Risk capital allocation by coherent risk measures based on one-sided moments

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Abstract

This paper proposes differentiability properties for positively homogeneous risk measures which ensure that the gradient can be applied for reasonable risk capital allocation on non-trivial portfolios. It is shown that these properties are fulfilled for a wide class of coherent risk measures based on the mean and the one-sided moments of a risky payoff. In contrast to quantile-based risk measures like Value-at-Risk, this class allows allocation in portfolios of very general distributions, e.g. discrete ones. Two examples show how risk capital given by the VaR can be allocated by adapting risk measures of this class to the VaR.

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1 Introduction

From the works of Denault (2001) and Tasche (2000) it is known that differentiability of risk measures is crucial for risk capital allocation in portfolios. The reason is that in the case of differentiable positively homogeneous risk measures the gradient due to asset weights has figured out to be the unique reasonable per-unit allocation principle. After a short introduction to risk measures at the end of the present section, the approaches of Denault (2001) and Tasche (2000) to this result are briefly reviewed in Section 2 of this paper.

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However, in contrast to the mentioned result, it is known that in practice quantile-based risk measures like the widely used Value-at-Risk methodology or the so-called Expected Shortfall encounter situations, e.g. in the case of insurance claims, credit portfolios or digital options, where probability distributions are discrete and the risk measures are not differentiable anymore (cf. Tasche, 2000). Furthermore, Section 3 of this paper shows that at least in the case of subadditive positively homogeneous risk measures differentiability on all portfolios actually is not desirable since the risk measures become linear and minimal in this case. As a solution, we define weaker differentiability properties (also Section 3). For positively homogeneous (and in particular coherent) risk measures these properties allow allocation by the gradient on all relevant portfolios. Excluded are portfolios that contain only one type of assets. However, in these cases the allocation problem is trivial. In Section 4, we introduce a wide class of coherent risk measures based on the mean and the one-sided moments of a risky payoff. In order to construct the class, it is shown that weighted sums of coherent risk measures are again coherent. Hence, it is possible to "mix" coherent risk measures. For example, one could consider the arithmetic mean of the maximum-loss-principle and a semi-deviation-like risk measure - both are members of the given class. An important result of Section 4 is that the constructed risk measures (expected and maximum loss excluded) are examples for the weakened differentiability properties of Section 3. In contrast to quantile-based risk measures, members of this class allow allocation in portfolios of very general distributions, e.g. discrete ones. Furthermore, for any fixed random payoff X risk measures of this class can be chosen such that the risk capital due to X equals any value between the expected and the maximum loss of X. In Section 5, two numerical examples show how this property can be used to choose a particular risk measure of the class which assigns the same risk capital to a given portfolio as VaR does. As a consequence, the risk capital originally given by the VaR can be allocated by the gradient due to the chosen risk measure. Section 6 compares the notation of this paper with the one used in Tasche (2000), respectively Denault (2001). In addition to the mentioned results of the paper, some of the lemmas proven in the technical appendix could be interesting in themselves.

Given a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, we will consider the vector space $L^p(\Omega, \mathcal{A}, \mathbb{Q})$, or just $L^p(\mathbb{Q})$, for $1 \leq p \leq \infty$. Even though $L^p(\mathbb{Q})$ consists of equivalence classes of p-integrable random variables, we will often treat its elements as random variables. Due to the context, no confusion should arise. The notation will be as follows. We have $||X||_p = (\mathbf{E}_{\mathbb{Q}}|X|^p)^{\frac{1}{p}}$ and $||X||_\infty =$ ess.sup$\{|X|\}$. Recall, that $L^p(\mathbb{Q}) \subset L^q(\mathbb{Q})$ if $1 \leq q < p \leq \infty$, since $||.||^q \leq ||.||^p$. X^- is defined as $\max\{-X, 0\}$. We denote $\sigma_p^-(X) = ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p$. Now, let $U \subset \mathbb{R}^n$ for $n \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ be open and positively homogeneous, i.e. for $u \in U$ we have $\lambda u \in U$ for all $\lambda > 0$. A function $f : U \to \mathbb{R}$ is called positively homogeneous (or homogeneous of degree one) if $f(\lambda u) = \lambda f(u)$ for all $\lambda > 0$, $u \in U$. When f is also differentiable at every $u = (u_1, \ldots, u_n) \in U$, we obtain the well-known Euler Theorem

$$f(u) = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial u_i}(u). \tag{1}$$

We consider a one-period framework, that means we have the present time 0 and a future time horizon T. Between 0 and T no trading is possible. We assume "risk" to be given by a random payoff X, i.e. a random variable in $L^p(\mathbb{Q})$ representing a cash flow at T. We want to consider a risk measure $\rho(X)$ to be the extra minimum cash added to X that makes the position acceptable for the holder or a regulator. For this reason, we state the following definition.

DEFINITION 1.1. *A* **risk measure** *on $L^p(\mathbb{Q})$, $1 \leq p \leq \infty$, is defined by a functional $\rho : L^p(\mathbb{Q}) \to \mathbb{R}$.*

We now give a definition of coherent risk measures. For a further motivation and interpretation of this axiomatic approach to risk measurement we refer to the article of Artzner et al. (1999).

DEFINITION 1.2. *A functional $\rho : L^p(\mathbb{Q}) \to \mathbb{R}$, where $1 \leq p \leq \infty$, is called a* **coherent risk measure (CRM)** *on $L^p(\mathbb{Q})$ if the following properties hold.*

(M) *Monotonicity: If $X \geq 0$ then $\rho(X) \leq 0$.*

(S) *Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.*

(PH) *Positive homogeneity: For $\lambda \geq 0$ we have $\rho(\lambda X) = \lambda \rho(X)$.*

(T) *Translation: For constants a we have $\rho(a + X) = \rho(X) - a$.*

As we work without interest rates - in contrast to Artzner et al. (1999) - there is no discounting factor in Definition 1.2. A generalization of CRM to the space of all random variables on a probability space can be found in Delbaen (2000). However, having $p \geq 1$ prevents us from being forced to allow infinitely high risks. See Delbaen (2000) for details on this topic.

The scientific discussion about suitable properties of risk measures continues. Especially in the context of actuarial mathematics (a risk measure can be seen as an insurance premium principle and vice versa) alternative approaches exist (e.g. Goovaerts, Kaas and Dhaene, 2003). For the purposes of this paper, we stay in the framework of positively homogeneous or coherent risk measures. A deeper discussion about properties of risk measures in different economic contexts is beyond the scope of this thesis.

2 Risk capital allocation by the gradient

Let us consider the payoff $X(u) := \sum_{i=1}^{n} u_i X_i \in L^p(\mathbb{Q})$ of a portfolio $u = (u_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ consisting of assets (or subportfolios) with payoffs $X_i \in L^p(\mathbb{Q})$.

