Safeguarded Augmented Lagrangian Methods in Banach Spaces

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Abstract. This paper presents a state-of-the-art survey for safeguarded augmented Lagrangian methods for constrained optimization problems in Banach spaces. The difference between the classical augmented Lagrangian method and its safeguarded version lies in the update of the multiplier estimates. The safeguarded method has significantly stronger global convergence properties than the classical algorithm. Local and rate-of-convergence results are also summarized. Some numerical results illustrate the practical behaviour of the safeguarded augmented Lagrangian approach.

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1. Introduction

This paper is dedicated to a thorough discussion of the augmented Lagrangian method (ALM) for constrained minimization problems of the form

$$(P) \qquad \underset{x\in C}{\text{minimize}}\, f(x) \quad \text{subject to} \quad G(x) \in K, \tag{1.1}$$

where X, Y are real Banach spaces, $f : X \to \mathbb{R}$ and $G : X \to Y$ are continuously differentiable functions, and $C \subseteq X$ as well as $K \subseteq Y$ are nonempty closed convex sets. The feasible set of (P) will be denoted by

$$\Phi := \{x \in C : \; G(x) \in K\}.$$

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To facilitate the application of the augmented Lagrangian technique, we assume that $i : Y \hookrightarrow H$ densely for some real Hilbert space H. This implies that we are working in the *Gel'fand triple* framework

$$Y \overset{i}{\hookrightarrow} H \cong H^* \overset{i^*}{\hookrightarrow} Y^*. \tag{1.2}$$

Furthermore, we assume that there is a closed convex set $\mathcal{K} \subseteq H$ such that $i^{-1}(\mathcal{K}) = K$. This allows us to interpret the constraint $G(x) \in K$ equivalently as $G(x) \in \mathcal{K}$. Note that we will usually suppress the embedding for the sake of brevity.

It should be stressed that the above framework is extremely general, and the resulting augmented Lagrangian method therefore covers a very broad spectrum of applications. Moreover, many prominent problem classes can be recovered as special cases of (P). Here, we apply the safeguarded augmented Lagrangian method in order to solve (P).

Historically, the augmented Lagrangian technique was first developed for nonlinear programs (in finite dimension). Indeed, the algorithm goes back to the seminal works by Hestenes [33] and Powell [65], and in its early days it was commonly referred to as the *method of multipliers*. The technique was further developed by many authors in the later parts of the 20th century, including Rockafellar [68–70], Bertsekas [9], and Conn, Gould, and Toint [21–23], who created the well-known LANCELOT software package. The algorithm was rediscovered by Andreani, Birgin, Martínez, and co-authors in [1, 2, 11, 12], a series of publications which culminated in the book [13] and the corresponding ALGENCAN software package.

In today's nonlinear programming landscape, algorithms such as interior point methods [29, 31] or sequential quadratic programming [31, 45] are often preferred to methods of augmented Lagrangian type, mainly due to their fast local convergence characteristics. In contrast, the augmented Lagrangian method possesses very strong global convergence properties, and it has been found to work rather well on degenerate problem classes such as problems with complementarity constraints [46]. A state-of-the-art local convergence analysis of the ALM for nonlinear programming is given in [27]. More discussion on nonlinear programming in general, and on the corresponding algorithms, can be found in [9, 10, 24, 62], and in the encyclopedia [28].

One of the main motivations for the generalization of augmented Lagrangian methods to the level of generality represented by (P) is the advent of function space optimization problems. Some early references in this context include [6, 7, 39–42, 76], and the book [30]. Most of these publications are restricted to very specific problem settings such as convex optimization problems or finite-dimensional constraints. In [8, 43], an augmented Lagrangian-type penalty scheme was proposed, in combination with a semismooth Newton method, for the solution of state-constrained optimal control problems. The resulting method came to be known as *Moreau–Yosida regularization*; it was further developed in [34, 35], and it is today considered a standard approach for state-constrained optimal control [37, 44, 75]. Some other techniques for such problems include Lavrentiev regularization [36, 59], interior point methods [56, 72], and the so-called virtual control approach [55], which is related to the augmented Lagrangian technique [54].

The purpose of this paper is to collect the recent developments and to summarize the convergence theory of the safeguarded method applied to Banach space optimization problems in a uniform framework. To this end, we first recall some background material and state some preliminary results in Section 2. We then provide a self-contained motivation of the augmented Lagrangian method in Section 3. A state-of-the-art summary of the global and local convergence properties of the augmented ALM is then provided in Sections 4 and 5, respectively. Numerical results for a variety of applications are given in Section 6. We then close the paper with some final remarks in Section 7.

2. Background Material

This chapter summarizes several concepts and results from optimization theory, Banach spaces and variational analysis which will be used later in our subsequent convergence theory. Most results are known, so we refer to the existing literature; occasionally, we provide a proof if either this proof is very short or we were not able to find an explicit reference.

2.1. Cones

This section is dedicated to the study of some basic objects which are useful when characterizing the geometric structure of sets in Banach spaces. Many aspects of the geometry of sets can be characterized through so-called cones (see below), and these play a major role in variational analysis, convex analysis, and optimization theory. The material discussed here incorporates elements from multiple books, e.g., [5, 14, 16].

Let $S \subseteq X$ be a nonempty set. We say that S is a *cone* if $\alpha S \subseteq S$ for all $\alpha > 0$. Given an arbitrary set $S \subseteq X$, we denote by

$$S^\circ := \{\phi \in X^* : \langle \phi, s \rangle \leq 0 \text{ for every } s \in S\}$$

the *polar cone* of S. Note that $S^\circ \subseteq X^*$. If X is a real Hilbert space, we treat S° as a subset of X.

Definition 2.1 (Tangent and normal cones). Let $C \subseteq X$ be an arbitrary set and $x \in C$. Then we define

(a) the *tangent cone* $\mathcal{T}_C(x)$ as

$$\mathcal{T}_C(x) := \big\{d \in X : \exists \{x^k\} \subseteq C,\ t_k \downarrow 0 \text{ such that } x^k \to x \text{ and } (x^k - x)/t_k \to d\big\}.$$

(b) the *normal cone* $\mathcal{N}_C(x)$ as

$$\mathcal{N}_C(x) := \{\phi \in X^* : \langle \phi, y - x \rangle \leq 0 \ \forall y \in C\}.$$

For $x \notin C$, both cones are defined as the empty set.

If X is a real Hilbert space, we treat $\mathcal{N}_C(x)$ as a subset of X instead of X^*. The normal cone is always a closed set, and satisfies the polarity relation

$$\mathcal{N}_C(x) = \mathcal{T}_C(x)^\circ$$

which is sometimes also taken as the definition of the normal cone and makes sense also for possibly nonconvex sets C. It should be noted, however, that there are a variety of different normal cones for general sets (see, for instance, [60]). Therefore, to avoid any ambiguity, we will reserve the symbol $\mathcal{N}_C$ for the case where C is convex.

The normal cone can be used to formulate a simple Fermat-type optimality condition.

Theorem 2.2 (Necessary optimality condition, [16]). *Let $f : X \to \mathbb{R}$ be a continuously differentiable mapping, and $C \subseteq X$ a nonempty closed convex set. If $\bar{x}$ is a local minimizer of f on C, then $0 \in f'(\bar{x}) + \mathcal{N}_C(\bar{x})$.*

The following is a famous decomposition theorem involving a closed convex cone in a Hilbert space and its polar.

Lemma 2.3 (Moreau decomposition, [61]). *Let H be a real Hilbert space and $K \subseteq H$ a nonempty closed convex cone. Then every $y \in H$ admits a unique decomposition $y = y_1 + y_2$ with $K \ni y_1 \perp y_2 \in K^\circ$. Indeed, $y_1 = P_K(y)$ and $y_2 = P_{K^\circ}(y)$.*

We now turn to another object which describes some aspects of the geometric structure of convex sets.

Definition 2.4 (Recession cone). Let $C \subseteq X$ be a nonempty convex set. Then the *recession cone* of C is the set $C_\infty := \{x \in X : x + C \subseteq C\}$.

The recession cone is always nonempty (since $0 \in C_\infty$) and a convex cone. Moreover, if C is closed, then so is C_∞. If the set C is a convex *cone*, then it is easy to see that $C_\infty = C$. On the other hand, if C is not a cone, then the recession cone can often be used as a substitute for C in situations where a conical structure is necessary. This is the case, for instance, in the context of (partial) order relations, which closely correspond to convex cones, see Section 2.2.

The following result provides some information on the polar cone $C_\infty^\circ := (C_\infty)^\circ$.

Lemma 2.5. *Let H be a real Hilbert space and $C \subseteq H$ a nonempty convex set. Then $\{y \in H : \sup_{w \in C}(w, y) < +\infty\} \subseteq C_\infty^\circ$. In particular, $\mathcal{N}_C(y) \subseteq C_\infty^\circ$ for all $y \in C$.*

Proof. Let $y \in H$ be a point with $(w, y) \le c$ for some $c \in \mathbb{R}$ and all $w \in C$. Let $x \in C_\infty$, and choose an arbitrary $x_0 \in C$. Then $x_0 + tx \in C$ for all $t > 0$, and hence $(x_0 + tx, y) \le c$. This cannot hold for all $t > 0$ if $(x, y) > 0$. Hence, $(x, y) \le 0$, and $y \in C_\infty^\circ$. □

The set $\{y \in H : \sup_{w \in C}(w, y) < +\infty\}$ in the statement of Lemma 2.5 is often called the *barrier cone* of C. Note that the inclusion stated in the lemma can be strict. In particular, there are situations where the barrier cone is not closed, and this makes it a priori impossible for it to equal C_∞°, which is always a closed cone by virtue of polarity. An example for this phenomenon can be found in [5, Exercise 6.23].

2.2. Convex Functions and Concave Operators

Convex functions play a central role in optimization theory. Occasionally, we write $\partial f(x)$ for the subdifferential of a convex function f in x, but most of the time the underlying mapping f will be differentiable. One of the most fundamental examples of a convex function is the distance function $d_C : X \to \mathbb{R}$ to a convex set $C \subseteq X$. Note that the following result holds for an arbitrary Banach space X, not necessarily a Hilbert space.

Lemma 2.6 (Distance function, [5,64]). *Let $C \subseteq X$ be a nonempty convex set. Then the function $d_C : X \to \mathbb{R}$, $d_C(x) := \inf_{y \in C} \|x - y\|_X$, is convex and nonexpansive.*

It is easy to see that the square of a nonnegative convex function is again convex. Thus, in the setting of Lemma 2.6, the squared distance function d_C^2 is also a convex function. If the space X is a real Hilbert space, then the squared distance function enjoys a much stronger form of regularity.

Lemma 2.7 ([5, Cor. 12.31]). *Let X be a real Hilbert space and $C \subseteq X$ a nonempty closed convex set. Then the squared distance function d_C^2 is convex and continuously differentiable on X with $(d_C^2)'(x) = 2(x - P_C(x))$ for all $x \in X$.*

Recall that there exist several different continuity notions in infinite-dimensional spaces, based on the topology used within theses space or whether a (weak) sequential continuity or (weak) lower semicontinuity is considered. The following well-known result states that several continuity properties coincide within the class of convex functions.

Proposition 2.8 ([5, Thm. 9.1]). *Let $C \subseteq X$ be a closed convex set and $f : C \to \mathbb{R}$ a convex function. Then the following are equivalent: (i) f is lower semicontinuous, (ii) f is weakly lower semicontinuous, and (iii) f is weakly sequentially lower semicontinuous.*

The theory of convex functions is useful for a wide variety of application problems. There are, however, certain practical scenarios where convexity properties of nonlinear *operators* $G : X \to Y$ are necessary, with X and Y real Banach spaces. More specifically, assume that we are dealing with an inclusion of the form

$$G(x) \in K, \quad K \subseteq Y \text{ a closed convex set.} \tag{2.1}$$

Ideally, we would like to work with a generalized notion of convexity which takes into account the mapping G and the geometry of the set K. To this end, assume for the moment that the set K in (2.1) is a closed convex *cone*. Then K induces the order relation

$$a \leq_K b \;:\Longleftrightarrow\; b - a \in K, \tag{2.2}$$

and K itself can be regarded as the nonnegative cone with respect to $\leq_K$. Thus, (2.1) can be rewritten as $G(x) \geq_K 0$, which suggests that the appropriate convexity notion in this case is a generalized type of concavity with respect to the order relation $\leq_K$. This property takes on the form

$$G\big((1-t)x+ty\big) \geq_K (1-t)G(x)+tG(y) \quad \text{for all } x,y\in X,\ t\in[0,1].$$

The above property is usually called K-*concavity*, and it is in fact a special case of the general concept which we define below. In the case where K is not a cone, the recession cone K_∞ turns out to be a useful substitute to define the order relation (2.2).

Definition 2.9 (Concave operator). Let $G: X\to Y$ be an arbitrary mapping and $K\subseteq Y$ a closed convex set with recession cone K_∞. We say that G is K_∞-*concave* if

$$G\big((1-t)x+ty\big) \geq_K (1-t)G(x)+tG(y) \quad \text{for all } x,y\in X,\ t\in[0,1],$$

where $\leq_K$ is the order relation defined by $a\leq_K b :\Longleftrightarrow b-a\in K_\infty$.

Let us now discuss the analytical consequences of generalized convexity (or concavity) in the sense of Definition 2.9. The resulting properties can be deduced by discussing situations in which the K_∞-concavity of G yields the (ordinary) convexity of a suitable composite mapping involving G.

We say that a mapping $m: Y\to\mathbb{R}$ is K_∞-*decreasing* if it is monotonically decreasing with respect to the order $\leq_K$, i.e., if $m(y_1)\leq m(y_2)$ whenever $y_1\geq_K y_2$.

