Generalized Krasnoselskii-Mann-type Iterations for Nonexpansive Mappings in Hilbert Spaces

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Abstract

The Krasnoselskii-Mann iteration plays an important role in the approximation of fixed points of nonexpansive operators; it is is known to be weakly convergent in the infinite dimensional setting. In this present paper, we provide a new inexact Krasnoselskii-Mann iteration and prove weak convergence under certain accuracy criteria on the error resulting from the inexactness. We also show strong convergence for a modified inexact Krasnoselskii-Mann iteration under suitable assumptions. The convergence results generalize existing ones from the literature. Applications are given to the Douglas-Rachford splitting method, the Fermat-Weber location problem as well as the alternating projection method by John von Neumann.

1 Introduction

Let X be a normed space and $T : X \to X$ be a nonexpansive mapping, i.e., T satisfies

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in X.$$

We further denote the set of fixed points of T by

$$F(T) := \{x \in X \mid Tx = x\}.$$

Note that, if T is actually a mapping from X to a subset $K \subseteq X$, then $F(T)$ automatically belongs to K. Prominent examples for nonexpansive mappings from a Hilbert space X to a nonempty, closed, and convex set $K \subseteq X$ are, for example, the projection map, the proximal point map, and several composite maps which involve at least one of these two mappings, see, e.g., [2] for more details.

Throughout this paper, we consider the real Hilbert space setting: H denotes a real Hilbert space with scalar product $\langle .,. \rangle$ and induced norm $\|\cdot\|$. Our aim is to find

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a fixed point of a nonexpansive mapping T defined on H. Existence and uniqueness results as well as many iterative schemes are well-known from the literature, cf. [3, 5, 6, 7] and references therein for some relevant results in this direction. In particular, one of the most famous fixed point methods is the Krasnoselskii-Mann iteration from [19, 24] that starts at some given point $x_1 \in H$ and uses the recursion

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n \quad \forall n = 1, 2, \ldots \tag{1}$$

for some suitably chosen scalars $\lambda_n \in [0, 1]$. The most general convergence result for this procedure is due to Reich [26] and assumes that $F(T)$ is nonempty and the scalars λ_n satisfy the condition

$$\sum_{n=1}^{\infty} \lambda_n(1 - \lambda_n) = \infty, \tag{2}$$

then the iterates $\{x_n\}$ converge weakly to a fixed point of T. This statement remains true if $T : K \to K$ with a nonempty, closed, and convex set $K \subseteq H$, in which case it follows immediately from (1) that the whole sequence $\{x_n\}$ remains in K provided that the starting point x_1 is chosen from K.

Strong convergence of the Krasnoselskii-Mann iteration cannot be expected in general, as noted by a counterexample in [15]. On the other hand, there exist a couple of modified schemes which guarantee strong convergence results, see [5, 8] for different examples. One of these schemes uses the recursion

$$x_{n+1} := \alpha_n x_n + \beta_n T x_n + \delta_n u, \tag{3}$$

where $T : H \to K$ is nonexpansive, $K \subseteq H$ is nonempty, closed, and convex, $\alpha_n, \beta_n, \delta_n \in [0, 1]$ are suitably chosen scalars satisfying $\alpha_n + \beta_n + \delta_n = 1$, and u denotes a fixed element from K; for more details and conditions on the choice of $\alpha_n, \beta_n, \delta_n$, we refer to [9, 10, 18, 28] and the discussion in Section 4.

The overall convergence behaviour of the Krasnoselskii-Mann iteration from (1) is therefore well-investigated and yields very satisfactory global convergence results. On the other hand, this theory requires that T can be evaluated exactly. In general, this is an unrealistic assumption because the evaluation of T might involve the computation of a projection or the solution of a nonlinear (convex) program. Combettes [11] therefore considers the convergence of the inexact Krasnoselskii-Mann iteration

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n(T x_n + e_n) \tag{4}$$

with a given starting point $x_1 \in H$, where e_n represents an error in the evaluation of $T x_n$. He proves weak convergence of the sequence $\{x_n\}$ under the assumptions that $F(T)$ is nonempty, $\lambda_n \in (0, 1)$ satisfies (2), and the additional error condition

$$\sum_{n=1}^{\infty} \lambda_n \|e_n\| < \infty.$$

The same inexact Krasnoselskii-Mann scheme has been investigated recently by Liang et al. [20] where additional results are presented, in particular, suitable rate of convergence results are provided.

Apart from the error due to the inexact evaluation of T, implementations of the Krasnoselskii-Mann iteration produce an additional error due to the finite precision arithmetic of the computer. To get a complete picture of the practical numerical behaviour of the Krasnoselskii-Mann iteration, we are therefore forced to analyse the convergence properties of a scheme like

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n(Tx_n + e_n) + \tilde{e}_n,$$

where, again, e_n represents the error in the evaluation of Tx_n, whereas $\tilde{e}_n$ denotes the error resulting from the finite precision arithmetic. To keep the notation simple, we can write this as

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n Tx_n + r_n$$

for some vector r_n that we call the *residual* since it represents the difference between the exact Krasnoselskii-Mann iteration and its inexact counterpart.

Here we consider the more general inexact scheme

$$x_{n+1} := \alpha_n x_n + \beta_n Tx_n + r_n, \tag{5}$$

where $\alpha_n, \beta_n \in [0, 1]$ are suitable numbers satisfying $\alpha_n + \beta_n \leq 1$, hence these two numbers do not necessarily sum up to one, and r_n is again called the residual vector. Despite the fact that this generalizes existing choices, it turns out in our subsequent analysis that, to some extent, the particular choice $\alpha_n + \beta_n < 1$ also bridges the gap between weak and strong convergence results.

Sometimes it is more convenient to consider the recursion

$$x_{n+1} := \alpha_n x_n + \beta_n Tx_n + \gamma_n e_n, \tag{6}$$

where

$$\alpha_n, \beta_n, \gamma_n \in [0, 1] \quad \text{satisfy} \quad \alpha_n + \beta_n + \gamma_n = 1 \tag{7}$$

and the vector e_n is called the *error*. Note that this iterative scheme is a special case of (5) simply by setting $r_n = \gamma_n e_n$. Our aim is to prove weak and strong convergence results for these modified inexact Krasnoselskii-Mann iterations under suitable assumptions which generalize the conditions known for the previously mentioned exact and inexact versions of this fixed-point method.

The paper is therefore organized as follows: We first recall some basic definitions and results in Section 2. The weak convergence of the iterative scheme from (5) (and its special instance from (6), (7)) is then investigated in Section 3. Strong convergence of a modified version is shown in Section 4. An application to the Douglas-Rachford splitting method, the Fermat-Weber location problem, and the alternating projection method by John von Neumann can be found in Section 5. We conclude with some final remarks in Section 6.

Notation: Given a Hilbert space H, we denote by 2^H the power set of H. An operator $A : H \to 2^H$ is sometimes called a multi-function. Given such a multi-function, we write $\text{zer}(A)$ for the set $\{x \in H \mid 0 \in Ax\}$. The projection of an element $x \in H$ onto a nonempty, closed, and convex set $C \subseteq H$ is denoted by $P_C x$.

2 Preliminaries

Here we state some basic properties that will be used in our convergence theorems. We begin with the following lemma whose proof is elementary and therefore omitted.

Lemma 2.1. *Let X be a real inner product space. Then the following statements hold:*

(a) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X.$

(b) $\|tx+sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x-y\|^2, \quad \forall x, y \in X, \forall s, t \in \mathbb{R}.$

The following result is also well-known, see, e.g., [1]. It plays a central role in our weak convergence result.

Lemma 2.2. *Let $\{\sigma_n\}$ and $\{\gamma_n\}$ be nonnegative sequences satisfying $\sum\limits_{n=1}^{\infty} \sigma_n < \infty$ and $\gamma_{n+1} \leq \gamma_n + \sigma_n, n = 1, 2, \ldots$. Then, $\{\gamma_n\}$ is a convergent sequence.*

The next result comes from [30] and will be exploited in our strong convergence result.

Lemma 2.3. *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 1,$$

where

(a) $\{\alpha_n\} \subset [0,1], \sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\limsup \sigma_n \leq 0$;

(c) $\gamma_n \geq 0 \ (n \geq 1), \sum_{n=1}^{\infty} \gamma_n < \infty$.

Then, $a_n \to 0$ as $n \to \infty$.

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$, and let K be a nonempty, closed, and convex subset of H.

For any point $u \in H$, there exists a unique point $P_K u \in K$ such that

$$\|u - P_K u\| \leq \|u - y\|, \quad \forall y \in K.$$

P_K is called the *metric projection* of H onto K. We know that P_K is a nonexpansive mapping of H onto K. More precisely, P_K is known to be *firmly nonexpansive* in the sense that

$$\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2 \tag{8}$$

for all $x, y \in H$. Furthermore, $P_K x$ is characterized by the properties $P_K x \in K$ and

$$\langle x - P_K x, P_K x - y\rangle \geq 0, \quad \forall y \in K. \tag{9}$$

We finally restate an important result which is due to Opial [25] and characterizes the weak limit of a weakly convergent sequence in a Hilbert space.

Theorem 2.4. *(Opial)*
Let H be a Hilbert space and $\{x_n\}$ be any sequence in H converging weakly to x. Then the strict inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$$

holds for all $y \neq x$.

An operator $A : H \to 2^H$ with domain $D(A)$ is said to be monotone if

$$\langle u - v, x - y\rangle \geq 0 \quad \forall x, y \in D(A), \quad u \in Ax, v \in Ay.$$

We say that the monotone operator A is maximal monotone if its graph

$$G(A) := \{(x, y) : x \in D(A), y \in Ax\}$$

is not properly contained in the graph of any other monotone operator.

3 Weak Convergence

This section investigates the weak convergence properties of the generalized inexact Krasnoselskii-Mann iteration from (5). The following is the main convergence result and shows that we re-obtain the classical weak convergence of the exact Krasnoselskii-Mann iteration under suitable conditions on the choice of α_n, β_n and the behaviour of r_n. These conditions will be discussed in some more detail after the proof of this result.

Theorem 3.1. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Suppose that $T : H \to K$ is a nonexpansive mapping such that its set of fixed points $F(T)$ is nonempty. Let the sequence $\{x_n\}$ in H be generated by choosing $x_1 \in H$ and using the recursion*

$$x_{n+1} := \alpha_n x_n + \beta_n T x_n + r_n, \quad \forall n \geq 1, \tag{10}$$

where r_n denotes the residual vector. Here we assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$ for all $n \geq 1$ and the following conditions hold:

(a) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$;

(b) $\sum_{n=1}^{\infty} \|r_n\| < \infty$;

(c) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Then the sequence $\{x_n\}$ generated by (10) converges weakly to a fixed point of T.

Proof. We divide the proof into four steps.