DEFINITION 2.1. *A* **portfolio base** *in $L^p(\mathbb{Q})$ is a vector $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$. The components of B do not have to be linearly independent.*

Having $B = (X_1, \ldots, X_n)$, a risk measure ρ on the payoffs $L^p(\mathbb{Q})$ implies a risk measure ρ_B on the portfolios $\mathbb{R}^n$. In particular, we define $\rho_B : \mathbb{R}^n \to \mathbb{R}$ by

$$\rho_B : u \mapsto \rho(X(u)). \tag{2}$$

If ρ_B is obtained from a CRM ρ on $L^p(\mathbb{Q})$ and X_n is the only constant component in B and not equal zero, ρ_B is also called coherent (cf. Denault, 2001). If ρ fulfills axiom (S) and (PH) in Definition 1.2, ρ_B is subadditive and positively homogeneous on $\mathbb{R}^n$.

Due to diversification effects (or subadditivity of the risk measure), the total risk of a portfolio is usually assumed to be less then the sum of the risks of each subportfolio, i.e. we often have $\rho_B(u) < \sum_{i=1}^{n} \rho_B(u_i e_i)$, where e_i is the i-th canonical unit vector in $\mathbb{R}^n$. The so-called allocation problem is the question, how much risk capital should be allocated to each of the subportfolios $u_i e_i$ and hence how the subportfolios should benefit from the diversification. However, as identical payoffs should be treated identically, this question is equivalent to the search for a reasonable per-unit allocation principle.

DEFINITION 2.2. *Given a portfolio base B and a risk measure ρ_B on $\mathbb{R}^n$ a* **per-unit allocation** *in $u \in \mathbb{R}^n$ is a vector $(a_i(\rho_B, u))_{1\leq i\leq n}$, such that*

$$\sum_{i=1}^{n} u_i a_i(\rho_B, u) = \rho_B(u). \tag{3}$$

In Denault (2001) the author drives the attention of the reader to a result of Aubin in the theory of coalitional games with fractional players. Aubin's theorem states that in the case of a positively homogeneous, convex and differentiable cost function the core of such a game (Aubin uses the prefix *fuzzy*) consists of one element: the gradient of the cost function due to the normed weights of the players (Aubin, 1979). From this result, it is immediate that in the case of a subadditive and positively homogeneous risk measure (e.g. a coherent one), which is differentiable at a portfolio $u \in \mathbb{R}^n$, the gradient $(\frac{\partial \rho_B}{\partial u_i}(u))_{1\leq i\leq n}$ is the *unique fair* per-unit allocation. To derive this statement from Aubin's result, the notion of cost functions in game theory has to be replaced by our notion of a risk measure. The players of the game are given by the certain $u_i X_i$, coalitions of fractional players are given by portfolios v with $0 \leq v \leq u$, where the given portfolio u can without loss of generality be assumed to be positive. Note that convexity and subadditivity are equivalent under positive homogeneity. The core of such a game contains all per-unit allocations $(a_i(\rho_B, u))_{1\leq i\leq n}$, such that for all coalitions v with $0 \leq v \leq u$ we have $\sum_{i=1}^{n} v_i a_i(\rho_B, u) \leq \rho_B(v)$. That means, no sub-coalition v of u features less stand-alone risk than the risk the coalition v would have been charged by the respective per-unit allocation due to u. In this sense, the elements of the core are fair allocations. For the sake of completeness, it should be mentioned that in the case of a positively homogeneous risk measure the core of the game is identical to the subdifferential of ρ_B at u. If ρ_B is also convex or subadditive, the core is nonempty, convex and compact (Aubin, 1979). However, in this general case uniqueness of the core gets lost. For differentiable CRM Denault proved that the Aumann-Shapley value, which is the above gradient, features certain coherence properties (Denault, 2001). For a deeper study of the connections between the theory of convex games and coherent risk measures we refer to Delbaen (2002).

In the case of just positively homogeneous risk measures, the theory of convex games is no longer suitable to model the allocation problem. However, it is still possible to talk about reasonable allocations. Tasche (2000) considers the so-called return on risk-adjusted capital (RORAC) of the payoff $X(u)$

of a portfolio u, which he defines by $f(u) = \mathbf{E}_{\mathbb{Q}}[X(u)]/\rho_B(u)$. Note, that what we called risk measure is denoted economic capital by Tasche, whereas he defines risk as fluctuation risk from the mean. Now, the idea is to call a per-unit allocation *suitable for performance measurement* with ρ_B, when $(a_i(\rho_B, u))_{1\le i\le n}$ gives the right signals for local changes in the portfolio. More precise, if $\mathbf{E}_{\mathbb{Q}}[X_i]/a_i(\rho_B, u) > f(u)$, there should be an $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have $f(u - \varepsilon e_i) < f(u) < f(u + \varepsilon e_i)$. Analogously, for $\mathbf{E}[X_i]/(a_i(\rho_B, u)) < f(u)$ we demand $f(u - \varepsilon e_i) > f(u) > f(u + \varepsilon e_i)$. Tasche shows that in the case of differentiable positively homogeneous risk measures the unique per-unit allocation $(a_i(\rho_B, u))_{1\le i\le n}$ that is continuous on $\mathbb{R}^n$ and suitable for performance measurement due to the risk adjusted return function is the gradient $(\frac{\partial \rho_B}{\partial u_i}(u))_{1\le i\le n}$ (Tasche, 2000).

In both approaches, Denault's and Tasche's, the relationship between total risk and risk contribution per unit is established by the Euler Theorem

$$\rho_B(u) = \sum_{i=1}^{n} u_i \frac{\partial \rho_B}{\partial u_i}(u). \tag{4}$$

The per-unit risk contribution equals the marginal risk. So, concerning risk capital allocation due to a (subadditive) positively homogeneous risk measure on $L^p(\mathbb{Q})$, it would be desirable to have ρ_B to be differentiable on $\mathbb{R}^n$ for every portfolio base $B \in (L^p(\mathbb{Q}))^n$ for all $n \in \mathbb{N}^+$.

3 Differentiability properties

As the Value-at-Risk methodology is widely used in practice, marginal risks of VaR have been considered in several papers. In the Gaussian case we refer to the works of Garman (1996) and (1997), in the general case of continuous distributions to Laurent, Gouriéroux and Scaillet (2000). The perhaps more sophisticated (but also quantile-based) expected shortfall (called Tail-VaR by some authors) is considered in Scaillet (2000). Despite of the results in the case of continuous distributions, having a quantile-based risk measure ρ like VaR or expected shortfall, it is known that ρ_B is not differentiable on $\mathbb{R}^n$ in general. Roughly speaking, for differentiability at least one of the X_i has to possess a continuous density (Tasche, 2000). Hence, it is a problem to deal with discrete spaces $(\Omega, \mathcal{A}, \mathbb{Q})$ like e.g. in the case of credit portfolios, insurance claims or digital options. It will be shown in Section 4 that the step to moment based risk measures avoids this difficulty. Beside the differentiability difficulties, it is also known that VaR is not subadditive (Artzner et al., 1999). As diversification is not rewarded, this is a major drawback.

