

**MATHEMATICAL PROGRAMS WITH VANISHING
CONSTRAINTS: OPTIMALITY CONDITIONS
AND CONSTRAINT QUALIFICATIONS**

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Abstract. We consider a difficult class of optimization problems that we call a mathematical program with vanishing constraints. Problems of this kind arise in various applications including optimal topology design problems of mechanical structures. We show that some standard constraint qualifications like LICQ and MFCQ usually do not hold at a local minimum of our program, whereas the Abadie constraint qualification is sometimes satisfied. We also introduce a suitable modification of the standard Abadie constraint qualification as well as a corresponding optimality condition, and show that this modified constraint qualification holds under fairly mild assumptions. Finally, we discuss the relation between our class of optimization problems with vanishing constraints and a mathematical program with equilibrium constraints.

Key Words. Constrained optimization, vanishing constraints, structural optimization, constraint qualifications, optimality conditions, mathematical programs with equilibrium constraints.

1 Introduction

The paper deals with optimization problems of the form

$$\begin{aligned}
 \min \quad & f(x) \\
 \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \dots, m, \\
 & h_j(x) = 0 \quad \forall j = 1, \dots, p, \\
 & H_i(x) \geq 0 \quad \forall i = 1, \dots, l, \\
 & G_i(x)H_i(x) \leq 0 \quad \forall i = 1, \dots, l,
 \end{aligned} \tag{1}$$

where all functions $f, g_i, h_j, G_i, H_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are assumed to be continuously differentiable. We call (1) a *mathematical program with vanishing constraints*. This terminology comes from the fact that, for certain applications (see Section 2), some of the constraints vanish, i.e., may not be considered at certain points of the feasible region. For example, consider the following prototype of an optimization problem with vanishing constraints:

$$\begin{aligned}
 \min \quad & f(x) \\
 \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \dots, m, \\
 & h_j(x) = 0 \quad \forall j = 1, \dots, p, \\
 & G_i(x) \leq 0 \quad \text{if } x \in \mathcal{H}_i \quad \forall i = 1, \dots, l.
 \end{aligned} \tag{2}$$

Here we assume that the sets $\mathcal{H}_i \subset \mathbb{R}^n$ are non-empty and, say, open for all $i = 1, \dots, l$. In this formulation, the constraint “ $G_i(x) \leq 0$ ” vanishes from the problem at points $x \notin \mathcal{H}_i$. In other words, we do not care about the sign of $G_i(x)$ at points $x \notin \mathcal{H}_i$. Vanishing constraints of this kind are typical, e.g., in design problems or structural optimization, see Section 2 for more details.

Note that (2) is an optimization problem with some nonstandard constraints. In order to reformulate (2) in a suitable way, we assume that there exist continuously differentiable functions $H_i : \mathbb{R}^n \rightarrow \mathbb{R}$ characterizing the sets \mathcal{H}_i through the identities

$$\mathcal{H}_i = \{x \in \mathbb{R}^n \mid H_i(x) > 0\} \quad \text{for all } i = 1, \dots, l. \tag{3}$$

Note that the strict inequality in “ $H_i(x) > 0$ ” corresponds to the fact that \mathcal{H}_i is an open set. The latter, indeed, causes some troubles in view of the existence of solutions to problem (2). But this is the situation in interesting applications (cf., e.g., Example 2.1 below).

Moreover, without loss of generality, we may assume that $H_i(x) \geq 0$ for all x which are feasible for (2). Otherwise, we may formally replace H_i by H_i^2 , although this is not recommended in practice. Fortunately, in most applications, such a transformation is not necessary since the constraints “ $H_i(x) \geq 0$ ” are typically part of the problem (2) (within the group of constraints “ $g_i(x) \leq 0$ ”). In any case, problem (1) is therefore a natural reformulation of (2). To this end, note that the sign of $G_i(x)$ in (1) is not relevant at those points x where $H_i(x) = 0$. This implicitly models the effect of vanishing constraints as in formulation (2). This explains the reason for our choice of a general problem in the form (1).

The advantage of formulation (1) (in contrast to (2)) is that (1) is an optimization problem with standard equality and inequality constraints only. Hence we may try to apply standard constraint qualifications in order to get suitable optimality conditions for our mathematical program with vanishing constraints (1). However, it turns out that these standard constraint qualifications are usually not satisfied for our problem (1), and we therefore have to find more specialized constraint qualifications and/or suitable optimality conditions for problem (1).

The paper is organized in the following way: Section 2 describes two applications from structural optimization which, in a very natural way, lead to mathematical programs with vanishing constraints. In Section 3, we then show that several standard constraint qualifications are usually violated for problem (1). Section 4 gives a more detailed discussion of the standard Abadie constraint qualification since this condition has a chance of being satisfied. A modification of the Abadie constraint qualification, which takes into account the special structure of mathematical programs with vanishing constraints, as well as a corresponding optimality condition are the subject of Section 5. We close this manuscript with some final remarks in Section 6 where, in particular, we discuss the relation between problem (1) and so-called mathematical programs with equilibrium constraints. In particular, we show that optimization problems with vanishing constraints may, in principle, be reformulated as mathematical programs with equilibrium constraints, but that such a reformulation causes some additional troubles and is therefore not recommended.

2 Examples: Topology Optimization of Mechanical Structures

In order to motivate mathematical programs with vanishing constraints as an interesting class of optimization problems, we present some applications immediately leading to problems of this kind. We concentrate on examples from structural optimization. One of the classical problems in this field are design problems. Modern approaches do not predefine any shape of the structure yet to be designed. For example, the number, the location, and the shape and size of holes in the structure are solely determined by the optimization process. In contrast to traditional “shape optimization”, this new and free design optimization is referred to as “topology optimization”. Due to the immense freedom of the problem in the design space, formulations of topology optimization problems are usually abstract or large-scaled. Calculations in this field started with the development of optimization algorithms and codes running on computers. An early paper on topology optimization of truss structures is [7] from 1964 using linear programming. Numerical topology optimization of continuum structures started in the late 1980’s with the idea of regarding design problems as a problem of material distribution [9, 4]. During the last decade, many extensions have been made to various problem formulations and solution methods. An overview of the state of the art is given in the monograph [5].

Meanwhile topology optimization started to become an accepted tool in industrial ap-

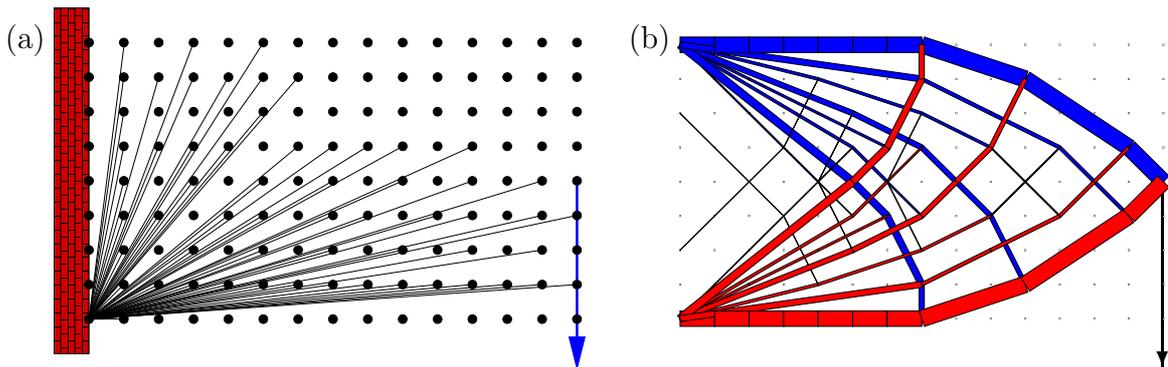


Figure 1: (a) Ground structure; (b) Optimal truss structure

plications like airplane and car manufacturing. The results from topology optimization, however, still suffer from the fact that many realistic side constraints cannot be incorporated into the mathematical problem formulations because of complexity, nonlinearities, and singularities. As a consequence, the obtained results are partly doubtful, and must be substantially post-processed. Many publications in current research activities deal with the incorporation of stress constraints into topology optimization problems since this is an urgent necessity from the engineering point of view. As will be clear below, however, stress constraints in a topology context immediately lead to singularities because stresses are not defined at points of the design domain where material is not present, i.e., where “the structure has a hole”. This difficulty is closely related to the phenomenon of so-called “singular optimizers” [1, 5]. The following two examples illustrate the modeling difficulty using two typical problems from structural design. The first example deals with the topology design problem of trusses, the second one considers its counterpart for continuum structures.

Example 2.1 We want to find the optimal design of a truss structure. We use the so-called “ground structure approach” introduced in [7]. To this end, consider a given set of M so-called “potential bars” which are defined by the coordinates of their end nodes (in \mathbb{R}^2 or in \mathbb{R}^3). Moreover, for each potential bar, material parameters are given (Young’s modulus E_i , relative moment of inertia s_i , stress bounds $\sigma_i^t > 0$ and $\sigma_i^c < 0$ for tension and compression, respectively). These parameters are needed for the formulation of constraints preventing structural failure in the case when the potential bar is realized as a real bar. The latter is the case if the calculated cross-sectional area a_i is positive. Finally, boundary conditions (i.e., fixed nodal coordinates) and external loads (i.e., loads applying at some of the nodes) are given. Such a scenario is called a “ground structure”. The problem (“optimal truss topology design problem”) is to find cross-sectional areas a_i^* for each potential bar such that failure of the whole structure is prevented, the external load is carried by the structure, and a suitable objective function is minimal. The latter is usually the total weight of the structure or its deformation energy (“compliance”).

In order to obtain a good resulting structure after optimization, the ground structure should be “rich” enough, i.e., should consist of many potential bars. Figure 1 (a) illustrates a ground structure in 2D in a standard design scenario. The structure (yet to be designed)

is fixed to the left (indicated by a wall). On the right hand side, the given external load applies (vertical arrow) which must be carried by the structure. We have discretized a 2D rectangular design area by 15×9 nodal points. All nodal points are pair-wise connected by potential bars. After the deletion of long potential bars which are overlapped by shorter ones, we end up with 5614 potential bars. Very few of these potential bars are depicted in Figure 1 (a) by black lines (Plotting all bars would result in a completely black picture, and hence only very few potential bars are shown).

Of course, in view of a practical realization of the calculated structure after optimization, one hopes that the optimal design a^* will make use of only a few of the potential bars, i.e., $a_i^* > 0$ for a small number of indices i only, whereas most of the (many) optimal cross-sectional areas a_i^* are zero. Figure 1 (b) shows the optimized structure based on the ground structure indicated in Figure 1 (a). Indeed, most of the potential bars are not realized as real bars. Such a behaviour is typical in applied truss topology optimization problems.

