LOTTERY VERSUS ALL-PAY AUCTION CONTESTS: A REVENUE DOMINANCE THEOREM

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Abstract

We allow a contest organizer to bias a contest in a discriminatory way; i.e., she can favor specific contestants by designing the contest rule in order to maximize total equilibrium effort (resp. revenue). The two predominant contest regimes are considered, all-pay auctions and lottery contests. For all-pay auctions the optimal bias is derived in closed form: It implies extreme competitive pressure among active contestants and low endogenous entry. Moreover, the exclusion principle advanced by Baye et al. (1993) becomes obsolete in this case. In contrast, the optimally biased lottery induces larger entry of contestants due to softer competition. Our main result regarding total revenue comparison under the optimal biases reveals that the all-pay auction revenue dominates the lottery contest for all levels of heterogeneity among contestants. The incentive effect due to a strongly discriminating contest rule (all-pay auction) dominates the participation effect due to a weakly discriminating contest rule (lottery).

Key Words: All-pay auction, lottery contest, optimal bias, revenue, exclusion principle.

JEL classification: C72; D72

1 Introduction

The all-pay auction and the lottery contest game are the most frequently used setups to model strategic competition among agents that exert non-refundable effort to influence their respective chances to win a fixed prize. Both types of models have been extensively used in applied analysis, for instance, in the areas of R&D competition, lobbying, sports, rent-seeking, procurement, etc., see Konrad (2009) [11] for a survey. One of the reasons for the popularity of these models might be the analytical tractability, especially if employed under the assumptions that the rules that govern the competition are anonymous and that agents are homogeneous. Recently, there is a growing interest in relaxing these limiting assumptions: The heterogeneity of contestants comes into the focus of analysis and, as a consequence, also the question of how the contest organizer should exploit heterogeneity among contestants by treating different contestants differently. Recent examples that follow this approach are Siegel (2012, 2010), [12, 13], Kirkegaard (2012), [10], Epstein et al. (2011), [4], Szech (2011), [15], and Franke et al. (2011), [6].

Due to the prominence of the rent-seeking interpretation in this literature an important aspect in the strategic analysis is the relation between aggregate equilibrium efforts of the agents (i.e., the revenue of the auction or contest) and the underlying institutional rules and characteristics that govern the specific form of the contest. In Baye et al. (1993), [2], for instance, an analysis of the all-pay auction with heterogeneous players established the so called exclusion principle, which implies that a revenue-maximizing contest organizer might optimally exclude strong agents from the competition ex-ante. This result is in contrast to the symmetric lottery case considered in Fang (2002), [5], where it is shown that exclusion of strong players is never optimal for the contest organizer. Moreover, the direct comparison between these two contest regimes reveals that neither revenue-dominates the other a priori. The intuition for this result can be attributed to the trade-off between competitive pressure and entry which is differently resolved in the two regimes: Competitive pressure in an all-pay auction is primarily generated by the institution itself. Its highly discriminative, all-pay deterministic winner-takes-it-all nature endogenously restricts entry in equilibrium to (generically) just two contestants. However, competition between those two is so intense, that in (mixed strategy) equilibrium only one has a positive payoff in expectation (while both have a positive probability of winning). In contrast, a lottery contest with its characteristic probabilistic contest rule is much less discriminative as an institution because it does not require to be the highest bidder in order to win. This characteristic is highly conducive for attracting entry; i.e., competitive pressure in a lottery contest is primarily generated by the interaction of many active contestants in equilibrium. Fang (2002), [5] shows that from a revenue maximizing contest designer's point of view it depends on degree and nature of heterogeneity of contestants whether it is better to ignite competitive pressure *ex ante* (through the choice of a very discriminative contest success function like the all-pay auction), which is reduced endogenously *ex post* due to a minimal amount of entry, or to opt for weaker competitive pressure *ex ante* (by choosing a less discriminative contest success function like the lottery contest) which is endogenously reinforced *ex post* due to entry of more contestants.

Importantly, both of these models are based on the assumption that the contest organizer is neutral with respect to the contestants; that is, she chooses among contest regimes, which treat contestants anonymously. This is certainly not the case in many real world contests (just think of the contest rules for a job opening of a professorship), where the contest organizer has control over some variables, which bias the contest systematically (and legally) in favor of certain contestants. Further examples are provided by biased contests in affirmative action contexts, public procurement practices, which favor local or national firms over others, sport tournaments with handicap schemes, and litigation law, which allocates the costs between the parties involved asymmetrically. A detailed account of biased public procurement in Israel is provided by Epstein et al. (2011), [4]. Biasing the contest rule gives the contest organizer additional power to promote her interests, in particular in the presence of heterogeneous contestants. This situation is analyzed for the case of two contestants in Epstein et al. (2011), [4], where the contest organizer can specify individual weights for each of two contestants. Setting individual weights reflects her potential for discriminating between the two contestants which has consequences for the revenue comparison between all-pay auction and lottery contest: The optimally biased all-pay auction revenue-dominates any biased lottery contest, independently of heterogeneity between the two contestants. However, the restriction to the two-player case is particularly severe for at least two reasons: Firstly, it ignores the basic trade-off with regard to competitive pressure as described above. The ”minimal entry” feature of the all-pay auction is eliminated, likewise the scope of the lottery contests for increased competitive pressure through additional entry. Secondly, the solution theory of the biased lottery contest with only two contestants is a degenerate case of the general n-player solution, see Franke et al. (2011), [6]. More precisely, the optimal weight for a contestant in the two-player case only depends on his own characteristics, whereas with three or more players any optimal individual bias weight depends on the characteristics of *all* contestants.

The objective of this paper is to determine a revenue (or total effort) maximizing contest organizer's choice of contest, when she is faced with n heterogeneous contestants. Her choice set consists of a set of (potentially biased) contest success function, which contains lotteries and all-pay auctions. For lottery contests we can rely on Franke et al. (2011), [6], who analyze the optimal choice of the contest organizer if her choice set is restricted to biased lotteries. However, the optimal choice from the set of biased all-pay auctions has not been determined so far. This derivation is challenging due to the fact that, depending on the choice of the all-pay auction contest rule, multiple mixed strategy equilibria might exist which are not revenue equivalent. Nevertheless, we derive a simple expression of the optimal bias in closed form and the corresponding revenue for any finite set of contestants with heterogeneous valuations. This result allows us then to compare the induced revenue in the two regimes under the respective optimal biases. Our second main result (Theorem 4.3) states that revenue dominance of an optimally biased all-pay auction over the optimally biased lottery holds for any given set of heterogeneous contestants. This result is far from trivial, but has a clear intuition: The ability of the contest organizer to discriminate between contestants in the all-pay auction is used to make the exclusion principle obsolete (an alternative approach is the modified all-pay auction rule in Gale and Stegeman (1994), [8], which gives only the strongest contestant a special status). Under the optimal bias it will always be the two strongest contestants who choose to be active, and they are made to compete with each other in the strongest possible way, i.e., in a playing field that is completely leveled due to the bias. No strong player is excluded a priori by the organizer. As expected, the discriminatory power of the contest organizer in the lottery contest is used to encourage further entry: In any optimally biased lottery contest at least the three strongest contestants are active. However, the playing field among active contestants is not completely leveled in the optimally biased lottery contest because balancing the playing field negatively affects incentives for strong contestants. Moreover, the optimal bias is specified such that not all contestants might be induced to become active. This incompatibility of high entry and high competitive pressure due to a leveled field *given* entry in the lottery contest contributes to its inferiority with respect to the optimally biased all-pay auction. Our theoretical results therefore provide a new explanation for the often observed phenomenon that only two strong contestants endogenously decide to participate in contests although the potential field of contestants is substantially larger (see the introduction of Fullerton and McAfee (1999), [7], for some real world examples of this phenomenon in research contests). Our revenue dominance theorem demonstrates that the reason for this observation might not be the irrational manipulation of the contest design from the side of the contest organizer (or outright illegal collusion with specific contestants) but instead her motivation to maximize total efforts.