However, even if risk measures are differentiable on $\mathbb{R}^n$, this can imply some problems. To understand what kind of problems can arise, we state a proposition which connects differentiability with linearity and minimality of subadditive positively homogeneous risk measures:
We have seen that it would be desirable to have ρ_B to be differentiable on $\mathbb{R}^n$ for every portfolio base $B \in (L^p(\mathbb{Q}))^n$ for all $n \in \mathbb{N}^+$. Considering the initial ρ on $L^p(\mathbb{Q})$, this implies the existence of Gâteaux-derivatives, i.e. derivatives due to directions on $L^p(\mathbb{Q})$.

PROPOSITION 3.1. *Let $\mathcal{S}$ be a subset of the four axioms given in Definition 1.2,* (PH) *and* (S) *being contained in $\mathcal{S}$. For a risk measure ρ on $L^p(\mathbb{Q})$, $1 \leq p \leq \infty$, that fulfills the axioms $\mathcal{S}$, the following properties are equivalent: (i) ρ is Gâteaux-differentiable on $L^p(\mathbb{Q})$, (ii) ρ is linear, (iii) ρ is minimal due to $\mathcal{S}$, i.e. there is no risk measure $\rho' \neq \rho$ fulfilling $\mathcal{S}$ such that $\rho'(X) \leq \rho(X)$ for all $X \in L^p(\mathbb{Q})$. Differentiability of ρ on $L^p(\mathbb{Q})$ implies (i), (ii) and (iii).*

COROLLARY 3.2. *A continuous coherent risk measure ρ on $L^p(\mathbb{Q})$ is Gâteaux-differentiable on $L^p(\mathbb{Q})$, $1 < p < \infty$, if and only if there exists a probability measure $\mathbb{Q}_\rho \sim \mathbb{Q}$ on Ω, such that $\rho(X) = -\mathbf{E}_{\mathbb{Q}_\rho}[X]$.*

In particular, Proposition 3.1 is true for coherent risk measures. The proof of 3.1 is omitted since equivalence of (i) and (ii) can be shown by a simple application of the axioms (PH) and (S). Since subadditive positively homogeneous risk measures are sub-linear functionals, the well-known proof for equivalence of (ii) and (iii) in the general sub-linear case can easily be adapted to our cases. The corollary follows from the duality of the $L^p(\mathbb{Q})$ spaces.

As the two statements are also true for subspaces of $L^p(\mathbb{Q})$, we face the following problem: If ρ_B is a differentiable risk measure on $\mathbb{R}^n$ which fulfills $\mathcal{S}$ (e.g. coherence), it is easy to show that ρ_B is linear. Therefore, ρ_B features no diversification effects. We also obtain that ρ is linear on the linear span $\langle B \rangle$ of the components of B, which implies that ρ is minimal on $\langle B \rangle$ due to $\mathcal{S}$ (coherence). Hence, differentiability on the whole $\mathbb{R}^n$ might be not useful.

Now, consider a portfolio base $B = (X_1, \ldots, X_n)$ and a portfolio $u = u_i e_i = (0, \ldots, 0, u_i, 0, \ldots, 0)$, $u_i \in \mathbb{R}$, $1 \leq i \leq n$. In this case the allocation problem is trivial, since by (3) the risk capital allocated to X_i - which is the only asset - is simply $\rho_B(u)/u_i$. The following definition is motivated by this consideration.

DEFINITION 3.3. *Consider a portfolio base* $B = (X_1, \ldots, X_n) \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$, $1 \leq p \leq \infty$, *and a portfolio* $u \in \mathbb{R}^n$. *Define* $U_e = \bigcup_{i=1}^n \langle e_i \rangle$, *where* $\langle e_i \rangle \subset \mathbb{R}^n$ *is the linear span of* e_i. *We propose to call a (subadditive) positively homogeneous risk measure* ρ *on* $L^p(\mathbb{Q})$ **suitable for risk capital allocation** *by the gradient due to the portfolio base* B *if the function* $\rho_B : \mathbb{R}^n \to \mathbb{R}$ *with* $\rho_B : u \mapsto \rho(X(u))$ *is differentiable on the open set* $\mathbb{R}^n \setminus U_e$.

4 A class based on one-sided moments

We define a class of coherent risk measures which depend on the mean and the one-sided higher moments of a risky position.

LEMMA 4.1. *For* $1 \leq p \leq \infty$ *and* $0 \leq a \leq 1$, *the risk measure* $\rho_{p,a}$ *with*

$$\rho_{p,a}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot \sigma_p^-(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p \qquad (5)$$

is coherent on $L^p(\mathbb{Q})$.

Delbaen (2002) shows that these risk measures can be obtained by the set of probability measures (also called *generalized scenarios*, compare Artzner et al. (1999)) $P = \{1 + a(g - \mathbf{E}[g]) \mid g \geq 0; ||g||_q \leq 1\}$, where $q = p/(p-1)$ and probability measures are identified with their densities. In Delbaen (2000) we find another type of risk measures that are connected to higher moments.

Proof of Lemma 4.1. The L^p-norm on the right side of (5) is finite, since $X \in L^p(\mathbb{Q})$. Axiom (T) and (PH) are obvious. From Minkowski's inequality and the inequality $(a+b)^- \leq a^- + b^-$ for $a, b \in \mathbb{R}$, we obtain axiom (S). Axiom (M): Let $X \geq 0$. We have $X - \mathbf{E}_{\mathbb{Q}}[X] \geq -\mathbf{E}_{\mathbb{Q}}[X]$, therefore $(X - \mathbf{E}_{\mathbb{Q}}[X])^- \leq \mathbf{E}_{\mathbb{Q}}[X]$ and hence $||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_\infty = \text{ess.sup}\{(X - \mathbf{E}_{\mathbb{Q}}[X])^-\} \leq \mathbf{E}_{\mathbb{Q}}[X]$. Since $||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p \leq ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_\infty$ for $p \in [1, \infty]$, we get $||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p \leq \mathbf{E}_{\mathbb{Q}}[X]$. Remembering $0 \leq a \leq 1$, this completes the proof. □

The L^p-norms imply that $\rho_{q,a} \leq \rho_{p,a}$ if $q < p$. The following result is on weighted sums of coherent risk measures and generalizes the trivial fact that convex sums of CRM are again CRM .