Step 1: We show that the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists for any given fixed point $x^* \in F(T)$. To this end, choose an arbitrary $x^* \in F(T)$. Then we obtain from (10) and the nonexpansiveness of T that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n x_n + \beta_n T x_n + r_n - x^*\| \\
&= \|\alpha_n(x_n - x^*) + \beta_n(Tx_n - x^*) + r_n - (1-\alpha_n-\beta_n)x^*\| \\
&\le \alpha_n\|x_n - x^*\| + \beta_n\|Tx_n - x^*\| + \|r_n - (1-\alpha_n-\beta_n)x^*\| \\
&\le \alpha_n\|x_n - x^*\| + \beta_n\|x_n - x^*\| + \|r_n - (1-\alpha_n-\beta_n)x^*\| \\
&= (\alpha_n+\beta_n)\|x_n - x^*\| + \|r_n - (1-\alpha_n-\beta_n)x^*\| \\
&\le (\alpha_n+\beta_n)\|x_n - x^*\| + (1-\alpha_n-\beta_n)\|r_n - x^*\| + (\alpha_n+\beta_n)\|r_n\| \\
&\le \|x_n - x^*\| + (1-\alpha_n-\beta_n)M + \|r_n\|,
\end{aligned}$$

for some $M > 0$ whose existence follows from condition (b) (note that, in general, the exact value of M depends on the given fixed point x^*, but that our subsequent analysis only requires that there exists such a constant for any given fixed point of T). Applying Lemma 2.2 and using conditions (b) and (c), we have that $\lim\limits_{n\to\infty}\|x_n - x^*\|$ exists. In particular, this implies that $\{x_n\}$ is bounded.

Step 2: Here we show that $\liminf\limits_{n\to\infty}\|x_n - Tx_n\| = 0$ holds. Using Lemma 2.1, we obtain for an arbitrary $x^* \in F(T)$ that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad= \|\alpha_n(x_n - x^*) + \beta_n(Tx_n - x^*) + r_n - (1-\alpha_n-\beta_n)x^*\|^2 \\
&\quad\overset{\text{Lem. 2.1}(a)}{\le} \|\alpha_n(x_n - x^*) + \beta_n(Tx_n - x^*)\|^2 + 2\langle r_n - (1-\alpha_n-\beta_n)x^*, x_{n+1} - x^*\rangle \\
&\quad\overset{\text{Lem. 2.1}(b)}{=} \alpha_n(\alpha_n+\beta_n)\|x_n - x^*\|^2 + \beta_n(\alpha_n+\beta_n)\|Tx_n - x^*\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 \\
&\qquad +2\big\langle r_n - (1-\alpha_n-\beta_n)x^*, x_{n+1} - x^*\big\rangle \\
&\quad\le (\alpha_n+\beta_n)^2\|x_n - x^*\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 \\
&\qquad +2\big\langle r_n - (1-\alpha_n-\beta_n)x^*, x_{n+1} - x^*\big\rangle \\
&\quad\le \|x_n - x^*\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 + 2\big\langle r_n - (1-\alpha_n-\beta_n)x^*, x_{n+1} - x^*\big\rangle \\
&\quad= \|x_n - x^*\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 + 2(1-\alpha_n-\beta_n)\big\langle r_n - x^*, x_{n+1} - x^*\big\rangle \\
&\qquad +2(\alpha_n+\beta_n)\big\langle r_n, x_{n+1} - x^*\big\rangle \\
&\quad\le \|x_n - x^*\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 + 2\big[(1-\alpha_n-\beta_n)\|r_n - x^*\| \\
&\qquad +(\alpha_n+\beta_n)\|r_n\|\big]\,\|x_{n+1} - x^*\| \\
&\quad\le \|x_n - x^*\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 + M_1(1-\alpha_n-\beta_n) + M_2\|r_n\|,
\end{aligned}$$

for some $M_1, M_2 > 0$ (recall that $\{x_n\}$ is bounded in view of Step 1). This implies that

$$\alpha_n\beta_n\|x_n - Tx_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M_1(1-\alpha_n-\beta_n) + M_2\|r_n\|, \quad n \ge 1.$$

Therefore, by conditions (b) and (c), we have

$$\sum_{n=1}^{\infty}\alpha_n\beta_n\|x_n - Tx_n\|^2 \le \|x_1 - x^*\|^2 + M_1\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) + M_2\sum_{n=1}^{\infty}\|r_n\| < \infty.$$

Using assumption (a), we obtain $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.

Step 3: We now show that we actually have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. To this end, first observe that

$$x_{n+1} - x_n = \beta_n(Tx_n - x_n) + r_n - (1 - \alpha_n - \beta_n)x_n. \tag{11}$$

This implies

$$\begin{aligned}
&\|Tx_{n+1} - x_{n+1}\| \\
&= \|Tx_{n+1} - Tx_n + Tx_n - (\alpha_n x_n + \beta_n Tx_n + r_n)\| \\
&= \|Tx_{n+1} - Tx_n + (1-\beta_n)(Tx_n - x_n) - r_n + (1 - \alpha_n - \beta_n)x_n\| \\
&\leq \|Tx_{n+1} - Tx_n\| + (1-\beta_n)\|Tx_n - x_n\| + \|(1 - \alpha_n - \beta_n)x_n - r_n\| \\
&\leq \|x_{n+1} - x_n\| + (1-\beta_n)\|Tx_n - x_n\| + \|(1 - \alpha_n - \beta_n)x_n - r_n\| \\
&\overset{(11)}{=} \|Tx_n - x_n\| + 2\|(1 - \alpha_n - \beta_n)x_n - r_n\| \\
&\leq \|Tx_n - x_n\| + 2(1 - \alpha_n - \beta_n)\|x_n\| + 2\|r_n\| \\
&\leq \|Tx_n - x_n\| + 2\|r_n\| + 2M_3(1 - \alpha_n - \beta_n),
\end{aligned}$$

for some $M_3 > 0$. Observe from conditions (b) and (c) that $2\sum_{n=1}^{\infty}\big(\|r_n\| + M_3(1 - \alpha_n - \beta_n)\big) < \infty$. Applying Lemma 2.2 to the last chain of inequalities, we have that $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists. In view of Step 2, this yields $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Step 4: In this final step, we prove the weak convergence of the sequence $\{x_n\}$ to a fixed point of T. Since $\{x_n\}$ is bounded by Step 1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some element p. We first show that $p \in F(T)$. Assume the contrary that $p \neq Tp$. Using Opial's Theorem 2.4 and the fact that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ by Step 3, we get

$$\begin{aligned}
\liminf_{n\to\infty} \|x_{n_k} - p\| &< \liminf_{n\to\infty} \|x_{n_k} - Tp\| \\
&\leq \liminf_{n\to\infty} \big(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tp\|\big) \\
&= \liminf_{n\to\infty} \|Tx_{n_k} - Tp\| \\
&\leq \liminf_{n\to\infty} \|x_{n_k} - p\|.
\end{aligned}$$

This contradiction shows that $p \in F(T)$. Suppose that the whole sequence $\{x_n\}$ does not converge weakly to p. Then there exists another subsequence $\{x_{m_j}\}$ of $\{x_n\}$ which converges weakly to some $q \neq p$. As in in the case of p we must have $q \in F(T)$. It therefore follows from Step 1 that $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} \|x_n - q\|$ exist. Let us denote these limits by $d_1 := \lim_{n\to\infty} \|x_n - p\|$ and $d_2 := \lim_{n\to\infty} \|x_n - q\|$, respectively. Exploiting Opial's Theorem 2.4 once again, we obtain

$$\begin{aligned}
d_1 &= \liminf_{k\to\infty} \|x_{n_k} - p\| < \liminf_{k\to\infty} \|x_{n_k} - q\| = d_2 \\
&= \liminf_{j\to\infty} \|x_{m_j} - q\| < \liminf_{j\to\infty} \|x_{m_j} - p\| = d_1,
\end{aligned}$$

which is a contradiction. Therefore, $p = q$ and the entire sequence $\{x_n\}$ converges weakly to p. This completes the proof. □

Let us discuss Theorem 3.1 to some extent in the following remark.

Remark 3.2. (a) Consider the exact Krasnoselskii-Mann iteration from (1) which corresponds to the case $r_n = 0$ as well as $\alpha_n = 1 - \lambda_n, \beta_n = \lambda_n$ for suitable numbers $\lambda_n \in [0, 1]$. It then follows that conditions (b) and (c) in Theorem 3.1 are automatically satisfied. Furthermore, condition (a) reduces to (2), i.e. we re-obtain the usual assumption for the convergence of the Krasnoselskii-Mann iteration as a special case of our Theorem 3.1.

(b) Next consider the inexact Krasnoselskii-Mann iteration from (4) proposed by Combettes [11]. This is also a special case of our recursion (10) by setting $\alpha_n := 1 - \lambda_n, \beta_n := \lambda_n$, and $r_n := \lambda_n e_n$. It follows that condition (c) in Theorem 3.1 holds automatically, whereas conditions (a) and (b) become

$$\sum_{n=1}^{\infty} \lambda_n(1 - \lambda_n) = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n \|e_n\| < \infty,$$

which are precisely the convergence assumptions used by Combettes [11] (formally, Combettes assumes that $\lambda_n \in (0, 1)$, whereas here λ_n can be taken from the closed interval $[0, 1]$).

(c) Recall that our iterative scheme is more general than (1) and (4) because α_n and β_n do not necessarily sum up to one. Theorem 3.1 still yields global convergence provided that the sequences α_n, β_n satisfy the conditions (a) and (c), where (a) may be viewed as the counterpart of (2) and (c) tells us how fast $\alpha_n + \beta_n$ has to approach one, so that α_n and β_n asymptotically approach the numbers $1 - \lambda_n$ and λ_n, respectively, in the classical Krasnoselskii-Mann iteration. Besides being more general, the discussion in the subsequent section also shows that the possibility of allowing $\alpha_n + \beta_n < 1$ brings the Krasnoselskii-Mann iteration much closer to a strongly convergent modification.

(d) Formally, the standard Picard-iteration $x_{n+1} := Tx_n$ is a special case of the iterative scheme from (10). However, it is known that the Picard iteration is, in general, neither weakly nor strongly convergent for nonexpansive mappings (take, e.g., $H = \mathbb{R}$ and $Tx = -x$). This fact is reflected by condition (a) in Theorem 3.1 which implies that we cannot take $\alpha_n = 0$ for all or almost all $n \in \mathbb{N}$. This also indicates that an assumption like condition (a) is necessary to verify at least weak convergence of any Krasnoselskii-Mann-type iteration. $\diamondsuit$

We next present a couple of counterexamples to illustrate the necessity of the three conditions (a), (b), and (c) in Theorem 3.1. The first counterexample shows that Theorem 3.1 is not true if condition (a) fails, but conditions (b) and (c) are satisfied.

