GENERALIZED NASH EQUILIBRIUM PROBLEMS

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Abstract. The Generalized Nash Equilibrium Problem is an important model that has its roots in the economic sciences but is being fruitfully used in many different fields. In this survey paper we aim at discussing its main properties and solution algorithms, pointing out what could be useful topics for future research in the field.

1 Introduction

This is a survey paper on the Generalized Nash Equilibrium Problem (GNEP for short). Although the GNEP is a model that has been used actively in many fields in the past fifty years, it is only since the mid-nineties that research on this topic gained momentum, especially in the operations research (OR) community. This paper aims at presenting in a unified fashion the contributions that have been given over the years by people working in many different fields. In fact, the GNEP lies at the intersection of many different disciplines (e.g. economics, engineering, mathematics, computer science, OR), and sometimes researchers in different fields worked independently and unaware of existing results. We hope this paper will serve as a basis for future research and will stimulate the interest in GNEPs in the OR community. While we try to cover many topics of interest, we do not strive for maximum technical generality and completeness, especially when this would obscure the overall picture without bringing any real new insight on the problem.

As we already mentioned, many researchers from different fields worked on the GNEP, and this explains why this problem has a number of different names in the literature including *pseudo-game*, *social equilibrium problem*, *equilibrium programming*, *coupled constraint equilibrium problem*, and *abstract economy*. We will stick to the term generalized Nash equilibrium problem that seems the one favorite by OR researchers in recent years.

Formally, the GNEP consists of N players, each player ν controlling the variables $x^\nu \in \mathbb{R}^{n_\nu}$. We denote by $\boldsymbol{x}$ the vector formed by all these decision variables:

$$\boldsymbol{x} := \begin{pmatrix} x^1 \\ \vdots \\ x^N \end{pmatrix},$$

which has dimension $n := \sum_{\nu=1}^N n_\nu$, and by $\boldsymbol{x}^{-\nu}$ the vector formed by all the players' decision variables except those of player ν. To emphasize the ν-th player's variables within $\boldsymbol{x}$, we sometimes write $(x^\nu, \boldsymbol{x}^{-\nu})$ instead of $\boldsymbol{x}$. Note that this is still the vector $\boldsymbol{x} = (x^1, \dots, x^\nu, \dots, x^N)$ and that, in particular, the notation $(x^\nu, \boldsymbol{x}^{-\nu})$ does not mean that the block components of $\boldsymbol{x}$ are reordered in such a way that x^ν becomes the first block.

Each player has an objective function $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ that depends on both his own variables x^ν as well as on the variables $\boldsymbol{x}^{-\nu}$ of all other players. This mapping θ_ν is often called the *utility function* of player ν, sometimes also the *payoff function* or *loss function*, depending on the particular application in which the GNEP arises.

Furthermore, each player's strategy must belong to a set $X_\nu(\boldsymbol{x}^{-\nu}) \subseteq \mathbb{R}^{n_\nu}$ that depends on the rival players' strategies and that we call the *feasible set* or *strategy space* of player ν. The aim of player ν, given the other players' strategies $\boldsymbol{x}^{-\nu}$, is to choose a strategy x^ν that solves the minimization problem

$$\text{minimize}_{x^\nu}\ \theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) \quad \text{subject to} \quad x^\nu \in X_\nu(\boldsymbol{x}^{-\nu}). \tag{1}$$

For any $\boldsymbol{x}^{-\nu}$, the solution set of problem (1) is denoted by $\mathcal{S}_\nu(\boldsymbol{x}^{-\nu})$. The GNEP is the problem of finding a vector $\bar{\boldsymbol{x}}$ such that

$$\bar{x}^\nu \in \mathcal{S}_\nu(\bar{\boldsymbol{x}}^{-\nu}) \quad \text{for all } \nu = 1, \dots, N.$$

Such a point $\bar{\boldsymbol{x}}$ is called a *(generalized Nash) equilibrium* or, more simply, a *solution* of the GNEP. A point $\bar{\boldsymbol{x}}$ is therefore an equilibrium if no player can decrease his objective function by changing unilaterally $\bar{x}^\nu$ to any other feasible point. If we denote by $\mathcal{S}(\boldsymbol{x})$ the set $\mathcal{S}(\boldsymbol{x}) := \Pi_{\nu=1}^N \mathcal{S}_\nu(\boldsymbol{x}^{-\nu})$, we see that we can say that $\bar{\boldsymbol{x}}$ is a solution if and only if $\bar{\boldsymbol{x}} \in \mathcal{S}(\bar{\boldsymbol{x}})$, i.e. if and only if $\bar{\boldsymbol{x}}$ is a fixed point of the point-to-set mapping $\mathcal{S}$. If the feasible sets $X_\nu(\boldsymbol{x}^{-\nu})$ do not depend on the rival players' strategies, so we have $X_\nu(\boldsymbol{x}^{-\nu}) = X_\nu$ for some set $X_\nu \subseteq \mathbb{R}^{n_\nu}$ and all $\nu = 1, \ldots, N$, the GNEP reduces to the standard Nash equilibrium problem (NEP for short), cf. Section 2.

We find it useful to illustrate the above definitions with a simple example.

Example 1.1 Consider a game with two players, i.e. $N = 2$, with $n_1 = 1$ and $n_2 = 1$, so that each player controls one variable (for simplicity we therefore set $x_1^1 = x^1$ and $x_1^2 = x^2$). Assume that the players' problems are

$$\begin{array}{lll} \min_{x^1} & (x^1-1)^2 \\ \text{s.t.} & x^1 + x^2 \le 1, \end{array} \qquad\qquad \begin{array}{lll} \min_{x^2} & (x^2-\frac{1}{2})^2 \\ \text{s.t.} & x^1 + x^2 \le 1. \end{array}$$

The optimal solution sets are given by

$$\mathcal{S}_1(x^2) = \begin{cases} 1, & \text{if } x^2 \le 0, \\ 1 - x^2, & \text{if } x^2 \ge 0, \end{cases} \quad \text{and} \quad \mathcal{S}_2(x^1) = \begin{cases} \frac{1}{2}, & \text{if } x^1 \le \frac{1}{2}, \\ 1 - x^1, & \text{if } x^1 \ge \frac{1}{2}. \end{cases}$$

Then it is easy to check that the solutions of this problem are given by $(\alpha, 1-\alpha)$ for every $\alpha \in [1/2, 1]$. Note that the problem has infinitely many solutions.

In the example above the sets $X_\nu(\boldsymbol{x}^{-\nu})$ are defined explicitly by inequality constraints. This is the most common case and we will often use such an explicit representation in the sequel. More precisely, in order to fix notation, we will several times assume that the sets $X_\nu(\boldsymbol{x}^{-\nu})$ are given by

$$X_\nu(\boldsymbol{x}^{-\nu}) = \{x^\nu \in \mathbb{R}^\nu : g^\nu(x^\nu, \boldsymbol{x}^{-\nu}) \le 0\}, \tag{2}$$

where $g^\nu(\cdot, \boldsymbol{x}^{-\nu}) : \mathbb{R}^{n_\nu} \to \mathbb{R}^{m_\nu}$. Equality constraints can easily be incorporated, we omit them for notational simplicity. Furthermore, we won't make any distinction between constraints of player ν that depend on the player's variables x^ν only and those that depend also on the other players' variables; in fact, the former can formally be included in the latter without loss of generality.

The paper is organized as follows: In the next section, after some historical notes, we describe some relevant applications. In Section 3 we discuss relations of the GNEP with other problems and introduce an important subclass of the GNEP. In Section 4, existence results are presented along with KKT conditions and some further theoretical results. In Section 5, we analyze solution procedures, while in the final section we briefly discuss further topics of interest and draw some conclusions.

A few words regarding our notation and the necessary background. The Euclidean projection of a vector $\boldsymbol{x}$ onto a set $\boldsymbol{X}$ is denoted by $P_{\boldsymbol{X}}(\boldsymbol{x})$. We say that a function f is

C^0 if it is continuous, and C^k if it is k-times continuously differentiable. For a real-valued C^1-function f, we denote its gradient at a point $\boldsymbol{x}$ by $\nabla f(\boldsymbol{x})$. Similarly, for a vector-valued C^1-function F, we write $JF(\boldsymbol{x})$ for its Jacobian at a point $\boldsymbol{x}$. We assume that the reader is familiar with classical optimization concepts and has some basic notions about variational inequalities (VIs) and quasi-variational inequalities (QVIs).

2 Historical Overview and Examples

The celebrated Nash equilibrium problem (NEP), where $X_\nu(\boldsymbol{x}^{-\nu}) = X_\nu$ for all $\nu = 1, \ldots, N$, was formally introduced by Nash in his 1950/1 papers [75, 76], but the origins of the concept of equilibrium can be traced back to Cournot [22], in the context of an oligopolistic economy, and have obvious and closer antecedents in the work of von Neumann [77] and von Neumann and Morgenstern [78] on zero-sum two-person games. Nash papers [75, 76] are a landmark in the scientific history of the twentieth century and the notion of Nash equilibrium has extensively proved to be powerful, flexible, and rich of consequences.

However, the need of an extension of the NEP, where the players interact also at the level of the feasible sets, soon emerged as necessary. The GNEP was first formally introduced by Debreu [27] in 1952 (where the term social equilibrium was coined). This paper was actually intended to be just a mathematical preparation for the famous 1954 Arrow and Debreu paper [8] about economic equilibria. In this latter paper, Arrow and Debreu termed the GNEP "an abstract economy" and explicitly note that "... In a game, the pay-off to each player depends upon the strategies chosen by all, but the domain from which strategies are to be chosen is given to each player independently of the strategies chosen by other players. An abstract economy, then, may be characterized as a generalization of a game in which the choice of an action by one agent affects both the pay-off and the domain of actions of other agents", cf. [8, p. 273]. It is safe to say that [8] and the subsequent book [28] provided the rigorous foundation for the contemporary development of mathematical economics.

The mathematical-economic origin of the GNEP explains why the GNEP has long been (let's say up to the beginning of the nineties) the almost exclusive domain of economists and game-theory experts. In truth, it must also be noted that in this community some reserves have been advanced on GNEPs, on the grounds that a GNEP is not a game. For example, Ichiishi states, in his influential 1983 book [57, p. 60], "It should be emphasized, however, that an abstract economy is *not* a game, ... since player j must know the others' strategies in order to know his own feasible strategy set ..., but the others cannot determine their feasible strategies without knowing j's strategy. Thus an abstract economy is a *pseudo-game* and it is useful only as a mathematical tool to establish existence theorems in various applied contexts."

The point here is that one cannot imagine a game where the players make their choices *simultaneously* and then, for some reason, it happens that the constraints are satisfied. But indeed, this point of view appears to be rather limited, and severely undervalues

(a) the descriptive and explanatory power of the GNEP model;

(b) its *normative* value, i.e., the possibility to use GNEPs to design rules and protocols, set taxes and so forth, in order to achieve certain goals, a point of view that has been central to recent applications of GNEPs outside the economic field (see below and Section 2.2);

(c) the fact that in any case different paradigms for games can and have been adopted, where it is possible to imagine that, although in a noncooperative setting, there are mechanisms that make the satisfactions of the constraints possible.

Following the founding paper [8], researchers dedicated most of their energies to the study of the existence of equilibria under weaker and weaker assumptions and to the analysis of some structural properties of the solutions (for example uniqueness or local uniqueness). The relevant literature will be discussed more in detail in Section 4. It was not until the beginning of the 1990s, however, that applications of the GNEP outside the economic field started to be considered along with algorithms for calculation of equilibria. In this respect, possibly one of the early contributions was given by Robinson in 1993 in [92, 93]. In these twin papers, Robinson considers the problem of measuring effectiveness in optimization-based combat models, and gives several formulations that are nothing else but, in our terminlogy, GNEPs. For some of these GNEPs, Robinson provides both existence results and computational procedures.

More or less at the same time, Scotti, see [97] and references therein, introduced GNEPs in the study and solution of complex structural design problems as an evolution of the more standard use of nonlinear programming techniques promoted by Schmit in the 1960s (see [96] for a review) and motivated by some early suggestions in the previous decade, see [90, 103].

After these pioneering contributions, in the last decade the GNEP became a relatively common paradigm, used to model problems from many different fields. In fact GNEPs arise quite naturally from standard NEPs if the players share some common resource (a communication link, an electrical transmission line, a transportation link etc.) or limitations (for example a common limit on the total pollution in a certain area). More in general the ongoing process of liberalization of many markets (electricity, gas, telecommunications, transportation and others) naturally leads to GNEPs. But GNEPs have also been employed to model more technical problems that do not fit any of the categories listed above, and it just seems likely that now that the model is winning more and more popularity, many other applications will be uncovered in the near future. It is impossible to list here all relevant references for these applications; we limit ourselves to a few that, in our view, are either particularly interesting or good entry points to the literature [1, 2, 4, 8, 11, 12, 17, 21, 31, 44, 45, 48, 49, 50, 53, 54, 58, 60, 62, 64, 82, 86, 87, 99, 101, 106, 107].