The paper is organized as follows. Section 2 contains the model, Section 3 derives the optimally biased all-pay auction. In Section 4 we recapitulate the optimally biased lottery contest as derived in Franke et al. (2011), [6], and compare it to the result from Section 3, which gives our Revenue Dominance Theorem. Section 5 concludes.

2 The Model

There are n agents $N = \{1, \ldots, n\}$, that participate either in a contest or in an all-pay auction which implies that they can influence the probability to win a non-divisible prize by exerting non-refundable effort. Agents are heterogeneous with respect to their valuation of the prize; that is, agent $i \in N$ values the prize at $v_i \in (0, \infty)$ and chooses a strategy (exerts effort) $x_i \in [0, \infty)$ to influence the probability $Pr_i(x_i, x_{-i}) : [0, \infty)^n \to [0, 1]$ of winning the prize, where $\sum_{i \in N} Pr_i(x_i, x_{-i}) = 1$ and $(x_i, x_{-i}) := (x_1, \ldots, x_n)$ for all $i = 1, \ldots, n$. Hence, the payoff function of agent i is:

$$\pi_i(x_i, x_{-i}) = Pr_i(x_i, x_{-i})v_i - x_i \text{ for all } i \in N.$$

The formal rule of a contest, which maps an individual's effort into his winning probability as a function of the other contestants' efforts is called a contest success function (CSF). We are going to consider deterministic and probabilistic CSFs; we refer to the former as all-pay auctions and to the latter as lotteries. Technically speaking, lotteries are logit CSFs with linear component functions.

We will assume without loss of generality that agents are ordered with respect to their valuations: $v_1 \geq v_2 \geq \ldots \geq v_n$. The contest organizer has the power to bias the contest outcome with respect to specific agents. This implies that the contest organizer can specify a vector of agent specific weights $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in (0, \infty)^n$ that affect the impact of the agents' effort on the win probability as specified below. We consider two different classes of contest success functions that govern the probability to win the prize for player i:

- The *biased all-pay auction (BAA)* framework:

$$Pr_i^{BAA}(x_i, x_{-i}) = \begin{cases} 1, & \text{if } \alpha_i x_i > \alpha_j x_j \text{ for all } j \neq i, \\ \frac{1}{k+1}, & \text{if } \alpha_i x_i = \alpha_j x_j \text{ for } k \text{ agents } j \neq i \text{ and } \alpha_i x_i > \alpha_l x_l \text{ for all other agents } l \neq i, \\ 0, & \text{if } \alpha_i x_i < \alpha_j x_j \text{ for some } j \neq i. \end{cases}$$

- The *biased lottery contest game (BLC)* framework:

$$Pr_i^{BLC}(x_i, x_{-i}) = \begin{cases} \frac{\alpha_i x_i}{\sum_{j=1}^n \alpha_j x_j}, & \text{if } \sum_{j=1}^n \alpha_j x_j \neq 0, \\ 0, & \text{if } \sum_{j=1}^n \alpha_j x_j = 0. \end{cases}$$

We are going to evaluate the two regimes with respect to the maximal total (expected) revenue

$X^{*,c}$ that they induce in equilibrium: $X^{*,c} = \sum_{j \in N} \mathrm{E}[x_j^{*,c}]$ with $c \in \{BAA, BLC\}$, where $x^{*,c}$ is the (potentially mixed) Nash equilibrium strategy under the respective optimal bias and $\mathrm{E}[x_j^{*,c}]$ is the corresponding expected equilibrium effort of agent j. Alternatively, we can formulate the following three stage game, where the objective of the contest organizer is to maximize total revenue:

1. Stage: The contest organizer chooses the competitive regime BAA or BLC.
2. Stage: The contest organizer specifies the optimal bias α; i.e., chooses the best CSF.
3. Stage: The agents choose the payoff maximizing strategies.

Note that the previous contributions by Baye et al. (1993), [2], who introduced the exclusion principle, and Fang (2002), [5], who compared the standard (unbiased) all-pay auction and lottery contest, can be viewed as restricting the contest organizer's choice of α_i to 0 or 1 for all $i = 1, \ldots, n$. The former means 'exclusion', the latter 'participation', i.e. 'becoming a finalist' in the language of these authors.

We will derive the subgame-perfect Nash-equilibrium by backward induction. The characterization and existence proof of the Nash equilibrium in the third stage given a fixed bias α is standard: For asymmetric lottery contest games the methods presented in Stein (2002), [14], as well as Cornes and Hartley (2005), [3], can be used; for the biased all-pay auction a transformation allows that similar arguments as in Baye et al. (1993), [2], can be applied. Hence, we directly concentrate on the second stage. In the following section we derive a closed form formula for maximal total revenue in the all-pay auction framework and provide an optimal bias. For the lottery contest we rely on the results in Franke et al. (2011), [6], where a closed form expression for total revenue under the optimal bias is provided. Note that the respective biases for the all-pay auction and the lottery contest framework are not unique which had to be expected because all contest success functions are homogeneous of degree zero in both frameworks.

3 Revenue Maximization in the All-Pay Auction

The scope of discrimination and the resulting amount of total revenue in the all-pay auction is derived as follows. In the first lemma, we are going to show that the biased all-pay auction is strategically equivalent to a standard unbiased all-pay auction with transformed valuations. This allows us to use the results from this literature, e.g. Baye et al. (1993 and 1996), [2], [1], and Hillman and Riley (1989), [9]. We derive the maximal revenue and a corresponding bias in closed form.

Before we are going to present the equivalence result in the next lemma we introduce the following notation. Denote by $y_i = \alpha_i x_i$ and $\tilde{v}_i = \alpha_i v_i$ for all $i \in N$. In line with Baye et al. (1993), [2], the expected effort from agent i's (potentially mixed) strategy y_i is denoted by $E[y_i]$ for all $i \in N$.

Lemma 3.1 *The BAA framework is equivalent to a standard unbiased all-pay auction based on transformed valuations $\tilde{v} = \{\tilde{v}_1, \ldots, \tilde{v}_n\}$, where total (expected) equilibrium revenue is equal to:*

$$\tilde{X}^{BAA} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathrm{E}[y_i^*] \tag{1}$$

with y^* being an equilibrium of the unbiased all-pay auction.

Proof. We now use the transformed effort variable y_i for all $i \in N$. Then the contest success function can be formulated as a standard unbiased all-pay auction (AA):

$$Pr_i^{AA}(y_i, y_{-i}) = \begin{cases} 1, & \text{if } y_i > y_j \text{ for all } j \neq i, \\ \frac{1}{k+1}, & \text{if } y_i = y_j \text{ for } k \text{ agents } j \neq i \text{ and } y_i > y_l \text{ for all other agents } l \neq i, \\ 0, & \text{if } y_i < y_j \text{ for some } j \neq i, \end{cases}$$

while the payoff function of agent i can be expressed as:

$$\pi_i(y_i, y_{-i}) = Pr_i^{AA}(y_i, y_{-i}) v_i - \frac{y_i}{\alpha_i} \text{ for all } i \in N.$$

Multiplying the payoff function of agent $i \in N$ by the constant factor $\alpha_i > 0$ does not affect the equilibrium of the transformed game. Let $\tilde{\pi}_i = \alpha_i \pi_i$ for all $i \in N$. The transformed game is then equivalent to a standard all-pay auction with payoff-function:

$$\tilde{\pi}_i(y_i, y_{-i}) = Pr_i^{AA}(y_i, y_{-i}) \tilde{v}_i - y_i \text{ for all } i \in N,$$

where total revenue is calculated as: $\tilde{X}^{BAA} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathrm{E}[y_i^*]$ because $\mathrm{E}[x_i^*] = \frac{1}{\alpha_i} \mathrm{E}[y_i^*]$ for all $i \in N$. □

Note, that the bias weights $(\alpha_1, \ldots, \alpha_n)$, which transform original valuations $(v_1, \ldots, v_n)$ with $v_1 \geq \ldots \geq v_n$ into transformed valuations $(\tilde{v}_1, \ldots, \tilde{v}_n) = (\alpha_1 v_1, \ldots, \alpha_n v_n)$, need not preserve the order of the original valuations. Thus, it might be necessary to permute the contestants in order to reobtain ordered valuations. As the permutation depends on the respective bias, the contest organizer can induce each possible ordering of contestants and each possible constellation of transformed valuations by specifying an appropriate bias. However, our first result circumvents the problems posed by changed orderings of valuations after applying the bias by relying on a bias parameter which leaves the order of valuations *unchanged*. Moreover, this specific bias parameter will constitute a lower bound for total equilibrium revenue under the optimal bias.