Example 3.3. Take $T : \mathbb{R} \to \mathbb{R}, Tx := \max\{0, -x\}$. Then T is nonexpansive and has a unique fixed point $x = 0$. Consider the iteration

$$x_{n+1} := \alpha_n x_n + \beta_n Tx_n + r_n \quad \text{with} \quad \beta_n = \frac{1}{n^2}, \quad \alpha_n = 1 - \frac{1}{n^2} \quad \text{and} \quad r_n := 0,$$

so that we have

$$x_{n+1} = \big(1 - \frac{1}{n^2}\big)x_n + \frac{1}{n^2}\max\{0, -x_n\}.$$

Note that the choices of α_n, β_n, and r_n satisfy conditions (b), (c) from Theorem 3.1, whereas assumption (a) is violated since

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \sum_{n=1}^{\infty} \frac{1}{n^2}\big(1 - \frac{1}{n^2}\big) = \sum_{n=2}^{\infty} \frac{1}{n^2}\big(1 - \frac{1}{n^2}\big) \leq \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.$$

Taking $x_1 = -1$, a simple induction shows that the iterates x_n are given by $x_n = \frac{n}{2(n-1)}$ for all $n \geq 2$. Hence $x_n = \frac{n}{2(n-1)} \to \frac{1}{2} \neq 0$, i.e. $\{x_n\}$ converges, but not to the unique fixed point of T. $\Diamond$

The next example establishes the fact that Theorem 3.1 is not true if condition (b) fails, but conditions (a) and (c) are satisfied.

Example 3.4. As in the previous example, we take $T : \mathbb{R} \to \mathbb{R}, Tx := \max\{0, -x\}$. We consider the recursion

$$x_{n+1} := \alpha_n x_n + \beta_n T x_n + r_n \quad \text{with} \quad \beta_n = \frac{1}{n}, \quad \alpha_n = 1 - \frac{1}{n} \quad \text{and} \quad r_n = \frac{1}{n},$$

so the iteration becomes

$$x_{n+1} := \big(1 - \frac{1}{n}\big)x_n + \frac{1}{n}\max\{0, -x_n\} + \frac{1}{n}.$$

The choice of α_n, β_n, and r_n guarantee that conditions (a), (c) of Theorem 3.1 hold, whereas condition (b) is obviously violated. Using the starting point $x_1 = -1$, it is not difficult to see that the sequence $\{x_n\}$ has the explicit representation $x_n = \frac{n}{n-1}$ for all $n \geq 2$. Clearly, we see that $x_n \to 1$, but the limit point is not a fixed point of T. $\Diamond$

The final counterexample shows that Theorem 3.1 may not hold if conditions (a), (b) are satisfied, whereas condition (c) is violated.

Example 3.5. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = -x + 2$ for all $x \in \mathbb{R}$. Then it is clear that T is nonexpansive and $F(T) = \{1\}$. Furthermore, let us take $\alpha_n := \beta_n := \frac{1}{\sqrt{n+3}}$, and $r_n := 0$ for all $n \geq 1$. Then it is easy to see that $\sum\limits_{n=1}^{\infty} \alpha_n \beta_n = \sum\limits_{n=1}^{\infty} \frac{1}{n+3} = \infty$, $\sum\limits_{n=1}^{\infty} |r_n| = 0 < \infty$, and $\sum\limits_{n=1}^{\infty} \big(1 - \alpha_n - \beta_n\big) = \sum\limits_{n=1}^{\infty} \big(1 - \frac{2}{\sqrt{n+3}}\big) = \infty$. This implies that conditions (a), (b) are satisfied, whereas condition (c) is violated in Theorem 3.1. Now, for any initial point $x_1 \in \mathbb{R}$, our iterative scheme (10) becomes

$$\begin{aligned} x_{n+1} &:= \alpha_n x_n + \beta_n T x_n + r_n \\ &= \frac{1}{\sqrt{n+3}} x_n + \frac{1}{\sqrt{n+3}}(-x_n + 2) \\ &= \frac{1}{\sqrt{n+3}}(x_n - x_n + 2) \\ &= \frac{2}{\sqrt{n+3}} \to 0, \quad n \to \infty. \end{aligned}$$

But $0 \notin F(T)$. Therefore, $\{x_n\}$ does not converge to a fixed point of T. $\Diamond$

As a direct consequence of Theorem 3.1, we obtain the following corollary for the iterative scheme from (6), (7).

Corollary 3.6. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Suppose that $T : H \to K$ is a nonexpansive mapping such that its set of fixed points $F(T)$ is nonempty. Let the sequence $\{x_n\}$ in H be generated by choosing $x_1 \in H$ and using the recursion*

$$x_{n+1} := \alpha_n x_n + \beta_n T x_n + \gamma_n e_n,$$

where e_n denotes the error, and $\alpha_n, \beta_n, \gamma_n$ are nonnegative sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$ such that the following conditions hold:

(a) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$;

(b) $\sum_{n=1}^{\infty} \gamma_n \|e_n\| < \infty$;

(c) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Note that, if we assume $\{e_n\}$ to be bounded, then assumption (c) from Corollary 3.6 implies (b), so there is no need to force an extra condition like (b). On the other hand, it is clear that the error e_n might be difficult to control in practice. While the scalars $\alpha_n, \beta_n, \gamma_n$ can always be chosen such that the corresponding assumptions hold, we usually cannot check whether assumption (b) is true. This might be possible if it is known that the underlying problem satisfies a (local) error bound, but in general this assumption just motivates that one has to evaluate the operator T at x_n with a sufficient accuracy.

A possible advantage of the iterative scheme from (6), (7) is outlined in the following remark.

Remark 3.7. The exact Krasnoselskii-Mann iteration (1) is often applied to a nonexpansive operator $T : K \to K$, where K denotes a nonempty, closed, and convex subset of a real Hilbert space H. The convergence proof coincides with the one for operators $T : H \to H$ simply because the iteration itself guarantees that the entire sequence $\{x_n\}$ remains in K provided that the starting point x_1 is chosen from K. For our inexact iteration (6), (7) (and similar for the inexact iteration from (4)), the situation is different: The new iterate x_{n+1} is defined as a convex combination of x_n, Tx_n, and e_n. But, in general, there is no reason why the error e_n should belong to K. Hence the sequence $\{x_n\}$ is usually not in K. Hence we have to assume that T is an operator defined on the whole space H. In some situations, however, the result might hold also for $T : K \to K$, e.g., if zero belongs to the interior of K and the error e_n is sufficiently small (which is not a completely unreasonable assumption), then it is likely that also e_n belongs to K, and then the whole sequence $\{x_n\}$ generated by our inexact scheme belongs to K. Though this does not hold in general, we stress that, of course, the weak limit always belongs to K simply because it was shown to be a fixed point of T in Theorem 3.1. $\Diamond$

4 Strong Convergence

The Krasnoselskii-Mann iteration, applied to nonexpansive operators, is known to be weakly convergent, but not strongly convergent in general. Strong convergence results can be obtained either under significantly stronger assumptions, or for suitably modified iteration schemes. Here we are interested in the latter approach. There exist different ways to modify the standard method in order to guarantee strong convergence, with different levels of generality. Since more general schemes do not lead to better convergence results, at least not theoretically, we follow one of the simplest approaches and consider an inexact version of the recursion

$$x_{n+1} := \alpha_n x_n + \beta_n T x_n + \delta_n u$$

for some fixed element $u \in K$ and suitable parameters $\alpha_n, \beta_n, \delta_n \in [0, 1]$, cf. (3). We postpone a discussion of existing results til the end of this section.

Here we obtain a strong convergence result for the approximation of fixed points of a nonexpansive mapping using an inexact form of (3) and give sufficient conditions on the iteration parameters. Before stating the formal result, let us recall that the fixed-point set $F(T)$ of a nonexpansive operator $T : H \to K$ is known to be a nonempty, closed, and convex set, see [2], so the projection onto this set is well-defined.

Theorem 4.1. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Suppose that $T : H \to K$ is a nonexpansive mapping such that its set of fixed points $F(T)$ is nonempty. Let the sequence $\{x_n\}$ in H be generated by choosing $x_1 \in H$ and using the recursion*

$$x_{n+1} = \delta_n u + \alpha_n x_n + \beta_n T x_n + r_n, \quad \forall n \geq 1, \tag{12}$$

where $u \in K$ denotes a fixed vector, r_n represents the residual, and the nonnegative real numbers $\alpha_n, \beta_n, \delta_n$ are chosen such that $\alpha_n + \beta_n + \delta_n \leq 1, n \geq 1$, and

(a) $\lim\limits_{n\to\infty} \delta_n = 0, \sum\limits_{n=1}^{\infty} \delta_n = \infty$;

(b) $\liminf\limits_{n\to\infty} \alpha_n \beta_n > 0$;

(c) $\sum\limits_{n=1}^{\infty} (1 - \alpha_n - \beta_n - \delta_n) < \infty$, *and*

(d) $\sum\limits_{n=1}^{\infty} \|r_n\| < \infty$.

Then the sequence $\{x_n\}$ generated by (12) strongly converges to a point in $F(T)$, which is the nearest point projection of u onto $F(T)$.

Proof. Let $x^* \in F(T)$. Then, we obtain from (12), the nonexpansiveness of T and $\alpha_n + \beta_n \leq 1 - \delta_n$ that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad = \|\delta_n(u - x^*) + \alpha_n(x_n - x^*) + \beta_n(T x_n - x^*) + r_n - (1 - \alpha_n - \beta_n - \delta_n)x^*\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \delta_n\|u - x^*\| + \alpha_n\|x_n - x^*\| + \beta_n\|Tx_n - x^*\| + \|r_n - (1-\alpha_n-\beta_n-\delta_n)x^*\| \\
&\leq \delta_n\|u - x^*\| + (\alpha_n+\beta_n)\|x_n - x^*\| + \|r_n - (1-\alpha_n-\beta_n-\delta_n)x^*\| \\
&\leq \delta_n\|u - x^*\| + (1-\delta_n)\|x_n - x^*\| + \|r_n - (1-\alpha_n-\beta_n-\delta_n)x^*\| \\
&\leq \delta_n\|u - x^*\| + (1-\delta_n)\|x_n - x^*\| + (1-\alpha_n-\beta_n-\delta_n)\|x^*\| + \|r_n\| \\
&\leq \max\{\|u - x^*\|, \|x_n - x^*\|\} + (1-\alpha_n-\beta_n-\delta_n)\|x^*\| + \|r_n\|.
\end{aligned}
$$

Using induction, it is not difficult to see that this implies

$$\|x_{n+1} - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\} + \sum_{k=1}^{n} \|r_k\| + \|x^*\| \sum_{k=1}^{n} (1-\alpha_k-\beta_k-\delta_k)$$

for all $n \in \mathbb{N}$. This, in turn, yields

$$\|x_{n+1} - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\} + \sum_{k=1}^{\infty} \|r_k\| + \|x^*\| \sum_{k=1}^{\infty} (1-\alpha_k-\beta_k-\delta_k). \quad (13)$$

In particular, it follows from assumptions (c), (d) that the sequence $\{x_n\}$ is bounded in H.