LEMMA 4.2. *Let* $I \subset \mathbb{R}$ *be an index set and* $(\rho_i)_{i \in I}$ *be a family of coherent risk measures respectively defined on* $L^{p(i)}(\mathbb{Q})$, *where* $p : I \to [1, \infty]$. *Let* $(\rho_i)_{i \in I}$ *be point-wise uniformly bounded on* $L^{\sup p(I)}(\mathbb{Q})$ *in the sense that there is a function* $b : L^{\sup p(I)}(\mathbb{Q}) \to \mathbb{R}_0^+$ *such that for each* $X \in L^{\sup p(I)}(\mathbb{Q})$ *we* *have* $|\rho_i(X)| \leq b(X)$ *for all* $i \in I$. *Let* R *be a random variable with range* I *that is defined on a probability space* Ω' *with measure* $\mathbb{P}$. *Now, if for all* $X \in L^{\sup p(I)}(\mathbb{Q})$ *the mapping* $\rho_{R(.)}(X) : \Omega' \to \mathbb{R}$ *is measurable,*

$$\rho(X) = \mathbf{E}_{\mathbb{P}}[\rho_R(X)] \tag{6}$$

defines a coherent risk measure on $L^{\sup p(I)}(\mathbb{Q})$.

Proof. ρ is well-defined, since for each $X \in L^{\sup p(I)}(\mathbb{Q})$ we know from $|\rho_i(X)| \leq b(X)$ and the measurability assumption, that $\rho_R(X)$ is a bounded random variable and therefore $\mathbb{P}$-integrable. Now, the coherence axioms are obvious by the properties of $\mathbf{E}_{\mathbb{P}}$. □

Using Lemma 4.2, the result of Lemma 4.1 can be generalized.

PROPOSITION 4.3. *Let* P *be a random variable on a probability space* $(\Omega', \mathbb{P})$ *with range* $P(\Omega') \subset [1, p]$ *and assume that* $1 \leq p \leq \infty$ *and* $0 \leq a \leq 1$. *The risk measure*

$$\rho(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot \mathbf{E}_{\mathbb{P}}[\sigma_P^-(X)] \tag{7}$$

is coherent on $L^p(\mathbb{Q})$. *We have* $-\mathbf{E}_{\mathbb{Q}}[X] \leq \rho(X) \leq \text{ess.sup}\{-X\}$.

Proof. Due to Lemma 4.1 we consider a family $(\rho_{i,a})_{i\in[1,p]}$ of coherent risk measures given by (5), respectively defined on $L^i(\mathbb{Q})$. Now, let $b(X) = |\mathbf{E}_{\mathbb{Q}}[X]| + ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p$. Clearly, $|\rho_i(X)| \leq b(X)$ for all $1 \leq i \leq p$. For all $X \in L^p(\mathbb{Q})$ the mapping $\rho_{P(.),a}(X) : \Omega' \to \mathbb{R}$ is measurable, since $P(.)$ is measurable and for all $Y \in L^p(\mathbb{Q})$ the mapping $q \mapsto ||Y||_q$ is measurable on $P(\Omega')$ as it is continuous due to the relative topology on $P(\Omega')$ in $\mathbb{R} \cup \{\infty\}$ with the canonical topology (cf. Lemma 7.1). We obtain coherence of (7) by Lemma 4.2. The last statement follows from $||.||_p \leq ||.||_\infty$ and $\sigma_\infty^- = \text{ess.sup}\{(X - \mathbf{E}_{\mathbb{Q}}[X])^-\} = \text{ess.sup}\{-X + \mathbf{E}_{\mathbb{Q}}[X]\}$. □

REMARK 4.4. An immediate consequence of Lemma 7.1 is that for any X the risk measure ρ can be chosen such that $\rho(X)$ equals any value $v \in [-\mathbf{E}_{\mathbb{Q}}[X], \text{ess.sup}\{-X\}]$, i.e. any value between the expected loss and the maximum loss. In particular, for $X \not\equiv const$ a.s. and $v \in [-\mathbf{E}_{\mathbb{Q}}[X] + \sigma_1^-(X), \text{ess.sup}\{-X\}]$ there is a unique $p^* = p^*(v) \in [1, \infty]$ such that $\rho_{p^*,1}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + \sigma_{p^*}^-(X) = v$.

EXAMPLE 4.5. $\rho(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a_1\sigma_1^- + a_2\sigma_2^- + \ldots + a_\infty\sigma_\infty^-$, where $a_p \geq 0$ for $p \in \{1, 2, 3, \ldots, \infty\}$ and $a_\infty + \sum_{p=1}^{\infty} a_p \leq 1$ is a coherent risk measure on

$L^q(\mathbb{Q})$, where $q := \sup\{p|a_p > 0\}$ (we use the convention $0\cdot(\pm\infty) = (\pm\infty)\cdot 0 = 0$). In particular, $a_2 = a_\infty = \frac{1}{2}$ could be interpreted as a coherent "mixture" of the semi-deviation and the maximum-loss-principle.

DEFINITION 4.6. *For $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$, $1 < p < \infty$, the set $U_C(B)$ denotes the set of all $u \in \mathbb{R}^n$ for which $\sum_{i=1}^n u_i X_i \equiv const.$*

LEMMA 4.7. *The set $\mathbb{R}^n \setminus U_C(B)$ is open in $\mathbb{R}^n$.*

Proof. The linear mapping $X(.) :\ \mathbb{R}^n \to L^p(\mathbb{Q})$, where $u \mapsto X(u)$, is bounded, since $||X(u)||_p \le \sum_{i=1}^n |u_i| \cdot ||X_i||_p \le ||u|| \cdot \sum_{i=1}^n ||X_i||_p$. Hence, $X(.)$ is continuous on $\mathbb{R}^n$. The set C of all constant elements of $L^p(\mathbb{Q})$ is closed, since $L^p(\mathbb{Q})$ is a Banach-space due to the theorem of Riesz-Fischer and every Cauchy-sequence of constant elements in $L^p(\mathbb{Q})$ converges to a constant limit in $L^p(\mathbb{Q})$ (due to L^p-norm). Since $X(.)$ is continuous, $[X(.)]^{-1}(C) = U_C(B)$ is closed and $\mathbb{R}^n \setminus U_C(B)$ open. □

We can now state a result on differentiability of the class of coherent risk measures that was introduced in Proposition 4.3.