Theorem 2.10. *Let X,Y be real Banach spaces, $K\subseteq Y$ a nonempty closed convex set, and $G: X\to Y$ a K_∞-concave operator. Then:*

(a) If $m: Y\to\mathbb{R}$ is convex and K_∞-decreasing, then $m\circ G$ is convex.
(b) The function $d_K\circ G: X\to\mathbb{R}$ is convex.
(c) If $\lambda\in K_\infty^\circ$, then $x\mapsto\langle\lambda,G(x)\rangle$ is convex.
(d) The set $M:=\{x\in X: G(x)\in K\}$ is convex.

Proof. Can be found in [49, Lemma 2.1]. □

2.3. Pseudomonotone Operators

We first recall the following notion of *pseudomonotonicity* in the sense of Brezis [17].

Definition 2.11 (Pseudomonotonicity). We say that an operator $F: X\to X^*$ is *pseudomonotone* if, whenever

$$\{x^k\}\subseteq X, \quad x^k\rightharpoonup x, \quad \text{and} \quad \limsup_{k\to\infty}\big\langle F(x^k),x^k-x\big\rangle\leq 0,$$

then

$$\langle F(x),x-y\rangle \leq \liminf_{k\to\infty}\big\langle F(x^k),x^k-y\big\rangle \quad \text{for all } y\in X.$$

Despite its somewhat peculiar appearance, the notion of pseudomonotonicity will play a fundamental role in the subsequent theory. Some sufficient conditions for pseudomonotone operators are summarized in the following lemma. This result illustrates that the class of pseudomonotone operators is quite large.

Lemma 2.12 (Sufficient conditions for pseudomonotonicity). *Let X be a real Banach space and $T, U : X \to X^*$ given operators. Then:*

(a) If T is monotone and continuous, then T is pseudomonotone.
(b) If, for every $y \in X$, the mapping $x \mapsto \langle T(x), x - y\rangle$ is weakly sequentially lsc, then T is pseudomonotone.
(c) If T is completely continuous, then T is pseudomonotone.
(d) If T is continuous and $\dim(X) < +\infty$, then T is pseudomonotone.
(e) If T and U are pseudomonotone, then $T + U$ is pseudomonotone.

Proof. (b) is obvious. The remaining assertions can be found in [77, Prop. 27.6]. □

It follows from the above observations that the concept of pseudomonotone operators provides a unified approach to different classes of operators, including monotone and completely continuous ones. Property (b) in the above lemma is occasionally referred to as *(Ky–)Fan-hemicontinuity.*

2.4. KKT-type Conditions

We define the *Lagrange function* or *Lagrangian* of (P) as the mapping

$$\mathcal{L} : X \times Y^* \to \mathbb{R}, \quad \mathcal{L}(x, \lambda) := f(x) + \langle \lambda, G(x)\rangle. \tag{2.3}$$

and denote by $\mathcal{L}'$ the derivative of the Lagrangian with respect to x alone. Note that we do not include the abstract constraint C into the Lagrangian. The Lagrangian can be used to formulate the KKT system of (P) in the following way.

Definition 2.13 (KKT point). A point $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ is a *KKT point* of (P) if

$$-\mathcal{L}'(\bar{x}, \bar{\lambda}) \in \mathcal{N}_C(\bar{x}) \quad \text{and} \quad \bar{\lambda} \in \mathcal{N}_K(G(\bar{x})).$$

We say that $\bar{x} \in X$ is a *stationary point* of (P) if $(\bar{x}, \bar{\lambda})$ is a KKT point for some multiplier $\bar{\lambda} \in Y^*$, and denote by $\Lambda(\bar{x})$ the set of such multipliers.

For the KKT conditions to be necessary optimality conditions of (P), certain *constraint qualifications* are required; they ensure that the feasible set is well-behaved and that, roughly speaking, the reconstruction of its geometry from first-order information is possible. One of the most fundamental constraint qualifications in infinite dimensions is the following one.

Definition 2.14 (Robinson constraint qualification). Let $x \in X$ be a feasible point for (P). We say that the *Robinson constraint qualification (RCQ)* holds in x if

$$0 \in \operatorname{int}\big[G(x) + G'(x)(C - x) - K\big].$$

The above condition was introduced by Robinson in [67] in the context of certain stability properties of nonlinear inclusions. In the context of finite-dimensional nonlinear programs, RCQ turns out to be equivalent to the well-known Mangasarian-Fromovitz constraint qualification. Under RCQ, the following first-order optimality condition holds.

Theorem 2.15 (KKT conditions under RCQ, [14, Thm. 3.9]). *Let $\bar{x}$ be a local minimizer of (P) and assume that RCQ holds in $\bar{x}$. Then the set of Lagrange multipliers $\Lambda(\bar{x})$ is nonempty, closed, convex, and bounded in Y^*.*

In order to verify feasibility of (weak) limit points in our global convergence analysis, we will also need the following straightforward generalization of RCQ to possibly infeasible points. To keep a clear distinction, we call the resulting condition the *extended* Robinson constraint qualification, though its definition is essentially the same as for RCQ itself.

Definition 2.16 (Extended Robinson constraint qualification). Let $x \in X$ be an arbitrary, not necessarily feasible point. We say that the *extended Robinson constraint qualification (extended RCQ, ERCQ)* holds in x if

$$0 \in \operatorname{int}\big[G(x) + G'(x)(C - x) - K\big].$$

An important property of ERCQ is that it guarantees that, whenever x is a stationary point of a certain measure of infeasibility, then x is actually a feasible point. We formulate this result in a slightly more general framework. The proof can be found in [15, Lemma 5.2].

Proposition 2.17. *Let $i : Y \hookrightarrow H$ densely for some real Hilbert space H, and let $\mathcal{K} \subseteq H$ be a closed convex set with $i^{-1}(\mathcal{K}) = K$. Let $\bar{x} \in X$ be a stationary point of the problem $\min_{x \in C} d_{\mathcal{K}}^2(G(x))$, and assume that ERCQ holds in $\bar{x}$ with respect to the constraint system of (P). Then $G(\bar{x}) \in K$.*

Assume now that we have a point $\hat{x}$ which is "almost" a solution of (P). A popular definition in this context is that of *ε-minimizers*: given $\varepsilon > 0$, we say that $\hat{x} \in \Phi$ is an ε-minimizer of (P) if $f(\hat{x}) \le f(x) + \varepsilon$ for all $x \in \Phi$. For such approximate minimizers, it is indeed possible to obtain an inexact analogue of the KKT conditions. This result is usually called Ekeland's variational principle.

Proposition 2.18 (Ekeland's variational principle, [14, Thm. 3.23]). *Let $\bar{x} \in \Phi$ be an ε-minimizer of (P), let $\delta := \varepsilon^{1/2}$, and assume that RCQ holds at every $x \in B_\delta(\bar{x}) \cap \Phi$. Then there exist another ε-minimizer $\hat{x}$ of (P) and $\lambda \in Y^*$ such that $\|\hat{x} - \bar{x}\|_X \le \delta$,*

$$\operatorname{dist}\big(-\mathcal{L}'(\hat{x}, \lambda), \mathcal{N}_C(\hat{x})\big) \le \delta, \quad \textit{and} \quad \lambda \in \mathcal{N}_K(G(\hat{x})).$$

Many practical algorithms for constrained optimization iteratively construct a primal-dual sequence $\{(x^k, \lambda^k)\}$ which satisfies the KKT conditions in an asymptotic sense. This motivates to analyze such "sequential" analogues of the KKT conditions in more detail. The subsequent notion is also used by similar approaches in finite dimensions, see [3, 4, 13].

Definition 2.19 (Asymptotic KKT sequence). We say that a sequence $\{(x^k, \lambda^k)\} \subseteq C \times Y^*$ is an *asymptotic KKT sequence* for (P) if there exist null sequences $\{\varepsilon^k\} \subseteq X^*$ and $\{r_k\} \subseteq \mathbb{R}$ such that, for all k,

$$\varepsilon^k - \mathcal{L}'(x^k, \lambda^k) \in \mathcal{N}_C(x^k) \quad \text{and} \quad \big\langle \lambda^k, y - G(x^k) \big\rangle \le r_k \ \forall y \in K. \tag{2.4}$$

Our main aim in this section is to give sufficient conditions which guarantee that, if $\{(x^k, \lambda^k)\}$ is an asymptotic KKT sequence and $\bar{x}$ is a (possibly weak) limit point of $\{x^k\}$, then $\bar{x}$ is a stationary point of (P). In this context, it is worth mentioning that Definition 2.19 imposes no conditions on the attainment of feasibility. This aspect is left unspecified for the sake of flexibility; indeed, we will mainly be concerned with scenarios where $\bar{x}$ is some kind of limit point of $\{x^k\}$ and we already know from a preliminary analysis that $\bar{x}$ is a feasible point.

Note that, while the conditions posed in Definition 2.19 seem reasonably weak, it is possible to generalize the asymptotic KKT concept even further. In particular, in our formulation, the second inequality in (2.4) is assumed to hold *uniformly* on K. If K is unbounded, then it may be more natural to require some kind of uniformness of the inequality on bounded subsets of K. In any case, however, the augmented Lagrangian method which we will discuss later satisfies the uniform bound from (2.4), and a more general analysis is therefore not necessary for our purposes.

3. Motivation and Statement of the Algorithm

This section first recalls the original method of multipliers for equality constraints. It then presents a self-contained and simple approach for its generalization to abstract inequality constraints (in a Banach space setting). Finally, we give a formal statement of the overall method for a general problem of the form (P) and prove some preliminary properties of this method.

3.1. The Original Method of Multipliers

In its initial form, the method of multipliers is an algorithm for the solution of equality-constrained minimization problems in finite dimensions. Here, we present this original method in a slightly more general framework. Consider an equality-constrained optimization problem of the form

$$\underset{x \in C}{\text{minimize}}\, f(x) \quad \text{subject to} \quad h(x) = 0, \tag{3.1}$$

where $f : X \to \mathbb{R}$, $C \subseteq X$ is a closed convex set, and $h : X \to H$. We assume that X is a real Banach space and H is a real Hilbert space. In the special case of the original method of multipliers, we have $X := \mathbb{R}^n$, $H := \mathbb{R}^m$ with $m, n \in \mathbb{N}$, and $C := X$.

The basic idea is to tackle (3.1) by combining elements of Lagrangian theory with a penalty-type scheme. Recall that the Lagrangian of the problem takes on the form $\mathcal{L}(x, \lambda) = f(x) + \big(\lambda, h(x)\big)$. By adding a positive multiple of $\|h(x)\|_H^2$, we penalize the violation of the equality constraint, thus ending up with the *augmented Lagrangian*

$$\mathcal{L}_\rho(x, \lambda) := f(x) + \big(\lambda, h(x)\big) + \frac{\rho}{2}\|h(x)\|_H^2. \tag{3.2}$$

From an algorithmic perspective, we now proceed as follows. Given a penalty parameter ρ_k and a current estimate λ^k of the Lagrange multiplier, we compute

x^{k+1} as a minimizer (or approximate minimizer) of (3.2) on C so that, ideally, x^{k+1} is close to feasibility (if ρ_k is large) and close to being a minimizer of the Lagrangian $\mathcal{L}(\cdot, \lambda^k)$. Let us assume, for the moment, that the functions f and h are continuously differentiable, and that x^{k+1} is an exact minimizer of $\mathcal{L}_{\rho_k}(\cdot, \lambda^k)$ on C. Then the standard first-order optimality conditions yield the inclusion

$$\mathcal{N}_C(x^{k+1}) \ni -\mathcal{L}'_{\rho_k}(x^{k+1}, \lambda^k) = -f'(x^{k+1}) - h'(x^{k+1})^*(\lambda^k + \rho_k h(x^{k+1})).$$

This immediately suggests $\lambda^{k+1} := \lambda^k + \rho_k h(x^{k+1})$ as the new estimate of the Lagrange multiplier, which is often called the *Hestenes–Powell multiplier update*.

After the above procedure is completed, the penalty parameter is updated based on a heuristic test. The most common option is to keep ρ_k if the constraint violation has decreased sufficiently, and to increase it otherwise. We thus end up with the following overall algorithm.

Algorithm 3.1 (Original method of multipliers). Let $(x^0, \lambda^0) \in X \times H$, $\rho_0 > 0$, let $\gamma > 1$, $\tau \in (0, 1)$, and set $k := 0$.

Step 1. If (x^k, λ^k) satisfies a suitable termination criterion: STOP.

Step 2. Compute an approximate solution x^{k+1} of the problem

$$\underset{x \in C}{\text{minimize}}\, \mathcal{L}_{\rho_k}(x, \lambda^k). \tag{3.3}$$

Step 3. Update the vector of multipliers to $\lambda^{k+1} := \lambda^k + \rho_k h(x^{k+1})$.

Step 4. If $\|h(x^{k+1})\|_H \le \tau \|h(x^k)\|_H$ holds, set $\rho_{k+1} := \rho_k$. Otherwise, set $\rho_{k+1} := \gamma \rho_k$.

Step 5. Set $k \leftarrow k + 1$ and go to Step 1.