Let $z := P_{F(T)}u$; recall that this projection exists since $F(T)$ is nonempty, closed, and convex. We now distinguish two cases.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}_{n=n_0}^{\infty}$ is non-increasing. Then $\{\|x_n - z\|\}_{n=1}^{\infty}$ converges, and we therefore obtain

$$\|x_n - z\| - \|x_{n+1} - z\| \to 0, \quad n \to \infty. \quad (14)$$

Then from (12) and Lemma 2.1 (a), (b), we obtain that

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad = \|\alpha_n(x_n - z) + \beta_n(Tx_n - z) + \delta_n(u - z) + r_n - (1-\alpha_n-\beta_n-\delta_n)z\|^2 \\
&\quad \overset{\text{Lem. 2.1 (a)}}{\leq} \|\alpha_n(x_n - z) + \beta_n(Tx_n - z)\|^2 \\
&\qquad + 2\big\langle \delta_n(u - z) + r_n - (1-\alpha_n-\beta_n-\delta_n)z, x_{n+1} - z \big\rangle \\
&\quad \overset{\text{Lem. 2.1 (b)}}{=} \alpha_n(\alpha_n+\beta_n)\|x_n - z\|^2 + \beta_n(\alpha_n+\beta_n)\|Tx_n - z\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 \\
&\qquad + 2\big\langle \delta_n(u - z) + r_n - (1-\alpha_n-\beta_n-\delta_n)z, x_{n+1} - z \big\rangle \\
&\quad \leq (\alpha_n+\beta_n)^2\|x_n - z\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 \\
&\qquad + 2\big\langle \delta_n(u - z) + r_n - (1-\alpha_n-\beta_n-\delta_n)z, x_{n+1} - z \big\rangle \\
&\quad \leq (1-\delta_n)^2\|x_n - z\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 \\
&\qquad + 2\big\langle \delta_n(u - z) + r_n - (1-\alpha_n-\beta_n-\delta_n)z, x_{n+1} - z \big\rangle \\
&\quad \leq (1-\delta_n)\|x_n - z\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 \\
&\qquad + 2\big\langle \delta_n(u - z) + r_n - (1-\alpha_n-\beta_n-\delta_n)z, x_{n+1} - z \big\rangle \\
&\quad = (1-\delta_n)\|x_n - z\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 + 2\delta_n\big\langle u - z, x_{n+1} - z \big\rangle \\
&\qquad + 2\big\langle r_n - (1-\alpha_n-\beta_n-\delta_n)z, x_{n+1} - z \big\rangle \quad (15) \\
&\quad \leq \|x_n - z\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 + 2\delta_n\big\langle u - z, x_{n+1} - z \big\rangle \\
&\qquad + 2\big\langle r_n - (1-\alpha_n-\beta_n-\delta_n)z, x_{n+1} - z \big\rangle.
\end{aligned}
$$

Using the boundedness of $\{x_n\}$, this implies that

$$\begin{aligned}\alpha_n\beta_n\|x_n - Tx_n\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \delta_n M_5 \\ &\quad +(1-\alpha_n-\beta_n-\delta_n)M_6 + \|r_n\|M_7 \qquad (16)\end{aligned}$$

for some $M_5, M_6, M_7 > 0$. By condition (b), we can assume without loss of generality that there exists $\epsilon > 0$ such that $\alpha_n\beta_n \geq \epsilon$ for all $n \geq 1$. Hence, we obtain from (16) together with (14) and conditions (a), (c), (d) that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Since $\{x_n\}$ is bounded, we can extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z, x_n - z\rangle = \lim_{k\to\infty}\langle u - z, x_{n_k} - z\rangle$$

and $\{x_{n_k}\}$ converges weakly to some element p. By following the same line of arguments as in Theorem 3.1 above, we can show that $p \in F(T)$. Hence, we obtain from (9) that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z, x_{n+1} - z\rangle &= \limsup_{n\to\infty}\langle u - z, x_n - z\rangle \\ &= \lim_{k\to\infty}\langle u - z, x_{n_k} - z\rangle \\ &= \langle u - z, p - z\rangle \\ &\leq 0.\end{aligned}$$

Now, we have from (15) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\delta_n)\|x_n - z\|^2 - \alpha_n\beta_n\|x_n - Tx_n\|^2 + 2\delta_n\langle u - z, x_{n+1} - z\rangle \\ &\quad +2\big\langle r_n - (1-\alpha_n-\beta_n-\delta_n)z, x_{n+1} - z\big\rangle \\ &\leq (1-\delta_n)\|x_n - z\|^2 + 2\delta_n\langle u - z, x_{n+1} - z\rangle \\ &\quad +(1-\alpha_n-\beta_n-\delta_n)M_6 + \|r_n\|M_7. \qquad (17)\end{aligned}$$

Applying Lemma 2.3 in (17) and using conditions (a), (c), (d), we have that $\lim_{n\to\infty}\|x_n - z\| = 0$. Thus, $x_n \to z = P_{F(T)}u$ for $n \to \infty$.

Case 2: Assume that there is no $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}_{n=n_0}^{\infty}$ is monotonically decreasing. The technique of proof used here is adapted from [23]. Set $\Gamma_n = \|x_n - z\|^2$ for all $n \geq 1$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\},$$

i.e. $\tau(n)$ is the largest number k in $\{1, \ldots, n\}$ such that Γ_k increases at $k = \tau(n)$; note that, in view of Case 2, this $\tau(n)$ is well-defined for all sufficiently large n. Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

After a conclusion similar to (16) (note that the first difference in that equation is nonpositive in our current situation), it is easy to see that $\|x_{\tau(n)} - Tx_{\tau(n)}\| \to 0$. Furthermore, using the boundedness of $\{x_n\}$ and conditions (a), (c), (d), we get

$$\begin{aligned}
\|x_{\tau(n)+1} - x_{\tau(n)}\| &= \|\delta_{\tau(n)}(u - x_{\tau(n)}) + \beta_{\tau(n)}(Tx_{\tau(n)} - x_{\tau(n)}) \\
&\quad + r_{\tau(n)} - (1 - \alpha_{\tau(n)} - \beta_{\tau(n)} - \delta_{\tau(n)})x_{\tau(n)}\| \\
&\leq \delta_{\tau(n)}\|u - x_{\tau(n)}\| + \beta_{\tau(n)}\|Tx_{\tau(n)} - x_{\tau(n)}\| \\
&\quad + \|r_{\tau(n)} - (1 - \alpha_{\tau(n)} - \beta_{\tau(n)} - \delta_{\tau(n)})x_{\tau(n)}\| \\
&\to 0, \quad n \to \infty. \qquad (18)
\end{aligned}$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$, which converges weakly to some $p \in F(T)$. Similarly, as in Case 1 above and exploiting (18), we can show that

$$\limsup_{n\to\infty} \langle u - z, x_{\tau(n)+1} - z\rangle \leq 0.$$

Following (17), we obtain

$$\begin{aligned}
\|x_{\tau(n)+1} - z\|^2 &\leq (1 - \delta_{\tau(n)})\|x_{\tau(n)} - z\|^2 + 2\delta_{\tau(n)}\langle u - z, x_{\tau(n)+1} - z\rangle \\
&\quad + (1 - \alpha_{\tau(n)} - \beta_{\tau(n)} - \delta_{\tau(n)})M_6 + \|r_{\tau(n)}\|M_7. \qquad (19)
\end{aligned}$$

By Lemma 2.3 and using conditions (a), (c), (d), we have from (19) that $\lim_{n\to\infty} \|x_{\tau(n)} - z\| = 0$ which, in turn, implies $\lim_{n\to\infty} \|x_{\tau(n)+1} - z\| = 0$. Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all sufficiently large n that $0 \leq \Gamma_n \leq \Gamma_{\tau(n)+1}$. Hence $\lim_{n\to\infty} \Gamma_n = 0$. Therefore, $\{x_n\}$ converges strongly to z. This completes the proof. □

Let us have a closer look at the iterative scheme from (12). The only difference compared to the recursion from (10) comes from the additional term $\delta_n u$ for some $u \in K$. Now assume that the zero vector belongs to K, and that we take $u := 0$ in (12). Then this term vanishes, and the iteration (12) looks identical to the one from (10), at least formally. However, it is important to note that these two schemes are different even in this particular case, since condition (a) from Theorem 4.1 implies that δ_n cannot be chosen to be equal to zero for all n, and, in fact, is not allowed to converge to zero too fast. This, in turn, has some influence on the choice of the scalars α_n and β_n which then have to be taken in a slightly different way as in the weak convergence result from Theorem 3.1. Nevertheless, this observation indicates that the choice $\alpha_n + \beta_n < 1$ that is explicitly allowed in Theorem 3.1, is much closer to the situation where we get strong convergence than the usual (and possibly more natural) choice where $\alpha_n + \beta_n = 1$ for all $n \in \mathbb{N}$.

We next have a closer look at conditions (a)–(d) from Theorem 4.1. Due to the previous discussion and the corresponding observations from the previous section, the only assumption that needs to be discussed in some more detail is condition (b). Using an idea of [27], we give the following example to establish that our Theorem 4.1 fails if condition (b) is violated, i.e., if $\liminf_{n\to\infty} \alpha_n\beta_n = 0$.

Example 4.2. Let H be any real Hilbert space. Choose an arbitrary element $y \in H$ such that $y \neq 0$ and $\|y\| = 1$, and define a subset K of H by $K := \{x \in H : x = \lambda y, \lambda \in [0,1]\}$, i.e., K is the connecting line from the origin to the given point y. Observe that K is a nonempty, closed, and convex subset of H. Then define the mapping $T : H \to K$ by $Tx = 0$ for all $x \in K$. Clearly, T is nonexpansive, and $F(T) = \{0\}$.