PROPOSITION 4.8. *Assume $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$, $1 < p < \infty$ and $0 \le a \le 1$. Let $1 < P \le p$ be a random variable on a probability space with measure $\mathbb{P}$. The risk measures ρ_B implied by (7) are differentiable on $\mathbb{R}^n \setminus U_C(B)$. The partial derivatives are*

$$\begin{aligned} \frac{\partial \rho_B}{\partial u_i}(u) &= -\mathbf{E}_{\mathbb{Q}}[X_i] + a \cdot \mathbf{E}_{\mathbb{P}}[\sigma_P^-(X(u))^{1-P} \cdot \\ &\qquad \mathbf{E}_{\mathbb{Q}}[(-X_i + \mathbf{E}_{\mathbb{Q}}[X_i]) \cdot ((X(u) - \mathbf{E}_{\mathbb{Q}}[X(u)])^-)^{P-1}]]. \end{aligned} \tag{8}$$

The proof of Proposition 4.8 is rather technical and therefore given in the appendix. We want to show that the risk measures (7) actually can not be differentiable at some $u \in U_C(B)$. Suppose $u \in U_C(B)$, $a > 0$ and the risk measure defined by (5), which is the special case $P \equiv const$. We have $\rho_{p,a}(u) = -\mathbf{E}_{\mathbb{Q}}[X(u)]$, since $X(u) \equiv \mathbf{E}_{\mathbb{Q}}[X(u)]$. Easily we obtain the two different one-sided partial derivatives $-\mathbf{E}_{\mathbb{Q}}[X_i] + a \cdot ||(\pm X_i \mp \mathbf{E}_{\mathbb{Q}}[X_i])^-||_p$ in u, but $||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])^-||_p \ne ||(-X_i + \mathbf{E}_{\mathbb{Q}}[X_i])^-||_p$ in general. So, we have no differentiability in general.

COROLLARY 4.9. *Under the assumptions of 4.8, the risk measures ρ implied by (7) are suitable for risk capital allocation by the gradient due to the portfolio base B if the components $X_1, \ldots, X_n$ of B are linearly independent and $X_n \not\equiv 0$ is constant. The per-unit allocations are given by (8).*

Proof. $U_C(B) = \langle(0, \ldots, 0, 1)\rangle \subset U_e$. □

Corollary 4.9 is the main result on risk capital allocation by the considered class of coherent risk measures. No assumptions concerning the underlying probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ have been made, discrete spaces can be taken into consideration. The assumption of linear independence is quite weak as it should be no problem to find a vector base in a real market. Even the particular choice of the portfolio base B is not important as the gradient is an aggregation invariant allocation principle (Denault, 2001). The reason is that if we have two different portfolio bases B and B' as given in Corollary 4.9 with $\langle B\rangle = \langle B'\rangle$, there exists a linear isomorphism A on $\mathbb{R}^n$ such that we have $X(u) \equiv X'(u')$ and $\rho_B(u) = \rho_{B'}(u')$ for every $u = Au' \in \mathbb{R}^n$. We therefore obtain from standard analysis for any two equivalent portfolios v and v' with $v = Av'$

$$\sum_{i=1}^{n} v'_i \frac{\partial \rho_{B'}}{\partial u'_i}(u') = \sum_{i=1}^{n} v_i \frac{\partial \rho_B}{\partial u_i}(u). \tag{9}$$

So, the risk capital allocated to equivalent subportfolios, i.e. subportfolios with the same payoff in $L^p(\mathbb{Q})$, is identical.

5 Application

In this section, two examples illustrate how risk capital given by the Value-at-Risk can be allocated using the risk measures from Section 4. In particular, we use a risk measure of type $\rho_{p,1}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + \sigma_p^-(X)$ as given in (5). We define the Value-at-Risk by

$$\mathrm{VaR}_\alpha(X) = -\inf\{x : \mathbb{Q}(X \leq x) > \alpha\}. \tag{10}$$

As long as $\mathrm{VaR}_\alpha(X) \geq -\mathbf{E}_{\mathbb{Q}}[X] + \sigma_1^-(X)$, we know from Remark 4.4 that there is a unique $p^* \in [1, \infty]$ such that $\rho_{p^*,1}(X) = \mathrm{VaR}_\alpha(X)$. Since the risk measure $\rho_{p^*,1}$ $(1 < p^* < \infty)$ is suitable for risk capital allocation (cf. Corollary 4.9), the amount $\mathrm{VaR}_\alpha(X)$ can be allocated by allocation due to $\rho_{p^*,1}$, i.e. for a portfolio base B as given in 4.9 and ρ_B^* corresponding to $\rho_{p^*,1}$ (cf. (2)), we have

$$\mathrm{VaR}_\alpha(X(u)) = \rho_B^*(u) = \sum_{i=1}^{n} u_i \frac{\partial \rho_B^*}{\partial u_i}(u). \tag{11}$$

EXAMPLE 5.1 (Discrete distributions). Suppose two stochastically independent payoff variables X_1, X_2 with discrete distributions as given in

x	$\mathbb{Q}(X_1 = x)$	$\mathbb{Q}(X_2 = x)$
0.0	0.78	0.96
-0.5	0.20	0.02
-1.0	0.02	0.02

Table 1: Distribution of X_1, X_2

Table 1. The portfolio base is given by $B = (X_1, X_2, 1)$. X_1 and X_2 could be interpreted as one unit of a credit engagement. Obviously, X_1 bears higher risks as losses are more probable. We consider the portfolio $u = (u_1, u_2, u_3) = (1000, 1000, 0)$. Easily we compute $\mathrm{VaR}_{0.05}(X(u)) = 500$. To allocate the given risk capital, we adjust $\rho_B(u)$ by choosing p^*, such that $\rho_{p^*,1}(X(u)) = \mathrm{VaR}_{0.05}(X(u)) = 500$. We obtain $p^* \approx 2.9157$. From the discrete version of (8) ($|\Omega| = 9$, $P \equiv p^*$, $a = 1$) we obtain $\frac{\partial \rho_B^*}{\partial u_1}(u) \approx 0.31504$ and $\frac{\partial \rho_B^*}{\partial u_2}(u) \approx 0.18496$. The risk capital allocated to $u_1 X_1$ is 315.04, for $u_2 X_2$ it is 184.96. To check what happens for a more conservative VaR, we compute $\mathrm{VaR}_{0.01}(X(u))$, which is 1000. We obtain $p^* \approx 9.4355$ and the risk capital allocated to $u_1 X_1$ is 477.98, for $u_2 X_2$ it is 522.02. It is interesting that in the second case more risk capital is allocated to X_2, which seems to bear less risk. However, the relative difference is quite small compared to the first case. This seems to be reasonable as we have $\mathrm{VaR}_{0.01}(u_1 X_1) = \mathrm{VaR}_{0.01}(u_2 X_2) = 1000$.