3.2. Inequality Constraints and Slack Variables

Having established the classical multiplier method for equality-constrained problems, we now outline how the algorithm can be extended to the inequality-constrained case. To this end, we consider an optimization problem of the form (P), that is,

$$\underset{x \in C}{\text{minimize}}\, f(x) \quad \text{subject to} \quad G(x) \in K,$$

where, as before, $f : X \to \mathbb{R}$ and $G : X \to Y$ are given mappings, and $C \subseteq X$ and $K \subseteq Y$ nonempty closed convex sets. Moreover, H is a real Hilbert space with $i : Y \hookrightarrow H$ densely, and $\mathcal{K} \subseteq H$ is a closed convex set with $i^{-1}(\mathcal{K}) = K$. In this setting, we can restate (P) as the problem

$$(P_H) \qquad \underset{x \in C}{\text{minimize}}\, f(x) \quad \text{subject to} \quad G(x) \in \mathcal{K}. \tag{3.4}$$

We can transform this problem into an equality-constrained problem by adding an artificial variable $s \in \mathcal{K}$, also called a *slack variable*. This results in the equality-constrained problem

$$\underset{(x,s) \in C \times \mathcal{K}}{\text{minimize}}\, f(x) \quad \text{subject to} \quad G(x) - s = 0.$$

In the context of the equality-constrained framework (3.1) from the previous section, this essentially amounts to defining the mapping $h : X \times H \to H$, $h(x, s) := G(x) - s$.

The new problem is now an equality-constrained optimization problem on the space $X \times H$, and its augmented Lagrangian in the sense of (3.2) is given by

$$\mathcal{L}_\rho^s(x,s,\lambda) = f(x) + \big(\lambda, h(x,s)\big) + \frac{\rho}{2}\|h(x,s)\|_H^2. \tag{3.5}$$

In order to transform the augmented Lagrangian into a form where s is eliminated, observe that we can rewrite $\mathcal{L}_\rho^s$ as

$$\mathcal{L}_\rho^s(x,s,\lambda) = f(x) + \frac{\rho}{2}\left\|G(x) + \frac{\lambda}{\rho} - s\right\|_H^2 - \frac{\|\lambda\|_H^2}{2\rho}. \tag{3.6}$$

Taking into account the constraint $s \in \mathcal{K}$, we can now minimize this formula with respect to s for each fixed $x \in X$. Since s occurs only in the middle term, the result involves, by definition, the squared distance function $d_\mathcal{K}^2$.

Definition 3.2 (Augmented Lagrange function). For $\rho > 0$, the *augmented Lagrange function* or *augmented Lagrangian* of (P) is the function

$$\mathcal{L}_\rho : X \times H \to \mathbb{R}, \quad \mathcal{L}_\rho(x,\lambda) := f(x) + \frac{\rho}{2} d_\mathcal{K}^2\left(G(x) + \frac{\lambda}{\rho}\right) - \frac{\|\lambda\|_H^2}{2\rho}. \tag{3.7}$$

Before discussing some other observations and consequences of the slack variable approach, we first give some general properties of the augmented Lagrangian.

Proposition 3.3. *Let $\mathcal{L}_\rho : X \times H \to \mathbb{R}$ be the augmented Lagrangian* (3.7)*. Then:*

(a) $\mathcal{L}_\rho$ is concave and continuously differentiable with respect to λ.
(b) If f is convex and G is $\mathcal{K}_\infty$-concave, then $\mathcal{L}_\rho$ is convex with respect to x.
(c) If f and G are continuously differentiable, then $\mathcal{L}_\rho$ is so with respect to x.
(d) If $x \in X$ is a feasible point, then $\mathcal{L}_\rho(x,\lambda) \le f(x)$ for all $x \in X$ and $\lambda \in H$.

Proof. (a): The concavity follows from the fact that $\mathcal{L}_\rho(x,\cdot)$ is an infimum of affine functions by (3.5), and the continuous differentiability follows from that of $d_\mathcal{K}^2$.

(b): This is a consequence of Theorem 2.10.

(c): This follows again from the continuous differentiability of $d_\mathcal{K}^2$.

(d): If $G(x) \in \mathcal{K}$, then $d_\mathcal{K}(G(x) + \lambda/\rho) \le \|\lambda\|_H/\rho$ by the nonexpansiveness of the distance function. Hence, $\mathcal{L}_\rho(x,\lambda) \le f(x) + (\rho/2)\|\lambda\|_H^2/\rho^2 - \|\lambda\|_H^2/(2\rho) = f(x)$. □

Let us close this section by mentioning some byproducts of the slack variable approach. For fixed λ and ρ, the minimizing value of s in (3.6) is given by $\bar{s}(x) := P_\mathcal{K}(G(x) + \lambda/\rho)$. It follows that

$$h(x, \bar{s}(x)) = G(x) - P_\mathcal{K}\left(G(x) + \frac{\lambda}{\rho}\right). \tag{3.8}$$

Recall that, in the original method of multipliers (Algorithm 3.1), the norm of the equality constraint was used to determine whether the penalty parameter ρ_k should be increased after a given iteration. The above calculations suggest that (3.8) should be used to control ρ_k in the general case.

Another byproduct of the slack variable technique is a natural candidate for the Lagrange multiplier update. Assume that $\lambda^k \in H$ is a given estimate of the Lagrange multiplier of (P_H), that $\rho_k > 0$, and x^{k+1} is the next primal iterate (typically, some kind of minimizer of $\mathcal{L}_{\rho_k}(\cdot, \lambda^k)$). Taking into account the update rule in Algorithm 3.1, the next dual iterate is given by

$$\lambda^{k+1} = \lambda^k + \rho_k h(x^{k+1}, \bar{s}(x^{k+1})) = \rho_k \left[G(x^{k+1}) + \frac{\lambda^k}{\rho_k} - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{\lambda^k}{\rho_k} \right) \right].$$

This formula will play a fundamental role in the subsequent algorithms. Note that the above updating scheme can also be motivated (in the differentiable case) by looking at the stationarity condition of $\mathcal{L}_{\rho_k}(\cdot, \lambda^k)$, evaluated in x^{k+1}.

3.3. The Algorithm

This section presents the main algorithmic framework for the remainder of this paper. It is based on the method of multipliers from Section 3.1 and the slack variable transformation from Section 3.2, but it differs from the original multiplier method in one key aspect: the use of a safeguarded multiplier sequence. This will be the main tool to obtain much sharper (global) convergence assertions than those which are possible for the traditional algorithm.

Recall that we are dealing with a problem of the form (P), that we are working in the Gel'fand triple framework (1.2), and that $\mathcal{K} \subseteq H$ is a nonempty closed convex set with $i^{-1}(\mathcal{K}) = K$. The algorithm now proceeds by augmenting the constraint $G(x) \in \mathcal{K}$ in the space H. This means that, in a sense, we are not really attempting to solve (P) but the transformed problem (P_H). Nevertheless, we will see that many convergence properties of the augmented Lagrangian method can be stated accurately in terms of (P) (using, for instance, constraint qualifications for that problem).

For the precise specification of the method below, we will need a means of controlling the penalty parameter ρ. Motivated by (3.8), it is natural to use the function

$$V(x, \lambda, \rho) = \left\| G(x) - P_{\mathcal{K}}\left(G(x) + \frac{\lambda}{\rho} \right) \right\|_H, \tag{3.9}$$

which can be seen as a composite measure of feasibility and complementarity at the current iterates. Using this function, the augmented Lagrangian method can be given as follows.

Algorithm 3.4 (ALM for constrained optimization). Let $(x^0, \lambda^0) \in X \times H$, $\rho_0 > 0$, let $B \subseteq H$ be a nonempty bounded set, $\gamma > 1$, $\tau \in (0, 1)$, and set $k := 0$.

Step 1. If (x^k, λ^k) satisfies a suitable termination criterion: STOP.

Step 2. Choose $w^k \in B$ and compute an approximate solution x^{k+1} of the problem

$$\underset{x \in C}{\operatorname{minimize}}\ \mathcal{L}_{\rho_k}(x, w^k). \tag{3.10}$$

Step 3. Update the vector of multipliers to

$$\lambda^{k+1} := \rho_k \left[G(x^{k+1}) + \frac{w^k}{\rho_k} - P_{\mathcal{K}}\left(G(x^{k+1}) + \frac{w^k}{\rho_k} \right) \right]. \tag{3.11}$$

Step 4. Let $V_{k+1} := V(x^{k+1}, w^k, \rho_k)$ and set

$$\rho_{k+1} := \begin{cases} \rho_k, & \text{if } k = 0 \text{ or } V_{k+1} \le \tau V_k, \\ \gamma \rho_k, & \text{otherwise.} \end{cases} \tag{3.12}$$

Step 5. Set $k \leftarrow k + 1$ and go to Step 1.

Some remarks are in order. First among them is the fact that we have not specified what constitutes an "approximate solution" in Step 2. There are multiple options in this regard. For instance, we could require that x^{k+1} is an (approximate) global minimizer of $\mathcal{L}_{\rho_k}(\cdot, w^k)$. This is probably the simplest assumption from a theoretical point of view, but it is effectively restricted to problems where some form of convexity is present. On the other hand, we could also require that x^{k+1} is some kind of approximate stationary point of (3.10). This is more realistic in the nonconvex case, but it is also more intricate to deal with in theoretical terms. We will analyze both these approaches individually in the subsequent sections.

In practical terms, the augmented subproblems are typically solved by applying an appropriate generalized Newton method. The necessity for such methods stems from the fact that the augmented Lagrangian is once but in general not twice continuously differentiable with respect to x.

The second remark pertains to the sequence $\{w^k\}$, which will occasionally be referred to as the *safeguarded (Lagrange) multiplier sequence*. The presence of w^k can be seen as the distinctive feature of the algorithm, and it separates the method from traditional augmented Lagrangian schemes. Indeed, in Algorithm 3.4, we use w^k in certain places where conventional algorithms simply use λ^k. The main motivation is that w^k is always a bounded sequence (it is specifically required to be so), and this is the main ingredient to obtain sharper global convergence results. As a consequence, the above algorithm has strictly stronger convergence properties than its traditional counterpart. An actual example demonstrating this fact is somewhat involved and given in [47], see also the discussion at the end of Section 4. Note that, despite the boundedness of $\{w^k\}$, the sequence $\{\lambda^k\}$ in Algorithm 3.4 can still be unbounded. The actual choice of w^k allows for a certain degree of freedom. For instance, we could always choose $w^k := 0$, thus obtaining an algorithm which is essentially a quadratic penalty method. In practice, it is usually advantageous to keep w^k as close as possible to λ^k, for instance, by choosing the set B as a simple but large bounded set, and taking

$$w^k := P_B(\lambda^k)$$

for all k. This choice has the advantage that, if the sequence $\{\lambda^k\}$ is indeed bounded and the set B is large enough, then we can expect to have $w^k = \lambda^k$ for all k. On the other hand, if $\{\lambda^k\}$ is unbounded, then the safeguarding scheme will prevent w^k from escaping to infinity.

Finally, let us remark that the penalty updating scheme in (3.12) makes a distinction between the cases $k = 0$ and $k \geq 1$. This is because the value V_0 is formally undefined since we do not have w^{-1} and ρ_{-1}. In practice, it is often beneficial to treat this initial step differently, for instance, by simply setting $w^{-1} := w^0$, $\rho_{-1} := \rho_0$, and performing the penalty update in the same way as for $k \geq 1$. In any case, the treatment of this initial step has no impact on the convergence theory. The nature of the multiplier update allows us to state two assertions which hold completely independently of x^{k+1}, cf. [50].

Lemma 3.5. *We have $\lambda^k \in \mathcal{K}_\infty^\circ$ for all k. Moreover, there is a null sequence $\{r_k\} \subseteq \mathbb{R}_+$ such that $\big(\lambda^k, y - G(x^k)\big) \leq r_k$ for all $y \in \mathcal{K}$ and $k \in \mathbb{N}$.*

Remark 3.6 (Cone constraints). If the set $\mathcal{K}$ is a closed convex *cone*, then the multiplier update (3.11) in Algorithm simplifies to $\lambda^{k+1} = P_{\mathcal{K}^\circ}(w^k + \rho_k G(x^{k+1}))$. This follows immediately from the Moreau decomposition, cf. Lemma 2.3.

Remark 3.7 (Dual interpretation). The dual update of the classical augmented Lagrangian is known to be equivalent to the proximal-point iteration applied to the dual optimization problem. A similar interpretation is possible for the safeguarded augmented Lagrangian where the dual update can be seen as a shifted Tikhonov regularization method, see [48] for more details.

Remark 3.8 (Nonlinear programs). Consider the nonlinear program

$$\min f(x) \quad \text{s.t.} \quad h(x) = 0, \; g(x) \leq 0$$

with continuously differentiable functions $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ for all components $i = 1, \dots, m$ and $j = 1, \dots, p$. This nonlinear program can be viewed as a special case of our general framework (P) by taking (e.g.)

$$X = \mathbb{R}^n, C = \mathbb{R}^n, G := \begin{pmatrix} h \\ g \end{pmatrix}, K := \{0\}^p \times (-\infty, 0]^m.$$

In this case, writing $\lambda =: (\mu, \eta)$ for the multipliers of the equality and inequality constraints, the squared distance function is given by

$$d_{\mathcal{K}}^2\left(G(x) + \frac{\eta}{\rho}\right) = \sum_{j=1}^{p} \left(h_j(x) + \frac{\mu_j}{\rho}\right)^2 + \sum_{i=1}^{m} \max{}^2\left\{0, g_i(x) + \frac{\eta_i}{\rho}\right\}.$$

Plugging this into the definition of the augmented Lagrangian, an elementary calculation shows that this function simplifies to

$$\mathcal{L}_\rho(x, \mu, \eta) = f(x) + \frac{\rho}{2}\|h(x)\|_2^2 + \mu^T h(x) + \frac{1}{2\rho} \sum_{i=1}^{m} \Big[\max{}^2\big\{0, \eta_i + \rho g_i(x)\big\} - \eta_i^2\Big],$$

which is the usual augmented Lagrangian for nonlinear programs with equality and inequality constraints.

