_m$ such that every component b^m_ν ($\nu \in K\backslash K^*$) either converges to a finite b_ν or diverges to $+\infty$. Denote by I_f the index set of the converging components and by I_∞ the index set of the diverging components. Then $I_\infty \neq \emptyset$. This, however, implies $d + i_f \leq 2$, where $i_f := |I_f|$. Otherwise, an argument similar to Lemma 4.10 would yield that all components of the limit point were bounded by the sum of the three smallest elements of $\{b_\nu \mid \nu \in K\backslash K^*\} \cup \{\beta_\nu \mid \nu \in K \cap K^*\}$, hence finite. Using this information, we can compare the infimum of the objective function, which is then given by

$$-\sum_{\mu\in I_f} \frac{1}{b_\mu}$$

with

$$b_\nu \geq \frac{\sum_{\mu\in K^*} \beta_\mu}{k^* - 2} \qquad \forall \nu \in I_f, \tag{25}$$

(by continuity) with the value of the objective function in the point $\left(\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\right)_{K\backslash K^*}$, which is feasible as we will show below. The value of the objective function corresponding to this vector is given by

$$\begin{aligned}
&-\sum_{\mu\in K\backslash K^*} \frac{k^*-2}{\sum_{j\in K^*}\beta_j} + \frac{(k-2)^2}{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\backslash K^*} \frac{\sum_{j\in K^*}\beta_j}{k^*-2}} \\
&= \; -(k-d)\frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k^*-2)\sum_{\mu\in K\cap K^*}\beta_\mu + (k-d)\sum_{\mu\in K^*}\beta_\mu} \\
&= \; -(k-d)\frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2)\sum_{\mu\in K^*}\beta_\mu + \delta}
\end{aligned}$$

To shorten the notation, we used the abbreviation

$$\delta := (k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\backslash K}\beta_\mu.$$

Using the definition of K^* together with $K^*\backslash K \neq \emptyset$, it is not difficult to see that $\delta > 0$. This yields

$$\begin{aligned}
&\left(-(k-d)\frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2)\sum_{\mu\in K^*}\beta_\mu + \delta}\right) - \left(-\sum_{\mu\in I_f}\frac{1}{b_\mu}\right)\\
\leq\ & -\frac{(k-d-i_f)(k^*-2)}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2)\sum_{\mu\in K^*}\beta_\mu + \delta}\\
\leq\ & -\frac{(k-2)(k^*-2)}{\sum_{\mu\in K^*}\beta_\mu} + \frac{(k-2)^2(k^*-2)}{(k-2)\sum_{\mu\in K^*}\beta_\mu + \delta}\\
=\ & \frac{(k-2)(k^*-2)}{\sum_{\mu\in K^*}\beta_\mu}\cdot\frac{-\delta}{(k-2)\sum_{\mu\in K^*}\beta_\mu + \delta}\\
<\ & 0,
\end{aligned}$$

where the first expression was motivated above, the first inequality follows by estimating the second term based on (25), the second inequality is a consequence of the fact that $d + i_f \leq 2$, the subsequent equation follows by direct calculation using some cancellations, and the final inequality uses the fact that $\delta > 0$.

It remains to prove the feasibility of $\left(\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\right)_{K\backslash K^*}$. Obviously, for all $\nu \in K\backslash K^*$

$$b_\nu = \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2} \geq \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}.$$

On the other hand, we have

$$\frac{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\backslash K^*} b_\mu}{k-2} = \frac{\sum_{\mu\in K^*}\beta_\mu - \sum_{\mu\in K^*\backslash K}\beta_\mu + (k-d)\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}}{k-2} > \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2} = b_\nu$$

for all $\nu \in K\backslash K^*$, where the strict inequality can be obtained using the fact $\delta > 0$. This proves the feasibility.

This, however, is a contradiction, as the objective function attains a smaller value in $\left(\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\right)_{K\backslash K^*}$ than in its infimum. Hence, our assumption was wrong and the objective function always attains a global minimum. In fact, we will prove in the next part that $\left(\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\right)_{K\backslash K^*}$ is this global minimum.

Calculation of the global minimum:
As all constraints in (24) are linear, the global minimum has to be a KKT-point. Therefore, our next step is to calculate all KKT-points of this problem.