In the remaining part of this section, we illustrate the scope of the GNEP by considering in some more detail three specific applications: The abstract economy by Arrow and Debreu, a power control problem in telecommunications, and a GNEP arising from the application of the Kyoto protocol. While the first application has a historical signification in that it constitutes the original motivation for the study of GNEPs, the other two problems described are examples of the contemporary use of GNEPs.

2.1 Arrow and Debreu Abstract Economy Model

The economic equilibrium model is a central theme to economics and deals with the problem of how commodities are produced and exchanged among individuals. Walras [105] was probably the first author to tackle this issue in a modern mathematical perspective. Arrow and Debreu [8] considered a general "economic system" along with a corresponding (natural) definition of equilibrium. They then showed that the equilibria of their model are those of a suitably defined GNEP (which they called an "abstract economy"); on this basis, they were able to prove important results on the existence of economic equilibria. Below we describe this economic model.

We suppose there are l distinct commodities (including all kinds of services). Each commodity can be bought or sold at a finite number of locations (in space and time). The commodities are produced in "production units" (companies), whose number is s. For each production unit j there is a set Y_j of possible production plans. An element $y^j \in Y_j$ is a vector in $\mathbb{R}^l$ whose h-th component designates the output of commodity h according to that plan; a negative component indicates an input. If we denote by $p \in \mathbb{R}^l$ the prices of the commodities, the production units will naturally aim at maximizing the total revenue, $p^T y^j$, over the set Y_j.

We also assume the existence of "consumptions units", typically families or individuals, whose number is t. Associated to each consumption unit i we have a vector $x^i \in \mathbb{R}^l$ whose h-th component represent the quantity of the h-th commodity consumed by the i-th individual. For any commodity, other than a labor service supplied by the individual, the consumption is non-negative. More in general, x^i must belong to a certain set $X_i \subseteq \mathbb{R}^l$. The set X_i includes all consumption vectors among which the individual could choose if there were no budgetary constraints (the latter constraints will be explicitly formulated below). We also assume that the i-th consumption unit is endowed with a vector $\xi^i \in \mathbb{R}^l$ of initial holdings of commodities and has a contractual claim to the share α_{ij} of the profit of the j-th production unit. Under these conditions it is then clear that, given a vector of prices p, the choice of the i-th unit is further restricted to those vectors $x^i \in X_i$ such that $p^T x^i \leq p^T \xi^i + \sum_{j=1}^{s} \alpha_{ij}(p^T y^j)$. As is standard in economic theory, the consumptions units aim is to maximize a utility function $u_i(x^i)$, which can be different for each unit.

Regarding the prices, obviously the vector p must be non-negative; furthermore, after normalization, it is assumed that $\sum_{h=1}^{l} p_h = 1$. It is also expected that free commodities, i.e. commodities whose price is zero, are only possible if the supply exceeds the demand; on the other hand, it is reasonable to require that the demand is always satisfied. These two last requirements can be expressed in the form: $\sum_{i=1}^{t} x^i - \sum_{j=1}^{s} y^j - \sum_{i=1}^{t} \xi^i \leq 0$ and $p^T(\sum_{i=1}^{t} x^i - \sum_{j=1}^{s} y^j - \sum_{i=1}^{t} \xi^i) = 0$.

With the above setting in mind Arrow and Debreu also make a series of further technical assumptions (which are immaterial to our discussion) on the properties of the sets Y_j, X_i, the functions u_i, etc., that correspond to rather natural economic features, and on this basis they define quite naturally a notion of an economic equilibrium. Essentially, an economic equilibrium is a set of vectors $(\bar{x}^1, \ldots, \bar{x}^t, \bar{y}^1, \ldots, \bar{y}^s, \bar{p})$ such that all the relations described above are satisfied. From our point of view, the interesting thing is that Arrow and Debreu show that the economic equilibria can also be described as the equilibria of a certain GNEP, and this reduction is actually the basis on which they can prove their key result: existence of equilibria. The GNEP they define has $s + t + 1$ players. The first s players correspond to the production units, the following t ones are the consumption units, and the final player is a fictious player who sets the prices and that is called "market participant". The j-th production player controls the variables y^j, and his problem is

$$\max_{y^j} p^T y^j \quad \text{s.t.} \quad y^j \in Y_j. \tag{3}$$

The i-th consumption player controls the variables x^i, and his problem is

$$\begin{array}{ll} \max_{x^i} & u_i(x^i) \\ \text{s.t.} & x^i \in X_i, \\ & p^T x^i \leq p^T \xi^i + \max\left\{0, \sum_{j=1}^{s} \alpha_{ij}(p^T y^j)\right\}. \end{array} \tag{4}$$

Finally, the market participant's problem is

$$\begin{array}{ll} \max_p & p^T \left(\sum_{i=1}^{t} x^i - \sum_{j=1}^{s} y^j - \sum_{i=1}^{t} \xi^i\right) \\ \text{s.t.} & p \geq 0, \\ & \sum_{h=1}^{l} p_h = 1. \end{array} \tag{5}$$

Altogether, (3)–(5) represent a GNEP with the joint constraints coming from (4).

2.2 Power Allocation in a Telecommunication System

The next example comes from the telecommunication field and is an example of the kind of applications of the GNEP that have flourished in the engineering world in the past ten years. The problem we consider is the power allocation in a Gaussian frequency-selective interference channel model [86]. In order to make the presentation self-contained and as much as possible clear, we consider a simplified setting which, however, captures all the technical issues at stake and is furthermore particularly significative.

Consider then the Digital Subscriber Line (DSL) technology, which is a very common method for broadband internet access. DSL customers use a home modem to connect to a Central Office through a dedicated wire. In a standard setting, the wires are bundled together in a common telephone cable, at least in proximity of the Central Office. Due to electromagnetic couplings, the DSL signal in the wires can interfere with one another, causing a degradation of the quality of the service. To complete the picture, one must take into account that the current standards prescribes the use of discrete multitone modulation which, in practice, divides the total available frequency band in each wire into a set of parallel subcarriers (typically either 256 for Asymmetric DSL (ADSL) and 4096 for Very high bit rate DSL (VDSL)). In this setting the parameter that can be controlled is, for each wire q and for each subcarrier k, the power p_k^q allocated for transmission. For each wire, the transmission quality is given by the maximum achievable transmission rate R_q.

This quantity depends both on the vector $(p_k^q)_{k=1}^N$ of power allocations across the N available subcarriers for wire q, and $\mathbf{p}^{-q} := (\mathbf{p}^r)_{r\neq q}^Q$, the vector representing the strategies of all the other wires. Under adequate technical assumptions it can be shown that

$$R_q(\mathbf{p}^q, \mathbf{p}^{-q}) = \sum_{k=1}^{N} \log\left(1 + \mathsf{sinr}_k^q\right),$$

with sinr_k^q denoting the Signal-to-Interference plus Noise Ratio (SINR) on the k-th carrier for the q-th link:

$$\mathsf{sinr}_k^q := \frac{|H_k^{qq}|^2\, p_k^q}{\sigma_q^2(k) + \sum_{r\neq q} |H_k^{qr}|^2\, p_k^r},$$

where σ_q^2 and H_k^{qr} are parameters describing the behavior of the communication system (see [86] for details).

In this setting, there is a single decision maker who must decide the power allocation. This decision maker, loosely speaking, on the one hand wants to minimize the power employed while guaranteeing to each wire q a transmission rate of at least R_q^*. For many reasons we cannot discuss here, that are, however, rather intuitive, the telecommunication engineers have come to the conclusion that a desirable way to choose the power allocation is to take it as the equilibrium of a GNEP we describe below. Each wire q is a player of the game, whose objective function is to minimize the total power used in transmission, with the constraint that the maximum transmission rate is at least R_q^*, i.e. the problem of the generic player q is

$$\begin{array}{ll} \max_{\mathbf{p}^q} & \sum_{k=1}^N p_k^q \\ \text{s.t.} & R_q(\mathbf{p}^q, \mathbf{p}^{-q}) \geq R_q^*, \\ & \mathbf{p}_q \geq 0. \end{array}$$

We stress that here the GNEP is used in a normative way. No one is really playing a game; rather, a single decision maker has established that the outcome of the GNEP is desirable and therefore (calculates and) implements it. This perspective is rather common in many modern engineering applications of the GNEP.

2.3 Environmental Pollution Control

1997 Kyoto agreements prescribe that the "Annex I Parties" (a list of developed countries and countries in transition to a market economy) must reduce by the year 2012 their overall emission of greenhouse gases 5 per cent below the 1990 levels. In order to reach this goal, various mechanisms are envisaged. One of the most interesting one is the so called "Joint Implementation"(JI). This mechanism is described in the Kyoto Protocol with the following words: "for the purpose of meeting its commitments ..., any Party included in Annex I may transfer to, or acquire from, any other such Party emission reduction units (ERU) resulting from projects aimed at reducing anthropogenic emissions by sources or enhancing anthropogenic removals by sinks of greenhouse gases ...". Said in other words, any country can invest in abroad projects in order to collect rewards in the form of ERU. It is expected that the JI mechanism will provide incentives for the development of environmental technologies and will channel physical and financial capitals to countries with in-transition economies thus promoting their sustainable economic growth. As we illustrate below, a GNEP can be used to asses the merits of the JI mechanism thus giving a valuable contribution to well founded strategies for the reduction of greenhouse gases. The following presentation is a modification of the one given in [17].

Let N be the number of countries (i.e. players) involved in the JI mechanism. For each country i, let e^i denote the emissions that result from its industrial production; we assume that these emissions are proportional to the industrial output of the country thus enabling us to express the revenue R of the country as a function of e^i. Emissions can be abated by investing in projects (e.g. installing filters, cleaning a river basin etc.) domestically or abroad. Let us indicate with I^i_j these environmental investments made by country i in country j. The benefit of this investment lies in the acquisition of ERUs, assumed here to be proportional to the investment, i.e. $\gamma_{ij} I^i_j$ (the coefficients γ_{ij} depend on both the investor, i, and the host country, j, because in general there is a dependence on both the investor's technologies and laws and the situation in the host country). The net emission in country i is given by $e^i - \sum_{j=1}^N \gamma_{ji} I^j_i$, which obviously cannot be negative. On the other hand, country i is accounted for the emission of $e^i - \sum_{j=1}^n \gamma_{ij} I^i_j$, that is, its own emissions minus the ERUs gained by investing in environmental projects; this quantity must be kept below a prescribed level E_i. To conclude the description of the problem, we also assume that pollution in one country can affect also other countries (for example pollution of a river in a country can affect another country which is crossed by the same river; acid rains are also influenced by air pollution in neighboring countries etc.). We therefore assume that damages from pollution in one country depend on the net emissions of all countries, according to a function $D_i\big(e^1 - \sum_{j=1}^N \gamma_{j1} I^j_1, \ldots, e^N - \sum_{j=1}^N \gamma_{jN} I^j_N\big)$. With this setting, the i-th player's problem becomes:

$$
\begin{array}{lll}
\min_{e^i, I^i_1, \ldots, I^i_N} & R_i(e^i) - \sum_{j=1}^N I^i_j - D_i\big(e^1 - \sum_{j=1}^N \gamma_{j1} I^j_1, \ldots, e^N - \sum_{j=1}^N \gamma_{jN} I^j_N\big) & \\
\text{s.t.} & e^i, I^i_1, \ldots, I^i_N \ \geq \ 0, & \\
& e^i - \sum_{j=1}^N \gamma_{ij} I^i_j \ \leq \ E_i, & \\
& e^i - \sum_{j=1}^N \gamma_{ji} I^j_i \ \geq \ 0, \quad j = 1, \ldots, N. &
\end{array}
$$

Note that in the resulting GNEP, the constraints of each problem involving other player's variables (the last N linear constraints) are the same for all players. This is precisely in the spirit of the JI mechanism.

3 Reformulations and the Jointly Convex Case

With this section, we begin a more in-depth examination of GNEPs. In our presentation, we will sometimes consider some special subclasses of GNEPs; note, however, that we won't explore in detail the properties of standard NEPs. Although this is obviously an extremely important subclass of GNEPs, the focus of this review is really on what happens in the case of "genuine" GNEPs, i.e. equilibrium problems where the feasible sets depend on the other player's decisions. Furthermore, the literature on pure NEPs is enormous and it would not be possible to review it in a single paper; we refer the interested reader to [9, 11, 24, 35, 41, 46, 74] as an entry point to this literature.

From now on, unless otherwise stated explicitly, we assume that all the objective functions satisfy the following continuity assumption.

Continuity Assumption For every player ν, the objective function θ_ν is C^0. □

3.1 Reformulations of the General GNEP

We begin our analysis by giving several equivalent formulations of the GNEP. On the one hand, these formulations shed some light on the connections between the GNEP and other better known problems while, on the other hand, they are often the basis for both theoretical and algorithmic developments. We note that probably all the reformulations given in this subsection are formally new, although they are the obvious extensions of results well known in the more specialized context of jointly convex GNEPs that will be introduced and discussed in Subsection 3.2.

We first introduce a function that historically played an important role in the study of the GNEP.