Remark 3.9 (Simplified augmented Lagrangian). In each iteration, Algorithm 3.4 minimizes the augmented Lagrangian with respect to x, for fixed w^k. Since this minimization procedure does not depend on the last term of our augmented Lagrangian, we would obtain the same sequence using the simplified Lagrangian

$$f(x) + \frac{\rho}{2} d_{\mathcal{K}}^2\left(G(x) + \frac{\lambda}{\rho}\right).$$

In fact, this is precisely the augmented Lagrangian used in [52]. On the other hand, this simplification changes the dual point of view completely and also gives a different function for finite-dimensional nonlinear programs, cf. Remarks 3.7 and 3.8.

Remark 3.10 (Moreau-Yosida regularization). Algorithm 3.4 allows to take $\{w^k\}$ as the null sequence. This choice corresponds to the classical quadratic penalty approach and is better known under the name *Moreau-Yosida regularization* in the current context, cf. [34, 35]. The multiplier update in the Moreau-Yosida regularization usually allows a shift. In any case, the subsequent convergence theory also covers this (shifted) Moreau-Yosida regularization.

4. Global Convergence Theory

In this section, we present the global convergence characteristics of Algorithm 3.4. To this end, we first establish a result regarding the existence of solutions of the penalized subproblems in Section 4.1. The next two sections consider the convergence to global minimizers and stationary points, respectively, depending on the degree by which we solve the penalized subproblems. The results are taken from the recent paper [15] and can be viewed as improvements from those presented in [52] where suitable feasibility and stationarity results were shown for strong limit points.

4.1. Existence of Penalized Solutions

In most situations, the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is bounded from below on C. This is satisfied, in particular, if f itself is already bounded from below on C, or if, roughly speaking, the penalty parameter is sufficiently large to make $\mathcal{L}_\rho$ coercive on the infeasible set. In any case, if $\mathcal{L}_\rho(\cdot, w)$ is bounded from below on C, then the augmented subproblems necessarily admit approximate minimizers. In the following, $\hat{x} \in C$ is called an ε-minimizer of a function $L : X \to \mathbb{R}$ on C if $L(\hat{x}) \le L(x) + \varepsilon$ for all $x \in C$.

Proposition 4.1. *Let $w \in H$, $\rho > 0$, and assume that the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is bounded from below on C. Then the following assertions hold:*

- *(a) For any $\varepsilon > 0$, there is an ε-minimizer $x_\varepsilon \in C$ of $\mathcal{L}_\rho(\cdot, w)$ on C.*
- *(b) If the functions f and G are continuously differentiable, then we can choose x_ε so that it additionally satisfies $\operatorname{dist}(-\mathcal{L}_\rho'(x_\varepsilon, w), \mathcal{N}_C(x_\varepsilon)) \le \varepsilon^{1/2}$.*

Proof. The first assertion follows from the lower boundedness assumption. The second property is a consequence of Ekeland's variational principle. □

We now discuss the existence of exact minimizers. The main proof technique is the direct method of the calculus of variations. For this, we need an appropriate kind of lower semicontinuity of the augmented Lagrangian. The following lemma provides two sufficient conditions for this property.

Lemma 4.2. *Assume that f is weakly sequentially lsc and G is either (i) continuous and $\mathcal{K}_\infty$-concave, or (ii) weakly sequentially continuous. Then, for each $\rho > 0$ and $w \in H$, the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is weakly sequentially lsc on X.*

Proof. Let $w \in H$ and $\rho > 0$. It suffices to verify the weak sequential lower semicontinuity of the function $h(x) := d_{\mathcal{K}}^2(G(x) + w/\rho)$. Observe that $d_{\mathcal{K}}$ is weakly sequentially lsc by Proposition 2.8. Hence, under (ii), we immediately obtain the same for h.

Consider now (i). In that case, the function h is convex (by Theorem 2.10) and continuous, thus again weakly sequentially lsc by Proposition 2.8. □

The weak sequential lower semicontinuity of the augmented Lagrangian yields the existence of penalized solutions if we assume either the weak compactness of the set C or an appropriate growth condition. We say that a function $J : X \to \mathbb{R}$ is *coercive* if $J(x^k) \to +\infty$ whenever $\{x^k\} \subseteq X$ and $\|x^k\|_X \to +\infty$.

Corollary 4.3. *Let $w \in H$, $\rho > 0$, and let one of the conditions in Lemma 4.2 be satisfied. If either (i) C is weakly compact, or (ii) X is reflexive and $\mathcal{L}_\rho(\cdot, w)$ is coercive, then the problem $\min_{x \in C} \mathcal{L}_\rho(x, w)$ admits a global minimizer.*

Clearly, a sufficient condition for the coercivity of the augmented Lagrangian is that of the objective function f. Even if this property does not hold, then it is common for $\mathcal{L}_\rho(\cdot, w)$ to be coercive if, roughly speaking, the objective function is coercive on the feasible set Φ and not too badly behaved outside of it. In that case, the penalty term in (3.7) yields the coercivity of $\mathcal{L}_\rho(\cdot, w)$ on the complement of Φ.

4.2. Convergence to Global Minimizers

In this section, we analyze the convergence properties of Algorithm 3.4 under the assumption that we can solve the subproblems in an (essentially) global sense. This is of course a rather restrictive requirement and can, in general, only be expected under certain convexity assumptions. However, the resulting theory is still appealing due to its simplicity. Indeed, the results below merely require some rather mild form of continuity (no differentiability), and can easily be extended to the case where the function f is extended-valued, i.e., it is allowed to take on the value $+\infty$.

Assumption 4.4 (Global minimization). We assume that f and $d_{\mathcal{K}} \circ G$ are weakly sequentially lsc on C and that $x^k \in C$ for all k. Moreover, for every $x \in C$, there is a null sequence $\{\varepsilon_k\} \subseteq \mathbb{R}$ such that $\mathcal{L}_{\rho_k}(x^{k+1}, w^k) \le \mathcal{L}_{\rho_k}(x, w^k) + \varepsilon_{k+1}$ for all k.

Recall that, for convex functions, weak sequential lower semicontinuity is implied by ordinary continuity. Thus, if f is a continuous convex function, then f is weakly sequentially lsc.

A similar comment applies to the weak sequential lower semicontinuity of the function $d_{\mathcal{K}} \circ G$. Indeed, there are two rather general situations in which this condition is satisfied: if G is weakly sequentially continuous, then $d_{\mathcal{K}} \circ G$ is weakly sequentially lsc since $d_{\mathcal{K}}$ is so by Proposition 2.8. On the other hand, if G is continuous and $\mathcal{K}_\infty$-concave in the sense of Definition 2.9, then $d_{\mathcal{K}} \circ G$ is a continuous convex function (by Theorem 2.10) and thus again weakly sequentially lsc. Let us also remark that, if G is continuous and affine, then both the above cases apply.

Finally, another salient feature of Assumption 4.4 is the dependence of the sequence $\{\varepsilon_k\}$ on the comparison point $x \in C$. The motivation behind this is that, if (P) is a smooth convex problem and the point x^{k+1} is "nearly stationary" in the sense that $\operatorname{dist}(-\mathcal{L}'_{\rho_k}(x^{k+1}, w^k), \mathcal{N}_C(x^{k+1})) \le \delta$ for some (small) $\delta > 0$, then, by convexity, we obtain an estimate of the form

$$\begin{aligned} \mathcal{L}_{\rho_k}(x, w^k) &\ge \mathcal{L}_{\rho_k}(x^{k+1}, w^k) + \mathcal{L}'_{\rho_k}(x^{k+1}, w^k)(x - x^{k+1}) \\ &\ge \mathcal{L}_{\rho_k}(x^{k+1}, w^k) - \delta \|x^{k+1} - x\|_X. \end{aligned}$$

This suggests that we should allow the sequence $\{\varepsilon_k\}$ in Assumption 4.4 to depend on the point x. In any case, the stated assumption is satisfied automatically if x^{k+1} is a global ε_{k+1}-minimizer of $\mathcal{L}_{\rho_k}(\cdot, w^k)$ for some null sequence $\{\varepsilon_k\}$.

We now turn to the convergence analysis of Algorithm 3.4 under Assumption 4.4. The theory is divided into separate analyses of feasibility and optimality. Since the augmented Lagrangian method is, at its heart, a penalty-type algorithm, the attainment of feasibility is particularly important for the success of the algorithm. A closer look at the definition of the augmented Lagrangian suggests that, if ρ is large, then the minimization of $\mathcal{L}_\rho$ essentially reduces to that of the infeasibility measure $d_{\mathcal{K}}^2(G(x))$. Hence, we can expect (weak) limit points of the sequence $\{x^k\}$ to be minimizers of this auxiliary function, which means that, roughly speaking, these points are "as feasible as possible". A precise statement of this assertion can be found in the following lemma.

Lemma 4.5. *Let $\{x^k\}$ be generated by Algorithm 3.4, let Assumption 4.4 hold, and let $\bar{x}$ be a weak limit point of $\{x^k\}$. Then $\bar{x}$ is a global minimizer of the function $d_{\mathcal{K}} \circ G$ on C. In particular, if the feasible set of (P) is nonempty, then $\bar{x}$ is feasible.*

Let us now turn to the optimality part.

Theorem 4.6. *Let $\{x^k\}$ be generated by Algorithm 3.4, let Assumption 4.4 hold, and assume that the feasible set of (P) is nonempty. Then $\limsup_{k\to\infty} f(x^{k+1}) \le f(x)$ for every $x \in \Phi$. Moreover, every weak limit point of $\{x^k\}$ is a global solution of (P).*

If the problem is convex with strongly convex objective, then it is possible to considerably strengthen the results of the previous theorem. Recall that, in this case, the weak sequential lower semicontinuity of f from Assumption 4.4 is implied by (ordinary) continuity. Recall also that a sufficient condition for the convexity of the feasible set Φ is the $\mathcal{K}_\infty$-concavity of G. Moreover, if G is $\mathcal{K}_\infty$-concave, then the distance function $d_{\mathcal{K}} \circ G$ is convex, and thus the weak sequential lower semicontinuity from Assumption 4.4 is implied by (ordinary) continuity of G.

Corollary 4.7. *Let $\{x^k\}$ be generated by Algorithm 3.4 and let Assumption 4.4 hold. Assume that X is reflexive, f is strongly convex on C, and the feasible set of (P) is nonempty and convex. Then $\{x^k\}$ converges strongly to the unique solution of (P).*

4.3. Stationarity of Limit Points

The theory on global minimization in the preceding section is certainly appealing from a theoretical point of view. However, the practical relevance of the corresponding results is essentially limited to problems where some form of convexity is present. It therefore seems natural to conduct a dedicated analysis for the augmented Lagrangian method which, instead of global minimization, takes into account suitable stationary concepts.

The present section is dedicated to precisely this approach. To that end, we assume that the functions defining the optimization problem are continuously differentiable and that we are able to compute local minimizers or stationary points of the subproblems (3.10) which occur in the algorithm. Recall that the first-order optimality conditions of these problems are given by

$$-\mathcal{L}'_{\rho_k}(x, w^k) \in \mathcal{N}_C(x).$$

Similarly to the previous section, we will allow for certain inexactness terms. A natural way of doing this is by considering the inexact first-order optimality condition

$$\varepsilon^{k+1} - \mathcal{L}'_{\rho_k}(x, w^k) \in \mathcal{N}_C(x),$$

where $\varepsilon^{k+1} \in X^*$ is an error term. For $k \to \infty$, the degree of inexactness should vanish in the sense that $\varepsilon^k \to 0$. Hence, we arrive at the following assumption.

Assumption 4.8 (Convergence to KKT points). We assume that

(i) f and G are continuously differentiable on X,
(ii) the derivative f' is bounded and pseudomonotone,
(iii) G and G' are completely continuous on C, and
(iv) $x^{k+1} \in C$ and $\varepsilon^{k+1} - \mathcal{L}'_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, where $\varepsilon^k \to 0$.

Recall that $\mathcal{L}_{\rho_k}$ is continuously differentiable by Proposition 3.3. The derivative $\mathcal{L}'_{\rho_k}$ (with respect to x) is given by

$$\mathcal{L}'_{\rho_k}(x, w^k) = f'(x) + \rho_k G'(x)^* \left[G(x) + \frac{w^k}{\rho_k} - P_{\mathcal{K}}\left(G(x) + \frac{w^k}{\rho_k} \right) \right]. \tag{4.1}$$

In particular, it holds that $\mathcal{L}'_{\rho_k}(x^{k+1}, w^k) = \mathcal{L}'(x^{k+1}, \lambda^{k+1})$.

As in the previous section, we treat the questions of feasibility and optimality in a separate manner. For the feasibility part, we relate the augmented Lagrangian to the infeasibility measure $d_{\mathcal{K}}^2 \circ G$.

Lemma 4.9. *Let $\{x^k\}$ be generated by Algorithm 3.4 under Assumption 4.8, and let $\bar{x}$ be a weak limit point of $\{x^k\}$. Then $\bar{x}$ is a stationary point of the problem* $\min_{x \in C} d_{\mathcal{K}}^2(G(x))$.

The above lemma indicates that weak limit points of the sequence $\{x^k\}$ have a strong tendency to be feasible points. Apart from the heuristic appeal of the result, there are several nontrivial cases where Lemma 4.9 automatically implies the feasibility of the limit point $\bar{x}$. Here, two cases in particular deserve a special mention: first, let us assume that the mapping G is $\mathcal{K}_\infty$-concave in the sense of Definition 2.9 (for instance, G could be affine). In this case, the function $d_{\mathcal{K}}^2 \circ G$ is convex by Theorem 2.10, and it follows that $\bar{x}$ is a global minimizer of this function. Hence, if the feasible set Φ is nonempty, then $\bar{x} \in \Phi$. The second interesting case arises if the point $\bar{x}$ satisfies the extended Robinson constraint qualification from Definition 2.16. In this case, the feasibility of $\bar{x}$ follows from Proposition 2.17.