The main difficulty in formulating (and solving) the problem lies in the fact that, generally speaking, constraints on structural failure can be formulated in a well-defined way only if there is some material giving mechanical response. As explained before, however, most potential bars will possess a zero cross-section at the optimizer. Hence, one option is the formulation of the problem as a problem with vanishing constraints. A simple formulation of the truss design problem with constraints on stresses and on local buckling takes the following form (compare to problem (2)):

$$\begin{aligned}
& \min_{a \in \mathbb{R}^M, u \in \mathbb{R}^d} && f(a, u) \\
& \text{s.t.} && g(a, u) \leq 0, \\
& && a_i \geq 0 && \forall i = 1, \dots, M, \\
& && K(a)u = f^{\text{ext}}, \\
& && \sigma_i^c \leq \sigma_i(a, u) \leq \sigma_i^t && \text{if } a_i > 0 \quad \forall i = 1, \dots, M, \\
& && f_i^{\text{int}}(a, u) \geq f_i^{\text{buck}}(a) && \text{if } a_i > 0 \quad \forall i = 1, \dots, M.
\end{aligned} \tag{4}$$

Here the vector $a \in \mathbb{R}^M$, $a \geq 0$, contains the vector of cross-sectional areas of the potential bars, and $u \in \mathbb{R}^d$ denotes the vector of nodal displacements of the structure under load, where d is the so-called degree of freedom of the structure, i.e., the number of free nodal displacement coordinates. The state variable u serves as an auxiliary variable. The objective function f often expresses structural weight or compliance but can also be any other measure evaluating a given design a and a corresponding state u . The nonlinear system of equations $K(a)u = f^{\text{ext}}$ symbolizes force equilibrium of (given) external loads $f^{\text{ext}} \in \mathbb{R}^d$ and internal forces (i.e., along the bars) expressed via Hooke's law in terms of displacements and cross-sections. The matrix $K(a) \in \mathbb{R}^{d \times d}$ is the global stiffness matrix corresponding to the structure a . This matrix is always symmetric and positive semidefinite. The constraint $g(a, u) \leq 0$ is a resource constraint, like on the total volume of the structure (e.g., if f denotes compliance) or on the compliance of the structure (e.g., if f denotes volume or weight). If $a_i > 0$, then $\sigma_i(a, u) \in \mathbb{R}$ is the stress along the i -th bar. Similarly, if $a_i > 0$, $f_i^{\text{int}}(a, u) \in \mathbb{R}$ denotes the internal force along the i -th bar, and $f_i^{\text{buck}}(a)$

corresponds to the permitted Euler buckling force. (We assume here that the geometry of the bar cross-section is given, e.g., as a circle or a square. Hence, the moment of inertia is a scaling of the cross-section, and the buckling force solely depends on a_i). Then the constraints on stresses and on local buckling make sense only if $a_i > 0$. Therefore, they must vanish from the problem if $a_i = 0$. Fortunately, the functions σ_i , f_i^{int} , and f_i^{buck} possess continuous extensions for $a_i \searrow 0$, and thus may be defined also for $a_i = 0$ (without any direct physical meaning, though). This allows a reformulation of the problem in the form (1). In this situation, the definitions $H_i(a, u) := a_i$ for all $i = 1, \dots, M$ will do the job. \diamond

Example 2.2 Here we consider the topology design problem of a continuum structure. Let $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ denote a so-called reference domain with boundary conditions and with given external loads applying at parts Γ_T of the boundary of Ω as traction forces. We seek for a structure contained in Ω satisfying the boundary and the force conditions. Moreover, we assume this structure to consist of elastic material described by a given material tensor E , and we look for the structure being as stiff as possible among all structures with a maximal total “volume” $V < \int_{\Omega} 1 \, d\Omega$. Moreover, in view of practical applications, we would like to include stress constraints at each point x of the structure. In the fashion of looking at a structure as a material distribution, a theoretical formulation of the problem is the following (cf. [4, 5] for details):

$$\begin{aligned} & \min_{\xi \in X, u \in U} l(u) \\ \text{s.t. } & a_{\xi}(u, v) = l(v) \quad \text{for all } v \in U, \\ & \sigma^c \leq \sigma(\xi, u) \leq \sigma^t \quad \text{for all } x \in \Omega, \\ & \int_{\Omega} \xi \, d\Omega \leq V. \end{aligned}$$

Here, as usual, l denotes the compliance of the structure depending on the displacement field u ,

$$l(u) := \int_{\Omega} f^{\text{ext}} u \, d\Omega + \int_{\Gamma_T} t u \, ds,$$

where U is the space of admissible displacement fields. The energy bilinear form, i.e., the internal virtual work of the structure at the equilibrium u , and with an arbitrary virtual displacement v , is denoted by

$$a_{\xi}(u, v) := \int_{\Omega} \xi(x) E_{ijkl}(x) \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, d\Omega,$$

as usual. Here we have used the standard index notation for tensors, we assume that E represents isotropic material with $E_{ijkl} \in L^{\infty}(\Omega)$, and we work with linearized strains $\varepsilon_{ij}(u) = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. The structure is hidden in the indicator function $\xi \in X := \{ \xi : \Omega \rightarrow \{0, 1\} \}$. The structure we are looking for is formed by the points $\{x \in \Omega \mid \xi(x) = 1\}$ where ξ (together with some $u \in U$) is a solution of the above problem. Due to the definition of

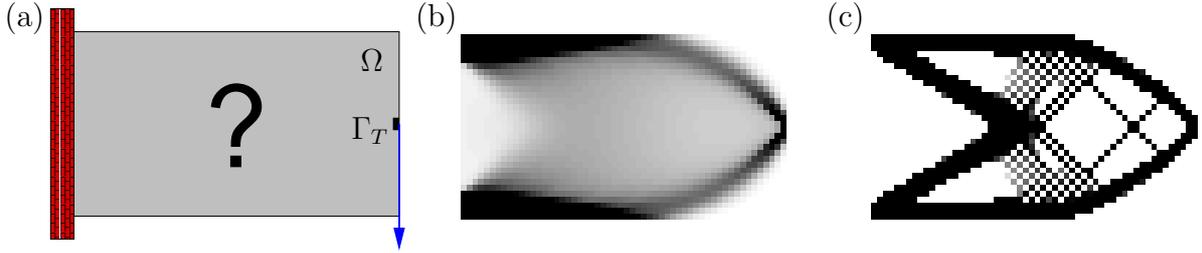


Figure 2: (a) Design domain; (b) Calculated structure, $p = 1$; (c) Calculated structure, $p = 2$

$a_\xi(\cdot, \cdot)$ the modeling is based on Hooke's law with material tensor $\xi(x)E(x)$ at all points $x \in \Omega$. Hence we have material E at points with $\xi(x) = 1$, and we have zero material (i.e., void) otherwise. For the modeling of local stresses $\sigma(\xi, x)$ there are a number of more or less sophisticated ways, and we are not going into the details here (cf. [5] for an overview). The functions σ^c, σ^t denote given functions for upper and lower stress bounds, respectively. We assume that $\sigma^c(x) \leq 0$ and $\sigma^t(x) \geq 0$ for all $x \in \Omega$, and we assume that $\sigma(\cdot, \cdot)$ is defined in a way that $\sigma(\xi, u) = 0$ if $\xi(x) = 0$.

Unfortunately, the design problem defined in this way does not necessarily possess a solution (ξ, u) , also in the absence of stress constraints. It may happen that in the limit ξ represents a structure with “arbitrarily small and infinitely many holes” (cf. [5] and the literature cited therein). Hence, a popular “approximation” of the problem is to replace ξ by a “density function” $\rho : \Omega \rightarrow [0, 1]$. This means, at each point $x \in \Omega$, now also material with tensor $\rho(x)^p E(x)$ is allowed, where $\rho(x) \in [0, 1]$, and $p \geq 1$ is a user-defined parameter (cf. below). In a 2D-setting, ρ may be interpreted as the thickness of a (yet 2D) structure (“variable thickness sheet problem” [5]).

After discretization using M finite elements, the problem now becomes very similar to the truss problem (4) from Ex. 2.1 (where f^{ext} now changes its meaning to its discretized counterpart $f^{\text{ext}} \in \mathbb{R}^d$; note also that here the objective function and the resource constraint have been chosen already as compliance and as volume, respectively):

$$\begin{aligned}
 & \min_{\rho \in \mathbb{R}^M, u \in \mathbb{R}^d} && f^{\text{ext}T} u \\
 \text{s.t.} &&& \sum_{i=1}^m \rho_i V_i \leq V, \\
 &&& 0 \leq \rho_i \leq 1 && \forall i = 1, \dots, M, \\
 &&& K(\rho)u = f^{\text{ext}}, \\
 &&& \sigma_i^c \leq \sigma_i(\rho, u) \leq \sigma_i^t \quad \text{if } \rho_i > 0 \quad \forall i = 1, \dots, M
 \end{aligned}$$

Here $K(\rho) = \sum_i \rho_i^p K_i$ denotes the total stiffness matrix of the structure, and K_i is the element stiffness matrix of the i th finite element in global reduced coordinates. Again we see the effect of vanishing stress constraints for elements with zero density ρ_i .

Figure 2 (a) shows the reference domain Ω , the boundary conditions, and the force

applied at a small piece Γ_T of the boundary. On purpose, the scenario was chosen analogously to the scenario in Fig. 1 (a) from above. Figure 2 (b) shows a solution of the problem for $p = 1$ where the optimal values $\rho_i^* \in [0, 1]$ for each finite element, respectively, are visualized through a grey-scale from white ($\rho_i^* = 0$) to black ($\rho_i^* = 1$). The volume bound V was chosen as 40% of the total area of Ω . We have used $56 \times 32 = 1792$ square finite elements for the discretization of both, ρ (piece-wise constant) and u (piece-wise bilinear and continuous). We mention that the stress constraints had to be appropriately chosen in order to get the problem solved. (We just want to give an illustration of the application here, not all the details themselves.) As immediately seen from the picture, the optimization makes very well use of the freedom in choosing intermediate densities, i.e., values $\rho_i^* \in]0, 1[$. In view of interpreting ρ^* as a material distribution, however, this is not desirable. Here the parameter p comes into play. By choosing $p > 1$, there is an effect “penalizing” intermediate densities ρ_i via the equilibrium $\sum_i \rho_i^p K_i u = f^{\text{ext}}$ in combination with the volume constraint. For $p > 1$, however, the precise meaning of either the material law or the volume constraint changes. It turns out, however, that the choice $p > 1$ results in much nicer structures from the realization point of view, i.e., solutions with intermediate densities are avoided. In the literature this “proportional stiffness model” is known as the SIMP model (Solid Isotropic Material with Penalization) [3, 15]. Figure 2 (c) shows the result of an optimization run for $p = 2$. It is obvious that this result may be clearly interpreted as a real structure. Moreover, this structure is similar to the solution truss in Fig. 1 (b). \diamond

3 Violation of Standard Constraint Qualifications

The aim of this section is to show that standard constraint qualifications usually do not hold for mathematical programs with vanishing constraints. In order to recall these constraint qualifications, we first consider the optimization problem

$$\begin{aligned} \min \quad & \tilde{f}(x) \\ \text{s.t.} \quad & \tilde{g}_i(x) \leq 0 \quad \forall i = 1, \dots, r, \\ & \tilde{h}_j(x) = 0 \quad \forall j = 1, \dots, s \end{aligned} \tag{5}$$

with continuously differentiable functions $\tilde{f}, \tilde{g}_i, \tilde{h}_j : \mathbb{R}^n \rightarrow \mathbb{R}$. Let

$$\tilde{X} := \{x \in \mathbb{R}^n \mid \tilde{g}_i(x) \leq 0 \ (i = 1, \dots, r), \tilde{h}_j(x) = 0 \ (j = 1, \dots, s)\}$$

denote the feasible set of the optimization problem (5).