Definition 3.1 The mapping

$$\Psi(\boldsymbol{x}, \boldsymbol{y}) := \sum_{\nu=1}^{N} \left[\theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) - \theta_\nu(y^\nu, \boldsymbol{x}^{-\nu})\right]$$

is called the *Nikaido-Isoda-function* (*NI-function* for short) or the *Ky Fan-function* of the GNEP.

Note that both names are used in the GNEP literature and that the function Ψ depends on the utility functions of each player, but not on the strategy spaces. In particular, the NI-function for GNEPs is identical to the NI-function for NEPs; and actually the NI-function was first introduced in [80] as a tool to improve on Nash's original existence result for NEPs. The NI-function has a simple interpretation: Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are two feasible points for the GNEP, each summand in the definition represents the improvement in the objective function of player ν when he changes his action from x^ν to y^ν while all the other players stick to the choice $\boldsymbol{x}^{-\nu}$. It is rather intuitive, and simple to prove, that equilibria of the GNEP are characterized by the impossibility to get any improvement for any feasible choice $\boldsymbol{y}$; this is essentially the content of the following theorem.

Theorem 3.2 Let Ψ be the NI-function of the GNEP, and define

$$\boldsymbol{X}(\boldsymbol{x}) := \Pi_{\nu=1}^{N} X_\nu(\boldsymbol{x}^{-\nu}), \qquad \hat{V}(\boldsymbol{x}) := \sup_{\boldsymbol{y}\in\boldsymbol{X}(\boldsymbol{x})} \Psi(\boldsymbol{x}, \boldsymbol{y}). \tag{6}$$

Then the following statements hold:

(a) $\hat{V}(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \boldsymbol{X}(\boldsymbol{x})$.

(b) $\hat{V}(\bar{\boldsymbol{x}}) = 0$ and $\bar{\boldsymbol{x}} \in \boldsymbol{X}(\bar{\boldsymbol{x}})$ if and only if $\bar{\boldsymbol{x}}$ is a solution of the GNEP.

Theorem 3.2 characterizes the solutions of a GNEP as the set of points $\bar{\boldsymbol{x}} \in \boldsymbol{X}(\bar{\boldsymbol{x}})$ such that $0 = \hat{V}(\bar{\boldsymbol{x}}) \leq \hat{V}(\boldsymbol{x})$, for all $\boldsymbol{x} \in \boldsymbol{X}(\bar{\boldsymbol{x}})$. With a little abuse of notation we can say that $\bar{\boldsymbol{x}}$ is a solution of the GNEP if and only if it is a global minimizer with zero objective value of the problem

$$\min \ \hat{V}(\boldsymbol{x}) \quad \text{s.t.} \quad \boldsymbol{x} \in \boldsymbol{X}(\boldsymbol{x}) \tag{7}$$

that we call "constrained quasi-optimization problem". The term "quasi-optimization" is used to emphasize the fact that this is not a standard optimization problem, since the feasible set depends on the variable $\boldsymbol{x}$, and also to highlight the parallelism to quasi-variational inequalities we will discuss shortly. Although this reformulation is possibly of not great practical interest in the general case, it turns out to be a useful tool in the case of "jointly convex" problems to be discussed in Subsection 3.2. Furthermore, it is also the basis for some more useful optimization reformulations that will be discussed in Subsection 5.3. If we consider a NEP, problem (7) becomes a real optimization problem (since the set $\boldsymbol{X}(\boldsymbol{x})$ does not depend on $\boldsymbol{x}$ and is therefore fixed). Note, however, that the minimization of the objective function is still a challenging task since $\hat{V}$ is, in general, nondifferentiable.

A different kind of reformulation can be established under the following additional convexity assumption.

Convexity Assumption For every player ν and every $\boldsymbol{x}^{-\nu}$, the objective function $\theta_\nu(\cdot, \boldsymbol{x}^{-\nu})$ is convex and the set $X_\nu(\boldsymbol{x}^{-\nu})$ is closed and convex.

This assumption is very common and is often satisfied, especially in the economic applications that originated the GNEP. The following theorem holds under this assumption.

Theorem 3.3 Let a GNEP be given, satisfying the Convexity Assumption, and suppose further that the θ_ν are C^1 for all ν. Then, a point $\boldsymbol{x}$ is a generalized Nash equilibrium if and only if it is a solution of the quasi-variational inequality QVI $(\boldsymbol{X}(\boldsymbol{x}), \mathbf{F}(\boldsymbol{x}))$[3], where

$$\boldsymbol{X}(\boldsymbol{x}) := \Pi_{\nu=1}^N X_\nu(\boldsymbol{x}^{-\nu}), \qquad \mathbf{F}(\boldsymbol{x}) := (\nabla_{x^\nu}\theta_\nu(\boldsymbol{x}))_{\nu=1}^N.$$

The connection stated in the previous theorem was first noted in [14], is certainly illuminating and parallels the classical one showing that under the same Convexity Assumption, a NEP is equivalent to a VI. Obviously this latter result is a particular case of Theorem 3.3 when the sets X_ν do not depend on $\boldsymbol{x}^{-\nu}$. We restate this result below for completeness.

[3] The quasi-variational inequality problem QVI $(\boldsymbol{X}(\boldsymbol{x}), \mathbf{F}(\boldsymbol{x}))$ consists in finding a vector $\bar{\boldsymbol{x}} \in \boldsymbol{X}(\bar{\boldsymbol{x}})$ such that $(\boldsymbol{y} - \bar{\boldsymbol{x}})^T\mathbf{F}(\bar{\boldsymbol{x}}) \geq 0$ for all $\boldsymbol{y} \in \boldsymbol{X}(\bar{\boldsymbol{x}})$.

Corollary 3.4 Let a NEP be given, satisfying the Convexity Assumption, and suppose further that the θ_ν are C^1 for all ν. Then a point $\boldsymbol{x}$ is an equilibrium if and only if it is a solution of the variational inequality VI $(\boldsymbol{X}, \mathbf{F}(\boldsymbol{x}))$[4], where

$$\boldsymbol{X} := \Pi_{\nu=1}^N X_\nu, \qquad \mathbf{F}(\boldsymbol{x}) := (\nabla_{x^\nu}\theta_\nu(\boldsymbol{x}))_{\nu=1}^N.$$

Unfortunately, while Corollary 3.4 turned out to be very useful in the study of the NEP (see for example [35]), Theorem 3.3 has less interesting consequences since the theory for QVIs (see, e.g., [20, 43, 48]) is far less advanced than that for VIs. It is therefore of interest to see whether it is possible to reduce a GNEP to a VI, at least under some suitable conditions. In this respect it turns out that valuable results can be obtained for a special class of GNEPs, the "jointly convex" GNEP, that will be discussed in the next subsection.

An alternative characterization of the solutions of a GNEP can be obtained by a fixed-point inclusion. To this end, let Ψ once again be the NI-function, let $\hat{V}$ be the corresponding merit function from (6), and let

$$\hat{\boldsymbol{Y}}(\boldsymbol{x}) := \{\boldsymbol{y}_x \in \boldsymbol{X}(\boldsymbol{x}) \mid \hat{V}(\boldsymbol{x}) = \Psi(\boldsymbol{x}, \boldsymbol{y}_x)\} \tag{8}$$

be the (possibly empty) set of vectors where the supremum is attained in the definition of $\hat{V}$. Note that $\boldsymbol{x} \mapsto \hat{\boldsymbol{Y}}(\boldsymbol{x})$ is a point-to-set mapping. The fixed points of this function are precisely the solutions of the GNEP according to the following result.

Theorem 3.5 A vector $\bar{\boldsymbol{x}}$ is a solution of the GNEP if and only if $\bar{\boldsymbol{x}} \in \hat{\boldsymbol{Y}}(\bar{\boldsymbol{x}})$ holds.

In the particular case where $\hat{\boldsymbol{Y}}(\boldsymbol{x}) = \{\boldsymbol{y}(\boldsymbol{x})\}$ is single-valued for each $\boldsymbol{x}$, it therefore follows that $\bar{\boldsymbol{x}}$ is a solution of the GNEP if and only if $\bar{\boldsymbol{x}}$ solves the fixed point equation $\boldsymbol{x} = \boldsymbol{y}(\boldsymbol{x})$. In general, however, unless relatively strong assumptions like uniform convexity of the utility functions θ_ν hold, the set $\hat{\boldsymbol{Y}}(\boldsymbol{x})$ does not reduce to a singleton, making the fixed point inclusion a rather difficult problem. Note that the very definition of a generalized Nash equilibrium is also given in terms of a fixed point inclusion (via the solution mapping $\mathcal{S}$). The advantage of the fixed point characterization discussed above is that, in some cases, it can be used to develop some algorithms for the solution of the GNEP, as we shall discuss in Section 5.

The fact that all reformulations we have presented reduce to extremely difficult problems is not a deficiency in our analysis; rather it is a manifestation of the fact that the GNEP, in its general form, is an extremely hard problem. It is then in order to investigate if suitable subclasses of GNEPs are more amenable to fruitful analysis. To this end, we introduce in the following subsection one such subclass. Further subclasses will be mentioned in the conclusions section.

[4]The variational inequality problem VI $(\boldsymbol{X}, \mathbf{F}(\boldsymbol{x}))$ consists in finding a vector $\bar{\boldsymbol{x}} \in \boldsymbol{X}$ such that $(\boldsymbol{y} - \bar{\boldsymbol{x}})^T\mathbf{F}(\bar{\boldsymbol{x}}) \geq 0$ for all $\boldsymbol{y} \in \boldsymbol{X}$.

3.2 The Jointly Convex Case

We consider here a special class of GNEPs that is important since it arises in some interesting applications and for which a much more complete theory exists than for the general GNEP.

Definition 3.6 Let a GNEP be given, satisfying the Convexity Assumption. We say that this GNEP is *jointly convex* if for some closed convex $\boldsymbol{X} \subseteq \mathbb{R}^n$ and all $\nu = 1, \ldots, N$, we have

$$X_\nu(\boldsymbol{x}^{-\nu}) = \{x^\nu \in \mathbb{R}^{n_\nu} : (x^\nu, \boldsymbol{x}^{-\nu}) \in \boldsymbol{X}\}. \tag{9}$$

Note that the Example 1.1 and the model described in Subsection 2.3 (under convexity assumptions on the objective functions) are instances of jointly convex GNEPs.

Remark 3.7 When the sets $X_\nu(\boldsymbol{x}^{-\nu})$ are defined explicitly by a system of inequalities as in (2), then it is easy to check that (9) is equivalent to the requirement that $g^1 = g^2 = \cdots = g^N := g$ and that $g(\boldsymbol{x})$ be (componentwise) convex with respect to all variables $\boldsymbol{x}$; furthermore, in this case, it obviously holds that $\boldsymbol{X} = \{\boldsymbol{x} \in \mathbb{R}^n : g(\boldsymbol{x}) \leq 0\}$.

This class of problems has been first studied in detail in a seminal paper by Rosen [95] and has been often identified with the whole class of GNEPs. Jointly convex GNEPs are also often termed as GNEPs with "coupled contraints"; however, we prefer the more descriptive definition of jointly convex.

The reformulations introduced in the previous subsection simplify in the case of jointly convex GNEPs. We first consider the quasi-optimization reformulation (7). Since, in the special case of jointly convex constraints, it is easy to see that (cf. [51])

$$\boldsymbol{x}^\nu \in X_\nu(\boldsymbol{x}^{-\nu}) \ \ \forall \nu = 1, \ldots, N \Longleftrightarrow \boldsymbol{x} \in \boldsymbol{X},$$

problem (7) simplifies to the following "real" optimization problem:

$$\min \ \hat{V}(\boldsymbol{x}) \quad \text{s.t.} \quad \boldsymbol{x} \in \boldsymbol{X}, \tag{10}$$

where $\hat{V}$ is defined as in (6). Theorem 3.2 can therefore be rewritten in the following way.

Theorem 3.8 A vector $\bar{\boldsymbol{x}}$ is a solution of the GNEP in the jointly convex case if and only if $\bar{\boldsymbol{x}}$ is a global minimum of the optimization problem (10) with zero objective function value.

Note that, in spite of the simplification, the objective function $\hat{V}$ is still hard to compute in general and nondifferentiable; we will deal further with these issues in Section 5.

We now consider the quasi-variational inequality reformulation of the GNEP to see how the jointly convex structure can help to simplify, at least to a certain extent, things.

Theorem 3.9 Let a jointly convex GNEP be given with C^1-functions θ_ν. Then, every solution of the VI $(\boldsymbol{X}, \mathbf{F})$ (where $\boldsymbol{X}$ is the set in the definition of joint convexity and, as usual, $\mathbf{F}(\boldsymbol{x}) := (\nabla_{x^\nu}\theta_\nu(\boldsymbol{x}))_{\nu=1}^N$), is also a solution of the GNEP.

We remark that the above theorem does not say that any solution of a jointly convex GNEP is also a solution of the VI $(\boldsymbol{X}, \mathbf{F})$, and actually in the passage from the GNEP to the VI it is not difficult to see that "most" solutions are lost. We illustrate this with a simple example.