Proposition 3.2 *Let $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$; then applying an optimal bias $\alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$ in the BAA framework yields total equilibrium revenue that satisfies*

$$X^{*,BAA} \geq \frac{v_1 + v_2}{2}.$$

Proof. To prove this result, it suffices to provide one bias α, which yields a corresponding equilibrium revenue $\tilde{X}^{BAA} = \frac{v_1+v_2}{2}$. For this reason we consider the special bias $\alpha \in (0, \infty)^n$, where $\alpha_1 = \frac{1}{v_1}$, $\alpha_2 = \frac{1}{v_2}$ and $\alpha_i = \frac{1}{2v_3}$ for all $i > 2$. The corresponding transformed valuations are then $\tilde{v}_1 = \tilde{v}_2 = 1$ and $\tilde{v}_i \leq \frac{1}{2} < 1$ for all $i > 2$. Note that this special bias preserves the ordering of the contestants, i.e. we have

$$\tilde{v}_1 = \tilde{v}_2 > \tilde{v}_3 \geq \ldots \geq \tilde{v}_n.$$

It is known that the equivalent unbiased all-pay auction with valuations $\tilde{v}$ has a unique and symmetric Nash equilibrium y^*, where

$$\mathrm{E}[y_1^*] = \mathrm{E}[y_2^*] = \tfrac{\tilde{v}_1}{2} = \tfrac{1}{2}$$

and $\mathrm{E}[y_i^*] = 0$ for all $i > 2$, see for example Theorem 1 in Baye et al. (1996) [1]. By Lemma 3.1 this yields an equilibrium revenue of

$$\tilde{X}^{BAA} = \frac{1}{\alpha_1}\mathrm{E}[y_1^*] + \frac{1}{\alpha_2}\mathrm{E}[y_2^*] = \frac{v_1 + v_2}{2}$$

and thus concludes the proof. □

The specific bias α in the previous proof is applied in such a way that the playing field among the two contestants with highest valuations is completely balanced; i.e. $\tilde{v}_1 = \alpha_1 v_1 = \alpha_2 v_2 = \tilde{v}_2$, and the remaining contestants are inactive. This levelled contest leads to payoff 0 for both contestants, cf. again Theorem 1 in Baye et al. (1996) [1]. This scenario of having the two contestants with the highest valuations compete against each other on equal terms (after applying the bias) has such a strong intuitive appeal, that one might conjecture that the lower bound provided in Proposition 3.2 is also an upper bound on total revenue. This would imply that the specific bias used in the proof is actually the optimal revenue maximizing bias of the biased all-pay auction in general. However, in order to prove this conjecture one has to consider all potential biases, in particular those which result in a changed ordering of valuations. The following example shows that applying such a bias induces total revenue which is actually less than the lower bound provided in Proposition 3.2. This insight will be generalized to all potential biases in the rest of this section.

Consider the case $v = (v_1, v_2, v_3) = (3, 2, 1)$ and apply the bias vector $\alpha = (\frac{1}{4}, \frac{1}{2}, \frac{4}{5})$. This yields the transformed valuations $\tilde{v} = (\alpha_1 v_1, \alpha_2 v_2, \alpha_3 v_3) = (\frac{3}{4}, 1, \frac{4}{5})$, which we have to reorder to $\bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3) = (1, \frac{4}{5}, \frac{3}{4}) = (\tilde{v}_2, \tilde{v}_3, \tilde{v}_1)$. This induces the permutation $(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (\alpha_{p^{-1}(2)} v_{p^{-1}(2)}, \alpha_{p^{-1}(3)} v_{p^{-1}(3)}, \alpha_{p^{-1}(1)} v_{p^{-1}(1)})$ with $p : \{1, 2, 3\} \to \{1, 2, 3\}$ and $p(1) = 2$, $p(2) = 3$, and $p(3) = 1$; i.e., $p(i)$ denotes the order index of the transformed value of v_i. For this specific combination of bias and valuations only the two contestants with the two highest transformed valuations will be active in equilibrium. Expected transformed equilibrium effort is then: $E[y^*_{p^{-1}(2)}] = \frac{\bar{v}_2}{2}$, $E[y^*_{p^{-1}(3)}] = \frac{\bar{v}_2^2}{2\bar{v}_1}$, and $E[y^*_{p^{-1}(1)}] = 0$, see for instance Hillman and Riley (1989), [9]. The contest organizer is interested in total revenue which is based on untransformed effort, see (1). For this specific bias total revenue can be calculated as follows: $\tilde{X}^{BAA} = \sum_{i=1}^{3} \frac{1}{\alpha_{p^{-1}(i)}} \mathrm{E}[y^*_{p^{-1}(i)}] = 0 + \frac{2}{5} + \frac{8}{5} = 2$, where the permutation and transformation had to be reversed to derive the last equality. Hence, total revenue for this bias specification is strictly less than the lower bound derived in Proposition 3.2 because $\frac{v_1+v_2}{2} = 2.5 > 2$ which directly implies that the considered bias cannot be optimal.

In general, let $p : N \to N, N = \{1, \ldots, n\}$ be defined by

$$(\bar{v}_1, \ldots, \bar{v}_n) = (\tilde{v}_{p^{-1}(1)}, \ldots, \tilde{v}_{p^{-1}(n)}) = (\alpha_{p^{-1}(1)} v_{p^{-1}(1)}, \ldots, \alpha_{p^{-1}(n)} v_{p^{-1}(n)})$$

such that the permuted valuations satisfy $\bar{v}_1 \geq \ldots \geq \bar{v}_n$. For notational convenience, let us write $\bar{\alpha}_i = \alpha_{p^{-1}(i)}$ and $\bar{y}_i = y_{p^{-1}(i)}$ for all $i = 1, \ldots, n$. Note then, that $\tilde{X}^{BAA} = \sum_{i=1}^{n} \frac{1}{\bar{\alpha}_i} \mathrm{E}[\bar{y}_i^*] = \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathrm{E}[y_i^*]$ because the terms of the sum are merely permuted.

We now prove that for $v = (v_1, \ldots, v_n)$ and $v_1 \geq \ldots \geq v_n$ the lower bound for an optimal bias α^* from Proposition 3.2 is also an upper bound for *all* biases $\alpha \in (0, \infty)^n$.