Consider the iterative scheme (12) with

$$\delta_n = \frac{1}{n},\ \alpha_n = \big(1 - \frac{1}{n}\big)\big(1 - \frac{1}{n^2}\big),\ \beta_n = \frac{1}{n^2}\big(1 - \frac{1}{n}\big),\ r_n = 0 \text{ for all } n \geq 1.$$

Then, it is clear that conditions (a), (c), and (d) are satisfied, but condition (b) is violated. Now, take $u := y \in K$, and let $x_1 := u$ be the starting point. Since $r_n = 0, \alpha_n + \beta_n + \delta_n = 1$, $x_1 \in K$, $u \in K$, and T maps into the convex set K, it follows by induction that the entire sequence $\{x_n\}$ generated by (12) belongs to K. Consequently, we have the representation $x_n = \lambda_n y$ for each $n \in \mathbb{N}$ for some $\lambda_n \in [0,1]$. Taking this into account, we can write (12) as

$$x_{n+1} = \frac{1}{n}y + \big(1 - \frac{1}{n}\big)\big(1 - \frac{1}{n^2}\big)x_n = \frac{1}{n}y + \big(1 - \frac{1}{n}\big)\big(1 - \frac{1}{n^2}\big)\lambda_n y.$$

Suppose $\|x_n\| < \frac{1}{2}$, and that x_n is not yet equal to the fixed point (this extra condition holds automatically, e.g., if $H = \mathbb{R}$ and $y = 1$, since then we obviously have $x_n > 0$ for all $n \in \mathbb{N}$). Then, we have

$$0 < \lambda_n = \|x_n\| < \frac{1}{2}.$$

Hence,

$$\begin{aligned}
\|x_{n+1}\| &= \left\| \Big(\frac{1}{n} + \big(1 - \frac{1}{n}\big)\big(1 - \frac{1}{n^2}\big)\lambda_n\Big) y \right\| \\
&= \Big(\frac{1}{n} + \big(1 - \frac{1}{n}\big)\big(1 - \frac{1}{n^2}\big)\lambda_n\Big)\|y\| \\
&= \frac{1}{n} + \big(1 - \frac{1}{n}\big)\big(1 - \frac{1}{n^2}\big)\lambda_n \\
&\geq \frac{1}{n} + \big(1 - \frac{1}{n}\big)\big(1 - \frac{1}{n}\big)\lambda_n \\
&= \frac{1}{n} + \big(1 - \frac{2}{n} + \frac{1}{n^2}\big)\lambda_n \\
&= \frac{1}{n} + \lambda_n - \frac{2}{n}\lambda_n + \frac{1}{n^2}\lambda_n \\
&> \frac{1}{n} + \lambda_n - \frac{2}{n}\lambda_n \\
&> \lambda_n \qquad \text{(since } \lambda_n < 1/2) \\
&= \|x_n\|.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ does not converge to 0, which is the unique fixed point of T. ◊

In view of the weak convergence result from Theorem 3.1, one might expect that condition (b) from Theorem 4.1 can be replaced by the weaker condition that $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. The following counterexample, however, shows that this is not possible in general.

Example 4.3. Let H be an arbitrary Hilbert space, and let y, K, and T be given as in Example 4.2, and let us again take $x_1 := u := y$. Consider the recursion (12) with the specifications

$$\delta_n = \frac{1}{2n}, \ \alpha_n = 1 - \frac{1}{n}, \ \beta_n = \frac{1}{2n}, \ r_n = 0, \ \forall n \geq 1.$$

Then we can clearly see that $\delta_n \to 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$, $\alpha_n + \beta_n + \delta_n = 1$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. Hence, conditions (a), (c), and (d) are satisfied, whereas (b) is violated, and the iterative scheme (12) becomes

$$x_{n+1} = \delta_n u + \alpha_n x_n + \beta_n T x_n + r_n = \frac{1}{2n} u + \big(1 - \frac{1}{n}\big) x_n.$$

Similar to the previous counterexample, one can argue that $x_n \in K$ for all $n \in \mathbb{N}$, so we can write $x_n = \lambda_n y$ for some number $\lambda_n \in [0, 1]$, and the recursion therefore yields

$$\begin{aligned} \|x_{n+1}\| &= \Big\| \frac{1}{2n} u + \big(1 - \frac{1}{n}\big) x_n \Big\| \\ &= \Big\| \frac{1}{2n} y + \big(1 - \frac{1}{n}\big) \lambda_n y \Big\| \\ &= \Big(\frac{1}{2n} + \big(1 - \frac{1}{n}\big) \lambda_n \Big) \|y\| \\ &= \frac{1}{2n} + \big(1 - \frac{1}{n}\big) \lambda_n. \end{aligned}$$

Suppose that $\|x_n\| < \frac{1}{2}$, then $\lambda_n = \|x_n\| < \frac{1}{2}$. So,

$$\|x_{n+1}\| = \frac{1}{2n} + \big(1 - \frac{1}{n}\big) \lambda_n > \lambda_n = \|x_n\|.$$

This implies that $\|x_{n+1}\| > \|x_n\|$ whenever $\|x_n\| < \frac{1}{2}$. Therefore, the sequence $\{x_n\}$ cannot converge to the unique fixed point 0. $\Diamond$

We close this section with a discussion of some related strong convergence results in the following remark.

Remark 4.4. (a) Strong convergence of the (unperturbed) iterative scheme (3) was shown by Yao et al. [31] under the following set of assumptions regarding the choice of the real numbers $\alpha_n, \beta_n, \delta_n \in (0, 1)$: (i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n\to\infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$; (iii) $\lim_{n\to\infty} \beta_n = 0$. Hence, using $r_n = 0$ in our setting, it follows that conditions (a), (c), and (d) hold, whereas (b) is violated. In fact, Example 4.2 satisfies all conditions from Yao et al. [31], but the corresponding sequence $\{x_n\}$ does not converge (strongly) to a fixed point. This shows that the main result from

[31] does not hold under the stated assumptions.

(b) The main result by Hu [17] also considers the (unperturbed) recursion (3) and proves strong convergence of the corresponding sequence $\{x_n\}$ under conditions on the scalars $\alpha_n, \beta_n, \delta_n$ which are even weaker than those noted in (a). In addition, [17] requires a certain property of the underlying space which, however, holds automatically in a Hilbert space. It therefore follows from Example 4.2 that the result from [17] cannot hold.

(c) Hu and Liu [16] consider the (unperturbed) iteration from (3) and verify strong convergence under the following conditions (adapted to our setting): (i) $\alpha_n + \beta_n + \gamma_n = 1$ (this condition is not stated explicitly in [16], but implicitly used within their proof); (ii) $\lim_{n\to\infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$; (iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Using $r_n := 0$ in our framework, we see that conditions (a), (c), and (d) in Theorem 4.1 obviously hold, whereas (b) is satisfied because we obtain from (i), (ii), and (iii) that $\liminf_{n\to\infty} \alpha_n\beta_n = \liminf_{n\to\infty} \alpha_n(1-\alpha_n-\delta_n) > 0$. Hence the assumptions used in [16] may be viewed as a special case of ours.

(d) The paper [10] by Cho et al. proves strong convergence of the slightly different iteration

$$x_{n+1} := \delta_n u + (1-\delta_n)\gamma_n x_n + (1-\delta_n)(1-\gamma_n)Tx_n$$

(adapted to our situation) under the assumptions that the real sequences $\{\gamma_n\}, \{\delta_n\}$ satisfy the conditions (i) $\lim_{n\to\infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$; (ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$. Setting $\alpha_n := (1-\delta_n)\gamma_n$, $\beta_n := (1-\delta_n)(1-\gamma_n)$, and $r_n := 0$, we may view this iteration as a special case of ours. Furthermore, it follows that conditions (a), (c), and (d) from Theorem 4.1 hold automatically, whereas (i) and (ii) also yield $\liminf_{n\to\infty} \alpha_n\beta_n = \liminf_{n\to\infty}(1-\delta_n)^2\gamma_n(1-\gamma_n) > 0$. Consequently, the convergence result from [10] is a another special case of Theorem 4.1. $\Diamond$

5 Applications

This section presents three applications of our general theory. Since the (simpler) Picard iteration $x_{n+1} := Tx_n$ is known to be (weakly) convergent for firmly nonexpansive operators T, while this is not true, in general, for nonexpansive mappings T, we are particularly interested in applications whose operator T is nonexpansive, but not firmly nonexpansive. We therefore consider the Peaceman-Rachford Splitting method in Section 5.1, whose relaxation leads to the Douglas-Rachford splitting method for the sum of two maximally monotone operators. This method can be generalized to finitely many maximally monotone operators in Section 5.2, which then is applied to the Fermat-Weber location problem in Section 5.3. Finally, we consider the alternating projection method by John von Neumann in Section 5.4.

5.1 Application to the Douglas-Rachford Splitting Method

This section presents an application of our theory to the Douglas-Rachford splitting method for finding zeros of an operator T such that T is the sum of two maximal monotone operators, i.e. $T = A + B$ with $A, B : H \to 2^H$ being maximal monotone multi-functions on a Hilbert space H. The method was originally introduced in [12] in a finite-dimensional setting, its extension to maximal monotone mappings in Hilbert spaces can be found in [21].

Before we specialize our results to this method, let us first recall the basics that are required to derive and analyze the Douglas-Rachford splitting method; for the corresponding details, we refer, e.g., to the monograph [2] by Bauschke and Combettes.

Let $\gamma > 0$ be a fixed parameter, and let us denote by

$$J_{\gamma A} := (I + \gamma A)^{-1} \quad \text{and} \quad J_{\gamma B} := (I + \gamma B)^{-1}$$

the *resolvents* of A and B, respectively, which are known to be firmly nonexpansive. Furthermore, let us write

$$R_{\gamma A} := 2J_{\gamma A} - I \quad \text{and} \quad R_{\gamma B} := 2J_{\gamma B} - I$$

for the corresponding *reflections* (also called *Cayley operators*), and note that the firm nonexpansiveness of the resolvents implies immediately that these reflections are nonexpansive operators.

Since one can show that $0 \in Tx$ for $T = A + B$ if and only if $x = J_{\gamma B}(y)$, where y is a fixed point of the nonexpansive mapping $R_{\gamma A} R_{\gamma B}$, a natural way to find a zero of $T = A + B$ is therefore to apply the Krasnoselskii-Mann iteration to this operator, which yields the iteration

$$y_{n+1} := (1 - \lambda_n) y_n + \lambda_n R_{\gamma A} R_{\gamma B} y_n, \quad n \geq 1, \tag{20}$$

which in turn gives an approximation in the original variables by setting $x_n := J_{\gamma B}(y_n)$. Note that this iteration requires only the evaluation of the resolvents of A and B separately, not of their sum $T = A + B$. Recall that (20) is known as the *Douglas-Rachford splitting method*, whereas the special case $\lambda_n = 1$ for all $n \in \mathbb{N}$ gives the *Peaceman-Rachford splitting method*.