Example 1.1 (continued) In the Introduction we have shown that this game has infinitely many solutions given by $(\alpha, 1-\alpha)$ for every $\alpha \in [1/2, 1]$. Consider now the VI $(\boldsymbol{X}, \mathbf{F})$ where

$$\boldsymbol{X} \;=\; \{\, (x^1, x^2) \,\in\, \mathbb{R}^2 \,:\, x^1 + x^2 \leq 1 \,\}, \qquad \mathbf{F} \;=\; \begin{pmatrix} 2x^1 - 2 \\ 2x^2 - 1 \end{pmatrix}.$$

$\mathbf{F}$ is clearly strictly monotone and therefore this VI has a unique solution which is given by $(3/4, 1/4)$ as can be checked by using the definition of VI. Note that, as expected, this is a solution of the original GNEP.

Definition 3.10 Let a jointly convex GNEP be given with C^1-functions θ_ν. We call a solution of the GNEP that is also a solution of VI$(\boldsymbol{X}, \mathbf{F})$ a *variational equilibrium*.

The alternative name *normalized equilibrium* is also frequently used in the literature instead of *variational equilibrium*. In view of its close relation to a certain variational inequality problem, however, we prefer to use the term "variational inequality" here.

With this new terminology, the point $(3/4, 1/4)$ is the (unique) variational equilibrium of the problem in Example 1.1. Note that, by Corollary 3.4, in the case of NEPs the set of solutions and of variational solutions coincide. For GNEPs, however, every variational equilibrium is a generalized Nash equilibrium, but Example 1.1 shows that the converse is not true in general.

It may be interesting to see whether variational equilibria enjoy some relevant structural properties. In order to investigate this point, we need the following definitions.

Definition 3.11 Let $\boldsymbol{X} \subseteq \mathbb{R}^n = \mathbb{R}^{n_1 + \cdots + n_N}$ be a closed and convex set. The ν-th section of $\boldsymbol{X}$ at a point $\bar{\boldsymbol{x}}$ is the set $S_\nu(\bar{\boldsymbol{x}}) := \{\boldsymbol{x} \in \boldsymbol{X} : \boldsymbol{x}^{-\nu} = \bar{\boldsymbol{x}}^{-\nu}\}$, while we define the section of $\boldsymbol{X}$ at $\bar{\boldsymbol{x}}$ as the set $S(\bar{\boldsymbol{x}}) := \cup_{\nu=1}^N S_\nu(\bar{\boldsymbol{x}})$. Finally, we define the *internal cone* $I_{\boldsymbol{X}}(\bar{\boldsymbol{x}})$ to $\boldsymbol{X}$ at $\bar{\boldsymbol{x}}$ as the smallest closed, convex cone with vertex at the origin such that $\bar{\boldsymbol{x}} + I_{\boldsymbol{X}}(\bar{\boldsymbol{x}})$ contains $S(\bar{\boldsymbol{x}})$.

The above definition of internal cone can be used to refine the QVI reformulation in the case of a jointly convex GNEP.

Proposition 3.12 ([19]) Let a jointly convex GNEP be given with C^1-functions θ_ν. Then a point $\bar{\boldsymbol{x}}$ is an equilibrium if and only if it is a solution of the quasi-variational inequality QVI $((\bar{\boldsymbol{x}} + I_{\boldsymbol{X}}(\bar{\boldsymbol{x}})) \cap \boldsymbol{X}, \mathbf{F})$.

Note that $\mathcal{S}(\bar{\boldsymbol{x}}) \subseteq \boldsymbol{X}$ and, therefore, $I_{\boldsymbol{X}}(\bar{\boldsymbol{x}}) \subseteq T_{\boldsymbol{X}}(\bar{\boldsymbol{x}})$ (where $T_{\boldsymbol{X}}(\bar{\boldsymbol{x}})$ is the usual tangent cone to $\boldsymbol{X}$ at $\bar{\boldsymbol{x}}$). Since we therefore have

$$\bar{\boldsymbol{x}} + I_{\boldsymbol{X}}(\bar{\boldsymbol{x}}) \cap \boldsymbol{X} \;\subseteq\; \big(\bar{\boldsymbol{x}} + T_{\boldsymbol{X}}(\bar{\boldsymbol{x}})\big) \cap \boldsymbol{X} \;=\; \boldsymbol{X},$$

we see from the definition of variational equilibria and Proposition 3.12 that variational equilibria are "more socially stable" than other equilibria of the GNEP. In other words, given an equilibrium, no deviation from $\bar{\boldsymbol{x}}$ in the internal cone will be acceptable for the players, while if $\bar{\boldsymbol{x}}$ is a variational equilibrium, no deviation in the (possibly) larger tangent cone will we acceptable.

The results above indicate that the calculation of a variational equilibrium could be a valuable target for an algorithm. Furthermore, in some applicative contexts, variational equilibria can also have further practical interest, see, for example, the comments in [48].

In order to state a useful characterization of a variational equilibrium, let us introduce a variant of the function $\hat{V}$, that will also be useful later, in the development of algorithms. Set

$$V(\boldsymbol{x}) := \sup_{\boldsymbol{y} \in \boldsymbol{X}} \Psi(\boldsymbol{x}, \boldsymbol{y}). \tag{11}$$

Note that V is nonnegative over $\boldsymbol{X}$ and the difference between $\hat{V}$ and V is simply in the set over which the sup in the definition is taken. Furthermore, since for any $\boldsymbol{x} \in \boldsymbol{X}$ we have $\boldsymbol{X} \subseteq \boldsymbol{X}(\boldsymbol{x})$, it is obvious that $V(\boldsymbol{x}) \geq \hat{V}(\boldsymbol{x})$ for any $\boldsymbol{x} \in \boldsymbol{X}$. Therefore if for some $\bar{\boldsymbol{x}}$ we have $V(\bar{\boldsymbol{x}}) = 0$ then $\bar{\boldsymbol{x}}$ is a solution of the GNEP by Theorem 3.8. In fact, this property fully characterizes the set of variational equilibria.

Proposition 3.13 ([51]) Let a jointly convex GNEP be given with C^1-functions θ_ν. Then a point $\bar{\boldsymbol{x}}$ is a variational equilibrium if and only if $\bar{\boldsymbol{x}} \in \boldsymbol{X}$ and $V(\bar{\boldsymbol{x}}) = 0$ holds.

Proposition 3.13 motivates to call any point $\bar{\boldsymbol{x}} \in \boldsymbol{X}$ satisfying $V(\bar{\boldsymbol{x}}) = 0$ a variational equilibrium of a jointly convex GNEP. This definition is slightly more general than the previous one since it does not require any smoothness of the objective functions θ_ν.

Note that the set of solutions of a GNEP is unaffected if we scale the utility functions θ_ν by a positive number r_ν. This, however, is not true for the variational equilibria. To see this, consider Example 1.1 again, but with the objective function of the second player multiplied with the factor $r_2 := 2$ (whereas θ_1 remains unchanged). Then an easy calculation shows that the unique variational equilibrium is given by $\tilde{\boldsymbol{x}} = \left(\frac{2}{3}, \frac{1}{3}\right)$ which is different from the one given in Example 1.1. This observation immediately leads to the slightly different, original definition of a normalized equilibrium given in Rosen's paper [95].[5]

We conclude this subsection by observing that the jointly convex GNEP has been the subject of much analysis and certainly it covers some very interesting applications. However, it should be noted that the jointly convex assumption on the constraints is strong, and practically is likely to be satisfied only when the joint constraints $g^\nu = g$, $\nu = 1, \ldots, N$ are linear, i.e. of the form $A\boldsymbol{x} \leq b$ for some suitable matrix A and vector b.

[5]Rosen calls a vector $\bar{\boldsymbol{x}}$ a normalized equilibrium of a jointly convex GNEP if there exist positive numbers $r_\nu > 0$ such that, in our terminology, $\bar{\boldsymbol{x}}$ is a variational equilibrium of the game that is obtained from our jointly convex GNEP by multiplying the utility functions θ_ν by the factor r_ν.

4 Theory

4.1 Basic Existence Results

Existence of solutions has been the main focus of early research in GNEPs. The 1952 Debreu paper [27], where the GNEP was formally introduced, also gives the first existence theorem. This existence result was based on fixed-point arguments, and this turned out to be the main proof tool used in the literature. Essentially this approach is based on the very definition of equilibrium that states that a point $\boldsymbol{x}$ is an equilibrium if $\boldsymbol{x} \in \mathcal{S}(\boldsymbol{x})$, where $\mathcal{S} := \Pi_{\nu=1}^{N} \mathcal{S}_\nu$ with the solution mappings $\mathcal{S}_\nu$ of problem (1) as introduced in Section 1. This shows clearly that $\boldsymbol{x}$ is a fixed point of $\mathcal{S}$, thus paving the way to the application of the fixed-point machinery to establish existence of an equilibrium. There also exist some other approaches, an interesting one being the one presented in [46], where a continuation approach is used. The main existence result is probably the one established in [8]. We report below a slightly simplified version given by Ichiishi [57]. Recall that, as usual, the blanket Continuity Assumption is supposed to hold.

Theorem 4.1 Let a GNEP be given and suppose that

(a) There exist N nonempty, convex and compact sets $K_\nu \subset \mathbb{R}^{n_\nu}$ such that for every $\boldsymbol{x} \in \mathbb{R}^n$ with $x^\nu \in K_\nu$ for every ν, $X_\nu(\boldsymbol{x}^{-\nu})$ is nonempty, closed and convex, $X_\nu(\boldsymbol{x}^{-\nu}) \subseteq K_\nu$, and X_ν, as a point-to-set map, is both upper and lower semicontinuous [6].

(b) For every player ν, the function $\theta_\nu(\cdot, \boldsymbol{x}^{-\nu})$ is quasi-convex on $X_\nu(\boldsymbol{x}^{-\nu})$[7].

Then a generalized Nash equilibrium exist.

Remark 4.2 When the sets X_ν are defined by inequality constraints as in (2), the lower and upper semicontinuity requirements translate into reasonably mild conditions on the functions g^ν. See for example [10, 94].

The relaxation of the assumptions in the previous theorem has been the subject of a fairly intense study. Relaxations of the (a) continuity assumptions; (b) compactness assumptions and (c) quasi-convexity assumption have all been considered in the literature. The relaxation of the continuity assumption is the most interesting one, since it is peculiar to GNEPs. In fact, a number of classical problems in economics can be formulated as games with discontinuous objective functions. The best known of these are probably Bertrand's

[6] A point-to-set mapping $G : Y \rightrightarrows Z$ (with Y and Z metric spaces, for example closed subsets of an Euclidean space) is upper semicontinuous at $y \in Y$ if for every sequence $\{y^k\}$ in Y converging to $y \in Y$, and for every neighborhood U of $G(y)$ in Z, there exist $\bar{k}$ such that $G(y^k) \subseteq U$ for all $k \geq \bar{k}$. G is lower semicontinuous at $y \in Y$ if for every sequence $\{y^k\}$ in Y converging to $y \in Y$, and for every open subset $U$of Z, with $G(y) \cap U \neq \emptyset$, there exists $\bar{k}$ such that $G(y^k) \cap U \neq \emptyset$ for all $k \geq \bar{k}$. G is upper (lower) semicontinuous on Y if it is upper (lower) semicontinuous at each point of Y.

[7] A function $f : \mathbb{R}^t \to \mathbb{R}$ is quasi-convex if the level sets $\mathcal{L}(\alpha) := \{x \in \mathbb{R}^t : f(x) \leq \alpha\}$ are convex for every $\alpha \in \mathbb{R}$.

model of duopolistic price competition [16] and Hotelling's model of duopolistic spatial competition [55]. In the Bertrand model, firms choose prices, and the firm that charges the lower price supplies the whole market. In the Hotelling model, instead, firms choose locations and each firm monopolizes the part of the market closer to that firm than to the others. In each case, discontinuities arise when firms charge the same price or locate at the same point. There are, however, a host of other problems that give rise to games with discontinuous objective functions; good entry points to the literature on the subject are [13, 25, 26]. There are several papers where the relaxation of continuity is pursued; the seminal one is the 1986 paper by Dasgupta and Maskin [25], further developments and applications are discussed [13, 26, 91, 100, 104] and references therein. However, with the (partial) exception of [13], where jointly convex GNEPs are discussed, all these papers deal only with pure NEPs. The most general result for GNEPs seems to be the one in [72]. In order to present the main result in [72], we need the following definition.

Definition 4.3 Let $f : \mathcal{F} \subseteq \mathbb{R}^t \to \mathbb{R}$ be a function.

1. f is said to be *upper pseudocontinuous at* $x \in \mathcal{F}$, if for all $y \in \mathcal{F}$ such that $f(x) < f(y)$, we have $\limsup_{z \to x} f(z) < f(y)$, and f is said to be *upper pseudocontinuous on* $\mathcal{F}$ if it is upper pseudocontinuous at every point $x \in \mathcal{F}$.

2. f is said to be *lower pseudocontinuous* at $x \in \mathcal{F}$ (on $\mathcal{F}$) if $-f$ is upper pseudocontinuous at x (on $\mathcal{F}$).

3. f is said to be *pseudocontinuous* at $x \in \mathcal{F}$ (on $\mathcal{F}$) if it is both upper and lower pseudocontinuous at x (on $\mathcal{F}$).