Proposition 3.3 *Let $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$; then applying any bias $\alpha = (\alpha_1, \ldots, \alpha_n)$ in the BAA framework yields total equilibrium revenue that satisfies*

$$\tilde{X}^{BAA} \leq \frac{v_1 + v_2}{2}.$$

Proof. Consider an arbitrary bias $\alpha \in (0, \infty)^n$, the corresponding transformed valuations $\tilde{v}_i = \alpha_i v_i$ and a permutation p such that the permuted valuations satisfy $\bar{v}_1 \geq \ldots \geq \bar{v}_n$. Then these transformed valuations have to belong to one of the following three cases, which we will now consider separately using results from the solution theory of Baye et al. (1993,1996) [2, 1] for the standard unbiased all-pay auction (here we also follow the terminology from [1] and say that player i *randomizes continuously* on a set $A \subseteq \mathbb{R}$ if he plays a mixed strategy such that the corresponding cumulative distribution function (cdf) is continuous on A and strictly increasing almost everywhere on A):

Case 1: $\bar{v}_1 > \bar{v}_2 > \bar{v}_3 \geq \ldots \geq \bar{v}_n$ Then Theorem 3 in Baye et al. (1996) implies that there is a unique Nash equilibrium $\bar{y}^*$, which satisfies $E[\bar{y}_i^*] = 0$ for all $i = 3, \ldots, n$. Lemma 1 in Baye et al. (1993) yields $E[\bar{y}_1^*] = \frac{\bar{v}_2}{2}$ and thus Theorem 1 in Baye et al. (1993) gives

$$E[\bar{y}_2^*] = \frac{\bar{v}_2^2}{\bar{v}_1} - \frac{\bar{v}_2}{\bar{v}_1} E[\bar{y}_1^*] = \frac{\bar{v}_2^2}{2\bar{v}_1} < \frac{\bar{v}_2}{2}.$$

Together, this implies

$$\tilde{X}^{BAA} = \sum_{i=1}^{n} \frac{E[\bar{y}_i^*]}{\bar{\alpha}_i} < \frac{\bar{v}_2}{2\bar{\alpha}_1} + \frac{\bar{v}_2}{2\bar{\alpha}_2} < \frac{\bar{v}_1}{2\bar{\alpha}_1} + \frac{\bar{v}_2}{2\bar{\alpha}_2} = \frac{v_{p^{-1}(1)} + v_{p^{-1}(2)}}{2} \leq \frac{v_1 + v_2}{2}.$$

Case 2: $\bar{v}_1 = \bar{v}_2 = \ldots = \bar{v}_m > \bar{v}_{m+1} \geq \ldots \geq \bar{v}_n$ with $2 \leq m \leq n$ We can assume without loss of generality that the permutation p is chosen such that $\bar{\alpha}_1 \leq \bar{\alpha}_2 \leq \ldots \leq \bar{\alpha}_m$. Theorem 1 in Baye et al. (1996) implies that in all possible Nash equilibria $\bar{y}^*$ the following holds: $E[\bar{y}_i^*] = 0$ for all $i = m+1, \ldots, n$, $\sum_{i=1}^{m} E[\bar{y}_i^*] = \bar{v}_1$ and there are at least two players in the set $\{1, \ldots, m\}$ randomizing continuously on $[0, \bar{v}_1]$, whereas all other players $i \in \{1, \ldots, m\}$ randomize continuously on an interval $(b_i, \bar{v}_1]$ with a $b_i \in [0, \bar{v}_1]$. Moreover, by the same result, whenever two or more players randomize continuously on a common interval, their cdfs are identical on that interval. Consequently, the (at least) two players randomizing continuously on the whole interval $[0, \bar{v}_1]$ exert the highest expected equilibrium effort $E[\bar{y}_i^*]$. Since there are at least two of them, this implies $E[\bar{y}_i^*] \leq \frac{\bar{v}_1}{2}$ for all $i = 1, \ldots, m$. Putting all of these pieces together, we obtain

$$\begin{aligned} \tilde{X}^{BAA} &= \sum_{i=1}^{n} \frac{E[\bar{y}_i^*]}{\bar{\alpha}_i} = \frac{E[\bar{y}_1^*]}{\bar{\alpha}_1} + \sum_{i=2}^{m} \frac{E[\bar{y}_i^*]}{\bar{\alpha}_i} \leq \frac{E[\bar{y}_1^*]}{\bar{\alpha}_1} + \sum_{i=2}^{m} \frac{E[\bar{y}_i^*]}{\bar{\alpha}_2} = \frac{E[\bar{y}_1^*]}{\bar{\alpha}_1} + \frac{\bar{v}_1 - E[\bar{y}_1^*]}{\bar{\alpha}_2} \\ &= \frac{\bar{v}_1}{\bar{\alpha}_2} + \left[\frac{1}{\bar{\alpha}_1} - \frac{1}{\bar{\alpha}_2}\right] E[\bar{y}_1^*] \leq \frac{\bar{v}_1}{2\bar{\alpha}_1} + \frac{\bar{v}_1}{2\bar{\alpha}_2} = \frac{\bar{v}_1}{2\bar{\alpha}_1} + \frac{\bar{v}_2}{2\bar{\alpha}_2} \leq \frac{v_1 + v_2}{2}. \end{aligned}$$

Case 3: $\bar{v}_1 > \bar{v}_2 = \ldots = \bar{v}_m > \bar{v}_{m+1} \geq \ldots \geq \bar{v}_n$ with $3 \leq m \leq n$ By Theorem 2 in Baye et al. (1996) we know that there is a continuum of Nash equilibria $\bar{y}^*$. All those satisfy $E[\bar{y}_i^*] = 0$ for all $i = m+1, \ldots, n$. For technical reasons, we have to distinguish two different cases depending on the bias α:

Case 3a: $\frac{\bar{v}_1}{\bar{\alpha}_1} \geq \frac{\bar{v}_2}{\bar{\alpha}_2}$ Theorem 2 in Baye et al. (1996) implies $\sum_{i=2}^m E[\bar{y}_i^*] = \frac{\bar{v}_2^2}{\bar{v}_1} - \frac{\bar{v}_2}{\bar{v}_1} E[\bar{y}_1^*]$ and states that, among the infinitely many Nash equilibria that occur in this case, the expression $E[\bar{y}_1^*]$ is maximal when only one other player $i \in \{2, \ldots, m\}$ is active. In this case, by the formulae following Theorem 2 in Baye et al. (1996), the cdf of player 1 is given by $G_1(y) = \frac{y}{\bar{v}_2}$ and thus $E[\bar{y}_1^*] = \int_0^{\bar{v}_2} y G_1'(y)\, dy = \frac{\bar{v}_2}{2}$. Hence, for an arbitrary Nash equilibrium, the corresponding expectation is at most $\frac{\bar{v}_2}{2}$. Assuming without loss of generality that the permutation p is chosen such that $\bar{\alpha}_2 \leq \ldots \leq \bar{\alpha}_m$ holds, we therefore obtain

$$\begin{aligned} \tilde{X}^{BAA} &= \sum_{i=1}^n \frac{E[\bar{y}_i^*]}{\bar{\alpha}_i} \leq \frac{E[\bar{y}_1^*]}{\bar{\alpha}_1} + \sum_{i=2}^m \frac{E[\bar{y}_i^*]}{\bar{\alpha}_2} = \frac{E[\bar{y}_1^*]}{\bar{\alpha}_1} + \frac{\frac{\bar{v}_2^2}{\bar{v}_1} - \frac{\bar{v}_2}{\bar{v}_1} E[\bar{y}_1^*]}{\bar{\alpha}_2} \\ &= \frac{\bar{v}_2^2}{\bar{\alpha}_2 \bar{v}_1} + \left[\frac{1}{\bar{\alpha}_1} - \frac{\bar{v}_2}{\bar{\alpha}_2 \bar{v}_1} \right] E[\bar{y}_1^*] \leq \frac{\bar{v}_2}{2\bar{\alpha}_1} + \frac{\bar{v}_2^2}{2\bar{\alpha}_2 \bar{v}_1} < \frac{\bar{v}_1}{2\bar{\alpha}_1} + \frac{\bar{v}_2}{2\bar{\alpha}_2} \leq \frac{v_1 + v_2}{2}. \end{aligned}$$

Note that the second inequality in this chain holds because the corresponding term in brackets is nonnegative in the case considered here.