Now let x^* be a local minimum of (5) and suppose that a suitable constraint qualification holds (see the discussion below). Then it is possible to show that there exist Lagrange multipliers $\tilde{\lambda}_i \in \mathbb{R}$ and $\tilde{\mu}_j \in \mathbb{R}$ such that the following first order optimality conditions or

Karush-Kuhn-Tucker conditions (KKT conditions, for short) hold:

$$\begin{aligned} \nabla \tilde{f}(x^*) + \sum_{i=1}^r \tilde{\lambda}_i \nabla \tilde{g}_i(x^*) + \sum_{j=1}^s \tilde{\mu}_j \nabla \tilde{h}_j(x^*) &= 0, \\ \tilde{h}_j(x^*) &= 0 \quad \forall j = 1, \dots, s, \\ \tilde{\lambda}_i \geq 0, \quad \tilde{g}_i(x^*) \leq 0, \quad \tilde{\lambda}_i \tilde{g}_i(x^*) &= 0 \quad \forall i = 1, \dots, r, \end{aligned} \tag{6}$$

see, e.g., [2, 11]. These KKT conditions play a major role for the design and analysis of several optimization algorithms, and it is therefore of central importance that these conditions hold under appropriate assumptions.

Suitable conditions which guarantee that the KKT conditions are satisfied at a local minimum x^* of (5) are some constraint qualifications. Here we give a brief list with the most prominent constraint qualifications that may be found in the literature (see, e.g., the survey [14]):

- The *linear independence constraint qualification* (LICQ for short) is said to hold at a local minimizer x^* of (5) if the gradients

$$\nabla \tilde{g}_i(x^*) \quad (i : \tilde{g}_i(x^*) = 0), \quad \nabla \tilde{h}_j(x^*) \quad (j = 1, \dots, s)$$

are linearly independent.

- The *Mangasarian-Fromovitz constraint qualification* (MFCQ for short) is said to hold at a local minimizer x^* of (5) if the gradients $\nabla \tilde{h}_j(x^*)$ ($j = 1, \dots, s$) are linearly independent and there is a vector $d \in \mathbb{R}^n$ such that

$$\nabla \tilde{g}_i(x^*)^T d < 0 \quad (i : \tilde{g}_i(x^*) = 0), \quad \nabla \tilde{h}_j(x^*)^T d = 0 \quad (j = 1, \dots, s).$$

- The *Abadie constraint qualification* (ACQ for short) is said to hold at a local minimizer x^* of (5) if $\mathcal{T}(x^*) = \mathcal{L}(x^*)$, where

$$\mathcal{T}(x^*) := \left\{ d \in \mathbb{R}^n \mid \exists \{x^k\} \subseteq \tilde{X}, \exists \{t_k\} \downarrow 0 : x^k \rightarrow x^* \text{ and } \frac{x^k - x^*}{t_k} \rightarrow d \right\}$$

is the standard *tangent cone* of (5) at x^* , and

$$\mathcal{L}(x^*) := \left\{ d \in \mathbb{R}^n \mid \nabla \tilde{g}_i(x^*)^T d \leq 0 \quad (i : \tilde{g}_i(x^*) = 0), \quad \nabla \tilde{h}_j(x^*)^T d = 0 \quad (j = 1, \dots, s) \right\}$$

denotes the corresponding *linearized cone* of (5) at x^* .

The following implications are well-known:

$$\text{LICQ} \implies \text{MFCQ} \implies \text{ACQ}.$$

Moreover, if ACQ holds at a local minimum x^* of (5), then there exist Lagrange multipliers $\tilde{\lambda}_i$ and $\tilde{\mu}_j$ such that the KKT conditions (6) hold. In particular, the KKT conditions are necessary optimality conditions under both LICQ and MFCQ. Note also that ACQ is one of the weakest constraint qualifications, see, again, the paper [14] for a complete overview.

We now want to apply these standard constraint qualifications to our constrained optimization problem from (1). To this end, let x^* be a local minimum of (1), and let us introduce the following index sets that will be used frequently in the subsequent analysis:

$$\begin{aligned} I_g &:= \{i \mid g_i(x^*) = 0\}, \\ J &:= \{1, \dots, p\}, \\ I_+ &:= \{i \mid H_i(x^*) > 0\}, \\ I_0 &:= \{i \mid H_i(x^*) = 0\}. \end{aligned} \tag{7}$$

Furthermore, we divide the index set I_+ into the following subsets:

$$\begin{aligned} I_{+0} &:= \{i \mid H_i(x^*) > 0, G_i(x^*) = 0\}, \\ I_{+-} &:= \{i \mid H_i(x^*) > 0, G_i(x^*) < 0\}. \end{aligned} \tag{8}$$

Similarly, we partition the set I_0 in the following way:

$$\begin{aligned} I_{0+} &:= \{i \mid H_i(x^*) = 0, G_i(x^*) > 0\}, \\ I_{00} &:= \{i \mid H_i(x^*) = 0, G_i(x^*) = 0\}, \\ I_{0-} &:= \{i \mid H_i(x^*) = 0, G_i(x^*) < 0\}. \end{aligned} \tag{9}$$

Note that the first subscript (+ or 0) in these index sets indicates whether $H_i(x^*)$ is positive or zero, whereas the second subscript (+, 0 or $-$) indicates whether the sign of $G_i(x^*)$ is positive, zero, or negative. Further note that these index sets depend on the particular solution x^* of (1). However, this solution will always be clear from the context, so there is no need to make this dependence explicit in our notation.

In our first result, we show that LICQ does not hold for our optimization problem (1) under fairly mild assumptions.

Lemma 3.1 *Let x^* be a local minimum of (1) such that $I_0 \neq \emptyset$. Then LICQ is violated at the point x^* .*

Proof. Let us introduce the function

$$\theta_i(x) := G_i(x)H_i(x) \quad \forall i = 1, \dots, l, \tag{10}$$

and note that its gradient is given by

$$\nabla \theta_i(x) = G_i(x)\nabla H_i(x) + H_i(x)\nabla G_i(x) \quad \forall i = 1, \dots, l.$$

Hence the definition of the index sets from (8), (9) implies

$$\nabla \theta_i(x^*) = \begin{cases} 0, & \text{if } i \in I_{00}, \\ G_i(x^*)\nabla H_i(x^*), & \text{if } i \in I_{0+} \cup I_{0-}, \\ H_i(x^*)\nabla G_i(x^*), & \text{if } i \in I_{+0}. \end{cases} \tag{11}$$

Now assume that LICQ holds at x^* . Then the gradients

$$\nabla g_i(x^*) (i \in I_g), \quad \nabla h_j(x^*) (j \in J), \quad \nabla H_i(x^*) (i \in I_0), \quad \nabla \theta_i(x^*) (i \in I_0 \cup I_{+0}) \quad (12)$$

must be linearly independent. Since $I_0 \neq \emptyset$, we have $I_{00} \neq \emptyset$ or $I_{0+} \cup I_{0-} \neq \emptyset$. However, for $i \in I_{00}$, we get $\nabla \theta_i(x^*) = 0$ from (11), and this vector cannot be a member of a set of linearly independent vectors. On the other hand, if $i \in I_{0+} \cup I_{0-}$, it follows from (11) that $\nabla \theta_i(x^*)$ is a nonzero multiple of $\nabla H_i(x^*)$. Hence this vector together with the corresponding gradient $\nabla H_i(x^*)$ forms a linearly dependent subset of the vectors from (12). These contradictions show that LICQ is violated at x^* . \square

We next show that, under a slightly stronger assumption, MFCQ is also not satisfied at a local minimum of our special optimization problem from (1).

Lemma 3.2 *Let x^* be a local minimum of (1) such that $I_{00} \cup I_{0+} \neq \emptyset$. Then MFCQ is violated at the point x^* .*

Proof. Suppose that MFCQ holds at x^* . Then the gradients $\nabla h_j(x^*) (j \in J)$ are linearly independent, and there is a vector $d \in \mathbb{R}^n$ such that

$$\nabla g_i(x^*)^T d < 0 (i \in I_g), \quad \nabla h_j(x^*)^T d = 0 (j \in J)$$

and

$$\nabla H_i(x^*)^T d > 0 (i \in I_0), \quad \nabla \theta_i(x^*)^T d < 0 (i \in I_0 \cup I_{+0}). \quad (13)$$

The first set of conditions is not really important in our proof, since the second set alone gives a contradiction. In fact, if we take an index $i \in I_{00}$, we get the contradiction $0 = \nabla \theta_i(x^*)^T d < 0$ from (11) and (13). Otherwise, if we have an index $i \in I_{0+}$, we also get a contradiction, since, on the one hand, the vector d satisfies $\nabla H_i(x^*)^T d > 0$ in view of (13) and, on the other hand, we have

$$\nabla H_i(x^*)^T d = \frac{1}{G_i(x^*)} \nabla \theta_i(x^*)^T d < 0$$

because of (11) and (13). Hence, in any case, we get a contradiction. \square

Note the difference in the assumptions of Lemma 3.1 and Lemma 3.2: The first result states that LICQ has a chance to hold only if all H_i constraints are inactive, whereas the second result says that MFCQ may hold if some of the H_i constraints are active, namely those with indices $i \in I_{0-}$.

We next discuss the relevance of the assumptions in Lemmas 3.1 and 3.2 from the point of view of our truss topology optimization problem from Example 2.1.