We refer to [72] for a detailed discussion of pseudocontinuity, here we note only that upper and lower pseudocontinuity are a relaxation of upper and lower semicontinuity. The following result holds [72].

Theorem 4.4 Let a GNEP be given and suppose that the same assumptions of Theorem 4.1 hold, except that the objective functions are assumed to be pseudocontinuous instead of continuous. Then a generalized Nash equilibrium exists.

Relaxations of the compactness assumption in Theorem 4.1 have also been considered; this issue is rather slippery and although it can been expected that suitable coercivity assumptions on the objective functions can make up for the possible lack of (uniform) compactness of the feasible sets, caution must be exercised to avoid inappropriate generalizations (a good case in point being Corollary 4.2 of [11]).

Example 4.5 Take $N = 2, n_1 = n_2 = 1$ and set $(x, y) := (x_1^1, x_1^2)$ for simplicity. We consider a NEP where the objective functions of the two players are given by $\theta_1(x, y) := \frac{1}{2}x^2 - xy$ and $\theta_2(x, y) := \frac{1}{2}y^2 - (x + 1)y$ and the feasible sets are $X_1 = X_2 := \mathbb{R}$ (i.e. the player's subproblems are unconstrained). It is easy to see that in this case $\mathcal{S}_1(y) = y$ and $\mathcal{S}_2(x) = x + 1$. Therefore, by definition, a solution $(\bar{x}, \bar{y})$ of this game should satisfy the system $\bar{x} = \bar{y}$ and $\bar{y} = \bar{x} + 1$, which has no solutions. We then conclude that the NEP has no equilibria. Note that the two objective functions are strongly convex (and uniformly so with respect to the rival's variable).

Actually, already in [8] a relaxation of the compactness assumption was put forward that is rather peculiar to the economic model considered there. Further results on this topic can be found in [13, 19]; in both cases the NI-function plays a key role and, very roughly speaking, the condition that substitutes the compactness assumption in Theorem 4.1, is some kind of compactness of the level sets of the NI-function.

The relaxation of the quasi-convexity assumption is also of obvious interest. For example, if we make reference to economic applications, when noncompetitive markets are embedded into a general equilibrium framework, the quasi-convexity assumption stands out as an artificial addition extraneous to the basic nature of the model, see for example [66, 79]. There are not many results on this complex topic, relevant references are [13, 81].

4.2 KKT Conditions

It is not difficult to derive primal-dual conditions for the GNEP. Assume, for simplicity, that the problem is defined as in (1) with the sets $X_\nu(\boldsymbol{x}^{-\nu})$ given by (2). With this structure in place, and assuming all functions involved are C^1, we can easily write down the KKT conditions for each player's problem; the concatenation of all these KKT conditions gives us what we can call the KKT conditions of the GNEP. Let's make this more precise.

Suppose that $\bar{\boldsymbol{x}}$ is a solution of the GNEP. Then, if for player ν a suitable constraint qualification holds (for example, the Mangasarian-Fromovitz or the Slater constraint qualification), there is a vector $\bar{\lambda}^\nu \in \mathbb{R}^{m_\nu}$ of multipliers so that the classical Karush-Kuhn-Tucker (KKT) conditions

$$\begin{array}{rcl} \nabla_{x^\nu} L_\nu(x^\nu, \bar{\boldsymbol{x}}^{-\nu}, \lambda^\nu) & = & 0, \\ 0 \;\leq\; \lambda^\nu \;\perp\; -g^\nu(x^\nu, \bar{\boldsymbol{x}}^{-\nu}) & \geq & 0 \end{array}$$

are satisfied by $(\bar{x}^\nu, \bar{\lambda}^\nu)$, where $L_\nu(\boldsymbol{x}, \lambda^\nu) := \theta_\nu(\boldsymbol{x}) + g^\nu(\boldsymbol{x})^T \lambda^\nu$ is the Lagrangian associated with the ν-th player's optimization problem. Concatenating these N KKT systems, we obtain that if $\bar{\boldsymbol{x}}$ is a solution of the GNEP and if a suitable constraint qualification holds for all players, then a multiplier $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^m$ exists that together with $\bar{\boldsymbol{x}}$ satisfies the system

$$\begin{array}{rcl} \mathbf{L}(\boldsymbol{x}, \boldsymbol{\lambda}) & = & 0, \\ 0 \;\leq\; \boldsymbol{\lambda} \;\perp\; -\mathbf{g}(\boldsymbol{x}) & \geq & 0, \end{array} \qquad (12)$$

where

$$\boldsymbol{\lambda} := \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^N \end{pmatrix}, \quad \mathbf{g}(\boldsymbol{x}) := \begin{pmatrix} g^1(\boldsymbol{x}) \\ \vdots \\ g^N(\boldsymbol{x}) \end{pmatrix}, \text{ and } \quad \mathbf{L}(\boldsymbol{x}, \boldsymbol{\lambda}) := \begin{pmatrix} \nabla_{x^1} L_1(\boldsymbol{x}, \lambda^1) \\ \vdots \\ \nabla_{x^N} L_N(\boldsymbol{x}, \lambda^N) \end{pmatrix}.$$

Under a constraint qualification, system (12) can therefore be regarded as a first order necessary condition for the GNEP and indeed system (12) is akin to a KKT system. However, its structure is different from that of a classical KKT system. Under further convexity assumptions it can be easily seen that the $\boldsymbol{x}$-part of a solution of system (12) solves the GNEP so that (12) then turns out to be a sufficient condition as well.

Theorem 4.6 Let a GNEP be given defined by (1) and (2) and assume that all functions involved are continuously differentiable.

(a) Let $\bar{\boldsymbol{x}}$ be an equilibrium of the GNEP at which all the player's subproblems satisfy a constraint qualification. Then, a $\bar{\boldsymbol{\lambda}}$ exists that together with $\bar{\boldsymbol{x}}$ solves system (12).

(b) Assume that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}})$ solves the system (12) and that the GNEP satisfies the Convexity Assumption. Then $\bar{\boldsymbol{x}}$ is an equilibrium point of the GNEP.

Remark 4.7 The differentiability assumption on the problem functions involved can be relaxed by using some suitable notion of subdifferential. This is rather standard and we do not go into details on this point here.

Next consider the case of a jointly convex GNEP with the feasible set $\boldsymbol{X}$ having the explicit representation

$$\boldsymbol{X} = \{\boldsymbol{x} \in \mathbb{R}^n : g(\boldsymbol{x}) \leq 0\}$$

for some (componentwise) convex function $g : \mathbb{R}^n \to \mathbb{R}^m$, cf. Remark 3.7. Hence the strategy space for player ν is given by

$$X_\nu(\boldsymbol{x}^{-\nu}) = \{x^\nu : g(x^\nu, \boldsymbol{x}^{-\nu}) \leq 0\}$$

for all $\nu = 1, \ldots, N$. Similar to the previous discussion on general GNEPs, it follows that the KKT conditions of player ν-th optimization problem is given by

$$\begin{aligned} \nabla_{x^\nu}\theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) + \nabla_{x^\nu} g(x^\nu, \boldsymbol{x}^{-\nu})\lambda^\nu &= 0, \\ 0 \leq \lambda^\nu \perp -g(x^\nu, \boldsymbol{x}^{-\nu}) &\geq 0 \end{aligned} \tag{13}$$

for some multiplier $\lambda^\nu \in \mathbb{R}^m$. On the other hand, consider the corresponding VI $(\boldsymbol{X}, \mathbf{F})$ from Theorem 3.9. The KKT conditions of this VI (see [35]) are given by

$$\begin{aligned} \mathbf{F}(\boldsymbol{x}) + \nabla g(\boldsymbol{x})\lambda &= 0, \\ 0 \leq \lambda \perp -g(\boldsymbol{x}) &\geq 0 \end{aligned} \tag{14}$$

for some multiplier $\lambda \in \mathbb{R}^m$. The precise relation between these two KKT conditions and a GNEP solution is given in the following result which, basically, says that (14) holds if and only if (13) is satisfied with the same multiplier for all players ν or, in other words, that a solution of the GNEP is a variational equilibrium if and only if the shared constraints have the same multipliers for all the players.

Theorem 4.8 ([32, 48]) Consider the jointly convex GNEP with g, θ_ν being C^1. Then the following statements hold:

(a) Let $\bar{\boldsymbol{x}}$ be a solution of the VI $(\boldsymbol{X}, \mathbf{F})$ such that the KKT conditions (14) hold with some multiplier $\bar{\lambda}$. Then $\bar{\boldsymbol{x}}$ is a solution of the GNEP, and the corresponding KKT conditions (13) are satisfied with $\lambda^1 := \ldots := \lambda^N := \bar{\lambda}$.

(b) Conversely, assume that $\bar{\boldsymbol{x}}$ is a solution of the GNEP such that the KKT conditions (13) are satisfied with $\bar{\lambda}^1 = \ldots = \bar{\lambda}^N$. Then $(\bar{\boldsymbol{x}}, \bar{\lambda})$ with $\bar{\lambda} := \bar{\lambda}^1$ is a KKT point of VI $(\boldsymbol{X}, \mathbf{F})$, and $\bar{\boldsymbol{x}}$ itself is a solution of VI $(\boldsymbol{X}, \mathbf{F})$.

4.3 Uniqueness

Uniqueness of the solution is a classical topic in analyzing a mathematical programming problem, and is of obvious interest also in the case of GNEPs. In fact, in some applications, it may be claimed that a GNEP model makes sense only if it has a unique solution; for example, this is the position hold by many economists with respect to GNEPs. Unfortunately, GNEPs have the "tendency" to have non unique solutions and to present, in fact, manifolds of solutions; Example 1.1 is just a manifestation of this. This fact is part of the folklore on GNEPs and is well recognized by practitioners. To understand a bit more about this phenomenon, we can have a second look at the KKT conditions (12). It is well known that we can reformulate (12) as a (possibly nonsmooth) system of equations by using complementarity functions. A *complementarity function* $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a function such that

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a \geq 0,\ b \geq 0,\ ab = 0.$$

If ϕ is a complementarity function, then it is immediate to see that (12) can be reformulated as the *square* system

$$\mathbf{\Phi}(\boldsymbol{x}, \boldsymbol{\lambda}) := \begin{pmatrix} \mathbf{L}(\boldsymbol{x}, \boldsymbol{\lambda}) \\ \boldsymbol{\phi}(-\mathbf{g}(\boldsymbol{x}), \boldsymbol{\lambda}) \end{pmatrix} = 0, \tag{15}$$

where $\boldsymbol{\phi} : \mathbb{R}^{m+m} \to \mathbb{R}^m$ ($m = \sum_{\nu=1}^N m_\nu$) is defined, for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$, by

$$\boldsymbol{\phi}(\boldsymbol{a}, \boldsymbol{b}) := \begin{pmatrix} \phi(a_1, b_1) \\ \vdots \\ \phi(a_m, b_m) \end{pmatrix}.$$

Many complementarity functions are known (see, e.g., [35]). Probably the simplest one is the min function

$$\phi(a, b) := \min\{a, b\} \qquad \text{for all } a, b \in \mathbb{R},$$

which is the one we assume is used in the sequel. This obviously makes the function $\boldsymbol{\phi}$ nondifferentiable. Although it is possible to envisage differentiable complementarity functions ϕ, it is well known that the use of a nondifferentiable ϕ is advantageous from many points of view (see [35]; this topic will also be taken up again in Subsection 5.6). Assume now this simple setting: the GNEP we are considering is jointly convex (therefore $g^\nu = g$ for all players ν) and $\bar{\boldsymbol{x}}$ is a solution with $\bar{\boldsymbol{\lambda}}$ being a corresponding Lagrange multiplier. Assume further that the gradients of the active constraints are linearly independent and strict complementarity holds (i.e., for all players, if a constraint is active, the corresponding multiplier is positive). Note that it is difficult to think of a "better behaved" GNEP: we are in the particularly simple jointly convex case, and all kind of regularities one may wish for are satisfied. Assume now, to avoid a trivial case, that in $\bar{\boldsymbol{x}}$ at least one constraint $\mathbf{g}_i$ is active for two players, let's say players 1 and 2 (this means that, locally, the GNEP "does not behave" like a NEP). Straightforward calculations show that under these conditions, $\boldsymbol{\phi}$ is differentiable at $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}})$ and that its Jacobian is singular due to the presence of two rows that are equal in correspondence to the gradients of the constraint $\mathbf{g}_i$ for players 1 and 2. Some further, elementary elaboration based on the implicit function theorem leads easily to the following result.

Proposition 4.9 ([33]) In the setting described above, the solution $\bar{\boldsymbol{x}}$ is a non isolated solution of the GNEP and $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}})$ is a nonisolated solution of the system (12).

Global uniqueness results can certainly be obtained, but usually only in the context of specific applicative contexts where the structure of the problem can be suitably exploited; we do not go into detail on this here.

For jointly convex GNEPs one can hope for global uniqueness of a variational equilibrium. Example 1.1 (continued) illustrates such a case. The theory of VI indicates that a simple condition to have this is to require that $\mathbf{F}$ be strictly monotone; this is exactly the case of Example 1.1 continued. Note, however, that this is still a strong requirement.