We now analyze the optimality properties of limit points. The main result in this direction is the following.

Theorem 4.10. *Let $\{(x^k, \lambda^k)\}$ be generated by Algorithm 3.4 under Assumption 4.8, let $x^{k+1} \rightharpoonup_I \bar{x}$ for some index set $I \subseteq \mathbb{N}$, and let $\bar{x}$ satisfy ERCQ with respect to the constraint system of (P). Then $\bar{x}$ is a stationary point of (P), the sequence $\{\lambda^{k+1}\}_{k \in I}$ is bounded in Y^*, and each of its weak-* limit points belongs to $\Lambda(\bar{x})$.*

Observe that the sequence $\{\lambda^k\}$ is only bounded in Y^* and not necessarily in H. If the extended RCQ holds with respect to the transformed constraint $G(x) \in \mathcal{K}$ (instead of the original condition $G(x) \in K$), then the result remains true with Y^* replaced by H. However, this assumption is too restrictive for many applications, in particular those where (P) is regular (in the constraint qualification sense) with respect to the original space Y, but not with respect to the larger space H.

Remark 4.11. If we know from the specific problem structure or from some other convergence result (e.g., Corollary 4.7) that the sequence $\{x^k\}$ or one of its subsequences is strongly convergent, then we can dispense with the pseudomonotonicity and complete continuity assumptions. In this case, the assertions of Lemma 4.9 and Theorem 4.10 remain true under Assumption 4.8 (i) and (iv) only.

We now return to the general case and provide two additional results which can be useful to obtain convergence in certain special cases. First, let us consider the case of convex constraints. The resulting theorem requires neither the complete continuity of G or G' nor any constraint qualification.

Proposition 4.12. *Let $\{x^k\}$ be generated by Algorithm 3.4, let Assumption 4.8* (i), (ii), (iv) *hold, let G be $\mathcal{K}_\infty$-concave on C, and assume that Φ is nonempty. Then every weak limit point $\bar{x}$ of $\{x^k\}$ satisfies $\bar{x} \in \Phi$ and $f'(\bar{x})d \geq 0$ for all $d \in \mathcal{T}_\Phi(\bar{x})$.*

Proof. Let $x^{k+1} \rightharpoonup_I \bar{x}$ for some subset $I \subseteq \mathbb{N}$. The feasibility of $\bar{x}$ follows from Lemma 4.9 and the discussion below. For the optimality, let $y \in \Phi$ be any feasible point. Then $\langle \mathcal{L}'_{\rho_k}(x^{k+1}, w^k), y - x^{k+1}\rangle \geq \langle \varepsilon^{k+1}, y - x^{k+1}\rangle$ by Assumption 4.8 and, using $\mathcal{L}'_{\rho_k}(x^{k+1}, w^{k+1}) = \mathcal{L}'(x^{k+1}, \lambda^{k+1})$, we obtain

$$\begin{aligned} \left\langle \varepsilon^{k+1}, y - x^{k+1}\right\rangle &\leq \left\langle f'(x^{k+1}) + G'(x^{k+1})^* \lambda^{k+1}, y - x^{k+1}\right\rangle \\ &= \left\langle f'(x^{k+1}), y - x^{k+1}\right\rangle + \left(\lambda^{k+1}, G'(x^{k+1})(y - x^{k+1})\right) \\ &\leq \left\langle f'(x^{k+1}), y - x^{k+1}\right\rangle + \left(\lambda^{k+1}, G(y) - G(x^{k+1})\right), \end{aligned}$$

where we used the fact that $x \mapsto (\lambda^{k+1}, G(x))$ is convex by Theorem 2.10 and Lemma 3.5. Using again Lemma 3.5, we now obtain $\langle f'(x^{k+1}), y - x^{k+1}\rangle \geq \langle \varepsilon^{k+1}, y - x^{k+1}\rangle + r_{k+1}$ with a null sequence $\{r_k\} \subseteq \mathbb{R}$. Since $\bar{x} \in \Phi$, we obtain in particular that $\liminf_{k\to\infty} \langle f'(x^k), \bar{x} - x^k\rangle \geq 0$. The pseudomonotonicity of f' therefore implies that

$$\langle f'(\bar{x}), y - \bar{x}\rangle \geq \limsup_{k\to\infty} \langle f'(x^k), y - x^k\rangle \geq 0 \quad \forall y \in \Phi,$$

and the proof is complete. □

Another special case arises if $C = X$ and the operator $G'(\bar{x})$ is surjective, where $\bar{x}$ is again a weak limit point of the sequence $\{x^k\}$. If we already know (e.g., by Proposition 4.12) that $\bar{x}$ is a stationary point of (P), then it is possible to prove the weak-* convergence of a subsequence of $\{\lambda^k\}$ under weaker assumptions than those in Theorem 4.10. Indeed, it is possible to obtain a convergence result for asymptotic KKT sequences under only the convergence $G'(x^k) \to G'(x)$, with no convergence of the values $G(x^k)$. We will see later that this is crucial for obtaining convergence for Bratu's obstacle problem, see Section 6.2, where $G\colon H_0^1(\Omega) \to H_0^1(\Omega)$, $G(x) := x - \psi$, with $\psi \in H_0^1(\Omega)$. In particular, G is obviously not completely continuous, but for $x^k \rightharpoonup \bar{x}$ it holds $G'(x^k) \to G'(\bar{x})$. We need the following auxiliary results. The first theorem is a slightly more general version of the Banach open mapping theorem.

Theorem 4.13 (Uniform open mapping theorem). *Let X, Y be real Banach spaces and $A \in L(X, Y)$ a surjective linear operator. Then there exists $r > 0$ such that $B_r^Y \subseteq A(B_1^X)$ and, whenever $T \in L(X, Y)$ and $\delta := \|T - A\|_{L(X,Y)} < r$, then $B_{r-\delta}^Y \subseteq T(B_1^X)$.*

Proof. The first assertion is the Banach open mapping theorem. For the proof of the second assertion, we refer the reader to [25, Thm. 1.2] or [26, Thm. 5D.2]. □

The second theorem states a convergence result for asymptotic KKT sequences under only the convergence $G'(x^k) \to G'(x)$, with no convergence of the values $G(x^k)$. We state this result in a slightly more general framework.

Proposition 4.14. *Let $\{x^k\} \subseteq X$, $\{T_k\} \subseteq L(X, Y)$, and $\{\lambda^k\} \subseteq Y^*$ be sequences such that $F(x^k) + T_k^* \lambda^k \rightharpoonup^* 0$. Assume that $x^k \rightharpoonup \bar{x}$ for some $\bar{x} \in X$, $F(x^k) \rightharpoonup^* F(\bar{x})$, $T_k \to T$ for some $T \in L(X, Y)$, and that T is surjective. Then $\{\lambda^k\}$ converges weak-* in Y^* to the unique solution of $F(\bar{x}) + T^* \lambda = 0$.*

Proof. We first show that $\{\lambda^k\}$ is weak-* convergent. Let $\hat{y} \in Y$ be an arbitrary point. It suffices to show that $\langle \lambda^k, \hat{y}\rangle$ is convergent. Let $r > 0$ be as in the uniform version of the Banach open mapping theorem (Theorem 4.13), so that $B_r^Y \subseteq T(B_1^X)$. Assume, without loss of generality, that $\hat{y} \in B_r^Y$, and let $\hat{w} \in B_1^X$ be a point such that $T\hat{w} = \hat{y}$. Set $\delta_k := \|T_k - T\|_{L(X,Y)}$, and let k be sufficiently large so that $\delta_k < r$. Then $\|\hat{y} - T_k\hat{w}\|_Y \le \delta_k$ and, by Theorem 4.13, there are points $d^k \in X$ such that $T_k d^k = \hat{y} - T_k\hat{w}$ and

$$\|d^k\|_X \le \frac{\|\hat{y} - T_k\hat{w}\|_Y}{r - \delta_k} \le \frac{\delta_k}{r - \delta_k}.$$

Define $w^k := \hat{w} + d^k$. Then $w^k \to \hat{w}$ and $T_k w^k = \hat{y}$ by definition. Hence,

$$0 \leftarrow \left\langle F(x^k) + T_k^* \lambda^k, w^k \right\rangle = \langle F(\bar{x}), \hat{w}\rangle + o(1) + \left\langle \lambda^k, \hat{y}\right\rangle.$$

Thus, we obtain $\langle \lambda^k, \hat{y}\rangle \to -\langle F(\bar{x}), \hat{w}\rangle$. Since $\hat{y} \in Y$ was arbitrary, this implies that $\{\lambda^k\}$ is weak-* convergent in Y^*.

Let $\bar{\lambda}$ denote the weak-* limit of $\{\lambda^k\}$. Using $F(x^k) + T_k^*\lambda^k \rightharpoonup^* 0$, it follows that $F(\bar{x}) + T^*\bar{\lambda} = 0$, and $\bar{\lambda}$ is unique since T^* is injective. □

Proposition 4.15. *Let $\{x^k\}$ be generated by Algorithm 3.4 and let $x^{k+1} \rightharpoonup_I \bar{x}$ for some $I \subseteq \mathbb{N}$ and $\bar{x} \in X$. Assume that $\bar{x}$ is a stationary point of (P), that $C = X$, f' is weak-* sequentially continuous, G' is completely continuous, and that $G'(\bar{x})$ is surjective. Then $\{\lambda^{k+1}\}_{k\in I}$ converges weak-* to the unique element in $\Lambda(\bar{x})$.*

Proof. Recall that $\mathcal{L}'_{\rho_k}(x^{k+1}, w^k) = \mathcal{L}'(x^{k+1}, \lambda^{k+1})$. Combining Assumption 4.4 and Lemma 3.5 we obtain the asymptotic conditions (for $k \ge 1$).

$$\varepsilon^k - \mathcal{L}'(x^k, \lambda^k) \in \mathcal{N}_C(x^k) \quad \text{and} \quad \langle \lambda^k, y - G(x^k)\rangle \le r_k \quad \forall y \in K.$$

Hence, the result follows from Proposition 4.14. □

In the context of optimality properties, it is worthwhile to briefly discuss the case of bounded penalty parameters. This is particularly interesting because any assertion made under this assumption is a *necessary* condition for the boundedness of $\{\rho_k\}$. It turns out that no constraint qualifications are needed in the bounded case, and the algorithm produces a Lagrange multiplier in H.

Corollary 4.16. *Let $\{(x^k, \lambda^k)\}$ be generated by Algorithm 3.4, let Assumption 4.8 hold, and let $\bar{x}$ be a weak limit point of $\{x^k\}$. If $\{\rho_k\}$ remains bounded, then $\{\lambda^k\}$ has a bounded subsequence in H, and $\bar{x}$ satisfies the KKT conditions of (P) with a multiplier in H.*

The above result implies that $\{\rho_k\}$ can only remain bounded if (P) admits a multiplier in H.

We close this section by noting that the nice global convergence properties of the safeguarded augmented Lagrangian method do not hold for the classical augmented Lagrangian approach which is the main reason for the modification of the updating rule of the multipliers. In fact, a counterexample in [47] shows that the classical method may generate limit points which have no meaning from the point of view of satisfying a suitable stationarity measure, whereas the safeguarded method has the desired behaviour. The counterexample provided in [47] is one-dimensional and convex in the sense that its objective function is convex (even linear) and the feasible set is also convex, though represented by a nonconvex function. The authors are not aware of a "fully" convex counterexample where the objective function and the inequality constraints are all convex, and the equality constraints are linear. This leads to the following open problem.

Open Problem 4.17. Are the global convergence properties of the classical augmented Lagrangian method identical (or very similar) to the safeguarded Lagrangian method for fully convex problems?

5. Local Convergence

Here we discuss the local convergence properties of Algorithm 3.4. We first discuss in Section 5.1 the existence of local minima and the (strong!) convergence of such minima. These properties are based on a second-order sufficiency condition, whereas constraint qualifications are not required. This is interesting since it allows applications of our results to problems with a complicated structure of the feasible set. Additional conditions are necessary, however, in order to verify rate-of-convergence results, see Section 5.2. The results from this section are taken from the recent papers [15, 49].

5.1. Existence of Local Minima und Strong Convergence

Before we formulate the second-order sufficiency condition, we note that, as with constraint qualifications and KKT conditions, second-order conditions for (P) can be formulated either with respect to Y or H. In this section, to avoid unnecessary notational overhead, we will simply formulate the second-order condition and its consequences with respect to Y. The results below all remain true when Y is replaced by H (note that the choice $Y := H$ is even admissible in our framework).

Let $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ be a KKT point of (P). Throughout this section, we assume that f and G are twice continuously differentiable in a neighborhood of $\bar{x}$. Then consider, for $\eta > 0$, the *extended critical cone*

$$\mathcal{C}_\eta(\bar{x}) := \left\{ d \in \mathcal{T}_C(\bar{x}) : \begin{array}{l} f'(\bar{x})d \leq \eta \|d\|_X, \\ \operatorname{dist}(G'(\bar{x})d, \mathcal{T}_K(G(\bar{x}))) \leq \eta \|d\|_X \end{array} \right\}. \tag{5.1}$$

Note that $\mathcal{C}_\eta$ depends on $\bar{x}$ only. The following is the general form of a second-order sufficient condition which we will use throughout this section.

Definition 5.1 (Second-order sufficient condition). We say that the *second-order sufficient condition (SOSC)* holds in a KKT point $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ of (P) if there are $\eta, c > 0$ such that

$$\mathcal{L}''(\bar{x}, \bar{\lambda})(d, d) \geq c\|d\|_X^2 \quad \text{for all } d \in \mathcal{C}_\eta(\bar{x}).$$

As mentioned before, the extended critical cone and SOSC can also be formulated with respect to $\mathcal{K}$ and H for KKT pairs $(\bar{x}, \bar{y}) \in X \times H$.