Example 3.3 Consider the prototype application from truss topology optimization in Example 2.1. The assumption $I_0 \neq \emptyset$ of Lemma 3.1 is usually satisfied at a (locally) optimal structure a^* (with corresponding displacements u^*). To this end, recall that $H_i(a^*, u^*) = a_i^*$

denotes the cross-sectional area of the i -th bar. Hence I_0 will usually be a large set (cf. Figure 1 (b)). Consequently, LICQ has no chance to hold in this situation. Moreover, the assumption $I_{00} \cup I_{0+} \neq \emptyset$ from Lemma 3.2 is typically also satisfied at an optimizer (a^*, u^*) . To see this, we interpret the optimal structure a^* as a so-called “limiting structure” $a^* = \lim_{j \rightarrow +\infty} a^j$ with structures $a^j > 0$ (and corresponding displacements u^j). Then consider an index i with $a_i^* = 0$. For such a “vanishing bar” (i.e., $a_i^j \rightarrow a_i^* = 0$) the value of the stress $\sigma_i(a^j, u^j)$ typically increases up to a finite value, say $\lim_j \sigma_i(a^j, u^j) = \sigma_i(a^*, u^*)$ (independent of the convergence of u^j). Note that the value $\sigma_i(a^*, u^*)$ is a fictitious stress value (a “limiting stress”) because the i -th bar is not realized as a real bar ($a_i^* = 0$!). Typically, we have $\sigma_i(a^*, u^*) > \sigma_i^t$ or $\sigma_i(a^*, u^*) < \sigma_i^c$ because these values prevent a_i^* from being positive (i.e., optimization decides to choose $a_i^* = 0$ because otherwise the involved stresses would exceed the stress bounds). In this situation, we therefore have $i \in I_{0+}$. Numerical examples show that almost all indices i with $a_i^* = 0$ belong to I_{0+} . Hence MFCQ is unlikely to hold at local minimizers in Example 2.1. \diamond

Example 3.4 Let us study a truss design example of academic size. We consider the problem of minimizing the weight of a predefined part of the structure subject to constraints on total weight and total compliance of the structure, and on member stresses (in comparison to problem (4) in Ex. 2.1, we neglect the constraints on local buckling, for simplicity), i.e., with some given index set $I \subseteq \{1, \dots, M\}$,

$$\begin{aligned} & \min_{a \in \mathbb{R}^M, u \in \mathbb{R}^d} \sum_{i \in I} \kappa_i \ell_i a_i \\ \text{s.t. } & \sum_{i=1}^M \kappa_i \ell_i a_i - W \leq 0, \\ & f^{\text{ext}T} u - C \leq 0, \\ & K(a)u = f^{\text{ext}}, \\ & a_i \geq 0 \quad \text{for all } i = 1, \dots, M, \\ & a_i(\sigma_i^c - \sigma_i(a, u)) \leq 0 \quad \text{for all } i = 1, \dots, M, \\ & a_i(\sigma_i(a, u) - \sigma_i^t) \leq 0 \quad \text{for all } i = 1, \dots, M. \end{aligned}$$

Here, ℓ_i denotes the length of the i th potential bar, and κ_i denotes its specific structural weight per volume. The constants W and C denote the permitted maximal weight and maximal compliance, respectively, of the total structure. Moreover, in this problem setting formulated in areas and displacements, we have used that $\sigma_i(a, u)$ can be written as a linear function of u , and thus is well-defined (as a mathematical function) even if $a_i = 0$ (while losing its physical meaning as a member stress).

Minimization of the weight of only a part of the structure makes sense if the decision must be made how to design a few “critical” and “expensive” elements of the structure while all other elements are cheap in manufacturing. Imagine, e.g., the scenario requires some “backbone parts” made from very expensive material like specially hardened steel while all other bars in the structure can be manufactured from cheap material, and thus are neglected in the objective function. Together with the other side constraints, the constraint on total weight will control whether “expensive” bars $i \in I$ are used at all in the final design.

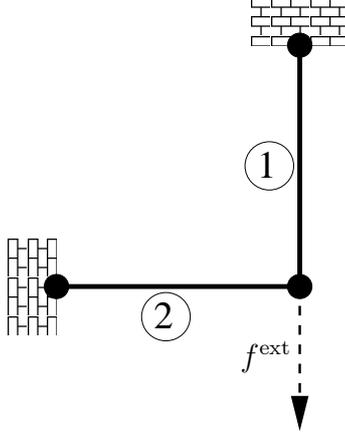


Figure 3: Ground structure of academic example (Ex. 3.4)

To be more concrete, consider the ground structure in Fig. 3 consisting of $M = 2$ potential bars with a vertical force applied at the single free nodal point, indicated by a dashed arrow. It is obvious that bar no. 1 is of paramount importance, and hence we put $I := \{1\}$. Let the length of both bars be 1, assume that the Young's moduli of the materials in both bars are $E_i := 1$, and let the specific weight factors κ_i be also 1, again for simplicity. Then, in global reduced coordinates (i.e., after deletion of fixed nodal displacement coordinates), for any $a \in \mathbb{R}^2$, $a \geq 0$, the global stiffness matrix is given by

$$K(a) = a_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $a_1, a_2 \geq 0$ denote the cross-sectional areas of bar 1 and bar 2, respectively. Moreover, let $f^{\text{ext}} := (0, -1)^T$ denote the given external force in reduced nodal coordinates, i.e., $f^{\text{ext}T}u = -u_2$ expresses total compliance of the structure, where $u = (u_1, u_2)^T$ is the displacement vector of the free bottom right nodal point with u_1, u_2 being the displacement in horizontal and vertical direction, respectively. With the stress bounds σ_i^c, σ_i^t as ∓ 1 , and with the bounds $W := 2$ and $C := 2$, we arrive at the following problem of type (1):

$$\begin{array}{l|l} \min_{a, u \in \mathbb{R}^2} a_1 & f(a, u) := a_1 \\ \text{s.t. } a_1 + a_2 - 2 \leq 0, & g_1(a, u) := a_1 + a_2 - 2, \\ -u_2 - 2 \leq 0, & g_2(a, u) := -u_2 - 2, \\ a_2 u_1 = 0, & h_1(a, u) := a_2 u_1, \\ a_1 u_2 + 1 = 0, & h_2(a, u) := a_1 u_2 + 1, \\ a_1 \geq 0, & H_{1,2}(a, u) := a_1, \\ a_2 \geq 0, & H_{3,4}(a, u) := a_2, \\ a_1(-1 + u_2) \leq 0, & G_1(a, u) := -1 + u_2, \\ a_1(-1 - u_2) \leq 0, & G_2(a, u) := -1 - u_2, \\ a_2(-1 + u_1) \leq 0, & G_3(a, u) := -1 + u_1, \\ a_2(-1 - u_1) \leq 0. & G_4(a, u) := -1 - u_1. \end{array} \quad (14)$$

Let us calculate all solutions of this problem. The second equilibrium constraint requires that neither a_1 nor u_2 can be zero. (This is also clear from the geometry: The structure must carry the external load, and this will lead to a displacement in vertical direction.) Moreover, $a_1 = -\frac{1}{u_2}$. Hence, u_2 is always negative, and minimization of a_1 means maximization of $|u_2|$. The stress constraints on bar no. 1 reduce to $-1 \leq u_2 \leq 1$ because $a_1 > 0$. Moreover, the compliance constraint says that $-u_2 \leq 2$, and thus is always satisfied. Hence, we obtain that $u_2^* := -1$ and $a_1^* := 1$ are optimal together with all other choices for (a_2, u_1) which are feasible (notice that the constraint on total weight $a_1^* + a_2 \leq 2$ is satisfied for all $a_2 \in [0, 1]$).

By this, we obtain the following set of optimal solutions of our problem:

$$\begin{aligned} & \left\{ (a_1, a_2, u_1, u_2)^T \mid a_1 = 1, a_2 = 0, u_1 \in \mathbb{R}, u_2 = -1 \right\} \\ \cup & \left\{ (a_1, a_2, u_1, u_2)^T \mid a_1 = 1, a_2 \in]0, 1], u_1 = 0, u_2 = -1 \right\}. \end{aligned}$$

From an engineering point of view, in this example, bar no. 1 must carry the total external load because bar no. 2 is perpendicular to f^{ext} (by the way, such a situation often occurs in truss design problems formulated on ground structures; cf. Fig. 1 (a)). Hence, in the above problem, we seek a design where a_1 is as slim as possible, nevertheless, carrying the load. Since the load is constant, making the bar slim, however, increases the absolute stress $|\sigma_1(a, u)| = \left| \frac{E_1}{\ell_1}(-u_2) \right| = |u_2|$. Hence, the optimal design is completely determined by the stress constraint for bar no. 1 because the side constraints on total weight or compliance do not become active.

Consider the particular solution

$$x^* := (a_1^*, a_2^*, u_1^*, u_2^*)^T = (1, 0, 1, -1)^T. \quad (15)$$

Then

$$\begin{aligned} I_g &= \emptyset, \quad J = \{1, 2\}, \quad I_+ = \{1, 2\}, \quad I_0 = \{3, 4\}, \\ I_{+0} &= \{2\}, \quad I_{+-} = \{1\}, \quad I_{0+} = \emptyset, \quad I_{00} = \{3\}, \quad I_{0-} = \{4\}. \end{aligned} \quad (16)$$

By Lemmas 3.1 and 3.2, the LICQ as well as the MFCQ is violated since $I_0 \supset I_{00} \neq \emptyset$. Of course, these facts also can be directly checked by the explicit problem formulation in (14).

Alternatively, one may consider the solution

$$\tilde{x}^* := (a_1^*, a_2^*, \tilde{u}_1^*, u_2^*)^T = (1, 0, 0, -1)^T \quad (17)$$

(differing from x^* in the 3rd component) with the corresponding index sets (here and in the sequel indicated by an additional “ \sim ”)

$$\begin{aligned} \tilde{I}_g &= \emptyset, \quad \tilde{J} = \{1, 2\}, \quad \tilde{I}_+ = \{1, 2\}, \quad \tilde{I}_0 = \{3, 4\}, \\ \tilde{I}_{+0} &= \{2\}, \quad \tilde{I}_{+-} = \{1\}, \quad \tilde{I}_{0+} = \emptyset, \quad \tilde{I}_{00} = \emptyset, \quad \tilde{I}_{0-} = \{3, 4\}. \end{aligned} \quad (18)$$

At the point \tilde{x}^* we have $\nabla h_1(\tilde{x}^*) = 0_{\mathbb{R}^4}$, and thus neither LICQ nor MFCQ have a chance to hold. \diamond

We next discuss the Abadie constraint qualification. As a first step in this direction, we give a representation of the linearized cone of (1) in our next result.

Lemma 3.5 *Let x^* be a local minimum of (1). Then the linearized cone of (1) at x^* is given by*

$$\begin{aligned} \mathcal{L}(x^*) = \{d \in \mathbb{R}^n \mid & \nabla g_i(x^*)^T d \leq 0 \quad (i \in I_g), \\ & \nabla h_j(x^*)^T d = 0 \quad (j \in J), \\ & \nabla H_i(x^*)^T d = 0 \quad (i \in I_{0+}), \\ & \nabla H_i(x^*)^T d \geq 0 \quad (i \in I_{00} \cup I_{0-}), \\ & \nabla G_i(x^*)^T d \leq 0 \quad (i \in I_{+0})\}. \end{aligned}$$

Proof. Let θ_i denote the function from (10). Then, using the definition of the index sets from (7)–(9), it follows that the linearized cone of the program (1) at x^* is given by

$$\begin{aligned} \mathcal{L}(x^*) = \{d \in \mathbb{R}^n \mid & \nabla g_i(x^*)^T d \leq 0 \quad (i \in I_g), \\ & \nabla h_j(x^*)^T d = 0 \quad (j \in J), \\ & \nabla H_i(x^*)^T d \geq 0 \quad (i \in I_0), \\ & \nabla \theta_i(x^*)^T d \leq 0 \quad (i \in I_0 \cup I_{+0})\}. \end{aligned}$$

Now, using the expression of the gradient $\nabla \theta_i(x^*)$ for $i \in I_0 \cup I_{+0}$ as given in (11), it follows that

$$\begin{aligned} \nabla \theta_i(x^*)^T d \leq 0 & \iff \nabla H_i(x^*)^T d \leq 0 \quad \forall i \in I_{0+}, \\ \nabla \theta_i(x^*)^T d \leq 0 & \iff 0 \leq 0 \quad \forall i \in I_{00}, \\ \nabla \theta_i(x^*)^T d \leq 0 & \iff \nabla H_i(x^*)^T d \geq 0 \quad \forall i \in I_{0-}, \\ \nabla \theta_i(x^*)^T d \leq 0 & \iff \nabla G_i(x^*)^T d \leq 0 \quad \forall i \in I_{+0}. \end{aligned}$$

The first equivalence, together with $\nabla H_i(x^*)^T d \geq 0$ for all $i \in I_0$, gives $\nabla H_i(x^*)^T d = 0$ for all $i \in I_{0+}$, whereas the second and third equivalences do not provide any new information. Putting together all these pieces of information, we immediately get the desired representation of the linearized cone. \square

The following example shows that ACQ may not hold if $I_{00} \neq \emptyset$.