Local uniqueness could also be of interest. This issue has not been much considered in the literature, though, with the significant exception of [29]. Note that Proposition 4.9 shows that even local uniqueness can easily be in jeopardy. In any case, it should be possible to derive some sensible conditions using, for example, the KKT conditions or some suitable conditions on any of the reformulations we described in Section 3.1.

4.4 Stability

Stability of the solution, when data are varied, is another classical topic in mathematical programming. This issue has been analyzed in some detail for NEPs, see for example [3, 18, 23, 39, 68, 69, 70]. Obviously, for NEPs one can use its reduction to VI and then apply the well-developed sensitivity theory existing for the latter class of problems [35].

However, when it comes to GNEPs in their full generality, very few results are available. Let a GNEP be parametrized by a parameter $p \in \mathbb{R}^t$; by this we mean that we have a GNEP(p) for each value of p in a suitable set $P \subseteq \mathbb{R}^t$, which is defined by the functions $\theta_\nu(p, \cdot)$ and $X_\nu(p, \cdot)$. Denote by $\mathcal{S}(p)$ the solution set of GNEP(p). The following result is proved in [71]. It is rather intuitive and its main interest lies in the minimal continuity assumptions adopted.

Theorem 4.10 Let a family of GNEP(p) be given, satisfying the following assumptions for every $\nu = 1, \ldots, N$ and for some $\bar{p} \in P$:

(a) θ_ν is pseudocontinuous at $(\bar{p}, \boldsymbol{x})$ for every $\boldsymbol{x}$ such that $x^\nu \in X_\nu(\bar{p}, \boldsymbol{x}^{-\nu})$;

(b) X_ν is upper and lower semicontinuous at $(\bar{p}, \boldsymbol{x}^{-\nu})$ for every $\boldsymbol{x}^{-\nu}$.

Let $\{p^k\}$ be a sequence such that $p^k \in P$ for every k and $\{p^k\} \to \bar{p}$, and let $\{\boldsymbol{x}^k\} \to \bar{\boldsymbol{x}}$, with $\boldsymbol{x}^k \in \mathcal{S}(p^k)$ for every k. Then $\bar{\boldsymbol{x}}$ is a solution of GNEP($\bar{p}$).

Note that even if we assume that $\mathcal{S}(\bar{p}) \neq \emptyset$, the theorem above neither says anything about the solvability of GNEP(p) when p is close to $\bar{p}$, nor gives any quantitative result about the solutions of the unperturbed problem and of the perturbed ones. These are usually the difficult issues one has to deal with in analyzing sensitivity results. It seems there is huge room for improvements in this respect; we are only aware of some partial results in this direction that, however, can only be applied to GNEPs with a very particular structure, see [85]. See also [59] and references therein for some related work.

5 Algorithms

In this section, we discuss algorithms for the solution of GNEPs. This topic is currently a very active research field, and there are many proposals. Our focus will be on methods for general GNEPs and jointly convex GNEPs, not on specific methods that can be developed for particular applications of GNEPs by taking into account their more specialized structure. We make clear from the outset that, in spite of the many proposals, it is probably safe to say that, at present, almost no algorithm can be shown to be globally convergent under clear or reasonable assumptions; certainly, there is still a lot of theoretical work needed in order to develop a reliable convergence theory for GNEPs. The only case for which some more interesting results have been obtained is when the GNEP has a jointly convex structure. Numerical experience with all these algorithms is still very limited, and it is not easy to assess which method is more promising in practice.

5.1 Practitioners Methods

Under this heading we present some methods that are most popular among practitioners and whose rationale is particular simple to grasp. They are “natural” decomposition method, be it of Jacobi- or Gauss-Seidel-type (see, e.g., [98] for the well-known counterparts of these methods in the case of systems of linear equations). Consider the general GNEP where the subproblem of player ν is given by

$$\min_{x^\nu} \theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) \quad \text{s.t.} \quad x^\nu \in X_\nu(\boldsymbol{x}^{-\nu}).$$

We first describe the nonlinear Jacobi-type method.

Algorithm 5.1 (Nonlinear Jacobi-type Method)

(S.0) Choose a starting point $\boldsymbol{x}^0 = \big(x^{0,1}, \dots, x^{0,N}\big)$, and set $k := 0$.

(S.1) If $\boldsymbol{x}^k$ satisfies a suitable termination criterion: STOP.

(S.2) `FOR` $\nu = 1, \dots, N$

Compute a solution $\boldsymbol{x}^{k+1,\nu}$ of

$$\min_{x^\nu} \ \theta_\nu(x^\nu, \boldsymbol{x}^{k,-\nu}) \quad \text{s.t.} \quad x^\nu \in X_\nu(\boldsymbol{x}^{k,-\nu}).$$

`END`

(S.3) Set $\boldsymbol{x}^{k+1} := \big(x^{k+1,1}, \dots, x^{k+1,N}\big)$, $k \leftarrow k+1$, and go to (S.1).

At each iteration k, Algorithm 5.1 has to solve N optimization problems in (S.2): For each $\nu \in \{1, \dots, N\}$ the objective function

$$\theta_\nu \left(x^{k,1}, \dots, x^{k,\nu-1}, x^\nu, x^{k,\nu+1}, \dots, x^{k,N}\right) \tag{16}$$

has to be minimized over all $x^\nu \in X_\nu(\boldsymbol{x}^{-\nu})$, whereas all block variables $x^{k,\mu}$ of the other players $\mu \neq \nu$ are fixed. However, this version does not use the newest information, since, when computing x^ν, we already have the new variables $x^{k+1,1}, \dots, x^{k+1,\nu-1}$ and may use them instead of $x^{k,1}, \dots, x^{k,\nu-1}$. In fact, we can use these variables both in θ_ν and in the feasible sets. In this way, we obtain the following Gauss-Seidel-type method.

Algorithm 5.2 (Nonlinear Gauss-Seidel-type Method)

(S.0) Choose a starting point $\boldsymbol{x}^0 = \big(x^{0,1}, \dots, x^{0,N}\big)$, and set $k := 0$.

(S.1) If $\boldsymbol{x}^k$ satisfies a suitable termination criterion: STOP.

(S.2) `FOR` $\nu = 1, \dots, N$

Compute a solution $x^{k+1,\nu}$ of

$$\begin{array}{ll} \min_{x^\nu} & \theta_\nu \left(x^{k+1,1}, \dots, x^{k+1,\nu-1}, x^\nu, x^{k,\nu+1}, \dots, x^{k,N}\right) \\ \text{s.t.} & x^\nu \in X_\nu(x^{k+1,1}, \dots, x^{k+1,\nu-1}, x^{k,\nu+1}, \dots x^{k,N}). \end{array} \tag{17}$$

`END`

(S.3) Set $\boldsymbol{x}^{k+1} := \big(x^{k+1,1}, \dots, x^{k+1,N}\big)$, $k \leftarrow k+1$, and go to (S.1).

While conceptually quite simple, the convergence properties of both Algorithm 5.1 and Algorithm 5.2 are not well-understood. Even in the simplest case of a standard NEP, it is known and easy to prove that $\bar{\boldsymbol{x}}$ is a Nash equilibrium if the entire sequence $\{\boldsymbol{x}^k\}$ (provided that it exists!) generated by one of these methods converges to this point $\bar{\boldsymbol{x}}$. This result is not necessarily true if $\bar{\boldsymbol{x}}$ is only an accumulation point of such a sequence. Conditions which guarantee the convergence of the whole sequence $\{\boldsymbol{x}^k\}$, however, are typically not known or extremely restrictive. The situation becomes even more complicate for GNEPs where additional properties of the constraints are required in order to prove suitable convergence results.

In some applications, however, convergence of these methods can be shown, see [86] for an example. The special case in which the objective functions of the GNEP do not depend on the other players' variables (and a few more technical assumptions hold) is analyzed in [37]. It is shown there that a modification of the Gauss-Seidel method from Algorithm 5.2, where proximal terms are added in the objective functions of the subproblems (17), so that the subproblems solved in step (S.2) become

$$\begin{array}{ll} \min_{x^\nu} & \theta_\nu\left(x^{k+1,1},\ldots,x^{k+1,\nu-1},x^\nu,x^{k,\nu+1},\ldots,x^{k,N}\right)+\tau^k\|x^\nu-x^{k,\nu}\|^2 \\ \text{s.t.} & x^\nu \in X_\nu(x^{k+1,1},\ldots,x^{k+1,\nu-1},x^{k,\nu+1},\ldots x^{k,N}), \end{array}$$

(with $\tau^k > 0$ and possibly tending to 0), has significant convergence properties. Among others, under a convexity assumption, every limit point of the sequence produced by Algorithm 5.2 is a solution of the GNEP (no need for convergence of the whole sequence).

Comments. The methods described in this subsection are most straightforward and easy to implement and this explains their popularity among practitioners. However, at present, they can be considered, at most, good and simple heuristics.

5.2 VI-type Methods

First consider the GNEP with general constraints given by (2). Theorem 4.6 (b) shows that, under convexity and differentiability assumptions, a KKT point, i.e. a solution of system (12), will yield a solution of the GNEP. Note that (12) is a mixed complementarity problem, i.e. a special variational inequality, for which efficient solvers are available [30, 35, 73]. This paves the way to several solution approaches to the solution of GNEPs. Unfortunately this statement has to be immediately qualified, since the convergence requirements for these methods are not easily applicable to the KKT system (12), and the conditions one obtains this way are rather unnatural in terms of the original GNEP and are not at all clear. Garcia and Zangwill [46] advocate the use of homotopy methods for the solution of the KKT conditions (12). While this approach is theoretically well founded and enjoys strong global convergence properties, it is well known that in practice homotopy methods fail to solve problems as soon as the dimension of the problem itself becomes realistic. In this context, it should also be mentioned that the VI approach increases the dimension from n to $n+m$ (recall that $m = \sum_{\nu=1}^N m_\nu$), so the VI problem could be significantly larger depending on the overall number of constraints.

The situation is somewhat better for jointly convex GNEPs. In this case we can apply Theorem 3.9. According to this result, we can calculate a solution of the GNEP by finding a solution of the corresponding VI$(\boldsymbol{X},\mathbf{F})$ (where $\boldsymbol{X}$ is the set in the definition of joint convexity and $\mathbf{F}(\boldsymbol{x}) := (\nabla_{x^\nu}\theta_\nu(\boldsymbol{x}))_{\nu=1}^N$). In principle then a (variational) equilibrium can be found by solving the variational inequality problem. Since there are plenty of algorithms available for VIs (see, e.g., [35]), we therefore obtain a whole bunch of methods for the solution of jointly convex GNEPs. However, the conditions for convergence derived in this way are very restrictive at best. To give a feel of what one can expect, consider the case in which $\mathbf{F}$ is monotone on $\boldsymbol{X}$[8]. It is known [35] that this is one of weakest conditions under which global convergence can be proved for the VI($\boldsymbol{X}, \mathbf{F}$). The monotonicity of the defining function is a standard and well accepted assumption in the VI theory, and it is satisfied in many practical applications. However, when one looks at the specific structure of $\mathbf{F}$, it is easy to see that the monotonicity assumption implies a connection among the θ_ν that cannot be expected to hold in general. To see this better assume that $\mathbf{F}$ is continuously differentiable. Then it is well known that $\mathbf{F}$ is monotone if and only if its Jacobian is positive semidefinite on $\boldsymbol{X}$. We have

$$J\mathbf{F} = \begin{pmatrix} \nabla_{x^1}\theta_1 & \nabla_{x^2}\theta_1 & \cdots & \nabla_{x^N}\theta_1 \\ \nabla_{x^1}\theta_2 & \nabla_{x^2}\theta_2 & \cdots & \nabla_{x^N}\theta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \nabla_{x^1}\theta_N & \nabla_{x^2}\theta_N & \cdots & \nabla_{x^N}\theta_N \end{pmatrix}.$$

It should be clear that requiring the positive semidefiniteness of this matrix amounts to making a very strong assumption on the structure and relations of the objective functions θ_ν. Note that the diagonal blocks of this matrix are positive semidefinite under the Convexity Assumption. Therefore, roughly speaking, diagonal dominance of these blocks would ensure positive semidefiniteness of the whole matrix. This can be interpreted as the fact that player ν has "more influence" on his objective function than the other players do, and can therefore be expected to hold is some applications. But it should be clear that, in general, this is a very strong requirement.

A possible disadvantage of this VI-approach is the fact that one can compute variational equilibria only, which excludes possible other solutions that might be of interest from a practical point of view. To circumvent this problem, one can alternatively use the characterization of all solutions of the GNEP as a QVI from Theorem 3.3. This characterization is true for a general (not necessarily jointly convex) GNEP. However, although there do exist a few ideas for solving QVIs (see, e.g., [42, 61]), none of these ideas can be viewed as an efficient and robust tool for solving GNEPs since the numerical solution of QVIs itself is still a highly difficult problem.