The above should be considered the "basic" second order condition which can be stated without any assumptions on the specific structure of (P). For many problem classes, it is possible to state more refined second-order conditions which are either equivalent to Definition 5.1 or turn out to have similar implications. Some information in this direction can be found, for instance, in [14, Section 3.3].

It turns out that SOSC implies the existence of local minimizers of the penalized subproblems in Algorithm 3.4 as well as strong convergence of the corresponding iterates. Our approach is motivated by a recent analysis in [27] for finite-dimensional nonlinear programming. Here, we extend the corresponding results to our general setting from (P) and show the existence of minimizers using only the proximity of x^k to $\bar{x}$, whereas no assumption regarding the proximity of the multipliers λ^k is required.

As a first step in the local convergence analysis, we consider a local minimizer of (P) and ask whether the augmented Lagrangian admits local minimizers near this point. As we shall see, the answer to this question is closely linked to the fulfillment of second-order sufficient conditions (SOSC) of the form given in Definition 5.1. When using the second-order condition, special care needs to be taken because the embedding $Y \hookrightarrow H$ allows us to interpret the constraint in (P) either in Y or in H. We have already seen that this makes a strong difference for constraint qualifications, and the situation for SOSC is quite similar. The second-order condition in H, for instance, requires the existence of Lagrange multipliers in H, which in itself is already a restriction. Nevertheless, this is in a sense the more "natural" second-order condition for the augmented Lagrangian method since the augmentation is performed in H. Thus, for the most part of this section (with the exception of Proposition 5.4), we will make the following assumption.

Assumption 5.2 (Local convergence). There is a KKT point $(\bar{x}, \bar{\lambda}) \in X \times H$ of (P) which satisfies the SOSC from Definition 5.1 with respect to the space H.

This assumption yields the following local existence and (strong) convergence result.

Theorem 5.3. *Let Assumption 5.2 hold and let $B \subseteq H$ be a bounded set. Then there are $\bar{\rho}, \bar{\varepsilon}, r > 0$ such that, for all $w \in B$, $\rho \geq \bar{\rho}$, and $\varepsilon \in (0, \bar{\varepsilon})$, there is a point $x = x_{\rho,\varepsilon}(w) \in C$ with $\|x - \bar{x}\|_X < r$ and the following properties:*

(i) *x is an ε-minimizer of $\mathcal{L}_\rho(\cdot, w)$ on $B_r(\bar{x}) \cap C$,*
(ii) *x satisfies* $\operatorname{dist}\big(-\mathcal{L}'_\rho(x, w), \mathcal{N}_C(x)\big) \leq \varepsilon^{1/2}$*, and*
(iii) *$x = x_{\rho,\varepsilon}(w) \to \bar{x}$ uniformly on B as $\rho \to \infty$ and $\varepsilon \to 0$.*

If X is reflexive and the augmented Lagrangian $\mathcal{L}_\rho(\cdot, w)$ is weakly sequentially lsc, then the assertions of the above theorem remain valid if we replace the ε-minimizers by exact minimizers. In this case, we obtain points $x = x_\rho(w)$ which satisfy (i) and (ii) with $\varepsilon := 0$ and which converge to $\bar{x}$ uniformly on B as $\rho \to \infty$. Sufficient conditions for the weak sequential lower semicontinuity of $\mathcal{L}_\rho(\cdot, w)$ were given in Lemma 4.2.

If the mapping G is completely continuous, then it is possible to prove a similar result under the second-order sufficient condition with respect to the space Y. This result is a generalization of a theorem from [53].

Proposition 5.4. *Let $(\bar{x}, \bar{\lambda}) \in X \times Y^*$ be a KKT point of (P) which satisfies SOSC with respect to the space Y, and $B \subseteq H$ a bounded set. Assume that*

(i) the space X is reflexive,
(ii) f is weakly sequentially lsc on X, and
(iii) G is completely continuous from X into Y.

Then there are $\bar{\rho}, r > 0$ such that, for every $w \in B$ and $\rho \geq \bar{\rho}$, the problem $\min_{x \in C} \mathcal{L}_\rho(x, w)$ admits a local minimizer $x = x_\rho(w)$ in $B_r(\bar{x}) \cap C$, and $x_\rho \to \bar{x}$ uniformly on B as $\rho \to \infty$.

5.2. Rate of Convergence

We are now in a position to discuss the convergence of Algorithm 3.4 from a quantitative point of view. Throughout this section, we assume that the space X is a real Hilbert space, that there is a local minimizer $\bar{x} \in X$ of (P) with a unique Lagrange multiplier $\bar{\lambda} \in H$, and that the following local error bound condition

$$c_1 \Theta(x, \lambda) \leq \|x - \bar{x}\|_X + \|\lambda - \bar{\lambda}\|_H \leq c_2 \Theta(x, \lambda) \tag{5.2}$$

holds for all $(x, \lambda) \in X \times H$ with x near $\bar{x}$ and $\Theta(x, \lambda)$ sufficiently small, where Θ is the residual

$$\Theta(x, \lambda) := \|x - P_C(x - \mathcal{L}'(x, \lambda))\|_X + \|G(x) - P_{\mathcal{K}}(G(x) + \lambda)\|_H.$$

The regularity assumptions mentioned above may seem rather stringent in view of the Gel'fand triple framework $Y \hookrightarrow H \hookrightarrow Y^*$. Indeed, a sufficient condition for the local error bound is a combination of the second-order sufficient condition and the strict Robinson condition (SRC), both with respect to the space H. This effectively rules out certain applications where the embedding $Y \hookrightarrow H$ is too weak, but the underlying issue is that we simply cannot expect the results in this section to hold if the constraint system of (P) is only regular with respect to the space Y. This is also evidenced by the fact that the rate-of-convergence analysis will enable us to prove the boundedness of the penalty sequence $\{\rho_k\}$, and this actually *implies* the existence of a Lagrange multiplier in H under certain assumptions, see Corollary 4.16 and the discussion after Corollary 5.8 below.

Despite these restrictions, the theory we develop here is still applicable to a fair amount of nontrivial problems such as control-constrained optimal control, elliptic parameter estimation problems, and of course optimization in finite dimensions.

Assumption 5.5 (Rate of convergence). We assume that

(i) X is a real Hilbert space with f and G continuously differentiable on X,
(ii) $(\bar{x}, \bar{\lambda}) \in X \times H$ is a KKT point of (P) which satisfies the error bound (5.2),
(iii) the primal-dual sequence $\{(x^k, \lambda^k)\}$ converges strongly to $(\bar{x}, \bar{\lambda})$ in $X \times H$,
(iv) the safeguarded multiplier sequence satisfies $w^k := \lambda^k$ for k sufficiently large, and

(v) $x^{k+1} \in C$ and $\varepsilon^{k+1} - \mathcal{L}'_{\rho_k}(x^{k+1}, w^k) \in \mathcal{N}_C(x^{k+1})$ for all k, where $\varepsilon^k \to 0$.

Two assumptions which may require some elaboration are (iii) and (iv). Note that we already know, by Theorem 5.3, that the augmented Lagrangian admits approximate local minimizers and stationary points in a neighborhood of $\bar{x}$. We shall now see that, if the algorithm chooses these local minimizers (or any other points sufficiently close to $\bar{x}$), then we automatically obtain the convergence $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ in $X \times H$. In this case, the sequence $\{\lambda^k\}$ is necessarily bounded in H, so it is reasonable to assume that the safeguarded multipliers are eventually chosen as $w^k = \lambda^k$. The following result can therefore be considered as (retrospective) justification for Assumption 5.5.

Proposition 5.6. *Let Assumption 5.5* (i), (ii), (v) *hold, and let RCQ hold in $\bar{x}$ with respect to the space H. Then there exists $r > 0$ such that, if $x^k \in B_r(\bar{x})$ for sufficiently large k, then $\Theta(x^k, \lambda^k) \to 0$ and $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ strongly in $X \times H$.*

We will now state convergence rates for the primal-dual sequence $\{(x^k, \lambda^k)\}$.

Theorem 5.7. *Let Assumption 5.5 hold and assume that $\varepsilon^{k+1} = o(\theta_k)$. Then:*

(a) For every $q \in (0,1)$, there exists $\bar{\rho}_q > 0$ such that, if $\rho_k \geq \bar{\rho}_q$ for sufficiently large k, then $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ Q-linearly in $X \times H$ with rate q.

(b) If $\rho_k \to \infty$, then $(x^k, \lambda^k) \to (\bar{x}, \bar{\lambda})$ Q-superlinearly in $X \times H$.

The assumption $\varepsilon^{k+1} = o(\theta_k)$ in the above theorem says that, roughly speaking, the degree of inexactness should be small enough to not affect the rate of convergence. Note that we are comparing ε^{k+1} to the optimality measure θ_k of the previous iterates (x^k, λ^k). Hence, it is easy to ensure this condition in practice, for instance, by always computing the next iterate x^{k+1} with a precision $\|\varepsilon^{k+1}\|_X \leq z_k \theta_k$ for some fixed null sequence z_k.

Corollary 5.8. *Let Assumption 5.5 hold and assume that the subproblems occurring in Algorithm 3.4 are solved exactly, i.e., that $\varepsilon^k = 0$ for all k. Then $\{\rho_k\}$ remains bounded.*

The boundedness of $\{\rho_k\}$ obviously rules out the Q-superlinear convergence of Theorem 5.7 (b). However, the former is usually considered more significant in practice since it prevents the subproblems from becoming excessively ill-conditioned.

Remark 5.9. If inexact solutions are allowed for the augmented Lagrangian subproblems, then the boundedness of $\{\rho_k\}$ requires a slightly modified updating rule for the penalty parameter since the one used in Algorithm 3.4 does not take into account the current measure of optimality. Indeed, if we replace the function V from (3.9) by

$$\tilde{V}(x, \lambda, \rho) := V(x, \lambda, \rho) + \|x - P_C(x - \mathcal{L}'(x, \lambda))\|_X,$$

then it is possible to show that $\{\rho_k\}$ remains bounded under the assumptions of Theorem 5.7. A proof for the case $C = X$ can be found in [49], and the extension to the general case is straightforward (see also [11, 13]).

Remark 5.10. In the case of finite-dimensional nonlinear programming, it is possible to obtain similar rate of convergence results to those above under the second-order sufficient condition only. In this case, one obtains that $(x^k, \lambda^k) \to (\bar{x}, \lambda)$ Q-linearly for some $\lambda \in \Lambda(\bar{x})$ which is not necessarily equal to $\bar{\lambda}$. This result can be found in [27]. The reason why this is possible is that, for nonlinear programming, the set $\mathcal{K}$ is polyhedral and, therefore, the second-order condition implies a local primal-dual error bound without any constraint qualification.

Remark 5.11. A specification of the previous results in the Banach space setting to nonlinear semi-definite programs, second-order cone programs and related problems is given in [51]. Though these results were essentially obtained from the general theory, the resulting convergence conditions may still be viewed as generalizations of previous results known for semi-definite programs etc., cf. [73]. Though these problems are finite-dimensional, they have a non-polyhedral feasible set, hence SOSC-type conditions alone were not enough in order to establish rate-of-convergence results.

6. Numerical Results

Since the safeguarded augmented Lagrangian method discussed in this paper is identical to the one from the recent paper [15] and since that paper already presents numerical results on a variety of different optimization problems, there is, formally, no need to provide additional material here. For illustrative reasons, however, we report some numerical results also in this paper using some other test examples. The implementation of our numerical examples has been done with FEniCS [57] using the DOLFIN [58] Python interface.

6.1. State-Constrained Optimal Control Problems

PDE-constrained optimal control problems describe a rather popular class of optimization problems. For our numerical test we adapted a linear elliptic example with known solution from [71] to the semilinear setting, see also [53].