Case 3b: $\frac{\bar{v}_1}{\bar{\alpha}_1} < \frac{\bar{v}_2}{\bar{\alpha}_2}$ This case, although intuitively being nonoptimal since it implies $v_{p^{-1}(1)} < v_{p^{-1}(2)}$, requires a surprisingly involved analysis. We consider an auxiliary bias $\hat{\alpha}$, which, under the *same* permutation p used in the definition of $\bar{\alpha}$ and $\bar{v}$, is of the form $\hat{\alpha} = \left(\frac{\bar{v}_2}{\bar{v}_1} \bar{\alpha}_1, \frac{\bar{v}_1}{\bar{v}_2} \bar{\alpha}_2, \bar{\alpha}_3, \ldots, \bar{\alpha}_n \right)$ and thus yields the transformed (and permuted) valuations

$$\hat{v} := (\hat{\alpha}_1 v_{p^{-1}(1)}, \ldots, \hat{\alpha}_n v_{p^{-1}(n)}) = (\bar{v}_2, \bar{v}_1, \bar{v}_3, \ldots, \bar{v}_n).$$

So essentially $\hat{v}$ is the same as $\bar{v}$ with only the first two components swapped. Note that $\hat{v}$ is not in descending order since $\bar{v}_2 < \bar{v}_1$. However, noting that we also have

$$\frac{\hat{v}_2}{\hat{\alpha}_2} = \frac{\bar{v}_1 \bar{v}_2}{\bar{v}_1 \bar{\alpha}_2} = \frac{\bar{v}_2}{\bar{\alpha}_2} > \frac{\bar{v}_1}{\bar{\alpha}_1} = \frac{\bar{v}_2 \bar{v}_1}{\bar{v}_2 \bar{\alpha}_1} = \frac{\hat{v}_1}{\hat{\alpha}_1},$$

we recognize that, after another permutation which switches the first two entries so that the components of $\hat{v}$ are in descending order, the bias $\hat{\alpha}$ belongs to Case 3a.

Hence, it suffices to show that any equilibrium effort $\tilde{X}^{BAA}$ generated by the bias $\bar{\alpha}$ is less or equal to an equilibrium effort $\hat{X}^{BAA}$ generated by the bias $\hat{\alpha}$, since this implies by Case 3a

$$\tilde{X}^{BAA} \leq \hat{X}^{BAA} < \frac{v_1 + v_2}{2}.$$

Now consider an arbitrary Nash equilibrium of the all-pay auction with valuations $\bar{v}$ and denote the corresponding expected equilibrium efforts by $E[\bar{y}_i^*]$ for $i = 1, \ldots, n$. Obviously, for each of these equilibria there exists a corresponding Nash equilibrium of the all-pay auction with valuations $\hat{v}$ such that the expected equilibrium efforts $E[\hat{y}_i^*]$ satisfy

$$E[\hat{y}_1^*] = E[\bar{y}_2^*], \quad E[\hat{y}_2^*] = E[\bar{y}_1^*] \quad \text{and} \quad E[\hat{y}_i^*] = E[\bar{y}_i^*] \quad \text{for all } i = 3, \ldots, n.$$

Hence, these two Nash equilibria satisfy

$$\tilde{X}^{BAA} - \hat{X}^{BAA} = \sum_{i=1}^{n} \frac{E[\bar{y}_i^*]}{\bar{\alpha}_i} - \sum_{i=1}^{n} \frac{E[\hat{y}_i^*]}{\hat{\alpha}_i} = \left[\frac{E[\bar{y}_1^*]}{\bar{v}_1} - \frac{E[\bar{y}_2^*]}{\bar{v}_2}\right]\left[\frac{\bar{v}_1}{\bar{\alpha}_1} - \frac{\bar{v}_2}{\bar{\alpha}_2}\right],$$

and thus

$$\tilde{X}^{BAA} - \hat{X}^{BAA} \leq 0 \quad \Longleftrightarrow \quad \frac{E[\bar{y}_1^*]}{\bar{v}_1} - \frac{E[\bar{y}_2^*]}{\bar{v}_2} \geq 0.$$

Therefore, it remains to verify the right inequality. By Theorem 2 in Baye et al. (1996), we know that player 1 is randomizing continuously on the interval $[0, \bar{v}_2]$ and that there is at least one player $i \in \{2, \ldots, m\}$ randomizing continuously on $(0, \bar{v}_2]$ and bidding zero with propability $G_i(0)$. All other players $j \in \{2, \ldots, m\}$ randomize continuously on an interval $(b_j, \bar{v}_2]$ with $b_j \in [0, \bar{v}_2]$ and bid zero with probability $G_j(b_j)$. Furtheremore, the same result guarantees that, whenever two or more players $i, j \in \{2, \ldots, m\}$ randomize continuously on the same interval, their cdfs G_i, G_j are identical on that interval. Since

$$E[\bar{y}_i^*] = 0 \cdot G_i(b_i) + \int_{b_i}^{\bar{v}_2} y G_i'(y)\, dy$$

for $i = 2, \ldots, m$, it follows from the previous observations that the player randomizing continuously on $(0, \bar{v}_2]$ has the highest expected equilibrium effort $E[\bar{y}_i^*]$ among the players $2, \ldots, m$. Hence, if $\frac{E[\bar{y}_1^*]}{\bar{v}_1} - \frac{E[\bar{y}_2^*]}{\bar{v}_2} \geq 0$ holds in the case $b_2 = 0$, it holds for all $b_2 \in [0, \bar{v}_2]$. Thus, we consider only the case $b_2 = 0$ and assume without loss of generality that the permutation p is chosen such that $0 = b_2 = \ldots = b_h < b_{h+1} \leq \ldots \leq b_m \leq \bar{v}_2$ with $2 \leq h \leq m$. Then, by the formulae following

Theorem 2 in Baye et al. (1996), the cdfs of player 1 and 2 are of the form

$$G_1(y) = \begin{cases} \frac{y}{\bar v_2}\left[\frac{\bar v_1-\bar v_2+y}{\bar v_1}\right]^{\frac{2-h}{h-1}}\left[\prod_{k=h+1}^m G_k(b_k)\right]^{\frac{-1}{h-1}} & \text{if } y\in[0,b_{h+1}),\\ \frac{y}{\bar v_2}\left[\frac{\bar v_1-\bar v_2+y}{\bar v_1}\right]^{\frac{2-j}{j-1}}\left[\prod_{k=j+1}^m G_k(b_k)\right]^{\frac{-1}{j-1}} & \text{if } y\in[b_j,b_{j+1}),\quad \text{for } j=h+1,\dots,m-1\\ \frac{y}{\bar v_2}\left[\frac{\bar v_1-\bar v_2+y}{\bar v_1}\right]^{\frac{2-m}{m-1}} & \text{if } y\in[b_m,\bar v_2],\end{cases}$$

$$G_2(y) = \begin{cases} \left[\frac{\bar v_1-\bar v_2+y}{\bar v_1}\right]^{\frac{1}{h-1}}\left[\prod_{k=h+1}^m G_k(b_k)\right]^{\frac{-1}{h-1}} & \text{if } y\in[0,b_{h+1}),\\ \left[\frac{\bar v_1-\bar v_2+y}{\bar v_1}\right]^{\frac{1}{j-1}}\left[\prod_{k=j+1}^m G_k(b_k)\right]^{\frac{-1}{j-1}} & \text{if } y\in[b_j,b_{j+1}),\quad \text{for } j=h+1,\dots,m-1\\ \left[\frac{\bar v_1-\bar v_2+y}{\bar v_1}\right]^{\frac{1}{m-1}} & \text{if } y\in[b_m,\bar v_2].\end{cases}$$