Example 3.6 Consider the optimization problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & H_1(x) := x_1 + x_2 \geq 0, \\ & G_1(x)H_1(x) := x_1(x_1 + x_2) \leq 0, \end{aligned}$$

which is of the form (1) with $n = 2, m = p = 0$, and $l = 1$. Its unique solution is given by $x^* = (0, 0)^T$. A simple calculation shows that the tangent cone of this program is given by

$$\mathcal{T}(x^*) = \{d \in \mathbb{R}^2 \mid d_1 + d_2 \geq 0, d_1 \leq 0\} \cup \{d \in \mathbb{R}^2 \mid d_1 + d_2 = 0\}$$

$$= \{d \in \mathbb{R}^2 \mid d_1 + d_2 \geq 0, d_1(d_1 + d_2) \leq 0\},$$

whereas Lemma 3.5 shows that the corresponding linearized cone has the representation

$$\mathcal{L}(x^*) = \{d \in \mathbb{R}^2 \mid d_1 + d_2 \geq 0\}.$$

Hence the linearized cone is strictly larger than the tangent cone, i.e., ACQ is violated in this example. \diamond

In the next section, we show that ACQ holds under reasonable conditions provided that $I_{00} = \emptyset$. In fact, looking at Lemmas 3.1, 3.2 and Example 3.6, the reader may ask whether ACQ is always violated if $I_{00} \neq \emptyset$. The following example shows, however, that this is not true in general.

Example 3.7 Consider the problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & H_1(x) := x_1 + x_2 \geq 0, \\ & G_1(x)H_1(x) := (-x_1 - x_2)(x_1 + x_2) \leq 0, \end{aligned}$$

whose unique solution is the origin $x^* := (0, 0)^T$. Hence we have $I_{00} = \{1\}$, in particular, this set is nonempty. Nevertheless, Lemma 3.5 and an elementary calculation shows that $\mathcal{T}(x^*) = \{d \in \mathbb{R}^2 \mid d_1 + d_2 \geq 0\} = \mathcal{L}(x^*)$, hence ACQ holds in this example. \diamond

Similarly, we may consider the academic truss design example from above.

Example 3.8 (Ex. 3.4, cont'd) Consider the optimizer x^* from (15) in Ex. 3.4. Then Lemma 3.5 yields

$$\mathcal{L}(x^*) = \{d \in \mathbb{R}^4 \mid d_1 = d_4 \geq 0, d_2 = 0, d_3 \in \mathbb{R} \text{ arbitrary}\}.$$

We claim that ACQ holds at x^* , i.e., that $\mathcal{T}(x^*) = \mathcal{L}(x^*)$. To see this, first recall that we always have $\mathcal{T}(x^*) \subseteq \mathcal{L}(x^*)$. To prove the other inclusion, take an arbitrary $d \in \mathcal{L}(x^*)$. Then $d = (d_1, 0, d_3, d_1)^T$ for some $d_1 \geq 0$ and d_3 arbitrary. Now let $\{t_k\} \downarrow 0$ be any given sequence and choose $\{x^k\}$ as follows:

$$x^k := \begin{pmatrix} 1 + t_k d_1 \\ 0 \\ 1 + t_k d_3 \\ -1 + \frac{t_k d_1}{1 + t_k d_1} \end{pmatrix} \quad \forall k \in \mathbb{N}.$$

Let X denote the feasible set of our problem. Then it is easy to see that $\{x^k\} \subseteq X$, $x^k \rightarrow x^*$, and $\frac{x^k - x^*}{t_k} \rightarrow d$. This shows that $d \in \mathcal{T}(x^*)$ and, therefore, ACQ holds at x^* (note that $I_{00} \neq \emptyset$ in this case).

On the other hand, consider the optimizer \tilde{x}^* from (17). Lemma 3.5 yields

$$\mathcal{L}(\tilde{x}^*) = \{d \in \mathbb{R}^4 \mid d_1 = d_4 \geq 0, d_2 \geq 0, d_3 \in \mathbb{R} \text{ arbitrary}\}. \quad (19)$$

We claim that the inclusion $\mathcal{T}(\tilde{x}^*) \subseteq \mathcal{L}(\tilde{x}^*)$ is strict, so that ACQ is violated at \tilde{x}^* . To this end, consider the particular vector $d := (0, 1, 1, 0)^T$. We obviously have $d \in \mathcal{L}(\tilde{x}^*)$, and want to show that $d \notin \mathcal{T}(\tilde{x}^*)$. Suppose, by contradiction, that there are sequences $\{t_k\} \downarrow 0$ and $\{x^k\} \subseteq X$ such that $x^k \rightarrow \tilde{x}^*$ and $\frac{x^k - \tilde{x}^*}{t_k} \rightarrow d$. Let us write $x^k = (a_1^k, a_2^k, u_1^k, u_2^k)^T$ for all $k \in \mathbb{N}$. Since $x^k \in X$, we have $a_2^k u_1^k = 0$ for all $k \in \mathbb{N}$. If there are infinitely many $k \in \mathbb{N}$ with $a_2^k = 0$, we get the contradiction

$$0 = \frac{0 - 0}{t_k} = \frac{x_2^k - \tilde{x}_2^*}{t_k} \rightarrow 1 = d_2.$$

On the other hand, if there are only finitely many $k \in \mathbb{N}$ with $a_2^k = 0$, we have $u_1^k = 0$ for all sufficiently large $k \in \mathbb{N}$. This, however, gives the contradiction

$$0 = \frac{0 - 0}{t_k} = \frac{x_3^k - \tilde{x}_3^*}{t_k} \rightarrow 1 = d_3.$$

Together this shows that $d \notin \mathcal{T}(\tilde{x}^*)$. Consequently, ACQ does not hold at \tilde{x}^* (note that $I_{00} = \emptyset$ in this case). In fact, by a similar argument, one can show that $\mathcal{T}(\tilde{x}^*)$ is equal to the non-convex cone

$$\mathcal{T}(\tilde{x}^*) = \{d \mid d_1 = d_4 \geq 0, d_2 = 0, d_3 \in \mathbb{R} \text{ arbitrary}\} \cup \{d \mid d_1 = d_4 \geq 0, d_2 \geq 0, d_3 = 0\}$$

which is obviously a proper subset of $\mathcal{L}(\tilde{x}^*)$. \diamond

4 Standard Abadie Constraint Qualification

The aim of this section is to show that the Abadie constraint qualification holds at a local minimum x^* of (1) under certain assumptions. To this end, we begin with the following simple but important result.

Theorem 4.1 *Let x^* be a local minimum of (1) such that ACQ holds at x^* . Then there exist Lagrange multipliers $\lambda_i \in \mathbb{R}$ ($i = 1, \dots, m$), $\mu_j \in \mathbb{R}$ ($j \in J$), $\eta_i^H, \eta_i^G \in \mathbb{R}$ ($i = 1, \dots, l$) such that*

$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{j \in J} \mu_j \nabla h_j(x^*) - \sum_{i=1}^l \eta_i^H \nabla H_i(x^*) + \sum_{i=1}^l \eta_i^G \nabla G_i(x^*) = 0 \quad (20)$$

and

$$\begin{aligned} h_j(x^*) &= 0 \quad \forall j \in J, \\ \lambda_i &\geq 0, \quad g_i(x^*) \leq 0, \quad \lambda_i g_i(x^*) = 0 \quad \forall i = 1, \dots, m, \\ \eta_i^H &= 0 \ (i \in I_+), \quad \eta_i^H \geq 0 \ (i \in I_{00} \cup I_{0-}), \quad \eta_i^H \text{ free} \ (i \in I_{0+}), \\ \eta_i^G &= 0 \ (i \in I_0 \cup I_{+-}), \quad \eta_i^G \geq 0 \ (i \in I_{+0}). \end{aligned} \quad (21)$$

Proof. Since ACQ holds at x^* , the standard KKT conditions of (1) are satisfied, i.e., there exist Lagrange multipliers $\lambda_i \in \mathbb{R}$ ($i = 1, \dots, m$), $\mu_j \in \mathbb{R}$ ($j \in J$) and $\rho_i, \nu_i \in \mathbb{R}$ ($i = 1, \dots, l$) such that the following conditions hold:

$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{j \in J} \mu_j \nabla h_j(x^*) - \sum_{i=1}^l \rho_i \nabla H_i(x^*) + \sum_{i=1}^l \nu_i \nabla \theta_i(x^*) = 0$$

and

$$\begin{aligned} g_i(x^*) &\leq 0, \quad \lambda_i \geq 0, \quad \lambda_i g_i(x^*) = 0 \quad \forall i = 1, \dots, m, \\ h_j(x^*) &= 0 \quad \forall j \in J, \\ H_i(x^*) &\geq 0, \quad \rho_i \geq 0, \quad \rho_i H_i(x^*) = 0 \quad \forall i = 1, \dots, l, \\ \theta_i(x^*) &\leq 0, \quad \nu_i \geq 0, \quad \nu_i \theta_i(x^*) = 0 \quad \forall i = 1, \dots, l, \end{aligned}$$

where, again, θ_i denotes the function from (10). Now, taking into account the representation (11) of the gradient of θ_i , and setting

$$\eta_i^H := \rho_i - \nu_i G_i(x^*) \quad \text{and} \quad \eta_i^G := \nu_i H_i(x^*) \quad \forall i = 1, \dots, l,$$

we immediately obtain the desired conditions (20), (21). \square

For obvious reasons, we call (20), (21) the *KKT conditions* of the optimization problem (1). Note that there is no sign restriction on the multipliers η_i^H whose components i belong to the index set I_{0+} .

We next state a technical lemma that will play a major role in developing new (specialized) constraint qualifications for our optimization problem (1).