Let $\Omega := (-1, 2)^2$. We aim at minimizing the objective function $f\colon L^2(\Omega) \to \mathbb{R}$

$$f(u) := \frac{1}{2}||Su - y_d||^2_{L^2(\Omega)} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)}, \tag{6.1}$$

subject to the pointwise inequality constraints

$$Su \le \psi \quad \text{in } \bar{\Omega}.$$

Here $\alpha > 0$ is a positive parameter and $y_d \in L^2(\Omega)$, $\psi \in C(\overline{\Omega})$ are given functions. The solution operator $S\colon L^2(\Omega) \to H^1_0(\Omega) \cap C(\overline{\Omega})$ maps the control u to the state $y := Su$, which is the uniquely determined weak solution of the underlying semilinear partial differential equation

$$\begin{aligned} -\Delta y + y^5 &= u + f && \text{in } \Omega, \\ y &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $f \in L^2(\Omega)$. In this setting, the operator S is completely continuous [19, Theorem 2.1] and Fréchet differentiable [19, Theorem 2.4]. We set

$$X := L^2(\Omega), \quad C := L^2(\Omega), \quad Y := C(\overline{\Omega}), \quad G(u) := Su - \psi, \quad K := C(\overline{\Omega})_-$$
$$H := L^2(\Omega), \quad \mathcal{K} := L^2(\Omega)_-$$

where $C(\overline{\Omega})_-$ denotes the closed convex cone of non-positive continuous functions and $L^2(\Omega)_-$ the non-positive functions in $L^2(\Omega)$. Applying standard arguments we obtain that problem (6.1) admits at least one solution, see for instance [38]. Let $\bar{u}$ denote a local solution and let us assume that there exists $\hat{u} \in L^2(\Omega)$ such that the linearized Slater condition

$$G(\bar{u}) + G'(\bar{u})(\hat{u} - u) \in \operatorname{int}\left(C(\overline{\Omega})_-\right) \quad \Leftrightarrow \quad S\bar{u} + S'(\bar{u})(\hat{u} - \bar{u}) \leq \psi - \sigma \text{ in } \bar{\Omega},\ \sigma > 0$$

is satisfied. Since the interior of $C(\overline{\Omega})_-$ is non-empty, the linearized Slater condition is, for feasible points, equivalent to the Robinson constraint qualification [14, Lemma 2.99] and we obtain existence of a multiplier $\lambda \in C(\overline{\Omega})_-^{\circ}$. Hence, λ is an element of the space of regular Borel measures $C(\overline{\Omega})^* = \mathcal{M}(\overline{\Omega})$ [19, Theorem 3.1]. Introducing the state $y = Su$ and the adjoint state $\bar{p} \in W_0^{1,s}(\Omega),\ s \in (1,2)$ it is well-known that first-order necessary optimality conditions for the original problem (6.1) are given by

$$\begin{cases} -\Delta \bar{y} + \bar{y}^5 = \bar{u} + f & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta \bar{p} + 5\bar{y}^4 \bar{p} = \bar{y} - y_d + \bar{\lambda} & \text{in } \Omega, \\ \bar{p} = 0 & \text{on } \partial\Omega, \end{cases}$$
$$\bar{p} + \alpha \bar{u} = 0, \tag{6.2}$$
$$\bar{\lambda} \in C(\overline{\Omega})_-^{\circ}, \quad \langle \bar{\lambda}, \psi - \bar{y} \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = 0.$$

The low regularity of $\bar{\lambda}$ complicates the direct numerical solution of the optimal control problem. However, by augmenting the objective function we eliminate the state constraints from the set of explicit constraints. Due to our choice of $\mathcal{K}$ we obtain $d_{\mathcal{K}}^2(\cdot) = ||(\cdot)_+||_{L^2(\Omega)}^2$, where $(\cdot)_+ := \max(0, \cdot)$. Following Algorithm 3.4 we have to solve a sequence of unconstrained subproblems of the type

$$\min_{u^k} f(u^k) + \frac{\rho_k}{2} \left|\left| \left(Su^k - \psi + \frac{w^k}{\rho_k} \right)_+ \right|\right|_{L^2(\Omega)}^2. \tag{6.3}$$

Since these problems are control-constrained only, it is straightforward to show existence of solutions and derive the corresponding optimality conditions [74, Theorem 4.20]. However, due to the nonlinearity of the solution operator S, the functional f is not convex. Accordingly, we can only expect to compute stationary points of the augmented subproblems which are not necessarily local or global minimizers. In order to apply our convergence results from Section 4.3, we need to verify Assumption 4.8.

- The mapping $G' \colon X \to L(X, Y)$ is completely continuous. In the present setting, since $X = L^2(\Omega)$ is reflexive and $G'(u) \in L(X, Y)$ is completely continuous for all u, this is equivalent to the following property: whenever

$u^k \rightharpoonup u$ and $h^k \rightharpoonup h$ in X, then $G(u^k)h^k \to G(u)h$ strongly in Y. A proof of this statement (for the Neumann case) can be found in [53, Lem. 4.7].
- The mapping $f' \colon X \to X^*$ is bounded and pseudomonotone. Note that $f'(u) := S'(u)(S(u) - y_d) + \alpha u$ for all $u \in X$. The operators S and S' are completely continuous, hence bounded (since X is reflexive). This implies the boundedness of f'. The pseudomonotonicity follows from the fact that the first term in f' is completely continuous and the second term is monotone and continuous, see Lemma 2.12.

In this scenario, it follows from Theorem 4.10 that every weak limit point u^* of the sequence $\{u^k\}$ is a stationary point of the problem. Moreover, the corresponding subsequence of multipliers $\{\lambda^k\}$ converges weak-* in $\mathcal{M}(\overline{\Omega})$ to a Lagrange multiplier in u^*.

For the sake of completeness let us state the optimality system of (6.3) that has to be solved in every iteration of Algorithm 3.4. Let $\bar{u}^k$ denote a local solution of the subproblem (6.3) and $\bar{y}^k \in H_0^1(\Omega) \cap C(\overline{\Omega})$ the corresonding state $\bar{y}^k := S\bar{u}^k$. Then, there exists an adjoint state $\bar{p}^k \in H_0^1(\Omega)$ such that the following system is satisfied:

$$\begin{cases} -\Delta \bar{y}^k + {\bar{y}^k}^5 = \bar{u}^k + f & \text{in } \Omega, \\ \bar{y}^k = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta \bar{p}^k + 5{\bar{y}^k}^4 \bar{p}^k = \bar{y}^k - y_d + \bar{\lambda}^k & \text{in } \Omega, \\ \bar{p}^k = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{aligned} \bar{p}^k + \alpha \bar{u}^k &= 0, \\ \bar{\lambda}^k &= (w^k + \rho_k(\bar{y}^k - \psi))_+. \end{aligned} \tag{6.4}$$

In this system, the approximation of the multiplier $\bar{\lambda}^k$ enjoys a much stronger regularity. In fact it is an $L^2(\Omega)$-function, which allows us to apply efficient solution algorithms. We use the notation $r := r(x_1, x_2) := \sqrt{x_1^2 + x_2^2}$ with $x_1, x_2 \in \Omega$ to set

$$\begin{aligned} \bar{y}(r) &:= -\frac{1}{2\pi\alpha}\chi_{r\le 1}\left(\frac{r^2}{4}(\log r - 2) + \frac{r^3}{4} + \frac{1}{4}\right), & \psi(r) &:= -\frac{1}{2\pi\alpha}\left(\frac{1}{4} - \frac{r}{2}\right) \\ \bar{u}(r) &:= \frac{1}{2\pi\alpha}\chi_{r\le 1}(\log r + r^2 - r^3), & y_d(r) &:= \tilde{y}_d(r) - 5\bar{y}^4\bar{p}, \\ \bar{p}(r) &:= -\alpha\bar{u}(r), & f(r) &:= \tilde{f}(r) - \bar{y}^5, \\ \bar{\lambda}(r) &:= \delta_0(r), \end{aligned}$$

where $\tilde{y_d}(r)$ and $\tilde{f}(r)$ are given auxiliary functions

$$\tilde{y_d}(r) := \bar{y}(r) - \frac{1}{2\pi}\chi_{r\le 1}(4 - 9r), \quad \tilde{f}(r) := -\frac{1}{8\pi}\chi_{r\le 1}(4 - 9r + 4r^2 - 4r^3).$$

Then, it can be show, that $(\bar{y}, \bar{u}, \bar{p}, \bar{\lambda})$ is a KKT point of (6.1). We used the parameters

$$\alpha := 1,\ \lambda^0 := 0,\ \rho_0 := 1,\ w_{\max} := 10^5,\ \gamma := 10,\ \tau := 0.2,$$

and initialized our starting-points equal to zero. To obtain a sequence of safeguarded multipliers $\{w^k\}$ we chose $w^k := \min(\lambda^k, w_{\max})$. We solved the arising subproblem with a semismooth Newton method up to the precision 10^{-6}. We stop the algorithm as soon as $\|\min\{\lambda^k, \psi - y^k\}\|_\infty \leq 10^{-6}$ was satisfied. The computed results can be seen in Figure 1 and Figure 2 for 256 gridpoints per dimension.

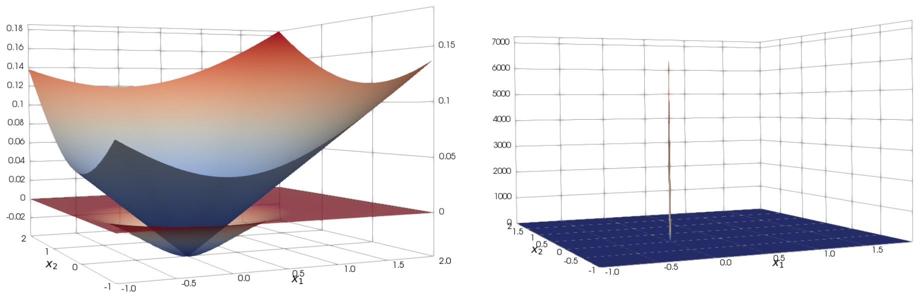

Figure 1. (Example 1) Left: Computed discrete optimal state y_h (transparent) with state constraint ψ. Right: Lagrange Multiplier μ_h.

The $L^2(\Omega)$-error of the computed solution (y_h, u_h) to the constructed solution $(\bar{y}, \bar{u})$ in dependence of the degrees of freedom is shown on the right hand side of Figure 2. Table 1 shows the iteration numbers of outer and inner iterations as well as the final value of the penalty parameter $\rho_{\max}$ with respect to the number of grid points per dimension.

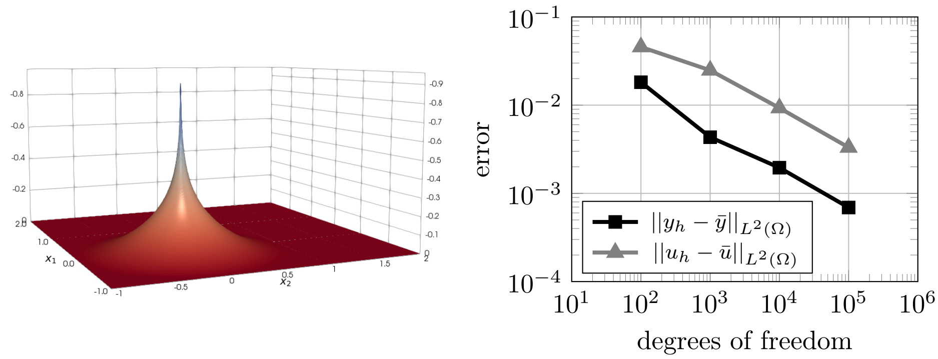

Figure 2. (Example 1) Left: Computed control u_h. Right: Errors $\|u_h - \bar{u}\|_{L^2(\Omega)}$ and $\|y_h - \bar{y}\|_{L^2(\Omega)}$ vs. dofs.

n	16	32	64	128	256
outer it.	10	9	10	11	12
inner it.	21	23	27	33	38
$\rho_{\max}$	10^5	10^5	10^7	10^7	10^8

Table 1. (Example 1) Iteration numbers.

6.2. Bratu's Obstacle Problem

Bratu's obstacle problem is a non-quadratic, nonconvex problem, which is an efficient tool to model nonlinear diffusion phenomena. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Bratu's obstacle problem is given by the minimization problem

$$\min_{u \in H_0^1(\Omega)} J(u) := ||\nabla u||^2_{L^2(\Omega)} - \alpha \int_\Omega \exp(-u(x)) \, \mathrm{d}x \quad \text{s.t.} \quad u \geq \psi, \tag{6.5}$$

where $\alpha > 0$ is a positive parameter and $\psi \in H_0^1(\Omega)$ denotes the given, fixed obstacle. To satisfy our general framework we set

$$X := Y := H_0^1(\Omega), \quad C := H_0^1(\Omega), \quad G(u) := u - \psi, \quad K := H_0^1(\Omega)_+,$$
$$H := L^2(\Omega), \quad \mathcal{K} := L^2(\Omega)_+.$$

Due to [52, Lemma 7.1] we know that J is well-defined, continuously Fréchet differentiable and weakly sequentially lower semicontinuous from $H_0^1(\Omega)$ into $\mathbb{R}$. Due to the constraint $u \geq \psi$, the functional J is coercive on the feasible set. By standard arguments we obtain existence of a solution $\bar{u} \in X$. Moreover, the surjectivity of the derivative $G'(\bar{u}) = \mathrm{Id}_X$ from X to Y implies the Robinson constraint qualification and, hence, the existence of a unique Lagrange multiplier $\bar{\lambda} \in H_0^1(\Omega)^* = H^{-1}(\Omega)$. The corresponding KKT system is given by

$$J'(\bar{u}) + \bar{\lambda} = 0$$
$$\langle \bar{\lambda}, \bar{u} - \psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0, \quad \bar{\lambda} \in \left(H_0^1(\Omega)_+\right)^\circ.$$

By definition of the polar cone we obtain that $\langle \bar{\lambda}, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \leq 0$ for all $u \in H_0^1(\Omega)$ with $u \geq 0$. Since the objective function J is not convex, one can only expect to compute stationary points of the augmented subproblems

$$\min_{u^k} J(u^k) + \frac{\rho_k}{2} || \left(u^k - \psi + \frac{w^k}{\rho_k} \right)_- ||^2_{L^2(\Omega)}$$

which are not necessarily local or global solutions.

Lemma 6.1. *If $\Omega \subseteq \mathbb{R}^2$, then the derivative $J' \colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is bounded and pseudomonotone.*

Proof. We split the objective function $J(u) := J_1(u) - J_2(u)$, where

$$J_1(u) := ||\nabla u||^2_{L^2(\Omega)}, \quad J_2(u) := \alpha \int_\Omega \exp(-u(x)) \, \mathrm{d}x.$$

The proof of [52, Lemma 7.1] shows that the integral term J_2 in the definition of J is weakly sequentially continuous, uniformly differentiable on bounded subsets of $H_0^1(\Omega)$, and J_2' is bounded on bounded subsets of $H_0^1(\Omega)$. It follows that J' is also a bounded operator. Since J_2 is completely continuous and uniformly differentiable on bounded subsets of X it follows that J_2' is completely continuous [63] and in particular pseudomontone. The monotonicity of $-\Delta$ yields that J_1' is monotone (and continuous). Thus, J' is pseudomonotone (Lemma 2.12). □

Due to Lemma 6.1 it follows from Proposition 4.12 that every weak limit point u^* of the sequence $\{u^k\}$ is a stationary point of the problem. Moreover, the corresponding subsequence of multipliers λ^k converges weak-* in $H^{-1}(\Omega)$ to the unique Lagrange multiplier in u^* (Proposition 4.15).