Using the definitions of the reflection operators, we may rewrite the iteration (20) as

$$\begin{aligned} y_{n+1} &:= (1 - \lambda_n) y_n + \lambda_n \big(2J_{\gamma A}(2J_{\gamma B} y_n - y_n) - 2J_{\gamma B} y_n + y_n \big) \\ &= y_n + 2\lambda_n \big(J_{\gamma A}(2J_{\gamma B} y_n - y_n) - J_{\gamma B} y_n \big). \end{aligned} \tag{21}$$

Similar to our previous sections, we now consider a more general setting and replace $1 - \lambda_n$ and λ_n by α_n and β_n, respectively. This yields the following iterative method:

$$\begin{aligned} y_{n+1} &:= \alpha_n y_n + \beta_n \big(2J_{\gamma A}(2J_{\gamma B} y_n - y_n) - 2J_{\gamma B} y_n + y_n \big) \\ &= (\alpha_n + \beta_n) y_n + 2\beta_n \big(J_{\gamma A}(2J_{\gamma B} y_n - y_n) - J_{\gamma B} y_n \big). \end{aligned}$$

Following Combettes [11], we also allow errors a_n and b_n in the evaluation of the resolvents $J_{\gamma A}$ and $J_{\gamma B}$, which finally gives our generalized Douglas-Rachford splitting method:

$$y_{n+1} := (\alpha_n + \beta_n) y_n + 2\beta_n \big(J_{\gamma A}(2(J_{\gamma B} y_n + b_n) - y_n) + a_n - (J_{\gamma B} y_n + b_n) \big).$$

We want to investigate the weak and strong convergence properties of this iterative scheme. We begin with the following weak convergence result.

Theorem 5.1. *Let H be a real Hilbert space. Let $\gamma \in (0,\infty)$, let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n \le 1$ for all $n \ge 1$. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences in H. Assume that $0 \in \operatorname{ran}(A+B)$. Let the sequence $\{y_n\}$ in H be generated by choosing $y_1 \in H$ and using the recursion*

$$y_{n+1} := \alpha_n y_n + 2\beta_n\big(J_{\gamma A}\big(2(J_{\gamma B}y_n + b_n) - y_n\big) + a_n\big) - 2\beta_n(J_{\gamma B}y_n + b_n) + \beta_n y_n \quad (22)$$

for all $n \ge 1$. Suppose the following conditions hold:

(a) $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n = \infty$;

(b) $\sum\limits_{n=1}^{\infty} \beta_n\big(\|a_n\| + \|b_n\|\big) < \infty$;

(c) $\sum\limits_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Then the sequence $\{y_n\}$ generated by (22) converges weakly to some point $y \in H$ such that $J_{\gamma B}y \in (A+B)^{-1}(0)$, i.e. $x := J_{\gamma B}y$ is a solution of the monotone inclusion problem for the operator $T := A+B$.

Proof. Using the notation of the reflection operator, we set

$$r_n := 2\beta_n a_n + \beta_n R_{\gamma A}(R_{\gamma B}y_n + 2b_n) - \beta_n R_{\gamma A}(R_{\gamma B}y_n), \quad \forall n \ge 1,$$

and define $T := R_{\gamma A}R_{\gamma B}$. Then we obtain

$$\begin{aligned}
y_{n+1} &= \alpha_n y_n + 2\beta_n\big[J_{\gamma A}(2(J_{\gamma B}y_n + b_n) - y_n) + a_n\big] - 2\beta_n(J_{\gamma B}y_n + b_n) + \beta_n y_n \\
&= \alpha_n y_n + \beta_n\big[2J_{\gamma A}(2(J_{\gamma B}y_n + b_n) - y_n) - 2(J_{\gamma B}y_n + b_n) + y_n + 2a_n\big] \\
&= \alpha_n y_n + \beta_n\big[R_{\gamma A}(2(J_{\gamma B}y_n + b_n) - y_n) + 2a_n\big] \\
&= \alpha_n y_n + \beta_n\big[R_{\gamma A}(2J_{\gamma B}y_n - y_n + 2b_n) + 2a_n\big] \\
&= \alpha_n y_n + \beta_n\big[R_{\gamma A}(R_{\gamma B}y_n + 2b_n) + 2a_n\big] \\
&= \alpha_n y_n + 2\beta_n a_n + \beta_n R_{\gamma A}(R_{\gamma B}y_n + 2b_n) \\
&= \alpha_n y_n + \beta_n R_{\gamma A}(R_{\gamma B}y_n) + 2\beta_n a_n + \beta_n R_{\gamma A}(R_{\gamma B}y_n + 2b_n) \\
&\quad -\beta_n R_{\gamma A}(R_{\gamma B}y_n) \\
&= \alpha_n y_n + \beta_n T y_n + r_n.
\end{aligned}$$

Therefore, our iterative scheme (22) can be re-written as in Theorem 3.1, cf. (10). Furthermore, by the nonexpansivity of $R_{\gamma A}$, we obtain that

$$\begin{aligned}
\sum_{n=1}^{\infty}\|r_n\| &= \sum_{n=1}^{\infty}\|2\beta_n a_n + \beta_n R_{\gamma A}(R_{\gamma B}y_n + 2b_n) - \beta_n R_{\gamma A}(R_{\gamma B}y_n)\| \\
&\le 2\sum_{n=1}^{\infty}\beta_n\|a_n\| + \sum_{n=1}^{\infty}\beta_n\|R_{\gamma A}(R_{\gamma B}y_n + 2b_n) - R_{\gamma A}(R_{\gamma B}y_n)\| \\
&\le 2\sum_{n=1}^{\infty}\beta_n\big(\|a_n\| + \|b_n\|\big) < \infty.
\end{aligned}$$

Using the fact that $0 \in \operatorname{ran}(A+B)$, we have that $F(T) \ne \emptyset$. Since T is nonexpansive as a composition of the two nonexpansive reflection operators, Theorem 3.1 implies that $\{y_n\}_{n=1}^{\infty}$ converges weakly to a fixed point of T, and this completes the proof. □

Taking A as the normal cone operator in Theorem 5.1, we obtain the following corollary from Theorem 5.1.

Corollary 5.2. *Let H be a real Hilbert space and C a closed affine subspace of H. Let $B : H \to 2^H$ be a maximal monotone operator. Let $\gamma \in (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$ for all $n \geq 1$. Suppose $\{a_n\}, \{b_n\}$ are sequences in H. Assume that $0 \in \operatorname{ran}(N_C + B)$. Let the sequence $\{y_n\}$ in H be generated by choosing $y_1 \in H$ and using the recursion*

$$y_{n+1} := \alpha_n y_n + 2\beta_n\big(P_C\big(2(J_{\gamma B}y_n + b_n) - y_n\big) + a_n\big) - 2\beta_n(J_{\gamma B}y_n + b_n) + \beta_n y_n \quad (23)$$

for all $n \geq 1$. Suppose the following conditions hold:

(a) $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n = \infty$;

(b) $\sum\limits_{n=1}^{\infty} \beta_n\big(\|a_n\| + \|b_n\|\big) < \infty$;

(c) $\sum\limits_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Then the sequence $\{P_C y_n\}$ converges weakly to $J_{\gamma B}y$.

Proof. Recall that N_C is maximally monotone with $J_{N_C} = P_C$, see, e.g., [2, Examples 20.41 and 23.4]. Now, set $A := N_C$ in Theorem 5.1 and note that $\gamma A = \gamma N_C = N_C$ due to the cone property. Hence we have $J_{\gamma A} = J_{N_C} = P_C$. Therefore, the recursion from (22) reduces to the one from (23). Consequently, it follows from Theorem 5.1 that the sequence $\{y_n\}$ generated by (23) converges weakly to some element $y \in H$ which is a fixed point of the operator $T := R_{\gamma A}R_{\gamma B}$. Since the projection operator onto closed affine subspaces in weakly continuous, cf. [2, Prop. 4.11], we obtain that the sequence $P_C y_n$ is weakly convergent to $P_C y = J_{\gamma A}y$. Since $y \in F(T)$, we may invoke [2, Prop. 4.21] to see that $P_C y = J_{\gamma B}y$, and this completes the proof. $\square$

By following the same line of arguments as in Theorem 5.1, we also obtain a strong convergence result.

Theorem 5.3. *Let H be a real Hilbert space. Let $\gamma \in (0, \infty)$, let $\{\alpha_n\}, \{\beta_n\}$, and $\{\delta_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \delta_n \leq 1$ for all $n \geq 1$. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences in H. Assume that $0 \in \operatorname{ran}(A + B)$. Let the sequence $\{y_n\}$ in H be generated by choosing $y_1 \in H$ and using the recursion*

$$y_{n+1} := \delta_n u + \alpha_n y_n + 2\beta_n\big(J_{\gamma A}\big(2(J_{\gamma B}y_n + b_n) - y_n\big) + a_n\big) - 2\beta_n(J_{\gamma B}y_n + b_n) + \beta_n y_n \quad (24)$$

for all $n \geq 1$, where $u \in H$ is a fixed vector. Suppose the following conditions hold:

(a) $\lim\limits_{n\to\infty} \delta_n = 0$, $\sum\limits_{n=1}^{\infty} \delta_n = \infty$;

(b) $\liminf\limits_{n\to\infty} \alpha_n\beta_n > 0$;

(c) $\sum\limits_{n=1}^{\infty} (1 - \alpha_n - \beta_n - \delta_n) < \infty$, *and*

(d) $\sum_{n=1}^{\infty} \beta_n \big(\|a_n\| + \|b_n\| \big) < \infty$.

Then the sequence $\{y_n\}$ *generated by* (24) *strongly converges to some point* $y \in H$ *such that* $J_{\gamma B} y \in (A+B)^{-1}(0)$.

We next relate the previous theorems to some existing results from the literature.

Remark 5.4. (a) Consider the standard Douglas-Rachford method as given, e.g., in (21). This corresponds to $\alpha_n = 1 - \lambda_n, \beta_n = \lambda_n$ and $a_n = b_n = 0$ in the setting of Theorem 5.1. Hence conditions (b), (c) hold automatically, whereas condition (a) becomes

$$\sum_{n=1}^{\infty} \lambda_n (1 - \lambda_n) = \infty. \tag{25}$$

Most references present the Douglas-Rachford splitting method in a slightly different way. In fact, setting $\nu_n := 2\lambda_n$ in (21), we obtain $\nu_n \in [0,2]$ for all $n \in \mathbb{N}$, and multiplying (25) by four, we see that (25) is equivalent to

$$\sum_{n=1}^{\infty} \nu_n (2 - \nu_n) = \infty,$$

which is the standard condition for the weak convergence of the Douglas-Rachford splitting method, see, e.g., [2]. In this re-scaled version, the choices $\nu_n < 1$ and $\nu_n > 1$ are called *underrelaxation* and *overrelaxation*, respectively.

(b) Combettes [11] considers the iterative scheme

$$y_{n+1} = y_n + 2\lambda_n \big(J_{\gamma A} \big(2(J_{\gamma B} y_n + b_n) - y_n \big) + a_n \big) - 2\lambda_n (J_{\gamma B} y_n + b_n),$$

which corresponds to the standard Douglas-Rachford method from (21) with errors a_n and b_n in the evaluation of the resolvents $J_{\gamma A}$ and $J_{\gamma B}$, respectively. The weak convergence result stated in [11] is a special case of Theorem 5.1 simply by setting $\alpha_n := 1 - \lambda_n$ and $\beta_n := \lambda_n$ (or slightly adapted in the re-scaled version outlined in comment (a)).