Comments. The direct solution of the KKT conditions in order to develop globally convergent algorithms seems very appealing and quite simple (actually, we could also have included this approach in the previous subsection). However the methods proposed in the literature are deficient either on the theoretical or on the practical side. The VI reduction of a jointly convex problem allows us to use well established methods for the solution of a VI. The disadvantage is that only variational equilibria can be computed this way and that the resulting assumptions are rather stringent. Technically speaking, probably the weakest assumption under which one can ensure convergence of an algorithm for the solution of

[8] $\mathbf{F}$ is monotone on the set $\boldsymbol{X}$ if, for any two $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$, it holds that $(\mathbf{F}(\boldsymbol{x}) - \mathbf{F}(\boldsymbol{y}))^T(\boldsymbol{x} - \boldsymbol{y}) \geq 0$.

the VI($\boldsymbol{X}, \mathbf{F}$) is that $\mathbf{F}$ be pseudo-monotone with respect to the solution set of the VI[9] (see [35, Chapter 12]). As observed in [32], although not mild, this is in any case a weaker assumption than those required by other methods for the solution of jointly convex GNEPs to be discussed in following subsections.

5.3 NI-Function-type Methods

Consider the general (not necessarily jointly convex) GNEP first. In principle, Theorem 3.2 allows us to apply optimization techniques to the constrained optimization problem (7) in order to solve the GNEP. Similarly, Theorem 3.5 also motivates the application of suitable fixed-point methods in order to solve the GNEP. However, due to the complications of these reformulations, none of these approaches has, so far, been investigated in the literature. In fact, all papers that we are currently aware of and that apply the NI-function in some way to solve the GNEP are dealing with the jointly convex case. Hence, in this subsection, we always assume that the GNEP is jointly convex.

In this situation, Theorem 3.8 guarantees that we have the reformulation (10) of the GNEP as a constrained optimization problem where the feasible set has a much simpler structure as in the reformulation (7). However, as noted before, the objective function of this program is (usually) still nonsmooth. In order to avoid this nonsmoothness, we first introduce a suitable modification of the NI-function that was proposed in [47] in the context of standard NEPs and later applied to GNEPs in [51].

Definition 5.3 Given a parameter $\gamma > 0$, the mapping

$$\Psi_\gamma(\boldsymbol{x}, \boldsymbol{y}) := \sum_{\nu=1}^{N} \left[\theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) - \theta_\nu(y^\nu, \boldsymbol{x}^{-\nu}) - \frac{\gamma}{2} \|x^\nu - y^\nu\|^2 \right]. \tag{18}$$

is called the *regularized Nikaido-Isoda-function* (*regularized NI-function* for short) of the GNEP.

Let Ψ denote the standard NI-function from Definition 3.1, then the regularized NI-function can be written as

$$\Psi_\gamma(\boldsymbol{x}, \boldsymbol{y}) = \Psi(\boldsymbol{x}, \boldsymbol{y}) - \frac{\gamma}{2} \|\boldsymbol{x} - \boldsymbol{y}\|^2.$$

Let us also denote by

$$\begin{aligned} V_\gamma(x) &:= \max_{\boldsymbol{y} \in \boldsymbol{X}} \Psi_\gamma(\boldsymbol{x}, \boldsymbol{y}) \\ &= \max_{\boldsymbol{y} \in \boldsymbol{X}} \sum_{\nu=1}^{N} \left[\theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) - \theta_\nu(y^\nu, \boldsymbol{x}^{-\nu}) - \frac{\gamma}{2} \|x^\nu - y^\nu\|^2 \right] \end{aligned} \tag{19}$$

[9]The function $\mathbf{F}$ is pseudo-monotone with respect to the solution set of the VI($\boldsymbol{X}, \mathbf{F}$), if the solution set of the VI is nonempty and for every solution $\bar{\boldsymbol{x}}$ it holds that $\mathbf{F}(\boldsymbol{y})^T(\boldsymbol{y} - \bar{\boldsymbol{x}}) \geq 0$ for all $\boldsymbol{y} \in \boldsymbol{X}$. Note that pseudo-monotonicity with respect to the solution set is obviously implied by the monotonicity of $\mathbf{F}$.

$$= \max_{\boldsymbol{y} \in \boldsymbol{X}} \sum_{\nu=1}^{N} \left[\theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) - \theta_\nu(y^\nu, \boldsymbol{x}^{-\nu})\right] - \frac{\gamma}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2$$

the corresponding merit function. Note that, for standard NEPs, this mapping V_γ coincides with the previously defined mapping V from (11) (but is different from the function $\hat{V}$ from (7)) when $\gamma = 0$.

The following result is the counterpart of Proposition 3.13 (for the mapping V) and Theorem 3.2 (for the mapping $\hat{V}$) and shows, in particular, that the mapping V_γ has similar properties as V and $\hat{V}$ and that, in addition, it turns out to be continuously differentiable for each $\gamma > 0$. On the other hand, note that it gives a characterization of variational equilibria only, whereas $\hat{V}$ provides a complete characterization of all solutions of a GNEP.

Theorem 5.4 ([51]) Consider a jointly convex GNEP. Then the regularized function V_γ has the following properties:

(a) $V_\gamma(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \boldsymbol{X}$.

(b) $\bar{\boldsymbol{x}}$ is a variational equilibrium if and only if $\bar{\boldsymbol{x}} \in \boldsymbol{X}$ and $V_\gamma(\bar{\boldsymbol{x}}) = 0$.

(c) For every $\boldsymbol{x} \in \boldsymbol{X}$, there exists a unique maximizer $\boldsymbol{y}_\gamma(\boldsymbol{x})$ such that

$$\operatorname{argmax}_{\boldsymbol{y} \in \boldsymbol{X}} \left[\Psi(\boldsymbol{x}, \boldsymbol{y}) - \frac{\gamma}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2\right] = \boldsymbol{y}_\gamma(\boldsymbol{x}),$$

and $\boldsymbol{y}_\gamma(\boldsymbol{x})$ is continuous in $\boldsymbol{x}$.

(d) The mapping V_γ is continuously differentiable if all θ_ν are continuously differentiable.

Note that statements (a), (b), and (c) hold without any smoothness of the mappings θ_ν. Using the first two statements of Theorem 5.4, we see that finding a solution of the GNEP is equivalent to computing a global minimum of the constrained optimization problem

$$\min V_\gamma(\boldsymbol{x}) \quad \text{s.t.} \quad \boldsymbol{x} \in \boldsymbol{X}.$$

The last statement of Theorem 5.4 shows that the new objective function overcomes one of the deficiencies of the mappings V and $\hat{V}$ that was used in the related optimization reformulations (7) and (10).

The following result shows that the definition of the mapping V_γ can also be used in order to get a fixed point characterization of the GNEP, cf. Theorem 3.5.

Theorem 5.5 ([51]) Let $\boldsymbol{y}_\gamma(\boldsymbol{x})$ be the vector defined in Theorem 5.4 (c) as the unique maximizer in the definition of the regularized function V_γ from (19). Then $\bar{\boldsymbol{x}}$ is a solution of GNEP if and only if $\bar{\boldsymbol{x}}$ is a fixed point of the mapping $\boldsymbol{x} \mapsto \boldsymbol{y}_\gamma(\boldsymbol{x})$, i.e., if and only if $\bar{\boldsymbol{x}} = \boldsymbol{y}_\gamma(\bar{\boldsymbol{x}})$.

Using the difference of two regularized NI-functions, it is also possible to reformulate the GNEP as an unconstrained optimziation problem. In fact, let $0 < \alpha < \beta$ be two given parameters, let Ψ_α, Ψ_β denote the corresponding regularized NI-functions, and let V_α, V_β be the corresponding merit functions. Then define

$$V_{\alpha\beta}(\boldsymbol{x}) := V_\alpha(\boldsymbol{x}) - V_\beta(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

This mapping has the following properties.

Theorem 5.6 ([51]) Under the assumption of Theorem 5.4, the following statements about the function $V_{\alpha\beta}$ hold:

(a) $V_{\alpha\beta}(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

(b) $\bar{\boldsymbol{x}}$ is a variational equilibrium of the GNEP if and only if $\bar{\boldsymbol{x}}$ is a global minimum of $V_{\alpha\beta}$ with $V_{\alpha\beta}(\bar{\boldsymbol{x}}) = 0$.

(c) The mapping $V_{\alpha\beta}$ is continuously differentiable if all θ_ν are continuously differentiable.

Similar to Theorem 5.4, we note that the differentiability assumption is needed in the previous result only for statement (c), whereas the other two statements hold without any smoothness assumption on the utility mappings θ_ν.

Theorem 5.6 shows that the variational equilibria of a GNEP are precisely the global minima of the *unconstrained* optimization problem

$$\min V_{\alpha\beta}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{20}$$

In view of Theorem 5.6 (c), this is a continuously differentiable optimization problem. In general, it is not twice continuously differentiable since V_γ is not twice continuously differentiable. However, under additional assumptions, one can show that the mapping V_γ (and, therefore, also the objective function $V_{\alpha\beta}$) is an SC^1-mapping, i.e. the gradient ∇V_γ is semismooth (see [84, 88, 89] for more details on semismoothness). This, in turn, allows the application of locally fast convergent Newton-type methods, cf. [52] for more details.

Another approach that is based on the NI-function and that exploits the fixed-point characterization from Theorem 3.5 is the *relaxation method* [102], that uses the recursion

$$\boldsymbol{x}^{k+1} := (1 - t_k)\boldsymbol{x}^k + t_k \boldsymbol{y}^k,$$

where $\boldsymbol{y}^k$ is an element of the set $\hat{\boldsymbol{Y}}(\boldsymbol{x}^k)$ defined in (8) and where the stepsize t_k satisfies the conditions

$$t_k \in (0,1] \ \forall k \in \mathbb{N}, \quad t_k \to 0, \quad \sum_{k=0}^{\infty} t_k = \infty,$$

or [65] the (optimal) choice

$$t_k := \arg \min_{t_k \in (0,1]} \hat{V}(1 - t_k)\boldsymbol{x}^k + t_k \boldsymbol{y}^k).$$

We see that, in the relaxation method, the new iterate is constructed as a weighted average of the "improved" point $\boldsymbol{y}^k$ and the current iteration. Under a number of technical assumptions, including the condition that the set $\hat{\boldsymbol{Y}}(\boldsymbol{x})$ is single-valued and continuous for all $\boldsymbol{x}$, it is shown in [102] that the relaxation method converges to a variational equilibrium of the GNEP. It is interesting to note that, from the theoretical point of view, the differentiability of the objective functions θ_ν is not required.

The relaxation method has been applied successfully to some applications of NEPs and GNEPs, see, e.g., [1, 2, 15, 21, 63, 65]. The relaxation method is well reviewed in [64].

Comments. The direct (constrained or uncontrained) optimization of one of the several variants of the function $\hat{V}$ introduced in this subsection is very appealing, since optimization methods are extremely reliable and well understood. Note, however, that, as usual, little is known regarding conditions on the GNEP that guarantee the verification of favorable conditions of one of these functions. Furthermore, the mere evaluation of any of these variants of $\hat{V}$ entails the solution of a constrained optimization problem. This last disadvantage is shared by the relaxation method. On the other hand, it must be said that the relaxation method is probably the only method for a sufficiently large class of problems for which, as indicated before, a certain practical experience has been gathered. As a final observation on this method, we remark that the conditions under which the method converges imply that the $\mathbf{F}$ is strictly monotone [64]. This shows that, from the theoretical point of view, the relaxation methods holds little advantage in comparison to the (in some cases much simpler) methods based on the VI-reformulation discussed in the previous subsection. It would be interesting to see a numerical comparison of these two classes of methods.

5.4 Penalty Methods

Another idea that comes from constrained optimization is to get rid of the complicated joint constraints in a GNEP and to solve a (possibly infinite sequence of) standard NEP(s) by adding the (difficult) joint constraints as a penalty term to the objective function of each player. This approach has been advocated for the first time in [43]. In this paper, a sequential penalty/augemnted Lagrangian-type method is analyzed whereas at each iteration a NEP is solved whose objective function is obtained by summing the original objective function and a smooth term involving the joint constraints and a penalty parameter that goes to infinity as the process progresses. Since the description of this method is rather complicated, we prefer to present its simpler "exact" counterpart, analyzed in [34, 36].

Consider the general GNEP

$$\min_{x^\nu} \ \theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) \quad \text{s.t.} \quad x^\nu \in X_\nu(\boldsymbol{x}^{-\nu})$$

with strategy spaces $X_\nu(\boldsymbol{x}^{-\nu})$ given by (2) for some smooth mappings $g^\nu : \mathbb{R}^n \to \mathbb{R}^{m_\nu}$. Let $\rho_\nu > 0$ be a penalty parameter and consider the penalized problem, where each player ν tries to solve the optimization problem

$$\min_{x^\nu} \ P_\nu(\boldsymbol{x}; \rho_\nu) := \theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) + \rho_\nu \big\| g^\nu_+(x^\nu, \boldsymbol{x}^{-\nu}) \big\|, \tag{21}$$

where $\| \cdot \|$ denotes the Euclidean norm. Note that this penalized problem is a standard (unconstrained) NEP (though having nonsmooth objective function). The corresponding penalty method then looks as follows.