Lemma 4.2 *Let $x^* \in \mathbb{R}^n$ be a local minimum of (1). Assume that the gradients*

$$\begin{aligned} \nabla h_j(x^*) &\quad (j \in J), \\ \nabla H_i(x^*) &\quad (i \in I_{00} \cup I_{0+}) \end{aligned}$$

are linearly independent, and that there is a vector \hat{d} satisfying

$$\nabla h_j(x^*)^T \hat{d} = 0 \quad (j \in J), \quad \nabla H_i(x^*)^T \hat{d} = 0 \quad (i \in I_{00} \cup I_{0+}) \quad (22)$$

and

$$\nabla g_i(x^*)^T \hat{d} < 0 \quad (i \in I_g), \quad \nabla G_i(x^*)^T \hat{d} < 0 \quad (i \in I_{+0}), \quad \nabla H_i(x^*)^T \hat{d} > 0 \quad (i \in I_{0-}). \quad (23)$$

Then there is an $\varepsilon > 0$ and a continuously differentiable curve $x : (-\varepsilon, +\varepsilon) \rightarrow \mathbb{R}^n$ such that $x(0) = x^$, $x'(0) = \hat{d}$ and $x(t) \in X$ for all $t \in [0, \varepsilon)$, where X denotes the feasible set of (1).*

Proof. Let us introduce the mapping $z : \mathbb{R}^n \rightarrow \mathbb{R}^q$, $q := |J| + |I_{00}| + |I_{0+}|$, defined by

$$z(x) := \begin{pmatrix} h_j(x) & (j \in J) \\ H_i(x) & (i \in I_{00}) \\ H_i(x) & (i \in I_{0+}) \end{pmatrix},$$

and let z_j denote the j th component function of z . Furthermore, let $\overline{H} : \mathbb{R}^{q+1} \rightarrow \mathbb{R}^q$ be the mapping defined by

$$\overline{H}_j(y, t) := z_j(x^* + t\hat{d} + z'(x^*)^T y) \quad \forall j = 1, \dots, q.$$

Then the (usually nonlinear) system of equations $\overline{H}(y, t) = 0$ has a solution $(y^*, t^*) := (0, 0)$, and the partial Jacobian

$$\overline{H}_y(0, 0) = z'(x^*)z'(x^*)^T \in \mathbb{R}^{q \times q}$$

is nonsingular since the Jacobian $z'(x^*)$ has full rank by assumption. Consequently, using the implicit function theorem, there is an $\varepsilon > 0$ and a continuously differentiable function $y : (-\varepsilon, +\varepsilon) \rightarrow \mathbb{R}^q$ such that $y(0) = 0$ and $\overline{H}(y(t), t) = 0$ for all $t \in (-\varepsilon, +\varepsilon)$. Moreover, its derivative is given by

$$y'(t) = -(\overline{H}_y(y(t), t))^{-1} \overline{H}_t(y(t), t) \quad \forall t \in (-\varepsilon, +\varepsilon).$$

In particular, this implies

$$y'(0) = -(\overline{H}_y(0, 0))^{-1} \overline{H}_t(0, 0) = -(\overline{H}_y(0, 0))^{-1} z'(x^*)\hat{d} = 0$$

in view of (22).

Now define

$$x(t) := x^* + t\hat{d} + z'(x^*)^T y(t).$$

Then $x(\cdot)$ is continuously differentiable on $(-\varepsilon, +\varepsilon)$, and we claim that $x(t)$ has all the desired properties (possibly on a slightly smaller interval). Since $y(0) = 0$ and $y'(0) = 0$, we immediately obtain $x(0) = x^*$ and $x'(0) = \hat{d} + z'(x^*)^T y'(0) = \hat{d}$. Hence it remains to show that $x(t) \in X$ for all sufficiently small $t \in [0, \varepsilon)$.

To this end, we first note that $\overline{H}(y(t), t) = 0$ implies $z_j(x(t)) = 0$ and, therefore,

$$\begin{aligned} h_j(x(t)) &= 0 \quad \forall j \in J, \\ H_i(x(t)) &= 0 \quad \forall i \in I_{00}, \end{aligned} \tag{24}$$

$$H_i(x(t)) = 0 \quad \forall i \in I_{0+} \tag{25}$$

for all $t \in (-\varepsilon, +\varepsilon)$. Furthermore, by continuity, we also have $H_i(x(t)) \geq 0$ for all $i \in I_+$ and all t sufficiently small. Next take an arbitrary index $i \in I_{0-}$, and define $\phi(t) := H_i(x(t))$. Then we have $\phi'(t) = \nabla H_i(x(t))^T x'(t)$ and, therefore, $\phi'(0) = \nabla H_i(x^*)^T \hat{d} > 0$ in view of

(23). Since $\phi(0) = 0$, this implies $H_i(x(t)) = \phi(t) > 0$ for all $t > 0$ sufficiently small. Consequently, we have shown that $H_i(x(t)) \geq 0$ for all $i = 1, \dots, l$ and all $t > 0$ sufficiently small. In a similar way, one can prove that $g_i(x(t)) \leq 0$ for all $i = 1, \dots, m$ and all $t > 0$ small. Hence it remains to show that the curve $x(t)$ stays feasible (locally) with respect to the constraints $\theta_i(x) \leq 0$.

In view of (24)–(25), this is certainly true for all $i \in I_{00} \cup I_{0+}$. Moreover, by continuity, this also holds for all $i \in I_{+-}$. Hence we only have to consider indices $i \in I_{0-} \cup I_{+0}$. To this end, define $\varphi(t) := G_i(x(t))H_i(x(t))$. Then an elementary calculation shows that

$$\varphi'(t) = H_i(x(t))\nabla G_i(x(t))^T x'(t) + G_i(x(t))\nabla H_i(x(t))^T x'(t).$$

This implies

$$\varphi'(0) = H_i(x^*)\nabla G_i(x^*)^T \hat{d} + G_i(x^*)\nabla H_i(x^*)^T \hat{d},$$

and, in view of (23), it is immediate to see that $\varphi'(0) < 0$ holds for all indices i belonging to one of the remaining index sets I_{+0} and I_{0-} . Consequently, we have $G_i(x(t))H_i(x(t)) = \varphi(t) < 0$ for all $i \in I_{+0} \cup I_{0-}$ and all $t > 0$ sufficiently small. This completes the proof. \square

Motivated by the assumptions used in Lemma 4.2, we now introduce a variant of the standard MFCQ condition that we call VC-MFCQ since it is a special constraint qualification tailored to optimization problems with vanishing constraints, i.e., optimization problems of type (1) (here and in the following, the abbreviation VC stands for “vanishing constraints”).

Definition 4.3 *We say that VC-MFCQ is satisfied at a local minimum x^* of (1) if the gradients*

$$\begin{aligned} \nabla h_j(x^*) & \quad (j \in J), \\ \nabla H_i(x^*) & \quad (i \in I_{00} \cup I_{0+}) \end{aligned}$$

are linearly independent, and if there is a vector \hat{d} satisfying

$$\nabla h_j(x^*)^T \hat{d} = 0 \quad (j \in J), \quad \nabla H_i(x^*)^T \hat{d} = 0 \quad (i \in I_{00} \cup I_{0+})$$

and

$$\nabla g_i(x^*)^T \hat{d} < 0 \quad (i \in I_g), \quad \nabla G_i(x^*)^T \hat{d} < 0 \quad (i \in I_{+0}), \quad \nabla H_i(x^*)^T \hat{d} > 0 \quad (i \in I_{0-}).$$

Note that VC-MFCQ is a reasonable assumption and that it is different from standard MFCQ (cf. the proof of Lemma 3.2). We now show that VC-MFCQ implies standard ACQ provided that the critical index set I_{00} is empty.

Theorem 4.4 *Let x^* be a local minimum of (1) with $I_{00} = \emptyset$ and such that VC-MFCQ holds. Then the standard Abadie constraint qualification holds at x^* .*

Proof. We have to show that $\mathcal{T}(x^*) = \mathcal{L}(x^*)$. It is well-known, however, that the inclusion $\mathcal{T}(x^*) \subseteq \mathcal{L}(x^*)$ always holds. Hence it remains to show that the linearized cone is a subset of the tangent cone. To this end, take any vector $d \in \mathcal{L}(x^*)$. Then Lemma 3.5 together with $I_{00} = \emptyset$ shows that we have

$$\begin{aligned}\nabla g_i(x^*)^T d &\leq 0 \quad \forall i \in I_g, \\ \nabla h_j(x^*)^T d &= 0 \quad \forall j \in J, \\ \nabla H_i(x^*)^T d &= 0 \quad \forall i \in I_{0+}, \\ \nabla H_i(x^*)^T d &\geq 0 \quad \forall i \in I_{0-}, \\ \nabla G_i(x^*)^T d &\leq 0 \quad \forall i \in I_{+0}.\end{aligned}$$

Now let $\hat{d} \in \mathbb{R}^n$ be a vector coming from our VC-MFCQ condition, and define

$$d(\delta) := d + \delta \hat{d}.$$

Then it is easy to see that $d(\delta)$ satisfies

$$\begin{aligned}\nabla g_i(x^*)^T d(\delta) &< 0 \quad \forall i \in I_g, \\ \nabla h_j(x^*)^T d(\delta) &= 0 \quad \forall j \in J, \\ \nabla H_i(x^*)^T d(\delta) &= 0 \quad \forall i \in I_{0+}, \\ \nabla H_i(x^*)^T d(\delta) &> 0 \quad \forall i \in I_{0-}, \\ \nabla G_i(x^*)^T d(\delta) &< 0 \quad \forall i \in I_{+0}.\end{aligned}$$

for all $\delta > 0$.

Let $\delta > 0$ be fixed for the moment. We then show that $d(\delta)$ belongs to the tangent cone $\mathcal{T}(x^*)$. Using the previous properties of $d(\delta)$, the assumption $I_{00} = \emptyset$, and the VC-MFCQ condition, it follows from Lemma 4.2 that there is an $\varepsilon > 0$ and a smooth curve $x : (-\varepsilon, +\varepsilon) \rightarrow \mathbb{R}^n$ (both depending on δ) such that $x(0) = x^*$, $x'(0) = d(\delta)$ and $x(t) \in X$ for all $t > 0$ sufficiently small. Now take an arbitrary sequence $\{t_k\} \downarrow 0$ and define $x^k := x(t_k)$. Then $\{x^k\} \subseteq X$, $x^k \rightarrow x^*$, and

$$d(\delta) = x'(0) = \lim_{k \rightarrow \infty} \frac{x(t_k) - x(0)}{t_k} = \lim_{k \rightarrow \infty} \frac{x^k - x^*}{t_k}.$$

This shows that $d(\delta) = d + \delta \hat{d} \in \mathcal{T}(x^*)$ for every $\delta > 0$.

Finally, taking $\delta_k \downarrow 0$ and noting that the tangent cone $\mathcal{T}(x^*)$ is closed, it follows that $d = \lim_{k \rightarrow \infty} d(\delta_k) \in \mathcal{T}(x^*)$. \square

As a consequence of Theorems 4.1 and 4.4, it follows that the KKT conditions (20), (21) are necessary optimality conditions at a local minimum x^* of (1) under the assumption that $I_{00} = \emptyset$ and that VC-MFCQ holds. Moreover, Theorem 4.4 implies that the tangent cone is polyhedral under these assumptions.

It is interesting to note that Theorem 4.4 does not hold without the assumption $I_{00} = \emptyset$, i.e., VC-MFCQ may not imply standard ACQ if this set is nonempty. This can be seen

by an inspection of Example 3.6 which obviously satisfies VC-MFCQ, whereas ACQ was violated. However, this is an example where $I_{00} \neq \emptyset$. The previous proof exploits the fact that $I_{00} = \emptyset$, since otherwise we would have $\nabla H_i(x^*)^T d(\delta) \geq 0$ for all $i \in I_{00}$, and then it is no longer possible to apply Lemma 4.2 in order to show that $d(\delta)$ belongs to the tangent cone $\mathcal{T}(x^*)$ (because this would require $\nabla H_i(x^*)^T d(\delta) = 0$ for all $i \in I_{00}$).