(c) Consider the Douglas-Rachford method from (21) in the re-scaled version with $\nu_n := 2\lambda_n$ as in comment (a). A typical implementation chooses an element $y_1 \in H$ as a starting point, and then computes $x_n := J_{\gamma B} y_n, z_n := J_{\gamma A}(2x_n - y_n), y_{n+1} := y_n + \nu_n (z_n - x_n)$ for all $n \geq 1$. The following inexact version (in finite-dimensional spaces) can be found in [14] (and similarly in [29] in Hilbert spaces): Choose $y_1 \in H$ and then

- compute x_n such that $\|x_n - J_{\gamma B}(y_n)\| \leq \varepsilon_n$;
- compute z_n such that $\|z_n - J_{\gamma A}(2x_n - y_n)\| \leq \tilde{\varepsilon}_n$;
- set $y_{n+1} := y_n + \nu_n (z_n - x_n)$

for all $n \geq 1$, where $\varepsilon_n, \tilde{\varepsilon}_n$ are nonnegative scalars which measure the degree of inexactness in the evaluation of the two resolvents. Global convergence of this inexact Douglas-Rachford method is shown under the assumptions

$$0 < \inf_{n\geq 1} \nu_n \leq \sup_{n\geq 1} \nu_n < 2, \quad \sum_{n=1}^{\infty} \varepsilon_n < \infty, \quad \sum_{n=1}^{\infty} \tilde{\varepsilon}_n < \infty. \tag{26}$$

Setting $a_n := z_n - J_{\gamma A}(2x_n - y_n), b_n := x_n - J_{\gamma B} y_n$, this method can be rewritten as

$$\begin{aligned} y_{n+1} &:= y_n + \nu_n (z_n - x_n) \\ &= y_n + \nu_n \big[J_{\gamma A}(2x_n - y_n) + a_n - x_n \big] \\ &= y_n + \nu_n \big[J_{\gamma A}\big(2(J_{\gamma B} y_n + b_n) - y_n \big) + a_n - (J_{\gamma B} y_n + b_n) \big], \end{aligned}$$

which fits precisely within our framework (with $\nu_n = 2\lambda_n$). Moreover, the assumptions (26) together with $\alpha_n := 1 - \lambda_n, \beta_n := \lambda_n$ obviously imply that all our convergence conditions hold, hence the inexact Douglas-Rachford method from [14] is also a special case of our framework.

(d) Consider our Theorem 3.1 and take $\alpha_n = 1 - \lambda_n, \beta_n = \lambda_n, r_n = 0$ for all $n \geq 1$, where $\lambda_n \in [0, 1]$, as well as the nonexpansive operator $T := R_{\gamma A} R_{\gamma B}$ for some $\gamma > 0$. Then our iterative scheme (10) reduces to the recursion (26) in [32], and our Theorem 3.1 reduces to Theorem 1 of that reference. Observe, however, that [32] use the much stronger assumption $0 < \inf_{n\geq 1} \lambda_n \leq \sup_{n\geq 1} \lambda_n < 1$ on the choice of the scalars λ_n as well as the maximal monotonicity of the operator $A + B$ (recall that the sum of two maximally monotone operators in monotone, but not necessarily maximal monotone). Hence our results may be viewed as improvements of those obtained in [32]. $\Diamond$

5.2 Extension to a Finite Number of Maximally Monotone Operators

We now extend the previous section by finding a zero of a finite number of maximally monotone operators; this is precisely the situation that occurs in our application to the Fermat-Weber location problem in the following section.

Hence let H be a real Hilbert space and let $A_i : H \to 2^H$ be maximally monotone for all $i = 1, \ldots, m$. Our aim is to find an element in $\text{zer}(A_1 + \ldots + A_m)$. We know how to do that for $m = 2$, and the basic idea for $m \geq 2$ is to re-cast this problem as a zero-finding problem of two operators in a suitable product space. To this end, we follow the idea described, e.g., in [2], and introduce the notation

$$\begin{aligned} &\mathbf{H} := H \times \ldots \times H \quad (m\text{-times}), \\ &\mathbf{D} := \big\{ (x, \ldots, x) \in \mathbf{H} \mid x \in H \big\}, \\ &\mathbf{j} : H \to \mathbf{D}, \; x \mapsto (x, \ldots, x), \\ &\mathbf{B} := A_1 \times \ldots \times A_m. \end{aligned}$$

Given any element $\mathbf{x} \in \mathbf{H}$, we write $\mathbf{x} = (x_1, \ldots, x_m)$ with the blocks x_i belonging to the Hilbert space H. Recall that $\mathbf{H}$ is also a Hilbert space with scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{H}} := \sum_{i=1}^{m} \langle x_i, y_i \rangle \quad \text{and induced norm} \quad \|\mathbf{x}\|_{\mathbf{H}} := \Big(\sum_{i=1}^{m} \|x_i\|^2 \Big)^{1/2}.$$

Further note that $\mathbf{D}$ is a subspace of $\mathbf{H}$. Then the following elementary properties play a central role, see [2, Ex. 23.4 and Prop. 25.4] for their proofs.

Lemma 5.5. *Let $A_i : H \to 2^H$ be maximally monotone for all $i = 1, \ldots, m$. Then, using the above notation, the following statements hold:*

(a) The orthogonal complement of $\mathbf{D}$ is given by $\mathbf{D}^\perp = \big\{\mathbf{u} \in \mathbf{H} \mid \sum_{i=1}^m u_i = 0\big\}$;

(b) The normal cone of $\mathbf{D}$ at a point $\mathbf{x} \in \mathbf{D}$ is given by $N_{\mathbf{D}}\mathbf{x} = \mathbf{D}^\perp$ (and the empty set if $\mathbf{x} \notin \mathbf{D}$);

(c) The projection of $\mathbf{x}$ onto $\mathbf{D}$ is given by $P_{\mathbf{D}}\mathbf{x} = \mathbf{j}\big(\frac{1}{m}(x_1 + \ldots + x_m)\big)$;

(d) The resolvent of $N_{\mathbf{D}}$ is given by $J_{N_{\mathbf{D}}}\mathbf{x} = P_{\mathbf{D}}\mathbf{x} \overset{(c)}{=} \mathbf{j}\big(\frac{1}{m}(x_1 + \ldots + x_m)\big)$;

(e) The resolvent $J_{\gamma\mathbf{B}}$ has the representation $J_{\gamma\mathbf{B}}\mathbf{x} = \big(J_{\gamma A_1}x_1, \ldots, J_{\gamma A_m}x_m\big)$;

(f) It holds that $\mathbf{j}\big(zer(A_1 + \ldots + A_m)\big) = zer\big(N_{\mathbf{D}} + \mathbf{B}\big)$.

Lemma 5.5 reduces our problem to finding a zero of the sum of two maximally monotone operators (note that both operators involved in statement (f) are known or easy to see to be maximally monotone). Therefore, setting $\mathbf{A} := N_{\mathbf{D}}$, we are exactly in the situation of the previous section. Assuming, for the sake of simplicity, that there are no errors a_n and b_n, the generalized Douglas-Rachford method in the product space can be written as

$$\begin{aligned}
\mathbf{x}_n &:= J_{\gamma\mathbf{B}}(\mathbf{y}_n),\\
\mathbf{z}_n &:= J_{\gamma\mathbf{A}}\big(2\mathbf{x}_n - \mathbf{y}_n\big),\\
\mathbf{y}_{n+1} &:= (\alpha_n + \beta_n)\mathbf{y}_n + 2\beta_n\big(\mathbf{z}_n - \mathbf{x}_n\big)
\end{aligned}$$

for all $n = 1, 2, \ldots$, where $\mathbf{y}_1$ is a suitable starting point. In view of Lemma 5.5 (d), we have $J_{\gamma\mathbf{A}} = P_{\mathbf{D}}$. Therefore, setting

$$\mathbf{p}_n := P_{\mathbf{D}}\mathbf{y}_n \qquad \mathbf{q}_n := P_{\mathbf{D}}\mathbf{x}_n,$$

and using the linearity of the projection onto the linear subspace $\mathbf{D}$, we have $\mathbf{z}_n = 2\mathbf{q}_n - \mathbf{p}_n$, hence the method can be rewritten as

$$\begin{aligned}
\mathbf{p}_n &:= P_{\mathbf{D}}\mathbf{y}_n,\\
\mathbf{x}_n &:= J_{\gamma\mathbf{B}}\mathbf{y}_n,\\
\mathbf{q}_n &:= P_{\mathbf{D}}\mathbf{x}_n,\\
\mathbf{y}_{n+1} &:= (\alpha_n + \beta_n)\mathbf{y}_n + 2\beta_n(2\mathbf{q}_n - \mathbf{p}_n - \mathbf{x}_n)
\end{aligned}$$

for $n = 1, 2, \ldots$. Noting that the block components of $\mathbf{p}_n$ and $\mathbf{q}_n$ are identical, and setting $\mathbf{y}_n =: \big(y_{n,1}, \ldots, y_{n,m}\big), \mathbf{x}_n =: \big(x_{n,1}, \ldots, x_{n,m}\big), \mathbf{p}_n =: \mathbf{j}(p_n), \mathbf{q}_n =: \mathbf{j}(q_n)$, we can exploit Lemma 5.5 and rewrite the iteration from the product space $\mathbf{H}$ in the following way as an iteration in the Hilbert space H itself:

$$\left.\begin{array}{rcl}
p_n &:=& \frac{1}{m}\sum_{i=1}^m y_{n,i},\\
x_{n,i} &:=& J_{\gamma A_i}y_{n,i} \quad \forall i = 1, \ldots, m,\\
q_n &:=& \frac{1}{m}\sum_{i=1}^m x_{n,i},\\
y_{n+1,i} &:=& (\alpha_n + \beta_n)y_{n,i} + 2\beta_n(2q_n - p_n - x_{n,i}) \quad \forall i = 1, \ldots, m.
\end{array}\right\} \tag{27}$$

Using Corollary 5.2, we then obtain the following convergence result.

Theorem 5.6. *Let H be a real Hilbert space, $A_i : H \to 2^H$ be maximally monotone operators for all $i = 1, \dots, m$ for some $m \geq 2$, and assume that $0 \in ran\big(A_1 + \ldots + A_m\big)$. Furthermore, let $\{\alpha_n\}, \{\beta_n\}$ be real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n \leq 1$ for all $n \in \mathbb{N}$ such that the following conditions hold:*

(a) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$;

(b) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Given any starting points $y_{1,i} \in H$ for all $i = 1, \dots, m$, the iteration (27) *generates a sequence $\{p_n\}$ which converges weakly to a point in $zer\big(A_1 + \ldots + A_m\big)$.*

We do not present a strongly convergent counterpart of the previous result since we will apply Theorem 5.6 to the finite-dimensional Fermat-Weber location problem.