Algorithm 5.7 (Exact Penalty-type Method)

(S.0) Choose $\boldsymbol{x}^0 \in \mathbb{R}^n$, $\rho_\nu > 0$ and $c_\nu \in (0, 1)$ for $\nu = 1, \dots, N$, set $k := 0$.

(S.1) If $\boldsymbol{x}^k$ satisfies a suitable termination criterion: STOP.

(S.2) Let $I^k := \{\nu : x^{k,\nu} \notin X_\nu(\boldsymbol{x}^{k,-\nu})\}$. For every $\nu \in I^k$, if

$$\|\nabla_{x^\nu} \theta_\nu(x^{k,\nu}, \boldsymbol{x}^{k,-\nu})\| > c_\nu \left[\rho_\nu \left\| \nabla_{x^\nu} \|g_+^\nu(x^{k,\nu}, \boldsymbol{x}^{k,-\nu})\| \right\| \right], \tag{22}$$

then double ρ_ν.

(S.3) Compute a solution of the penalized problem (21), set $k \leftarrow k + 1$, and go to (S.1).

Note that other penalty updating schemes than those from (S.2) are possible, and indeed [34] uses another technique for jointly convex GNEPs in order to obtain stronger convergence results in this particular case. In any event, the key point is that, under suitable conditions, after a finite number of possible updates of the penalty parameter, the solution of the penalized problem (21) is also a solution of the original GNEP. Note that, although structurally non differentiable, the penalized NEPs are unconstrained.

Comments. The main drawback of the exact penalty method we just described is that the penalized problem (21) may be very difficult to solve in practice. Similar observations also hold for the sequential penalty method in [43], the difference being that in this latter case the penalized problems are differentiable (assuming θ_ν and g^ν are differentiable), but the penalty parameter has to go to infinity to have convergence. For these methods, little or none practical experience is available. On the plus side, in principle penalty methods can be applied to general GNEPs (and not just to jointly convex ones). Therefore, even if at present they are not true practical methods, in our opinion they hold the potential for interesting developments. As side remark, we may add that also these methods, as the relaxation method, can be in principle applied in the case of nondifferentiable GNEPs. The conditions required for the convergence of penalty methods do not seem too strong, but they are substantially different from and difficult to compare to those used in methods described in the previous subsections.

5.5 ODE-based Methods

It is possible to characterize the solutions of GNEPs as stationary points of a certain system of ordinary differential equations (ODEs for short). The crucial question which then arises is under which conditions such a stationary point is (asymptotically) stable.

In order to give at least a feel for the kind of results one can obtain, we consider once again only the jointly convex GNEP in this subsection. Then $\bar{\boldsymbol{x}}$ is a variational equilibrium of this GNEP if and only if $\bar{\boldsymbol{x}}$ solves the variational inequality VI $(\boldsymbol{X}, \mathbf{F})$ with $\boldsymbol{X}$ and $\mathbf{F}$ being defined as in Theorem 3.9. Using standard characterizations for variational inequality solutions (see [35], for example), it therefore follows that $\bar{\boldsymbol{x}}$ is a variational solution of the GNEP if and only if $\bar{\boldsymbol{x}}$ satisfies the fixed point equation

$$\boldsymbol{x} = P_{\boldsymbol{X}}\big(\boldsymbol{x} - \gamma \mathbf{F}(\boldsymbol{x})\big) \tag{23}$$

for some fixed parameter $\gamma > 0$. Hence $\bar{\boldsymbol{x}}$ is a variational equilibrium if and only if $\bar{\boldsymbol{x}}$ is a stationary point of the dynamical system

$$\boldsymbol{x}'(t) = P_{\boldsymbol{X}}\big(\boldsymbol{x}(t) - \gamma \mathbf{F}(\boldsymbol{x}(t))\big) - \boldsymbol{x}(t). \tag{24}$$

Assuming that all functions θ_ν are C^1 and have a locally Lipschitz continous partial derivative $\nabla_{x^\nu}\theta_\nu$, it follows from the Lipschitz-continuity of the projection operator that the right-hand side of (24) is locally Lipschitz, too. Hence the system (24), given any initial state $\boldsymbol{x}(0) = \boldsymbol{x}^0$ with some $\boldsymbol{x}^0 \in \boldsymbol{X}$, has a unique solution $\boldsymbol{x}(t)$. The idea of the ODE-based methods is then to follow the trajectory given by this solution $\boldsymbol{x}(t)$. The natural question arising in this context is then under which conditions (with respect to the initial value $\boldsymbol{x}^0$) the solution $\boldsymbol{x}(t)$ converges to the stationary point of (24). This leads to the question of (asymptotic) stability of the system (24).

A stability result for this ODE system is as follows.

Theorem 5.8 ([19]) Consider the jointly convex GNEP, and let $\boldsymbol{X}$ and $\mathbf{F}$ be defined as in Theorem 3.9. Suppose that $\mathbf{F}$ is Lipschitz continuous with Lipschitz constant $L > 0$ and uniformly monotone with modulus $\mu > 0$. Then, for all $\gamma \in \big(0, \frac{2\mu}{L^2}\big)$, there is a constant $c > 0$ such that every solution of (24) with $\boldsymbol{x}(0) \in \boldsymbol{X}$ satisfies

$$\|\boldsymbol{x}(t) - \bar{\boldsymbol{x}}\| \leq \|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| \exp(-ct) \quad \forall t \geq 0,$$

i.e., $\boldsymbol{x}(t)$ converges exponentially to $\bar{\boldsymbol{x}}$.

The conditions used in Theorem 5.8 are relatively strong and correspond precisely to the convergence conditions for the standard projection method for the solution of variational inequalities that is also based on the fixed point characterization (23) of a solution of VI$(\boldsymbol{X}, \mathbf{F})$. Since there exists many modifications of this standard projection method (see, e.g., [35]) that are known to work under much weaker assumptions, we believe that one can also show suitable stability results for related ODE approaches.

Other ODE-approaches, always under rather stringent convexity/monotonicity assumptions, are given in the paper [95] by Rosen, which is based on the corresponding KKT conditions, in [38], and in [19] with a right-hand side that is possibly not continuous (hence it is not guaranteed that a solution $\boldsymbol{x}(t)$ of an initial value problem exists). Related ideas are also used in a series of papers by Antipin [5, 6, 7] in a slightly different context that can also be applied to GNEPs.

Comments. We believe that, at the current state-of-the-art, ODE-based methods are really non competitive with the other methods discussed previously, neither from the theoretical point of view (as we explained above) nor from the practical point of view (although this latter belief is not based on practical experience, but rather suggested by the behavior of ODE methods in nonlinear optimization). The main interest of ODE methods is historical, since they were first proposed in the influential paper [95].

5.6 Local Newton Methods

All the methods we have considered so far are *global methods*, i.e. they are designed to converge to a solution when the starting point is possibly very distant from the solution itself. Local methods assume that the starting point of the algorithm is "close enough" to a solution and aim at proving convergence at a good rate. The prototype of a local algorithm is obviously Newton's method that exhibits a superlinear/quadratic convergence rate. The extension of this method (or of some suitable variant) to the GNEP has been investigated in [33, 83]. The approach of both papers is similar in that they are based of some kind of Newton method applied to the KKT conditions (12). As we discussed in Subsection 4.3, system (12) can be rewritten as a suitable system of nondifferentiable equations $\mathbf{\Phi}(\boldsymbol{x}, \boldsymbol{\lambda}) = 0$, to which nonsmooth methods can be applied. In particular, we could think to apply the famous semismooth Newton method. This is essentially the route taken in [33]. However, we saw in Subsection 4.3 that the solutions of the GNEP (and therefore also the solutions of (12)) are usually nonisolated. This fact is well known to cause severe difficulties to most Newton-type methods, and the semismooth Newton method is no exception. Therefore, one has to rely on some more sophisticated recent methods that are able, with some restrictions, to cope with nonisolated solutions and still guarantee a superlinear/quadratic convergence rate. We refer the reader to [33] for details. Here we only mention that the most general method in [33] is a semismooth Levenberg-Marquardt-type method applied to $\mathbf{\Phi}(\boldsymbol{x}, \boldsymbol{\lambda}) = 0$ whose iteration is

$$(\boldsymbol{x}^{k+1}, \boldsymbol{\lambda}^{k+1}) \;:=\; (\boldsymbol{x}^k, \boldsymbol{\lambda}^k) + \boldsymbol{d}^k,$$

where $\alpha(\boldsymbol{x}^k, \boldsymbol{\lambda}^k) := \|\mathbf{\Phi}(\boldsymbol{x}^k, \boldsymbol{\lambda}^k)\|$, $\boldsymbol{d}^k := (\Delta\boldsymbol{x}^k, \Delta\boldsymbol{\lambda}^k)$ solves the linear system

$$\big[J\mathbf{\Phi}(\boldsymbol{x}^k, \boldsymbol{\lambda}^k)^T J\mathbf{\Phi}(\boldsymbol{x}^k, \boldsymbol{\lambda}^k) + \alpha(\boldsymbol{x}^k, \boldsymbol{\lambda}^k)\big]\boldsymbol{d} \;=\; -J\mathbf{\Phi}(\boldsymbol{x}^k, \boldsymbol{\lambda}^k)^T \mathbf{\Phi}(\boldsymbol{x}^k, \boldsymbol{\lambda}^k) \qquad (25)$$

and $J\mathbf{\Phi}(\boldsymbol{x}^k, \boldsymbol{\lambda}^k)$ is a "generalized Jacobian" of $\mathbf{\Phi}$ at $(\boldsymbol{x}^k, \boldsymbol{\lambda}^k)$. We refer the reader to [35] for the necessary notions of nonsmooth analysis. Here we only remark that the generalized Jacobian used in the above Newton method is just a matrix that is easy to calculate in our setting and that reduces to the usual Jacobian if the function $\mathbf{\Phi}$ is continuously differentiable around the point of interest.

The critical assumption needed in establishing a fast convergence rate is an "error bound" conditions. Roughly speaking, this means that we must be able to estimate the distance to a solution based on the information at the current iteration. Error bound analysis is a well developed area in nonlinear programming; in the case of GNEPs, however, very little is known on this topic; [33] contains some preliminary results in this direction.

As an alternative to semismooth methods, [83] advocates the use of a Josephy-Newton method for the solution of system (12). This means that, at each iteration of the method, a linear complementarity problem has to be solved. This does not compare favorably with the method in [33]; furthermore, in this latter paper, it is pointed out that the assumptions used in the analysis of the method in [83] are somewhat restrictive.

6 Final Considerations

The GNEP is a very useful and flexible modelling tool and its use is increasing steadily. However, with possibly the exception of existence theory, the study of GNEPs in general form is still largely incomplete. Until now the emerging pattern for the study of the GNEP has been: (a) consider a GNEP; (b) transform it into another, better understood, problem (be it a VI, a QVI, a minimization problem etc.); (c) apply to the latter problem some known results. Unfortunately, this (classical) approach has had a somewhat limited success. The main reason being that once the conditions imposed on the transformed problem are "brought back" to the original problem, they turn out to be extremely demanding or of difficult interpretation.

We believe there are two avenues to overcome this state of affairs. The first one is studying problems with special structures emerging from some real-world applications. This has already been done for some of the recent applications of GNEPs, especially those coming from web and telecommunication applications. The other, parallel avenue, is to undertake the study of GNEPs that have some additional mathematical structure making the problem more amenable to analysis. The study of the jointly convex GNEP is certainly an extremely significative example of this second approach. There are other mathematical structures that also appear promising, we can mention (a) problems where the feasible sets are of the form $X_\nu(\boldsymbol{x}^{-\nu}) = f^\nu(\boldsymbol{x}^{-\nu}) + X_\nu$ for some $f_\nu : \mathbb{R}^{-n_\nu} \to \mathbb{R}^\nu$ and a fixed set X_ν (see [85]); (b) problems where the objective functions are independent of the other players' variables (see [37]); (c) problems where the coupling constraints have a kind of symmetric structure (see [5, 6, 7]); (d) problems where the NI-function is convex-concave [40]. But certainly, what is still lacking seems to be a method of analysis that is really tailored to the GNEP and fully takes into account its nature and peculiarities.

We believe the near future will witness a larger and larger diffusion of GNEP models and a parallel increase in the interest in their theoretical and algorithmic analysis. We hope we made plain that the study of GNEPs is still in its infancy, its age notwithstanding. There are still many stimulating open problems to be attacked, both on the theoretical side and on the algorithmic/numerical one. Furthermore, beyond those we listed, there are further interesting topics on which little is known and that yet are emerging as important. Among these we just mention two. First the problem of selecting one specific solution among the many the GNEP usually has (what criteria can we use to establish that a certain solution is preferable to another, how can we compute it?). Second the EPEC: Equilibrium Programming with Equilibrium Constraints. EPECs can be viewed as GNEPs whose constraints are in turn defined by some kind of equilibrium condition. These problems naturally arise when considering multi-leader-follower games used in modelling complex competition situations, see, e.g., [31, 43, 56, 67, 107]. These problems are challenging and extremely hard to analyze.

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