We next introduce a condition that we call VC-LICQ and which may be viewed as a modification of the standard LICQ condition, taking into account the special structure of the optimization problem (1).

Definition 4.5 *We say that VC-LICQ is satisfied at a local minimum x^* of (1) if the gradients*

$$\begin{aligned} \nabla h_j(x^*) & \quad (j \in J), \\ \nabla g_i(x^*) & \quad (i \in I_g), \\ \nabla G_i(x^*) & \quad (i \in I_{+0}), \\ \nabla H_i(x^*) & \quad (i \in I_0) \end{aligned}$$

are linearly independent.

Note that VC-LICQ is different from standard LICQ. Moreover, it is easy to see that VC-LICQ implies VC-MFCQ. VC-LICQ, however, might be easier to verify than VC-MFCQ. Moreover, it guarantees uniqueness of the Lagrange multipliers. More precisely, we have the following result.

Theorem 4.6 *Let x^* be a local minimum of (1) with $I_{00} = \emptyset$ and such that VC-LICQ is satisfied. Then the standard Abadie constraint qualification holds at x^* . Moreover, there exist unique Lagrange multipliers satisfying (20), (21).*

Proof. The first statement follows immediately from Theorem 4.4 and the fact that VC-LICQ implies VC-MFCQ. The second statement follows directly from the KKT conditions (20), (21) and the linear independence of all gradient vectors belonging to those terms which might have a nonzero multiplier. \square

Note that VC-LICQ is obviously satisfied in Example 3.6, whereas the Abadie constraint qualification does not hold. Hence an additional assumption like $I_{00} = \emptyset$ used in Theorem 4.6 (and, therefore, also in Theorem 4.4) is certainly needed.

Let us, finally, have a look at the above academic truss example.

Example 4.7 (Exs. 3.4 and 3.8, cont'd) Consider the optimal point \tilde{x}^* of the problem in Ex. 3.4 (cf. (17)). As noted in Ex. 3.8, ACQ does not hold, although $\tilde{I}_{00} = \emptyset$ (cf. (18)). Moreover, $\nabla h_1(\tilde{x}^*) = 0_{\mathbb{R}^4}$, and thus neither Lemma 4.2 applies, nor VC-MFCQ is satisfied (nor VC-LICQ). Moreover, in this example, the functions H_3 and H_4 coincide, and thus (note that $\tilde{I}_0 = \tilde{I}_{0-} = \{3, 4\}$; cf. (18)), trivially, the gradients $\nabla H_3(\tilde{x}^*)$, $\nabla H_4(\tilde{x}^*)$ are linearly dependent since they are identical. \diamond

5 A Modified Abadie Constraint Qualification

The aim of this section is to introduce a modified Abadie constraint qualification tailored to the special structure of the optimization problem (1). This constraint qualification will then be used in order to prove a necessary optimality condition that is different from the KKT conditions stated in Theorem 4.1. We also provide sufficient conditions for the modified Abadie constraint qualification to be satisfied.

In order to define our modified Abadie constraint qualification, let us introduce the *modified linearized cone*

$$\begin{aligned} \mathcal{L}^{\text{MOD}}(x^*) := \{ & d \in \mathbb{R}^n \mid \nabla g_i(x^*)^T d \leq 0 \ (i \in I_g), \\ & \nabla h_j(x^*)^T d = 0 \ (j \in J), \\ & \nabla H_i(x^*)^T d = 0 \ (i \in I_{0+}), \\ & \nabla H_i(x^*)^T d \geq 0 \ (i \in I_{00} \cup I_{0-}), \\ & \nabla G_i(x^*)^T d \leq 0 \ (i \in I_{00} \cup I_{+0}) \}. \end{aligned}$$

Note that we always have $\mathcal{L}^{\text{MOD}}(x^*) \subseteq \mathcal{L}(x^*)$ in view of Lemma 3.5. Using this modified linearized cone, we now define our modified Abadie constraint qualification.

Definition 5.1 *The modified Abadie constraint qualification (modified ACQ for short) is said to hold at a local minimizer of (1) if $\mathcal{L}^{\text{MOD}}(x^*) \subseteq \mathcal{T}(x^*)$.*

Note that Definition 5.1 only requires that the modified linearized cone is a subset of the tangent cone. Another idea would be to use equality of these two cones, however, this would be a much stronger condition since the modified linearized cone is polyhedral and, therefore, convex, whereas the tangent cone might be non-convex.

The following note says that the modified ACQ condition is strictly weaker than standard ACQ.

Remark. Let $x^* \in \mathbb{R}^n$ be a local minimizer of (1) such that standard ACQ is satisfied at x^* . Then modified ACQ also holds at x^* since $\mathcal{L}^{\text{MOD}}(x^*) \subseteq \mathcal{L}(x^*) = \mathcal{T}(x^*)$. Moreover, the modified Abadie CQ is strictly weaker than the standard Abadie CQ. To see this, let us consider Example 3.6 once again. There we have

$$\mathcal{L}^{\text{MOD}}(x^*) = \{d \in \mathbb{R}^n \mid d_1 + d_2 \geq 0, d_1 \leq 0\} \subseteq \mathcal{T}(x^*),$$

hence the modified ACQ holds whereas standard Abadie was violated.

Example 5.2 (Exs. 3.4, 3.8, 4.7, cont'd) Again consider the minimizers x^* and \tilde{x}^* of Ex. 3.4 (cf. (15) and (17)). The definition of \mathcal{L}^{MOD} gives

$$\mathcal{L}^{\text{MOD}}(x^*) = \{d \in \mathbb{R}^4 \mid d_1 = d_4 \geq 0, d_2 = 0, d_3 \leq 0\}$$

which is strictly smaller than the linearized cone $\mathcal{L}(x^*)$ (cf. (19) in Ex. 3.8 where d_3 is arbitrary). However, as already seen in Ex. 3.8, $\mathcal{T}(x^*) = \mathcal{L}(x^*)$, and thus the modified ACQ is trivially satisfied at x^* .

Now consider \tilde{x}^* . Since $\tilde{I}_{00} = \emptyset$ (cf. (18)), we have

$$\mathcal{L}^{\text{MOD}}(\tilde{x}^*) = \mathcal{L}(\tilde{x}^*).$$

Hence, the modified ACQ is not satisfied at \tilde{x}^* because $\mathcal{T}(\tilde{x}^*) \subsetneq \mathcal{L}(\tilde{x}^*)$ as already seen in Ex. 3.8. \diamond

Since modified ACQ is weaker than standard ACQ, we cannot expect the KKT conditions from Theorem 4.1 to hold at a local minimum x^* where the modified ACQ is satisfied. However, we get another optimality condition under modified ACQ as stated in our following result.

Theorem 5.3 *Let x^* be a local minimum of (1) such that the modified ACQ condition holds at x^* . Then there exist Lagrange multipliers $\lambda_i \in \mathbb{R}$ ($i = 1, \dots, m$), $\mu_j \in \mathbb{R}$ ($j \in J$), $\eta_i^H, \eta_i^G \in \mathbb{R}$ ($i = 1, \dots, l$) such that*

$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{j \in J} \mu_j \nabla h_j(x^*) - \sum_{i=1}^l \eta_i^H \nabla H_i(x^*) + \sum_{i=1}^l \eta_i^G \nabla G_i(x^*) = 0 \quad (26)$$

and

$$\begin{aligned} \lambda_i &\geq 0, \quad g_i(x^*) \leq 0, \quad \lambda_i g_i(x^*) = 0 \quad \forall i = 1, \dots, m, \\ h_j(x^*) &= 0 \quad \forall j \in J, \\ \eta_i^H &= 0 \ (i \in I_+), \quad \eta_i^H \geq 0 \ (i \in I_{00} \cup I_{0-}), \quad \eta_i^H \text{ free} \ (i \in I_{0+}), \\ \eta_i^G &= 0 \ (i \in I_{0+} \cup I_{0-} \cup I_{+-}), \quad \eta_i^G \geq 0 \ (i \in I_{00} \cup I_{+0}). \end{aligned} \quad (27)$$

Proof. The technique of proof is standard in optimization, and we present it here only for the sake of completeness. Since x^* is a local minimum of (1), it follows that

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in \mathcal{T}(x^*).$$

Using the modified ACQ condition, this implies

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in \mathcal{L}^{\text{MOD}}(x^*). \quad (28)$$

Using the fact that $\mathcal{L}^{\text{MOD}}(x^*)$ is a polyhedral cone, and splitting all equality constraints into two inequalities, we may rewrite (28) as

$$\nabla f(x^*)^T d \geq 0 \quad \text{for all } d \text{ with } Ad \leq 0, \quad (29)$$

where A denotes the matrix whose rows are given by the vectors

$$\nabla g_i(x^*)^T \quad (i \in I_g),$$

$$\begin{aligned}
& \nabla h_j(x^*)^T && (j \in J), \\
& -\nabla h_j(x^*)^T && (j \in J), \\
& \nabla H_i(x^*)^T && (i \in I_{0+}), \\
& -\nabla H_i(x^*)^T && (i \in I_{0+}), \\
& -\nabla H_i(x^*)^T && (i \in I_{00} \cup I_{0-}), \\
& \nabla G_i(x^*)^T && (i \in I_{00} \cup I_{+0}).
\end{aligned}$$

Farkas' Lemma applied to (29) shows that there is a vector y satisfying the linear system

$$A^T y = -\nabla f(x^*), \quad y \geq 0. \quad (30)$$

We now partition the vector y in the same way as the rows of the matrix A and denote the elements of y by

$$\begin{aligned}
\lambda_i && (i \in I_g), \\
\mu_j^+ && (j \in J), \\
\mu_j^- && (j \in J), \\
\eta_i^{H,+} && (i \in I_{0+}), \\
\eta_i^{H,-} && (i \in I_{0+}), \\
\eta_i^H && (i \in I_{00} \cup I_{0-}), \\
\eta_i^G && (i \in I_{00} \cup I_{+0}).
\end{aligned}$$

Finally, setting

$$\mu_j := \mu_j^+ - \mu_j^- \quad (j \in J) \quad \text{and} \quad \eta_i^H := \eta_i^{H,-} - \eta_i^{H,+} \quad (i \in I_{0+})$$

and

$$\lambda_i := 0 \quad (i \notin I_g), \quad \eta_i^H := 0 \quad (i \in I_+), \quad \eta_i^G := 0 \quad (i \in I_{0+} \cup I_{0-} \cup I_{+-}),$$

we immediately obtain the desired statement from (30). \square

The conditions (26), (27) will sometimes be called the *modified KKT conditions* of the optimization problem (1). Note that the only difference between the standard KKT conditions from (20), (21) and these modified KKT conditions is in the multipliers η_i^G for $i \in I_{00}$: In the KKT conditions, these multipliers are zero, whereas in the modified KKT conditions, they are only nonnegative. Hence the modified KKT conditions are weaker than the standard KKT conditions, however, they also hold under the weaker modified Abadie constraint qualification.