By the same formulae, we also get

$$G_j(b_j) = \begin{cases} \left[\frac{\bar v_1-\bar v_2+b_h}{\bar v_1}\right]^{\frac{1}{h-1}}\left[\prod_{k=h+1}^m G_k(b_k)\right]^{\frac{-1}{h-1}} & \text{if } j=h,\\ \left[\frac{\bar v_1-\bar v_2+b_j}{\bar v_1}\right]^{\frac{1}{j-1}}\left[\prod_{k=j+1}^m G_k(b_k)\right]^{\frac{-1}{j-1}} & \text{if } j=h+1,\dots,m-1,\\ \left[\frac{\bar v_1-\bar v_2+b_m}{\bar v_1}\right]^{\frac{1}{m-1}} & \text{if } j=m.\end{cases}$$

Exploiting all these formulae and the convention $b_{m+1} = \bar v_2$ and $\prod_{k=m+1}^m G_k(b_k) = 1$, we obtain using integration by parts and some elementary calculations

$$\begin{aligned}
\frac{E[\bar y_1^*]}{\bar v_1} - \frac{E[\bar y_2^*]}{\bar v_2} &= \frac{1}{\bar v_1}\int_0^{\bar v_2} yG_1'(y)\,dy - \frac{1}{\bar v_2}\left[0\cdot G_2(0) + \int_0^{\bar v_2} yG_2'(y)\,dy\right]\\
&= \frac{\bar v_2}{\bar v_1} - 1 - \int_0^{\bar v_2}\left[\frac{G_1(y)}{\bar v_1} - \frac{G_2(y)}{\bar v_2}\right]dy\\
&= \frac{\bar v_2}{\bar v_1} - 1 - \frac{1}{\bar v_2}\sum_{j=h}^m\left[\prod_{k=j+1}^m G_k(b_k)\right]^{\frac{-1}{j-1}}\int_{b_j}^{b_{j+1}}\left[\frac{\bar v_1-\bar v_2+y}{\bar v_1}\right]^{\frac{1}{j-1}}\left(\frac{y}{\bar v_1}\frac{\bar v_1}{\bar v_1-\bar v_2+y} - 1\right)dy\\
&= \frac{\bar v_2}{\bar v_1} - 1 + \frac{\bar v_1-\bar v_2}{\bar v_1\bar v_2}\sum_{j=h}^m\left[\prod_{k=j+1}^m G_k(b_k)\right]^{\frac{-1}{j-1}}\int_{b_j}^{b_{j+1}}\left[\frac{\bar v_1-\bar v_2+y}{\bar v_1}\right]^{\frac{2-j}{j-1}}dy\\
&= \frac{\bar v_2}{\bar v_1} - 1 + \frac{\bar v_1-\bar v_2}{\bar v_2}\sum_{j=h}^m(j-1)\left[\prod_{k=j+1}^m G_k(b_k)\right]^{\frac{-1}{j-1}}\left(\left[\frac{\bar v_1-\bar v_2+b_{j+1}}{\bar v_1}\right]^{\frac{1}{j-1}} - \left[\frac{\bar v_1-\bar v_2+b_j}{\bar v_1}\right]^{\frac{1}{j-1}}\right)\\
&= \frac{\bar v_2}{\bar v_1} - 1 + \frac{\bar v_1-\bar v_2}{\bar v_2}\sum_{j=h}^m(j-1)\big(G_{j+1}(b_{j+1}) - G_j(b_j)\big)\\
&= \frac{\bar v_2-\bar v_1}{\bar v_1} + \frac{\bar v_1-\bar v_2}{\bar v_2}\left((m-1) - \sum_{j=h+1}^m G_j(b_j) - (h-1)G_h(b_h)\right)
\end{aligned}$$

$$= \frac{\bar{v}_1 - \bar{v}_2}{\bar{v}_1 \bar{v}_2} \left[-\bar{v}_2 + \bar{v}_1 \left((m-1) - \sum_{j=2}^{m} G_j(b_j) \right) \right].$$

By Theorem 2 in Baye et al. (1996), we know

$$\prod_{j=2}^{m} G_j(b_j) = \frac{\bar{v}_1 - \bar{v}_2}{\bar{v}_1}. \tag{2}$$

A simple consideration reveals that this, together with $G_j(b_j) \in (0, 1]$ for all $j = 2, \ldots, m$, implies

$$\sum_{j=2}^{m} G_j(b_j) \le (m-2) \cdot 1 + 1 \cdot \prod_{j=2}^{m} G_j(b_j) = (m-2) + \frac{\bar{v}_1 - \bar{v}_2}{\bar{v}_1}$$

(essentially, the first inequality comes from the observation that the sum is maximized if $G_j(b_j) = 1$ for all but one index $j \in \{2, \ldots, m\}$, whereas the remaining player takes his value in such a way the product constraint (2) is satisfied). Plugging this into the previous result, we obtain the desired inequality

$$\frac{E[\bar{y}_1^*]}{\bar{v}_1} - \frac{E[\bar{y}_2^*]}{\bar{v}_2} \ge \frac{\bar{v}_1 - \bar{v}_2}{\bar{v}_1 \bar{v}_2} \left[-\bar{v}_2 + \bar{v}_1 \left[(m-1) - (m-2) - \frac{\bar{v}_1 - \bar{v}_2}{\bar{v}_1} \right] \right] = 0.$$

This completes the proof. □

The last two Propositions imply:

Theorem 3.4 *Let $v_1 \ge v_2 \ge v_3 \ge \ldots \ge v_n$; then applying an optimal bias $\alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$ in the BAA framework yields total equilibrium revenue that satisfies*

$$X^{*,BAA} = \frac{v_1 + v_2}{2}. \tag{3}$$

Moreover, the proof of Proposition 3.3 reveals the optimal biases:

Corollary 3.5 *Let $v_1 = v_2 = \ldots = v_m > v_{m+1} \ge v_3 \ge \ldots \ge v_n$ with $2 \le m \le n$. Then the optimal biases in the BAA framework are those $\alpha^* \in (0, \infty)^n$ satisfying the following conditions: There is a subset $I^* \subseteq \{1, \ldots, m\}$ such that $|I^*| \ge 2$, $\alpha_i^* v_i = \alpha_j^* v_j$ for all $i, j \in I^*$ and for all $i \in I^*$, $k \notin I^*$ it holds that $\alpha_i^* v_i > \alpha_k^* v_k$.*

Let $v_1 > v_2 = \ldots = v_m > v_{m+1} \ge v_3 \ge \ldots \ge v_n$ with $2 \le m \le n$. Then the optimal biases in the BAA framework are those $\alpha^* \in (0, \infty)^n$ satisfying the following conditions: There is a unique

$i^* \in \{2, \ldots, m\}$ such that $\alpha_1^* v_1 = \alpha_{i^*}^* v_{i^*}$ and for all $k \neq 1, i^*$ it holds that $\alpha_1^* v_1 > \alpha_k^* v_k$.

Proof. According to the proof of Proposition 3.3, optimal biases α^* are those, for which equality holds in Case 2. This implies that all active players have the same transformed valuation $\alpha_i^* v_i$. All active players $i \in \{3, \ldots, m\}$ have to satisfy $\bar{\alpha}_i = \bar{\alpha}_2$ or equivalently $v_{p^{-1}(i)} = v_{p^{-1}(2)}$. Additionally, it either has to hold that $E[\bar{y}_1^*] = \frac{\bar{v}_1}{2}$, which is equivalent to the fact that only two players are active, or $\bar{\alpha}_1 = \bar{\alpha}_2$ has to be true, which is equivalent to $v_{p^{-1}(1)} = v_{p^{-1}(2)}$. Finally, using the assumption $\bar{\alpha}_1 \leq \ldots \leq \bar{\alpha}_m$, we have $v_{p^{-1}(1)} = v_1$ and $v_{p^{-1}(2)} = v_2$.