EXAMPLE 5.2 (Continuous distributions). Although continuous distributions are considered in this example, we assume that (5) are the risk measures of choice. A possible scenario could be the situation where these risk measures are intended to be used internally where at the same time external regulatory requirements define the minimum risk capital by the VaR-method. We assume to be given a portfolio base $B = (X_1, X_2, 1)$ with

$$\begin{aligned} X_1 &\sim n_1 \cdot v_1 \cdot (\exp(\sigma_1 Z_1) - 1) \\ X_2 &\sim \sqrt{n_2} \cdot v_2 \cdot \sigma_2 Z_2, \end{aligned} \tag{12}$$

where Z_1, Z_2 are assumed to be standard normally distributed with correlation $r \geq 0$. X_1 could be interpreted as the log-normal payoff of a portfolio of n_1 (identical) financial assets minus the price $n_1 v_1$ at which they were bought. The expected value of one asset is $v_1 \cdot \exp(\sigma_1^2/2)$. X_2 could be interpreted as an approximation of the sum of n_2 i.i.d. payoffs with expectation 0 and standard deviation $v_2 \cdot \sigma_2$, e.g. coming from a balanced credit portfolio or the liabilities of an insurance company. In particular, we assume $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $n_1 = 10^6$,

$n_2 = 10000$, $v_1 = 200$, $v_2 = 10^6$ and $r = 0.8$ ($r > 0$ is reasonable in the case of a credit portfolio). The "external" risk measure is assumed to be given by the 5%-VaR. The portfolio base is $B = (X_1, X_2, 1)$ and the portfolio $(1, 1, 0)$, i.e. the considered overall payoff is the sum $X = X_1 + X_2$. The expectation of X_1 is $4.04 \cdot 10^6$ (i.e. a mean return of 2%) and the standard deviation $41.2 \cdot 10^6$ (rounded values). For X_2 we have expectation 0 and $10 \cdot 10^6$ for the standard deviation (also rounded). All non-trivial computation, e.g. for $\mathrm{VaR}_{0.05}(X)$ and $\sigma_p^-(X)$, is done by the classical Monte-Carlo method, i.e. Z_1 and Z_2 are simulated and the VaR-quantile and the non-trivial integrals in (5) and (8) are obtained from the simulated empirical distributions. We get $\mathrm{VaR}_{0.05}(X) \approx 70 \cdot 10^6$. The calibration of $\rho_{p,1}(X)$ is done by the bisection method ($\rho_{p,1}(X)$ is monotone in p). We start with the interval $[1, 30]$, where p^* is assumed to be contained in, and go on 16 steps which corresponds to a theoretical error for p^* of less then $(30 - 1) \cdot 2^{-16} \approx 0.44 \cdot 10^{-3}$ (neglecting the Monte-Carlo error). For each integral $200 \cdot 10^6$ pseudo-random values of Z_1, respectively Z_2, are computed. We obtain $p^* \approx 10.05$ and $\rho_{p^*,1}(X) \approx 70.01 \cdot 10^6$. Computation of the partial derivatives gives $\frac{\partial \rho_B^*}{\partial u_1}(u) \approx 53.55 \cdot 10^6$ and $\frac{\partial \rho_B^*}{\partial u_2}(u) \approx 16.38 \cdot 10^6$, i.e. a sum $69.93 \cdot 10^6 \approx 70 \cdot 10^6$. As we have assumed X_2 to be the sum of n_2 i.i.d. payoffs, we obtain the fair risk capital $\frac{\partial \rho_B^*}{\partial u_2}(u)/n_2 \approx 1638$ for each individual payoff.

6 Comparison of the notation of Denault, Fischer and Tasche

As each of the three papers uses a particular notation, it is useful to have a direct comparison of variables and expressions corresponding to each other (see Table 2).

A remark on Tasche's approach (in Tasches's notation, Fischer's notation in brackets): Please note, that

$$m_i r(u) > a_i(u) m' u \tag{13}$$

is equivalent to

$$\frac{m_i}{a_i(u) - m_i} > \frac{m' u}{r(u) - m' u} = g(u) \quad (= f(u)), \tag{14}$$

and

$$\sum_i u_i a_i(u) = r(u) \tag{15}$$

Denault (2001)	Fischer	Tasche (2000)
$\frac{X_i}{\Lambda_i}$	X_i	C_i
Λ_i	u_i	u_i
X_i	$u_i X_i$	$u_i C_i$
λ_i	v_i	
$\frac{\lambda_i}{\Lambda_i} X_i$	$v_i X_i$	
$r(\Lambda)$	$\rho_B(u)$	$r(u) - \sum_{i=1}^n u_i m_i$
	$\mathbf{E}_{\mathbb{Q}}[X_i]$	m_i
	$\mathbf{E}_{\mathbb{Q}}[X_i] - X_i$	X_i
	$\rho_B(u) + \mathbf{E}_{\mathbb{Q}}[X(u)]$	$r(u)$
k_i	$a_i(\rho_B, u)$	$a_i(u) - m_i$
	$f(u)$	$g(u)$

Table 2: Different notation

equivalent to

$$\sum_i u_i(a_i(u) - m_i) = r(u) - \sum_i u_i m_i \tag{16}$$

$$\left(\text{or } \sum_i u_i a_i(\rho_B, u) = \rho_B(u)\right). \tag{17}$$

7 Appendix

LEMMA 7.1. *Let $P \subset [1, \infty]$ and $X \in L^{\sup P}(\mathbb{Q})$. The mapping $||X||_{(.)} : P \to [0, \infty)$, $p \mapsto ||X||_p$, is continuous due to the relative topology on P in $\mathbb{R} \cup \{\infty\}$ with the canonical topology.*

Proof. The case $P \subset [1, \infty)$ *and* X essentially bounded can be deduced from results in Bourbaki (1965). However, a general proof is needed.

The case $X \equiv 0$ is trivial, therefore we assume $||X||_p > 0$. Since $||X||_{(.)}$ is a real function which is monotone on P, it suffices to show that from the convergence $p_n \to p$ of a sequence $(p_n)_{n \in \mathbb{N}}$ in P there follows $||X||_{p_n} \to ||X||_p$. We first prove the case $p = \infty$, where $\infty \in P$ is assumed. For any $\varepsilon > 0$ there exists some $A \in \mathcal{A}$ with $\mathbb{Q}(A) > 0$ such that

$$|X(\omega)| \geq \text{ess.sup}\{|X|\} - \varepsilon \tag{18}$$

for all $\omega \in A$. Now, as $||.||_\infty := \text{ess.sup}(.)$, we have

$$\begin{aligned} \text{ess.sup}\{|X|\} &\geq ||X||_{p_n} \\ &\geq \left(\int_A (\text{ess.sup}\{|X|\} - \varepsilon)^{p_n} d\mathbb{Q}\right)^{\frac{1}{p_n}} \\ &= (\text{ess.sup}\{|X|\} - \varepsilon)(\mathbb{Q}(A))^{\frac{1}{p_n}}. \end{aligned} \tag{19}$$