We next provide some sufficient conditions for the modified Abadie constraint qualification to hold. In particular, Theorem 5.3 then holds under these conditions. We first show that the modified ACQ holds if all constraint functions are linear.

Theorem 5.4 *Let x^* be a local minimum of (1), and suppose that all constraint functions g_i, h_j, G_i , and H_i are affine. Then modified ACQ holds at x^* .*

Proof. We have to show that the inclusion $\mathcal{L}^{\text{MOD}}(x^*) \subseteq \mathcal{T}(x^*)$ holds. To this end, take an arbitrary vector $d \in \mathcal{L}^{\text{MOD}}(x^*)$, and define $x^k := x^* + t_k d$ for some sequence $\{t_k\} \downarrow 0$. Then it follows immediately that $x^k \rightarrow x^*$ and $(x^k - x^*)/t_k \rightarrow d$. Moreover, taking into account the definition of $\mathcal{L}^{\text{MOD}}(x^*)$ and exploiting the fact that all constraint functions are assumed to be affine mappings, it is easy to see that $\{x^k\} \subseteq X$ for all $k \in \mathbb{N}$ sufficiently large. Hence it follows from the definition of the tangent cone $\mathcal{T}(x^*)$ that $d \in \mathcal{T}(x^*)$. \square

Note that the assumptions of Theorem 5.4 are satisfied, in particular, for Example 3.6.

In order to present some other sufficient conditions for the modified ACQ to hold, we first state the following technical result.

Lemma 5.5 *Let x^* be a local minimum of (1), and suppose that the gradients*

$$\nabla h_j(x^*) \quad (j \in J), \quad \nabla H_i(x^*) \quad (i \in I_{0+})$$

are linearly independent, and that there is a vector $\hat{d} \in \mathbb{R}^n$ satisfying

$$\nabla h_j(x^*)^T \hat{d} = 0 \quad (j \in J), \quad \nabla H_i(x^*)^T \hat{d} = 0 \quad (i \in I_{0+})$$

and

$$\nabla g_i(x^*)^T \hat{d} < 0 \quad (i \in I_g), \quad \nabla G_i(x^*)^T \hat{d} < 0 \quad (i \in I_{00} \cup I_{0+}), \quad \nabla H_i(x^*)^T \hat{d} > 0 \quad (i \in I_{00} \cup I_{0-}).$$

Then there is an $\varepsilon > 0$ and a continuously differentiable curve $x : (-\varepsilon, +\varepsilon) \rightarrow \mathbb{R}^n$ such that $x(0) = x^$, $x'(0) = \hat{d}$, and $x(t) \in X$ for all $t \in [0, \varepsilon)$.*

Proof. The proof is essentially the same as the one of Lemma 4.2. To see this, define $z : \mathbb{R}^n \rightarrow \mathbb{R}^q, q := |J| + |I_{0+}|$, by

$$z(x) := \begin{pmatrix} h_j(x) & (j \in J) \\ H_i(x) & (i \in I_{0+}) \end{pmatrix}$$

and proceed almost word-by-word as in the proof of Lemma 4.2. This gives us a suitable curve $x(t)$ with a number of desired properties. The only difference between the proof of Lemma 4.2 and the current proof is in showing that this curve stays feasible with respect to the constraints $\theta_i(x) \leq 0$. In Lemma 4.2, this was shown by using a linearization of the mapping θ_i . Here we cannot use a linearization argument. However, we still have $\theta_i(x(t)) \leq 0$ for all $i = 1, \dots, l$ because it is easy to see that the properties of the vector \hat{d} guarantees that $G_i(x(t)) < 0$ and $H_i(x(t)) > 0$ holds for all $i \in I_+ \cup I_{00} \cup I_{0-}$ and all $t > 0$ sufficiently small, whereas for $i \in I_{0+}$ we have $\theta_i(x(t)) = 0$ since $H_i(x(t)) = 0$ for all these indices. \square

As a consequence of Lemma 5.5, we obtain another sufficient condition for the modified ACQ condition to be satisfied.

Theorem 5.6 *Let x^* be a local minimum of (1), and suppose that the assumptions of Lemma 5.5 hold. Then the modified ACQ condition is satisfied at x^* .*

Proof. We have to verify the inclusion $\mathcal{L}^{\text{MOD}}(x^*) \subseteq \mathcal{T}(x^*)$. To this end, take an arbitrary $d \in \mathcal{L}^{\text{MOD}}(x^*)$, and let $\hat{d} \in \mathbb{R}^n$ be the vector having the properties from Lemma 5.5. Then define $d(\delta) := d + \delta\hat{d}$. Exploiting the definition of the modified linearized cone $\mathcal{L}^{\text{MOD}}(x^*)$, it follows that this vector has the subsequent properties:

$$\begin{aligned} \nabla g_i(x^*)^T d(\delta) &< 0 & (i \in I_g), \\ \nabla h_j(x^*)^T d(\delta) &= 0 & (j \in J), \\ \nabla H_i(x^*)^T d(\delta) &= 0 & (i \in I_{0+}), \\ \nabla H_i(x^*)^T d(\delta) &> 0 & (i \in I_{00} \cup I_{0-}), \\ \nabla G_i(x^*)^T d(\delta) &< 0 & (i \in I_{00} \cup I_{+0}). \end{aligned}$$

Using Lemma 5.5 and the technique of proof from Theorem 4.4, it is easy to see that $d(\delta) \in \mathcal{T}(x^*)$ for all $\delta > 0$ sufficiently small. This implies $d \in \mathcal{T}(x^*)$ since the tangent cone is closed. \square

Finally, we present a sufficient condition which uses an LICQ-type assumption.

Theorem 5.7 *Let x^* be a local minimum of (1), and suppose that the gradients*

$$\begin{aligned} \nabla g_i(x^*) & \quad (i \in I_g), \\ \nabla h_j(x^*) & \quad (j \in J), \\ \nabla H_i(x^*) & \quad (i \in I_0), \\ \nabla G_i(x^*) & \quad (i \in I_{00} \cup I_{+0}) \end{aligned}$$

are linearly independent. Then modified ACQ holds, and there exist unique Lagrange multipliers satisfying (26), (27).

Proof. The first statement follows from Theorem 5.6 by noting that the assumptions of that result are obviously satisfied in our case. The second statement (uniqueness of multipliers) follows immediately from the modified KKT conditions (26), (27) and the linear independence of all gradient vectors with possibly nonzero multipliers. \square

We believe that the linear independence assumption used in Theorem 5.7 is rather natural. In fact, if we view the two factors $G_i(x)$ and $H_i(x)$ of the mapping θ_i from (10) separately, then the assumptions of Theorem 5.7 just say that the gradients of all active constraints are linearly independent. Hence this condition is a natural modification of the standard LICQ assumption. Therefore, we also think that this modified LICQ condition can be exploited for the development of numerical algorithms for the solution of mathematical programs with vanishing constraints.

6 Comparison with MPECs and Final Remarks

There is another optimization problem that has received much attention during the last decade and that is closely related to problem (1), namely the *mathematical program with equilibrium constraints* (or *complementarity constraints*), MPEC for short, see, e.g., the two monographs [10, 12]. An MPEC looks like an ordinary constrained optimization problem, however, its feasible set has a very special structure: Besides some standard equality and inequality constraints, all feasible points also have to satisfy some complementarity conditions. More precisely, an MPEC has the following form:

$$\begin{aligned}
 \min \quad & \tilde{f}(z) \\
 \text{s.t.} \quad & \tilde{g}_i(z) \leq 0 \quad \forall i = 1, \dots, m, \\
 & \tilde{h}_j(z) = 0 \quad \forall j = 1, \dots, p, \\
 & \tilde{G}_i(z) \geq 0 \quad \forall i = 1, \dots, l, \\
 & \tilde{H}_i(z) \geq 0 \quad \forall i = 1, \dots, l, \\
 & \tilde{G}_i(z)\tilde{H}_i(z) = 0 \quad \forall i = 1, \dots, l.
 \end{aligned} \tag{31}$$

MPECs are difficult programs since most of the standard constraint qualifications are violated. In fact, standard LICQ and standard MFCQ never hold (see [6]), whereas the standard Abadie constraint qualification is satisfied only in some rare situations. In fact, if z^* denotes a local minimum of (31) and

$$\beta := \beta(z^*) := \{i \mid \tilde{G}_i(z^*) = 0, \tilde{H}_i(z^*) = 0\}$$

denotes the degenerate or bi-active index set, then the standard tangent cone at z^* is usually the union of finitely many polyhedral cones, each of this polyhedral cones is generated by a partitioning of the degenerate set β , see [13, 8] for more details. Being the union of finitely many cones, the tangent cone is therefore nonconvex in general, hence the usual Abadie constraint qualification is not satisfied. The situation is different if $\beta = \emptyset$ since then the above union of finitely many polyhedral cones reduces to the union over a single polyhedral cone.

Now let us come back to our mathematical program with vanishing constraints (1). We first show that this program may be rewritten as an MPEC. In fact, introducing “slack variables” s_i , $i = 1, \dots, l$, problem (1) is equivalent to the following MPEC in the variables $z := (x, s)$:

$$\begin{aligned}
 \min_{x,s} \quad & f(x) \\
 \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, \dots, m, \\
 & h_j(x) = 0 \quad \forall j = 1, \dots, p, \\
 & G_i(x) - s_i \leq 0 \quad \forall i = 1, \dots, l, \\
 & H_i(x) \geq 0 \quad \forall i = 1, \dots, l, \\
 & s_i \geq 0 \quad \forall i = 1, \dots, l, \\
 & H_i(x)s_i = 0 \quad \forall i = 1, \dots, l.
 \end{aligned} \tag{32}$$

More precisely, the relation between the two problems (1) and (32) is as follows.

Lemma 6.1 (a) If x^* is a local minimum of (1), then $z^* := (x^*, s^*)$ is a local minimum of (32), where s^* denotes any vector with components

$$s_i^* \begin{cases} = 0, & \text{if } H_i(x^*) > 0, \\ \geq \max\{G_i(x^*), 0\}, & \text{if } H_i(x^*) = 0. \end{cases}$$

(b) If $z^* = (x^*, s^*)$ is a local minimum of (32), then x^* is a local minimum of (1).

The proof of Lemma 6.1 follows from the fact that the corresponding vectors are feasible for the respective optimization problems, and by noting that the objective function is the same for both programs. Note that, in statement (a), we have some freedom in the choice of the components s_i^* with indices i such that $H_i(x^*) = 0$.

In principle, it is therefore possible to reformulate a mathematical program with vanishing constraints as an MPEC. We believe, however, that this reformulation is not useful from a practical point of view, and that one should try to deal with problem (1) directly. For example, our discussion in Section 4 clearly shows that the standard Abadie constraint qualification has a good chance to be satisfied at a local minimum of a mathematical program with vanishing constraints, whereas it is usually violated for MPECs. Moreover, the dimension of the MPEC formulation (32) is larger than the one of the original program (1), and the slack variables in the program (32) are not defined uniquely, which might cause some troubles when solving (32) by suitable algorithms.

Hence we believe that a mathematical program with vanishing constraints is an interesting class of optimization problems for its own that deserve further investigation in order to get a better understanding, both from a theoretical and a numerical point of view.

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