Hence, in the case where $v_1 = v_2$, the optimal biases α^* are those ensuring that an arbitrary number of players with valuation v_1 is active in a Nash equilibrium and that all those active players have the same transformed valuation $\alpha_i^* v_i$.

In the case, where $v_1 > v_2$, the optimal biases α^* are those that guarantee that player 1 and only one player out of those with valuation v_2 are active in a Nash equilibrium and that both have the same transformed valuation $\alpha_1 v_1$. However, this implies that there can only be one other player $i^* \neq 1$ with transformed valuation $\alpha_1^* v_1$, since otherwise there exist Nash equilibria in which more than two players are active. □

We are now in a position to compare total revenue under the derived optimal bias with that under the unbiased all-pay auction where $\alpha_1 = \alpha_2 = 1$. For the unbiased all-pay auction it is well-known that in equilibrium the (expected) payoff to contestant 1 is $v_1 - v_2 \geq 0$, while the payoff to contestant 2 is 0 ; the contest organizer can expect $\mathrm{E}[y_1] + \mathrm{E}[y_2] = \frac{v_2}{2} + \frac{v_2^2}{2v_1}$. Hence, the contestants, namely contestant 1, lose the entire rent of $(v_1 - v_2)$, while the contest organizer by biasing the contest in the prescribed way generates additional revenue of $\frac{v_1+v_2}{2} - (\frac{v_2}{2} + \frac{v_2^2}{2v_1}) = \frac{(v_1+v_2)}{2v_1}(v_1 - v_2)$. Since $\frac{v_1+v_2}{2v_1} \leq 1$ the loss in contestants' payoff is larger than the gain in revenue for the organizer who applies the optimal bias.

4 Lotteries Versus All-Pay Auctions

The optimal bias for the asymmetric lottery contest has been derived in Franke et al. (2011), [6], under the condition that heterogeneity affects marginal costs to exert effort. However, a simple transformation leads to the framework as presented here, see [6], section 2. We repeat the result in its transformed version in the following proposition to maintain a consistent notation.

Proposition 4.1 *There exists an optimal bias α^{BLC} in the BLC framework that is not unique.*

However, any optimal bias α^{BLC} leads to:

$$\begin{aligned} X^{*,BLC} &= \frac{1}{4}\left[\sum_{j\in K^*} v_j - \frac{(k^*-2)^2}{\sum_{j\in K^*}\frac{1}{v_j}}\right], \textit{ where} \\ K^* &= \left\{ i \in N \;\middle|\; \frac{k^*-2}{v_i} < \sum_{j\in K^*}\frac{1}{v_j} \right\} \textit{ with } k^* := |K^*|. \end{aligned} \tag{4}$$

It is shown in [6], Lemma 4.8, that K^*, the set of active contestants, is well-defined and unique and can equivalently be written as

$$K^* = \left\{ i \in N \;\middle|\; \frac{i-2}{v_i} < \sum_{j=1}^{i}\frac{1}{v_j} \right\}. \tag{5}$$

Theorem 3.4 and Proposition 4.1 facilitates the revenue comparison between the two contest regimes under the respective optimal bias because closed form expressions for total revenue are provided in (3) and (4). However, for the optimally biased lottery contest the set of active agents is only indirectly defined in (5), which impedes a direct comparison. Hence, we need one more auxiliary result before we can state the main result of our paper. In the following lemma we consider biased lottery contests, where the agents' valuations are given according to $V = \{v_1, \ldots, v_n\}$, and denote by $X^{*,BLC}(V)$ the revenue of the lottery contest as defined in (4) and by $K(V)$ the set of active agents under the optimal bias according to (5). The lemma basically says that maximal revenue $X^{*,BLC}(V)$ increases with the valuations $V = (v_1, \ldots, v_n)$.

Lemma 4.2 *The function $X^{*,BLC}(V)$ is continuously differentiable on $(0,\infty)^n$ and the partial derivatives are given by*

$$\frac{\partial X^{*,BLC}(V)}{\partial v_i} = \begin{cases} 0 & \textit{if } i \notin K^*(V), \\ \frac{1}{4}\left[1 - \frac{(k(V)-2)^2}{\left(\sum_{j\in K(V)}\frac{1}{v_j}\right)^2}\frac{1}{v_i^2}\right], & \textit{if } i \in K^*(V). \end{cases}$$

In particular, for all $i \in N$ it holds that $\frac{\partial X^{*,BLC}(V)}{\partial v_i} \geq 0$, i.e. $X^{*,BLC}(V)$ is monotonically increasing in v_i for all $i \in N$.

Proof. Recall that $X^{*,BLC}(V)$ gives the maximal total effort of all active contestants in equilibrium after the contest organizer has chosen the optimal bias α^* given valuations $V = (v_1, \ldots, v_n)$. Hence, $\alpha^* = \alpha^*(V)$; in the same vein, $K^* = K^*(V)$ denotes the set of active contestants in the optimally biased contest given $V = (v_1, \ldots, v_n)$. Denote by $k^*(V)$ the cardinality of $K^*(V)$, $k^*(V) = |K^*(V)|$.

We now define the index set

$$L^*(V) = \left\{ i \in N \,\middle|\, \frac{k^*(V) - 2}{v_i} = \sum_{j \in K^*(V)} \frac{1}{v_j} \right\}$$

and set $l^*(V) = |L^*(V)|$. $L^*(V)$ contains those indices - if any - which belong to contestants who are indifferent between becoming active (with a bid of 0) and staying inactive (recall the definition of $K^*(V)$ from (5)).

So let $V = (v_1, \ldots, v_n)$ be given and consider any ε-neighborhood of V, $U_\varepsilon(V)$, for $\varepsilon > 0$ sufficiently small. It is then true that for any $V' \in U_\varepsilon(V)$

$$K^*(V) \subseteq K^*(V') \subseteq K^*(V) \cup L^*(V)$$

holds; i.e., for all valuations V' sufficiently close to V the set of active contestants in the optimally biased lottery contest for V' consists of all contestants active in the optimally biased lottery contest for V plus - possibly - contestants from $L^*(V)$, who have become active in V'. Intuitively, since the participation condition in (5) for a contestant i is given by an *in*equality, an active contestant in V, who satisfies the inequality, must stay active for sufficiently small changes in V as those cannot lead to a violation of the inequality. For the same reason, inactive contestants, who even violate the condition in $L^*(V)$, must stay inactive for sufficiently small changes in V. Formally, this is proven in [6], Theorem 3.2.

So let $M \subseteq L^*(V)$ and $m = |M|$. An alternative representation of $X^{*,BLC}(V)$ then reads:

$$\begin{aligned}
X^{*,BLC}(V) &= \frac{1}{4}\left[\sum_{j \in K^*(V)} v_j - \frac{(k^*(V)-2)^2}{\sum_{j \in K^*(V)} \frac{1}{v_j}}\right] \\
&= \frac{1}{4}\left[\sum_{j \in K^*(V)} v_j + \frac{m(k^*(V)-2)}{\sum_{j \in K^*(V)} \frac{1}{v_j}} - \frac{(k^*(V)+m-2)(k^*(V)-2)}{\sum_{j \in K^*(V)} \frac{1}{v_j}}\right] \\
&= \frac{1}{4}\left[\sum_{j \in K^*(V)} v_j + \sum_{j \in M} v_j - \frac{(k^*(V)+m-2)^2}{\sum_{j \in K^*(V)} \frac{1}{v_j} + \frac{m}{k^*(V)-2} \sum_{j \in K^*(V)} \frac{1}{v_j}}\right] \\
&= \frac{1}{4}\left[\sum_{j \in K^*(V) \cup M} v_j - \frac{(k^*(V)+m-2)^2}{\sum_{j \in K^*(V) \cup M} \frac{1}{v_j}}\right].
\end{aligned}$$

The second equality results from a trivial split of the last term, the third and fourth equalities result from using the definition of $L^*(V)$ (and hence M).