We obtain

$$\text{ess.sup}\{|X|\} \geq \lim_{p_n \to \infty} ||X||_{p_n} \geq \text{ess.sup}\{|X|\} - \varepsilon \tag{20}$$

and hence

$$\text{ess.sup}\{|X|\} = ||X||_\infty = \lim_{p_n \to \infty} ||X||_{p_n} \tag{21}$$

by definition of $||.||_\infty$. Now, assume $1 \leq p < \infty$ and $p \in P$. We have

$$|X(\omega)|^{p_n} \leq \max\{|X(\omega)|^{\sup P}, 1\}. \tag{22}$$

By dominated convergence, we obtain

$$\int |X(\omega)|^{p_n} d\mathbb{Q}(\omega) \longrightarrow \int |X(\omega)|^p d\mathbb{Q}(\omega), \tag{23}$$

i.e. $||X||_{p_n}^{p_n} \longrightarrow ||X||_p^p$. The triangle inequality gives us

$$\begin{aligned} & |\, ||X||_{p_n} - ||X||_p| \\ \leq & \left| \sqrt[p_n]{||X||_{p_n}^{p_n}} - \sqrt[p]{||X||_{p_n}^{p_n}} \right| + \left| \sqrt[p]{||X||_{p_n}^{p_n}} - \sqrt[p]{||X||_p^p} \right|. \end{aligned} \tag{24}$$

The right part of the sum converges to zero as the p-th root is a continuous function. The left part converges to zero for the following reasons. As we know, $a_n := ||X||_{p_n}^{p_n}$ converges to $a := ||X||_p^p > 0$. Now,

$$\begin{aligned} & \left| \sqrt[p_n]{||X||_{p_n}^{p_n}} - \sqrt[p]{||X||_{p_n}^{p_n}} \right| \\ = & | \sqrt[p_n]{a_n} - \sqrt[p]{a_n}| \\ = & |\exp\{\ln\{a_n\}/p\}| \cdot |\exp\{(1/p_n - 1/p)\ln\{a_n\}\} - 1|. \end{aligned} \tag{25}$$

The first factor is bounded, since a_n converges to $a > 0$, the second one converges to zero as $(1/p_n - 1/p)\ln\{a_n\}$ converges to zero and the exponential function is continuous. □

The proof of Proposition 4.8 needs the following technical lemmas.

LEMMA 7.2. *Let U be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}^+$, and $f : U \times \Omega \to \mathbb{R}$ be a function with following properties:*

a) $\omega \mapsto f(u,\omega)$ *is* $\mathbb{Q}$*-integrable for all* $u \in U$.

b) $u \mapsto f(u,\omega)$ *is in any* $u \in U$ *partially differentiable with respect to* u_i.

c) There exists a $\mathbb{Q}$*-integrable function* $h_U \geq 0$ *on* Ω *with* $\left|\frac{\partial f}{\partial u_i}(u,\omega)\right| \leq h_U(\omega)$ *for all* $(u,\omega) \in U \times \Omega$.

The function $\varphi(u) = \int f(u,\omega)d\mathbb{Q}(\omega)$ *on* U *is partially differentiable with respect to* u_i*. The mapping* $\omega \mapsto \frac{\partial f}{\partial u_i}(u,\omega)$ *is* $\mathbb{Q}$*-integrable and for* $u \in U$

$$\frac{\partial \varphi}{\partial u_i}(u) = \int \frac{\partial f}{\partial u_i}(u,\omega)d\mathbb{Q}(\omega). \tag{26}$$

The proof by the dominated convergence theorem is well-known.

LEMMA 7.3. *Define* $U = \triangle u_1 \times \cdots \times \triangle u_n \subset \mathbb{R}^n$*, where for all* $i \in \{1,\dots,n\}$ $\triangle u_i$ *is a nonempty, bounded and open interval in* $\mathbb{R}$*. Let* $X(u) = \sum_{i=1}^n u_i X_i$ *be a sum of real-valued random variables* $X_i \in L^p(\mathbb{Q})$ *with* $u = (u_1,\dots,u_n) \in U$*,* $n \in \mathbb{N}^+$ *and* $1 < p < \infty$*. Let* $y(u)$ *be a real-valued function that is differentiable, bounded and for which* $y(u) < \text{ess.sup}\{-X(u)\}$ *on* U*. The partial derivatives* $\frac{\partial y}{\partial u_i}(u)$ *are also assumed to be bounded on* U*. Under this assumptions,* $||(X(u)+y(u))^-||_p$ *is differentiable on* U.

Proof. Define $g(u,\omega) = (X(u,\omega) + y(u))^-$. For $1 \leq i \leq n$ we will prove existence and continuity of the partial derivatives of $||g(u)||_p$.

Existence: We have $||g(u)||_p = \left(\int g(u,\omega)^p \, d\mathbb{Q}(\omega)\right)^{1/p}$. Now, if we can apply Lemma 7.2 to g^p (where f from 7.2 corresponds to g^p) and if $g(u)$ is *not* constant 0 for every $u \in U$, we obtain for every i

$$\frac{\partial ||g(u)||_p}{\partial u_i}(u) = \int \frac{\partial g^p}{\partial u_i}(u)\, d\mathbb{Q} \cdot \frac{1}{p} \cdot \left(\int g(u)^p \, d\mathbb{Q}\right)^{\frac{1}{p}-1}. \tag{27}$$

Note, that for $u \in U$ we have $g(u) > 0$ on a set of measure greater 0, since $y(u) < \text{ess.sup}\{-X(u)\}$. Therefore the right integral in (27) is greater 0 (no division by zero!). We are going to check the points a) to c) from Lemma 7.2. Ad a). $\omega \mapsto g(u,\omega)^p$ is $\mathbb{Q}$-integrable, since $X(u) \in L^p(\mathbb{Q})$ and $y(u) \in \mathbb{R}$. Ad b). First, we consider the function $[(.)^-]^p : \mathbb{R} \to \mathbb{R}_0^+$, $x \mapsto (x^-)^p$. Clearly, this function is differentiable for $1 < p < \infty$. Now, $g(u,\omega)^p = [(\sum_{i=1}^n u_i X_i(\omega) + y(u))^-]^p$ - as a combination of a differentiable and a partially differentiable function - is partially differentiable at u_i. We obtain

$$\frac{\partial g^p}{\partial u_i}(u,\omega) \quad = \quad -\left(X_i(\omega) + \frac{\partial y}{\partial u_i}(u)\right) \cdot p \cdot g(u,\omega)^{p-1}. \tag{28}$$

Ad c). There exist positive constants a and b, such that for all $j \in \{1, \dots, n\}$ we have $|\frac{\partial y}{\partial u_j}(u)| \leq a$ and $|y(u)| \leq b$ on U. Now, define

$$u_{\max}(U) = \sup\{|u'_j| : u'_j \in \triangle u_j,\ j \in \{1, \dots, n\}\}, \tag{29}$$

which is finite, and

$$k_U(\omega) = n \cdot u_{\max}(U) \cdot \max_j\{|X_j(\omega)|\} + b. \tag{30}$$

Clearly, $k_U(\omega) \geq g(u, \omega)$. Now define

$$h_U(\omega) = (|X_i(\omega)| + a) \cdot p \cdot (k_U(\omega))^{p-1}. \tag{31}$$

Comparing this to (28), we clearly obtain

$$0 \leq \left|\frac{\partial g^p}{\partial u_i}(u, \omega)\right| \leq h_U(\omega) \tag{32}$$

for all $(u, \omega) \in U \times \Omega$. Concerning integrability of (31), we know that $(|X_i(\omega)| + a) \cdot p$ is p-integrable, since X_i is. We also know that $(k_U(\omega))^{p-1}$ is $\frac{p}{p-1}$-integrable. The latter statement follows from the fact that every single $|X_j(\omega)|$ is p-integrable and therefore $k_U(\omega)$ - as a multiple of the maximum plus a constant - is p-integrable. We further have $1/p + (p-1)/p = 1$. As an immediate consequence of Hölder's inequality, the product $h_U(\omega)$ of $(|X_i(\omega)| + a) \cdot p$ and $(k_U(\omega))^{p-1}$ is integrable.