Assume that $b_{K\backslash K^*}$ is such a KKT-point. Then we know that the following equations hold for all $\nu \in K\backslash K^*$:

$$\frac{1}{b_\nu^2} - \frac{(k-2)^2}{(\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\backslash K^*} b_\mu)^2} - \lambda_\nu^l(k^*-2) + \lambda_\nu^u(k-2) - \sum_{\mu\in K\backslash K^*}\lambda_\mu^u = 0,$$
$$(k^*-2)b_\nu \geq \sum_{\mu\in K^*}\beta_\mu,\ \ \lambda_\nu^l \geq 0,\ \ \Big((k^*-2)b_\nu - \sum_{\mu\in K^*}\beta_\mu\Big)\lambda_\nu^l = 0,$$
$$(k-2)b_\nu \leq \sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\backslash K^*} b_\mu,\ \ \lambda_\nu^u \geq 0,\ \ \Big((k-2)b_\nu - \sum_{\mu\in K\cap K^*}\beta_\mu - \sum_{\mu\in K\backslash K^*} b_\mu\Big)\lambda_\nu^u = 0.$$

The feasibility of $b_{K\backslash K^*}$ implies that

$$\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2} \leq b_\nu \leq \frac{1}{k-2}\left(\sum_{\mu\in K\backslash K^*} b_\mu + \sum_{\mu\in K\cap K^*}\beta_\mu\right) \quad \forall \nu\in K\backslash K^*$$

(recall that $k^*-2>0$ and $k-2>0$, so the above denominators are well-defined). However, the lower estimate for b_ν ($\nu \in K\backslash K^*$) given in the previous formula is strictly smaller than the upper estimate. This can be seen in the following way: Using the definition of K^* together with $K^*\backslash K \neq \emptyset$ as well as the feasibility of b_ν ($\nu \in K\backslash K^*$), we obtain

$$b_\nu \geq \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\ \ \forall\nu\in K\backslash K^* \quad\Longrightarrow\quad \sum_{\nu\in K\backslash K^*} b_\nu \geq \frac{k-d}{k^*-2}\sum_{\mu\in K^*}\beta_\mu,$$
$$\beta_\nu < \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}\ \ \forall\nu\in K^* \quad\Longrightarrow\quad \sum_{\nu\in K^*\backslash K} \beta_\nu < \frac{k^*-d}{k^*-2}\sum_{\mu\in K^*}\beta_\mu,$$

where the implications follow by taking the summations over all $\nu \in K\backslash K^*$ and all $\nu \in K^*\backslash K$, respectively. Using these estimates, we indeed obtain

$$\begin{aligned}\frac{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\backslash K^*} b_\mu}{k-2} &= \frac{\sum_{\mu\in K^*}\beta_\mu - \sum_{\mu\in K^*\backslash K}\beta_\mu + \sum_{\mu\in K\backslash K^*} b_\mu}{k-2}\\ &> \frac{\sum_{\mu\in K^*}\beta_\mu((k^*-2)-(k^*-d)+(k-d))}{(k^*-2)(k-2)}\\ &= \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}.\end{aligned} \tag{26}$$

Hence, every $\nu \in K\backslash K^*$ belongs to exactly one of the following three cases:
Case 1: If $b_\nu = \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}$, then the KKT-conditions together with (26) imply $\lambda_\nu^u = 0$. We define I_l as the set of all indices $\nu \in K\backslash K^*$ that belong to this case.
Case 2: If $b_\nu = \frac{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\backslash K^*} b_\mu}{k-2}$, then the KKT-conditions together with (26) imply $\lambda_\nu^l = 0$ and

thus

$$\lambda^u_\nu(k-2) - \sum_{\mu\in K\setminus K^*} \lambda^u_\mu = 0. \tag{27}$$

We define I_u as the set of all indices $\nu \in K\setminus K^*$ that belong to this case.
Case 3: If $b_\nu \in \left(\frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}, \frac{\sum_{\mu\in K\cap K^*}\beta_\mu+\sum_{\mu\in K\setminus K^*}b_\mu}{k-2}\right)$, then the KKT-conditions together with (26) imply $\lambda^l_\nu = \lambda^u_\nu = 0$ and

$$\frac{1}{b_\nu^2} - \frac{(k-2)^2}{(\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu)^2} - \sum_{\mu\in K\setminus K^*}\lambda^u_\mu = 0. \tag{28}$$