In particular, for $\alpha := 0$, problem (6.5) is reduced to the very well known obstacle problem. Opposed to Bratu's problem this problems is linear quadratic with a (strongly) convex objective function. The strong convexity of J not only implies uniqueness of the solution of the obstacle problem and its corresponding subproblem, it also implies that the primal sequence $\{u^k\}$ converges strongly to $\bar{u}$ in X (Corollary 4.7) and the dual sequence $\{\lambda^k\}$ converges weak-* in $H^{-1}(\Omega)$ by Theorem 4.10 (see Remark 4.11) or Proposition 4.15.

In order to test our example we chose the domain $\Omega := (0,1)^2$. We implemented the Bratu problem for the obstacle

$$\psi(x_1, x_2) := \sum_{i=1}^{3} q_i \exp\left(-500\left((x_1 - z_i)^2 + (x_2 - z_i)^2\right)\right) - 1,$$

where $q := (60, 80, 60)$, $z := (0.25, 0.5, 0.75)$. We chose the parameters

$$\alpha := 2,\ \lambda^0 := 0,\ \rho_0 := 1,\ w_{\min} := -10^5,\ \gamma := 10,\ \tau := 0.1,$$

and initialized our starting-points equal to zero. We obtain a sequence of safeguarded multipliers $\{w^k\}$ by choosing $w^k := \max(\lambda^k, w_{\min})$. We solve the unconstrained subproblems with a semismooth Newton method with the precision 10^{-6} and stop the algorithm as soon as $||\max\{\lambda^k, \psi - u^k\}||_\infty \leq 10^{-6}$ is satisfied. The computed results can be seen for 128 gridpoints per dimension in Figure 3 below. Further, some iteration numbers are given in Table 2.

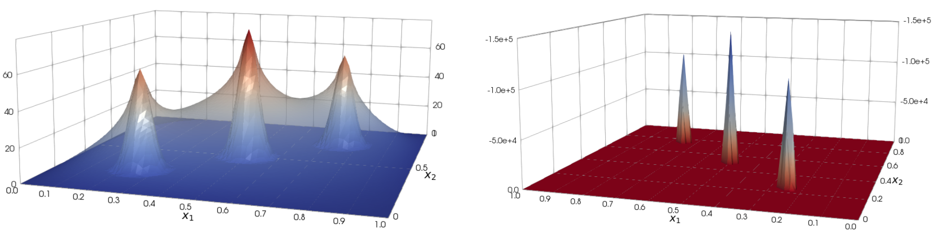

FIGURE 3. (Example 2) Left: Computed discrete optimal solution u_h (transparent) with constraint ψ. Right: Lagrange multiplier μ_h.

n	16	32	64	128	256
outer it.	9	9	12	12	13
inner it.	14	17	25	32	34
$\rho_{\max}$	10^4	10^5	10^{10}	10^{10}	10^{10}

TABLE 2. (Example 2) Iteration numbers.

6.3. $C(\overline{\Omega})$-Minimization

We consider an optimal control problem with an objective functional containing an $C(\overline{\Omega})$ norm term, namely

$$\underset{y\in H^1(\Omega)\cap C(\overline{\Omega}),\ u\in L^2(\Omega)}{\text{minimize}} \quad \frac{1}{2}||y-y_d||^2_{C(\overline{\Omega})} + \frac{\alpha}{2}||u||^2_{L^2(\Omega)}, \tag{6.6}$$

where the state y has to satisfy the semilinear partial differential equation

$$\begin{aligned} -\Delta y + \exp(y) &= u + f &&\text{in } \Omega \\ \partial y &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

where ∂y denotes the normal derivative of y on $\partial\Omega$ and f a function in $L^2(\Omega)$. Functionals including an $C(\overline{\Omega})$ norm term are not differentiable and therefore difficult to handle. We introduce the control-to-state mapping $S\colon L^2(\Omega) \to H^1(\Omega)\cap C(\overline{\Omega})$, which maps the control u on the associated, uniquely determined, state $S : u \mapsto y$ [18, Theorem 3.1]. The original problem is now substituted by an equivalent problem with a differentiable function given by

$$\min_{z\in\mathbb{R},u\in L^2(\Omega)} f(u,z) := \frac{1}{2}z^2 + \frac{\alpha}{2}||u||^2_{L^2(\Omega)} \quad \text{subject to} \quad |Su - y_d| \le z. \tag{6.7}$$

Clearly, problem (6.7) is related to state-constrained optimal control problems. However, now z is a free variable. Consequently, we aim at finding the smallest $z \in \mathbb{R}$ and $u \in L^2(\Omega)$ such that the pointwise inequality constraints are satisfied and the objective function f is minimized. Problems of this type have already been investigated in [32, 66]. Moreover, in [20] pointwise constraints on the state variable on a specified subdomain of Ω under piecewise constant controls were investigated. To satisfy our general framework we set $x := (u,z) \in L^2(\Omega)\times\mathbb{R}$, and

$$\begin{gathered} X := L^2(\Omega)\times\mathbb{R}, \quad C := L^2(\Omega)\times\mathbb{R}, \quad Y := C(\overline{\Omega})\times C(\overline{\Omega}), \\ K := C(\overline{\Omega})_- \times C(\overline{\Omega})_+, \quad H := L^2(\Omega)\times\mathbb{R}, \quad \mathcal{K} := L^2(\Omega)_- \times L^2(\Omega)_+ \end{gathered}$$

as well as

$$G(x) := \begin{pmatrix} Su - y_d - z \\ Su - y_d + z \end{pmatrix}.$$

This leads us again to a minimization problem of the type $\min_x f(x)$ such that $G(x) \in K$. Like in Example 1, the solution operator S [18], and hence G, is completely continuous and continuously Fréchet differentiable [74, Theorem 4.17]. Thus, we obtain by standard arguments the existence of an optimal solution $(\bar{y},\bar{u}) \in H^1(\Omega)\times L^2(\Omega)$ of (6.6). Hence, defining $\bar{z} := ||\bar{y}-y_d||_{C(\overline{\Omega})}$ we can conclude that $(\bar{u}, \bar{z})$ is a solution of (6.7). Let $\bar{x} := (\bar{u}, \bar{z}) \in (L^2(\Omega) \times \mathbb{R})$ denote a local solution. Then it is easy to see that the Robinson constraint qualification is satisfied. Indeed, the first line of the inclusion

$$0 \in \operatorname{int}\left[G(\bar{u}, \bar{z}) + G'(\bar{u}, \bar{z}) \begin{pmatrix} L^2(\Omega) - \bar{u} \\ \mathbb{R} - \bar{z} \end{pmatrix} - (K_- \times K_+)\right]$$

can be written as

$$0 \in \operatorname{int}\left[(S\bar{u} - y_d - \bar{z}) + S'(\bar{u})(L^2(\Omega) - \bar{u}) - (\mathbb{R} - \bar{z}) - K_-\right],$$

which is fulfilled as $\mathbb{R} + K_- = C(\bar{\Omega})$. Then, there exist Lagrange multipliers $\bar{\lambda}_1, \bar{\lambda}_2 \in C(\overline{\Omega})^* = \mathcal{M}(\overline{\Omega})$. Moreover, it is easy to see that $\bar{\lambda}_1 + \bar{\lambda}_2 \in \partial \left(\frac{1}{2}|| \cdot ||^2_{C(\overline{\Omega})}\right)(S\bar{u} - y_d)$, where ∂ denotes the convex subdifferential. It remains to verify Assumption 4.8. Following the same argumentation as in Example 1, we can deduce that $S'(u) \in L(L^2(\Omega), C(\overline{\Omega}))$ is completely continuous and, thus, $G' : X \to L(X, Y)$ is completely continuous. Further, the mapping $f' \colon X \to X^*$, $f'(x) = (\alpha u, z)^T$ is bounded and by Lemma 2.12 pseudomonotone.

According to Algorithm 3.4, we have to solve the following unconstrained subproblem in every iteration of the algorithm

$$\begin{aligned} \underset{u^k, z^k}{\text{minimize}}\ f(u^k, z^k) &+ \frac{\rho_k^1}{2} \left|\left| \left(Su^k - y_d - z^k + \frac{w_1^k}{\rho_k^1}\right)_+ \right|\right|^2_{L^2(\Omega)} \\ &+ \frac{\rho_k^2}{2} \left|\left| \left(Su^k - y_d + z^k + \frac{w_2^k}{\rho_k^2}\right)_- \right|\right|^2_{L^2(\Omega)}. \end{aligned}$$

Reintroducing the state $y = Su$ and the adjoint state $p \in H^1(\Omega)$, we obtain the corresponding optimality system by standard arguments

$$\begin{cases} -\Delta \bar{y}^k + e^{\bar{y}^k} = \bar{u}^k + f & \text{in } \Omega, \\ \partial \bar{y}^k = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta \bar{p}^k + e^{\bar{y}^k} \bar{p}^k = \lambda_1^k + \lambda_2^k & \text{in } \Omega, \\ \partial \bar{p}^k = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{aligned} \alpha \bar{u}^k + \bar{p}^k &= 0, \\ z^k - \int_\Omega \lambda_1^k + \int_\Omega \lambda_2^k &= 0, \end{aligned} \tag{6.8}$$

where

$$\lambda_1^k := (w_1 + \rho_k^1(\bar{y}^k - y_d - z^k))_+, \qquad \lambda_2^k := (w_2 + \rho_k^2(\bar{y}^k - y_d + z^k))_-.$$

To test our example we took $\Omega := (0, 1)^2$, set our starting-points equal to zero and chose the parameters

$$\alpha := 10^{-4}, \quad \lambda^0 := 0, \quad w_{\max} := 10^{-5}, \quad \gamma := 10, \quad \tau := 0.1.$$

Further, we chose $y_d := 0$ and $f := 8\sin(\pi x_1)\sin(\pi x_2) - 4$, where $(x_1, x_2) \in \Omega$. We solved the optimality system (6.8) with a semismooth Newton method with the precision 10^{-6} and stop the algorithm as soon as

$$|| \min\{\lambda_1^k, -Su^k + y_d + z^k\}||_\infty + || \max\{\lambda_2^k, -Su^k + y_d - z^k\}||_\infty \leq 10^{-6}$$

is satisfied. Figure 4 and 5 depict the computed results for $n = 128$ gridpoints per dimension. The corresponding optimal value of z has been computed as $\bar{z} = 6.7 \cdot 10^{-3}$. Some iteration numbers are shown in Table 3.

n	16	32	64	128
outer it.	8	7	8	8
inner it.	17	19	26	26
$\rho_{\max}$	10^4	10^5	10^6	10^6

TABLE 3. (Example 3) Iteration numbers.

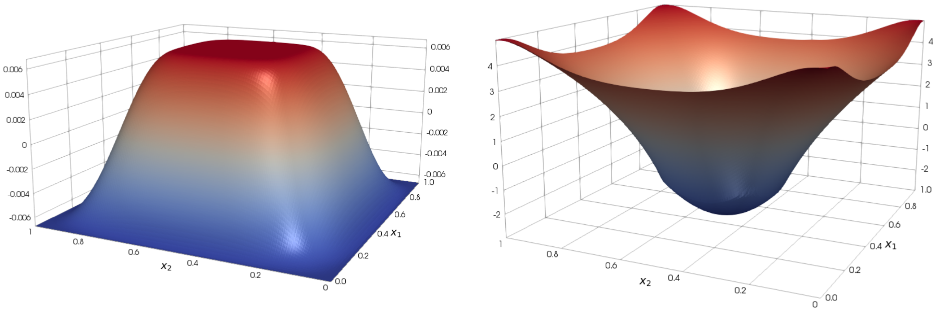

FIGURE 4. (Example 3) Computed discrete optimal state y_h (left) with optimal control u_h (right).

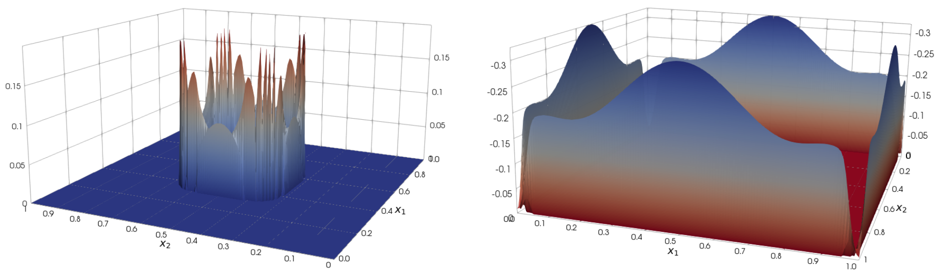

FIGURE 5. (Example 3) Computed discrete Lagrange multipliers $\mu_{h,1}$ and $\mu_{h,2}$.

7. Final Remarks

The previous survey shows that the safeguarded augmented Lagrangian approach has a very strong global and local convergence theory which allows its application to a wide variety of different applications. The numerical results in this and some related papers by the authors indicate that the approach also works quite successfully from a numerical point of view. Nevertheless, there are plenty of possible modifications which might be interesting to investigate. For example, in finite dimensions, the augmented Lagrangian approach converges under much weaker assumptions than the Robinson CQ, but these weaker assumptions currently do not exist in Banach spaces simply because there is not counterpart of the corresponding constraint qualifications in infinite dimensions. Another interesting generalization might be a relaxation of the second order sufficiency condition which currently is assumed to hold at a KKT point, but the existence of such a KKT point might be too strong an assumption for some difficult classes of optimization problems like mathematical programs of with complementarity constraints.

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