From the last expression of $X^{*,BLC}(V)$ we immediately see that $X^{*,BLC}(V)$ must be continuous at V: Any sequence $V^j \to V$ can be decomposed into - at most $2^{l^*(V)}$ - subsequences, such that each of these subsequences satisfies $K(V^j) = K^*(V) \cup M$ for all elements V^j of this subsequence with a fixed subset $M \subseteq L^*(V)$. Consequently $X^{*,BLC}(V^j)$ converges to $X^{*,BLC}(V)$.

In order to show continuous differentiability of $X^{*,BLC}(V)$ it suffices to show partial differentiability with respect to all $v_i,\ i = 1, \ldots, n$, and continuity of all the partial derivatives. For all $i \notin K^*(V) \cup L^*(V)$ we obviously have $\frac{\partial}{\partial v_i} X^{*,BLC}(V) = 0$ since $X^{*,BLC}(V)$ does not depend on v_i in a whole neighborhood. So let $i \in K^*(V) \cup L^*(V)$ and consider an arbitrary sequence $v_i^j \to v_i$. If we define $V^j = (v_1, \ldots, v_{i-1}, v_i^j, v_{i+1}, \ldots, v_n)$, then obviously $V^j \to V$. Again, consider any subsequence of V^j such that $K(V^j) = K^*(V) \cup M$ for a fixed $M \subseteq L^*(V)$ and consequently $\left|K(V^j)\right| = k^*(V) + m$ for all j in this subsequence. We then have on this subsequence:

$$\lim_{j\to\infty} \frac{X^{*,BLC}(V) - X^{*,BLC}(V^j)}{v_i - v_i^j} = \begin{cases} 0, & \text{for } i \in L^*(V) \setminus M, \\ \frac{1}{4}\left[1 - \frac{(k^*(V)+m-2)^2}{\left(\sum_{l \in K^*(V) \cup M} \frac{1}{v_l}\right)^2} \frac{1}{v_i^2}\right], & \text{for } i \in K^*(V) \cup M \end{cases}$$

$$= \begin{cases} 0, & \text{for } i \in L^*(V) \setminus M, \\ \frac{1}{4}\left[1 - \frac{(k^*(V)-2)^2}{\left(\sum_{l \in K^*(V)} \frac{1}{v_l}\right)^2} \frac{1}{v_i^2}\right], & \text{for } i \in K^*(V) \cup M \end{cases}$$

$$= \begin{cases} 0, & \text{for } i \in L^*(V) = M \cup L^*(V) \setminus M, \\ \frac{1}{4}\left[1 - \frac{(k^*(V)-2)^2}{\left(\sum_{l \in K^*(V)} \frac{1}{v_l}\right)^2} \frac{1}{v_i^2}\right], & \text{for } i \in K^*(V). \end{cases}$$

Here we have again made use of the definition of $L^*(V)$, which contains M.

Obviously, the above limit exists and is independent of the sequence V^j, or the respective subsequences. Hence, $X^{*,BLC}(V)$ is partially differentiable with respect to all $v_i,\ i = 1, \ldots, n$. Continuity of the partial derivatives $\frac{\partial}{\partial v_i} X^{*,BLC}(V)$ derived above can now be shown in the same way as we have shown continuity of $X^{*,BLC}(V)$. The nonnegativity of the partial derivatives is again a direct consequence of the definition of the set $K^*(V)$. $\square$

Now we can finally state the main result of this section. We shall prove that for any $V = (v_1, \ldots, v_n)$ the optimally biased BLC regime yields less total effort than $\frac{v_1+v_2}{2}$, which was shown to be the maximal revenue under the optimally biased BAA regime.

Theorem 4.3 *The optimal BAA regime induces higher total effort in equilibrium than the optimal BLC regime, i.e.* $X^{*,BAA} > X^{*,BLC}$.

Proof. We prove the theorem by first considering a special case, where the theorem can be verified by a simple calculation, and then reducing all other cases to the first one.

First we consider the case where all agents – possibly except the first one – have the same valuation, that is $v_1 \geq v_2 = v_3 = \ldots = v_n$. If we determine the set K^* of active agents in regime *BLC* based on (5), we immediately see that all agents are active in this case. Applying Theorem 3.4 the crucial inequality $X^{*,BAA} > X^{*,BLC}$ is thus satisfied if

$$\frac{v_1 + v_2}{2} > \frac{1}{4}\left[v_1 + (n-1)v_2 - \frac{(n-2)^2}{\frac{1}{v_1} + \frac{n-1}{v_2}}\right].$$

After some algebra this inequality can be simplified to:

$$(n-1)v_1^2 + 2v_1v_2 > (n-3)v_2^2$$

The first expression on the left hand side is larger than the expression on the right hand side which implies that the inequality holds.

Now consider arbitrary valuations $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$ and denote the corresponding optimal revenues by $X^{*,BAA}$ and $X^{*,BLC}$. To prove the assertion in this case, we construct auxiliary valuations by first substituting v_3 with v_2 and then compare the resulting revenue $X^{*,BLC}(v_1, v_2, v_2, v_4, \ldots, v_n)$ with $X^{*,BLC} = X^{*,BLC}(v_1, v_2, v_3, \ldots, v_n)$. Afterwards, we continue this procedure with $v_4, \ldots, v_n$. Lemma 4.2 and the previously considered special case imply

$$\begin{aligned}
X^{*,BLC} &= X^{*,BLC}(v_1, v_2, v_3, \ldots, v_n) \leq X^{*,BLC}(v_1, v_2, v_2, v_4, \ldots, v_n) \\
&\leq X^{*,BLC}(v_1, v_2, v_2, v_2, v_5 \ldots, v_n) \leq \ldots \leq X^{*,BLC}(v_1, v_2, v_2, v_2, \ldots, v_2) \\
&< \frac{v_1 + v_2}{2} = X^{*,BAA},
\end{aligned}$$

which completes the proof of this theorem. □

5 Concluding Remarks

We have shown that in the presence of heterogeneous contestants an optimally biased all-pay auction always revenue-dominates the optimally biased lottery contest. We have done so by providing the solution to a contest organizer's problem of optimally biasing all-pay auctions (Theorem 3.4) and comparing this solution to the known solution (Franke et al. (2011)) of finding an optimally biased lottery contest. Our revenue dominance result is in contrast to the comparison of the unbiased versions of these contest models if there are more than two contestants. The (unbiased) all-pay auction might yield less revenue (total effort), if in particular the exclusion principle applies; i.e., heterogeneity is such that it is revenue-enhancing to exclude the strongest player from participation. The two active but weaker contestants then may expend less effort than all the active players in the lottery contest. In contrast, we show that if the contest organizer has the ability to bias the contest, the exclusion principle of the all-pay auction becomes obsolete. No player is excluded. The contest organizer can always bias the all-pay auction in such a way that the two strongest players will be active and, moreover, compete on equal terms (the strongest player is therefore not excluded but sufficiently weakened in her effectiveness). All other players choose to be inactive. In short, the two strongest contestants are exposed on equal terms to the extremely discriminative all-pay auction. They are made to compete as two identical players with valuations $\frac{v_1+v_2}{2}$ would do, and this results in maximal (expected) revenue of $\frac{v_1+v_2}{2}$ for the contest organizer. Reducing the discriminativeness of the contest by using a lottery CSF will attract more entry into the contest; i.e., more contestants (at least three) will be active in equilibrium. But having more active contestants in the less discriminative contest does not pay off for the contest organizer: The increase in competitiveness due to a higher number of competitors cannot offset the loss of competitiveness due to a 'softer' contest. Economic policy instruments aimed at facilitating entry do not work for contests. If revenue maximization is the goal of the contest organizer then participation effects are not strong enough to outweigh incentive effects.

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