Our next step is to show that Case 3 cannot occur. To this end, let $i_u := |I_u|$ and add (27) for all $\nu \in I_u$. This yields (taking into account that $\lambda^u_\nu = 0$ for all $\nu \in I_u$)

$$(k-2-i_u)\sum_{\nu\in I_u}\lambda^u_\nu = 0.$$

We will show that $k-2-i_u > 0$. Then the nonnegativity of all λ^u_ν ($\nu \in I_u$) implies $\lambda^u_\nu = 0$ for all $\nu \in I_u$ and therefore $\lambda^u_\nu = 0$ for all $\nu \in K\setminus K^*$. But then (28) gives a formula for b_ν which is in contradiction to the value of b_ν in Case 3. Hence Case 3 cannot occur. To prove the assertion, assume that $k-2-i_u \leq 0$ or, equivalently, $i_u \geq k-2$. Summation over all b_ν ($\nu \in I_u$) with the expression for b_ν as in Case 2 and using the fact that $K\cap K^*$ is nonempty would then imply

$$\sum_{\nu\in I_u} b_\nu = \frac{i_u}{k-2}\left(\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu\right) \geq \sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu > \sum_{\mu\in K\setminus K^*} b_\mu \geq \sum_{\mu\in I_u} b_\mu$$

which gives the desired contradiction.

Hence, every KKT-point is of the form

$$b_\nu = \begin{cases} \frac{\sum_{\mu\in K^*}\beta_\mu}{k^*-2}, & \nu \in I_l, \\ \frac{\sum_{\mu\in K\cap K^*}\beta_\mu+\sum_{\mu\in K\setminus K^*}b_\mu}{k-2}, & \nu \in I_u, \end{cases}$$

with a characteristic partition $K\setminus K^* = I_l \cup I_u$. Using this and the abbreviation $i_l := |I_l|$, we can resolve the implicit definition of b_ν ($\nu \in I_u$) in the following way:

$$\begin{aligned} (k-2)b_\nu &= \sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu \\ &= \sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in I_u} b_\mu + \sum_{\mu\in I_l} b_\mu \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mu\in K^*}\beta_\mu - \sum_{\mu\in K^*\setminus K}\beta_\mu + \sum_{\mu\in I_u} b_\mu + \frac{i_l}{k^*-2}\sum_{\mu\in K^*}\beta_\mu \\
&= \sum_{\mu\in K^*}\beta_\mu - \sum_{\mu\in K^*\setminus K}\beta_\mu + i_u b_\nu + \frac{k-d-i_u}{k^*-2}\sum_{\mu\in K^*}\beta_\mu,
\end{aligned}
$$

where the first equation follows from the previous implicit expression of b_ν, the second equation takes into account the partition of the set $K\setminus K^*$ into the union $I_u \cup I_l$, the third equation uses a trivial identity together with the previous explicit represention of b_ν for all $\nu \in I_l$, and the fourth equation takes into account that all b_μ ($\mu \in I_u$) have a constant value (the same as b_ν) as well as the fact that $i_l + i_u = |I_l \cup I_u| = |K\setminus K^*| = k - d$. The representation we got in this way can now be solved for b_ν in order to get the explicit expression

$$
\begin{aligned}
b_\nu &= \frac{(k-d-i_u)\sum_{\mu\in K^*}\beta_\mu + (k^*-2)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu}{(k^*-2)(k-2-i_u)} \\
&= \frac{(k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + (k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu}{(k^*-2)(k-2-i_u)} \qquad (\nu \in I_u).
\end{aligned}
$$