Continuity: Consider a sequence $(u_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} u_n = u$ in $U = \triangle u_1 \times \cdots \times \triangle u_n$. Now, substitute u by u_n in (27). For fix $\omega \in \Omega$ it follows from the definition of $g(u)$ and (28) that the substituted expressions under the integrals in (27) converge (pointwise in ω) to the original expressions (in u). Now have in mind, that h_U (32) dominates the left integrand of (27) and $(k_U)^p$ (30) dominates the right one. As h_U and $(k_U)^p$ are integrable, it follows from the dominated convergence theorem that the substituted integrals themselves converge to the original integrals. Hence, (27) is continuous in u. □

LEMMA 7.4. *Assume $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$, $1 < p < \infty$. Suppose $0 \leq a \leq 1$. The risk measures $\rho_B(u)$ implied by (5) are differentiable on $\mathbb{R}^n \setminus U_C(B)$. The partial derivatives are*

$$\frac{\partial \rho_B}{\partial u_i}(u) = -\mathbf{E}_{\mathbb{Q}}[X_i] + a \cdot \sigma_p^-(X(u))^{1-p} \cdot \mathbf{E}_{\mathbb{Q}}[(-X_i + \mathbf{E}_{\mathbb{Q}}[X_i]) \cdot ((X(u) - \mathbf{E}_{\mathbb{Q}}[X(u)])^-)^{p-1}]. \tag{33}$$

Proof. As $\mathbb{R}\setminus U_C(B)$ is open, it can be seen as union of bounded n-dimensional open intervals U. We focus on the $L^p(\mathbb{Q})$-norm expression in $\rho_B(u)$. Define $y(u) = -\mathbf{E}_{\mathbb{Q}}[X(u)]$. Now, the requirements of Lemma 7.3 are satisfied, since $-\mathbf{E}_{\mathbb{Q}}[X(u)] < \text{ess.sup}\{-X(u)\}$ as long as $X(u) \not\equiv$ *const*. We obtain that the risk measure is differentiable in U and

$$\frac{\partial \rho_B}{\partial u_i}(u) = -\mathbf{E}_{\mathbb{Q}}[X_i] + \int \frac{\partial g^p}{\partial u_i}(u) d\mathbb{Q} \cdot a \cdot \frac{1}{p} \cdot ||g(u)||_p^{1-p} \quad . \tag{34}$$

As (34) does not depend on the choice of the particular $U \subset \mathbb{R}^n \setminus U_C(B)$, $\rho_B(u)$ is differentiable on $\mathbb{R}^n \setminus U_C(B)$. Since by definition $||g(u)||_p = \sigma_p^-(X(u))$, we obtain (33) by combining (28) with (34). □

Proof of Proposition 4.8. We use the notation from the proofs of the Lemmas 7.3 and 7.4. Assume $U = \triangle u_1 \times \cdots \times \triangle u_n$ to be a bounded nonempty n-dimensional open interval in $\mathbb{R}^n \setminus U_C(B)$, where for all $i \in \{1, \ldots, n\}$ $\triangle u_i$ is an open interval. Consider equation (7). We have

$$\mathbf{E}_{\mathbb{P}}[\sigma_P^-(X(u))] = \int ||g(u)||_{P(\omega')} d\mathbb{P}(\omega') \ . \tag{35}$$

We prove the existence and continuity of the partial derivatives of (35).

Existence: Again, we are going to check the points a) to c) from Lemma 7.2 (f corresponds to $||g(u)||_{P(\omega')}$). Ad a). $\omega' \mapsto ||g(u)||_{P(\omega')}$ is integrable, since $||g(u)||_{P(\omega')} \leq ||g(u)||_p < \infty$. Ad b). Since $P(\omega')$ is fix, it follows from the proof of Lemma 7.4 (Eq. (34)), that $u \mapsto ||g(u)||_{P(\omega')}$ is in every point $u \in U$ partially differentiable with respect to u_i. Ad c). From (34) we get

$$\frac{\partial f}{\partial u_i}(u, \omega') \quad = \quad \int \frac{\partial g^{P(\omega')}}{\partial u_i}(u) d\mathbb{Q} \cdot \frac{a}{P(\omega')} \cdot ||g(u)||_{P(\omega')}^{1-P(\omega')}. \tag{36}$$

From (28) we obtain

$$\frac{\partial g^{P(\omega')}}{\partial u_i}(u, \omega) = -(X_i(\omega) - \mathbf{E}_{\mathbb{Q}}[X_i]) \cdot P(\omega') \cdot g(u, \omega)^{P(\omega')-1}. \tag{37}$$

As $g(u, \omega)^{P(\omega')-1}$ is $\frac{P(\omega')}{P(\omega')-1}$-integrable, we get from Hölder's inequality

$$\begin{aligned} \left| \int \frac{\partial g^{P(\omega')}}{\partial u_i}(u, \omega) d\mathbb{Q} \right| \quad &\leq \quad \left|\left| \frac{\partial g^{P(\omega')}}{\partial u_i}(u, \omega) \right|\right|_1 \\ &\leq \quad ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_{P(\omega')} \cdot P(\omega') \cdot ||g(u)||_{P(\omega')}^{P(\omega')-1}. \end{aligned} \tag{38}$$

Combining this with (36), we obtain

$$\begin{aligned} \left| \frac{\partial f}{\partial u_i}(u, \omega') \right| \quad &\leq \quad ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_{P(\omega')} \cdot a \\ &\leq \quad ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_p \cdot a \equiv const. \end{aligned} \tag{39}$$

Choosing $h_U(\omega') = ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_p \cdot a$, this completes the proof of c). From the arbitrariness of $U \subset \mathbb{R}^n \setminus U_C(B)$, we obtain partial differentiability of ρ on $\mathbb{R}^n \setminus U_C(B)$. Equation (8) follows from the combination of Lemma (7.2) with the result (33) of Lemma 7.4.

Continuity: As we know from the proof of Lemma 7.4, expression (36) is continuous on $\mathbb{R}^n \setminus U_C(B)$. By (39), dominated convergence proves continuity of the partial derivatives. □

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