5.3 Application to the Fermat-Weber Location Problem

The classical Fermat-Weber problem in $H := \mathbb{R}^n$ is given by the optimization problem

$$\min f(x) := \sum_{i=1}^{m} \omega_i \|x - a_i\|, \tag{28}$$

where $\omega_i > 0$ are given weights, and $a_i \in H$ are pairwise disjoint points, sometimes called anchor points. Here, $\|\cdot\|$ stands for the Euclidean vector norm. The objective function f from (28) is convex and coercive, hence the problem always has a nonempty and convex solution set. Note, however, that f is not differentiable at the anchor points.

The Fermat-Weber problem is a famous model in location theory, and we refer to [13, 22] for suitable surveys with some historical background, several applications and generalizations. The most famous method for the solution of problem (28) is Weiszfeld's algorithm, see [4] for an extensive discussion. In principle, Weiszfeld's method is a fixed-point iteration, but not always well-defined and not necessarily convergent, at least not without suitable modifications. Here we present another fixed point method as a consequence of Theorem 5.6.

To this end, note that the objective function in (28) has the natural splitting

$$f(x) = f_1(x) + \ldots + f_m(x) \quad \text{with} \quad f_i(x) := \omega_i \|x - a_i\|.$$

Furthermore, x^* is a solution of (28) if and only if

$$0 \in \partial f(x^*) = \partial f_1(x^*) + \ldots + \partial f_m(x^*),$$

where

$$\partial f(x) := \big\{ s \mid f(y) \geq f(x) + \langle s, y - x \rangle \ \forall y \big\}$$

denotes the subdifferential of f at x, and the equation comes from the sum rule for this subdifferential. Since each subdifferential ∂f_i is maximally monotone, see, e.g., [2, Thm. 20.40], we see that we are in the situation discussed in Theorem 5.6

with $A_i := \partial f_i$ for $i = 1, \ldots, m$. In order to apply this result, first recall that the subdifferential of each mapping f_i is given by

$$\partial f_i(x) = \begin{cases} \{z \mid \|z\| \leq \omega_i\} & \text{if } x = a_i, \\ \left\{\omega_i \frac{x-a_i}{\|x-a_i\|}\right\} & \text{if } x \neq a_i. \end{cases} \tag{29}$$

In order to apply the generalized splitting method, we need to calculate the resolvents $J_{A_i} = J_{\partial f_i}$ (note that we take, without loss of generality, $\gamma = 1$ in the general splitting scheme, since other choices of γ can be incorporated by scaling the weights ω_i). The following result tells us that these resolvents are very easy to compute analytically. To this end, it is convenient to introduce the proximity operator Prox_g of a convex function g, defined by

$$\mathrm{Prox}_g(x) := \mathrm{argmin}_y \big\{ g(y) + \frac{1}{2}\|y - x\|^2 \big\}.$$

Then the following result holds.

Lemma 5.7. *Using the previous notation, we have*

$$J_{\partial f_i}(x) = Prox_{f_i}(x) = \begin{cases} a_i & \textit{if } \|x - a_i\| \leq \omega_i \\ x - \omega_i \frac{x-a_i}{\|x-a_i\|} & \textit{otherwise} \end{cases} \tag{30}$$

for all $i = 1, \ldots, m$.

Proof. Let $g := f_i$ for some $i \in \{1, \ldots, m\}$. Then the first equation is a general result, see, e.g., [2, Ex. 23.3]. Since Prox_g is the resolvent of a maximal monotone mapping ∂g, it follows that Prox_g is single-valued. Furthermore, observe that for all x, it holds that

$$x - y \in \partial g(y) \Leftrightarrow x \in (I + \partial g)y \Leftrightarrow y = \mathrm{Prox}_g(x).$$

Therefore, it suffices to show that the function from (30) satisfies the condition $x - \mathrm{Prox}_g(x) \in \partial g(\mathrm{Prox}_g(x))$, which is straightforward to see using the expression (29) for the subdifferential of $g = f_i$. This completes the proof. $\square$

Application of Theorem 5.6 to the Fermat-Weber problem (28) yields the following convergence result.

Theorem 5.8. *Let m be an integer such that $m \geq 2$, $a_1, \ldots, a_m \in \mathbb{R}^n$ be pairwise disjoint, and let $f_i(x) := \omega_i\|x - a_i\|, i = 1, 2, \ldots, m$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$ for all $n \geq 1$. For each $i = 1, 2, \ldots, m$, choose $y_{i,1} \in \mathbb{R}$., and for all $n \geq 1$, set*

$$\left.\begin{array}{rcl} p_n & := & \frac{1}{m}\sum\limits_{i=1}^{m} y_{n,i}, \\ x_{n,i} & := & Prox_{f_i} y_{n,i} \quad \forall i = 1, \ldots, m, \\ q_n & := & \frac{1}{m}\sum\limits_{i=1}^{m} x_{n,i}, \\ y_{n+1,i} & := & (\alpha_n + \beta_n) y_{n,i} + 2\beta_n(2q_n - p_n - x_{n,i}) \quad \forall i = 1, \ldots, m. \end{array}\right\} \tag{31}$$

Suppose the following conditions hold:

(a) $\sum\limits_{n=1}^{\infty} \alpha_n \beta_n = \infty$;

(b) $\sum\limits_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Then the sequence $\{p_n\}$ *generated by* (31) *converges to a solution of Fermat-Weber location problem* (28).

Note that all statements from this section can easily be extended to a Hilbert space, and the previous result then yields weak convergence to a solution. Note, however, that typical applications of the Fermat-Weber problem are in finite dimensions.

5.4 Application to the Alternating Projection Method by John von Neumann

Let $A, B \subseteq H$ be two nonempty, closed, and convex subsets of a real Hilbert space H, and suppose that $A \cap B \neq \emptyset$. Since the corresponding projection operators P_A and P_B are (firmly) nonexpansive, their composition

$$T := P_A P_B$$

is nonexansive (but not firmly nonexpansive unless the underlying sets have some additional properties, see, e.g., [6]). The fixed points of T are precisely the elements from the intersection $A \cap B$. The corresponding Picard iteration $x_{n+1} := Tx_n$ is known as the alternating projection method by John von Neumann. In order to obtain a globally convergent variant, one can use the relaxation approach

$$x_{n+1} := (1 - \lambda_n) x_n + \lambda_n T x_n = (1 - \lambda_n) x_n + \lambda_n P_A(P_B x_n).$$

Following our general scheme, we replace the scalars $1 - \lambda_n$ and λ_n by α_n and β_n, respectively, and allow errors a_n and b_n in the computation of the projections P_A and P_B. This results in the iteration

$$x_{n+1} := \alpha_n x_n + \beta_n \big(P_A(P_B x_n + b_n) + a_n \big), \quad n \geq 1. \tag{32}$$

Similar to the previous section, we now apply our general weak convergence result from Theorem 3.1 to this particular application.

Theorem 5.9. *Let* H *be a real Hilbert space, and let* $A, B \subseteq H$ *be nonempty, closed, and convex subsets such that* $A \cap B \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0, 1]$ *such that* $\alpha_n + \beta_n \leq 1$ *for all* $n \geq 1$. *Furthermore, let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences in* H. *Let the sequence* $\{x_n\}$ *in* H *be generated by using the recursion* (32) *with an arbitrary starting point* $x_1 \in H$. *Suppose the following conditions hold:*

(a) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$;

(b) $\sum_{n=1}^{\infty} \beta_n \big(\|a_n\| + \|b_n\| \big) < \infty$;

(c) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$.

Then the sequence $\{x_n\}$ converges weakly to an element of $A \cap B$.

Proof. Let $T := P_A P_B$. Then we can rewrite the iteration (32) as

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n \big(P_A(P_B x_n + b_n) + a_n \big) \\ &= \alpha_n x_n + \beta_n P_A(P_B x_n) + \beta_n \big(P_A(P_B x_n + b_n) - P_A(P_B x_n) + a_n \big) \\ &= \alpha_n x_n + \beta_n T x_n + r_n \end{aligned}$$

with

$$r_n := \beta_n \big(P_A(P_B x_n + b_n) - P_A(P_B x_n) + a_n \big).$$

Using the nonexpansiveness of the projection operator, we have

$$\begin{aligned} \|r_n\| &\leq \beta_n \big(\big\| P_A(P_B x_n + b_n) - P_A(P_B x_n) \big\| + \|a_n\| \big) \\ &\leq \beta_n \big(\big\| P_B x_n + b_n - P_B x_n \big\| + \|a_n\| \big) \\ &= \beta_n \big(\|b_n\| + \|a_n\| \big). \end{aligned}$$

Condition (b) therefore yields

$$\sum_{n=1}^{\infty} \|r_n\| < \infty.$$

Since T is nonexpansive, it follows that we are precisely in the situation of Theorem 3.1, so we obtain weak convergence of the sequence $\{x_n\}$ to a fixed point of T and, therefore, to an element in $A \cap B$. $\square$

Using a similar way of reasoning, we can also prove the following strong convergence result by applying Theorem 4.1.

Theorem 5.10. *Let H be a real Hilbert space, and let $A, B \subseteq H$ be nonempty, closed, and convex subsets such that $A \cap B \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$, and $\{\delta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \delta_n \leq 1$ for all $n \geq 1$. Furthermore, let $\{a_n\}$ and $\{b_n\}$ be sequences in H. Let the sequence $\{x_n\}$ in H be generated by*

$$x_{n+1} := \delta_n u + \alpha_n x_n + \beta_n \big(P_A(P_B x_n + b_n) + a_n \big), \quad n \geq 1,$$

using an arbitrary starting point $x_1 \in H$, where $u \in H$ denotes a fixed vector. Suppose the following conditions hold:

- *(a)* $\lim_{n \to \infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$;
- *(b)* $\liminf_{n \to \infty} \alpha_n \beta_n > 0$;
- *(c)* $\sum_{n=1}^{\infty} \beta_n \big(\|a_n\| + \|b_n\| \big) < \infty$;
- *(d)* $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n - \delta_n) < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element of $A \cap B$.

6 Final Remarks

This paper presents both weak and strong convergence results for a generalized Krasnoselskii-Mann iteration and applies the result to three particular applications. The convergence theorems can be used to re-cover existing results, but the examples and counterexamples provided as an illustration for our method also show that some of the existing results in the literature are erroneous. Part of our future reseach is devoted to the solution of (generalized) Nash equilibrium problems, where, under certain assumptions, suitable reformulations lead to nonexpansive fixed-point problems.

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