Using this and the abbreviations $b_l := b_\nu$ for $\nu \in I_l$ and $b_u := b_\nu$ for $\nu \in I_u$ (recall that both numbers are constant within their corresponding index sets), we can express the value of the objective function in a KKT-point as follows:

$$
\begin{aligned}
&-\sum_{\mu\in K\setminus K^*}\frac{1}{b_\mu} + \frac{(k-2)^2}{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu} \\
&= -\sum_{\mu\in I_u}\frac{1}{b_\mu} - \sum_{\mu\in I_l}\frac{1}{b_\mu} + (k-2)\underbrace{\frac{(k-2)}{\sum_{\mu\in K\cap K^*}\beta_\mu + \sum_{\mu\in K\setminus K^*} b_\mu}}_{=\frac{1}{b_u}} \\
&= \frac{k-2-i_u}{b_u} - (k-d-i_u)\frac{1}{b_l} \\
&= \frac{(k-2-i_u)^2(k^*-2)}{(k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + (k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu} - \frac{(k-d-i_u)(k^*-2)}{\sum_{\mu\in K^*}\beta_\mu} \\
&= \frac{k^*-2}{\sum_{\mu\in K^*}\beta_\mu}\cdot\frac{(k-2-i_u)(d-2)\sum_{\mu\in K^*}\beta_\mu - (k-d-i_u)\left[(k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu\right]}{(k-2-i_u)\sum_{\mu\in K^*}\beta_\mu + (k^*-d)\sum_{\mu\in K^*}\beta_\mu - (k^*-2)\sum_{\mu\in K^*\setminus K}\beta_\mu},
\end{aligned}
$$

here the first equation uses the partition of $K\setminus K^*$ into $I_u \cup I_l$, the second equation takes into account that b_μ is constant on the two index sets I_u and I_l, respectively, as well as the implicit representation of b_u, the third equation substitutes the explicit values for b_l and b_u, respectively, and the final equation can be verified by direct calculation.

We already know that, for every KKT-point, we have $i_u \in \{0, 1, \dots, k-3\}$. Hence, we are interested in the minimum of the term above for $i_u \in [0, k-3]$ (viewed as a continuous variable, for the moment). To this end, remember the abbreviation

$$\delta := (k^* - d) \sum_{\mu \in K^*} \beta_\mu - (k^* - 2) \sum_{\mu \in K^* \setminus K} \beta_\mu.$$

Obviously, δ does not depend on i_u and we have seen before that $\delta > 0$. Differentiation of the term above with respect to i_u then yields (after some algebraic manipulations)

$$\begin{aligned}
&\frac{\partial}{\partial i_u} \frac{k^* - 2}{\sum_{\mu \in K^*} \beta_\mu} \cdot \frac{(k-2-i_u)(d-2)\sum_{\mu \in K^*} \beta_\mu - (k-d-i_u)\delta}{(k-2-i_u)\sum_{\mu \in K^*} \beta_\mu + \delta} \\
&= \frac{k^* - 2}{\sum_{\mu \in K^*} \beta_\mu} \cdot \left(\frac{(-(d-2)\sum_{\mu \in K^*} \beta_\mu + \delta)((k-2-i_u)\sum_{\mu \in K^*} \beta_\mu + \delta)}{\left((k-2-i_u)\sum_{\mu \in K^*} \beta_\mu + \delta\right)^2} \right. \\
&\quad \left. - \frac{((k-2-i_u)(d-2)\sum_{\mu \in K^*} \beta_\mu - (k-d-i_u)\delta)(-\sum_{\mu \in K^*} \beta_\mu)}{\left((k-2-i_u)\sum_{\mu \in K^*} \beta_\mu + \delta\right)^2} \right) \\
&= \frac{k^* - 2}{\sum_{\mu \in K^*} \beta_\mu} \cdot \frac{\delta^2}{\left((k-2-i_u)\sum_{\mu \in K^*} \beta_\mu + \delta\right)^2},
\end{aligned}$$

which is strictly positive for all $i_u \in [0, k-3]$ because of $\delta > 0$. Hence the objective function is strictly increasing with respect to i_u. Therefore, the KKT-point corresponding to the global minimum is the one with the smallest i_u possible, i.e. the one with $i_u = 0$ (which, fortunately, turned out to be an integer, though within our intermediate calculations i_u was assumed to be a real number). While proving the existence of a global minimum, we have already shown that the vector $b_{K \setminus K^*}$ with $I_u = \emptyset$ and $I_l = K \setminus K^*$ is indeed feasible for (24) and thus the global minimum.

As we mentioned at the beginning of the proof, the vector $\beta_{K \setminus K^*}$ also is feasible for (24). Thus, we obtain the assertion by using the value of the objective function in its global minimum as a lower bound. □