A class of coherent risk measures based on one-sided moments

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Abstract

This brief paper explains how to obtain upper boundaries of shortfall probabilities for a class of coherent risk measures based on one-sided moments. The one-sided Chebyshev inequality is used for this purpose. By recurrent summation of one-sided moments, the class is further extended and features subclasses of risk measures which express discrete degrees of attitude towards risk. The members of such a subclass are coherent, converge to the maximum loss and are suitable for risk capital allocation by the gradient.

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1 Introduction

In Fischer (2003) a wide class of coherent risk measures depending on the mean and the one-sided moments of a risky position was introduced. The emphasis of the paper was on risk capital allocation and differentiability properties of risk measures which are suitable with respect to allocation by the gradient (cf. Tasche (2000), Denault (2001)). In contrast to quantile-based risk measures, the moment-based ones in Fischer (2003) figured out to have very appealing allocation (i.e. differentiability) properties. However, due to

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the nature of these measures, an obvious stochastic interpretation of determined risk capitals (e.g. by corresponding shortfall properties) as in the case of quantile-based ones is not possible.

Having a risky payoff X and a respective risk capital $\rho(X)$ one might be interested in the shortfall (or ruin) probability $\Pr(X + \rho(X) < 0)$ of the combined position "payoff + risk capital". This brief paper shows how upper (pointwise) boundaries for $\Pr(X + \rho(X) < 0)$ can be obtained for the one-sided moments measures (but not only for them) by the well-known one-sided Chebyshev inequality. The particular boundaries are simple explicit expressions using expectation, variance and risk capital due to the considered payoff. Compact proofs of the one-sided Chebyshev and a further Chebyshev-like inequality are given by Hölder's inequality. By recurrence, the class of measures of Fischer (2003) is further extended. The new measures are defined as finite sums of certain p-th moments of a risky payoff. The recurrent application of one and the same risk measurement principle (one-sided moments) defines discrete degrees of attitude towards risk. The generated risk measures are coherent, show reasonable convergence properties due to the number n of iteration steps, i.e. they converge to the maximum loss for $n \to \infty$, and finally they are suitable for risk capital allocation by the gradient.

The outline is as follows. In the present section we briefly introduce notation and define coherent risk measures (CRM) as well as suitability of risk measures due to risk capital allocation by the gradient. In Section 2 some results of Fischer (2003) on risk measures which depend on the mean and the absolute lower central, i.e. one-sided moments of a risky position are recalled. In Section 3 we applicate the one-sided Chebyshev inequality for the deduction of upper boundaries for shortfall probabilities. The last section extends the class of risk measures of Fischer (2003). Coherence, the convergence property and suitability for capital allocation as explained above are proven.

The following notation will be used. We will consider the vector space $L^p(\Omega, \mathcal{A}, \mathbb{Q})$, or just $L^p(\mathbb{Q})$, for $1 \leq p \leq \infty$ and a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$. $L^p(\mathbb{Q})$ consists of equivalence classes of p-integrable random variables, nonetheless we will treat its elements as random variables. Due to the context, no confusion should arise. As usual, $||X||_p = (\mathbf{E}_{\mathbb{Q}}[|X|^p])^{\frac{1}{p}}$ and $||X||_\infty = \text{ess.sup}\{|X|\}$.

$X^{\pm}$ is defined as $\max\{\pm X, 0\}$. We denote

$$\sigma_p^{\pm}(X) = ||(X - \mathbf{E}_{\mathbb{Q}}[X])^{\pm}||_p. \tag{1}$$

A one-period framework is considered, that means between the present time 0 and a future time horizon T no trading is possible. *Risk* is given by a random payoff X, i.e. a random variable in $L^p(\mathbb{Q})$ representing a cash flow at T. As usual, we consider a risk measure $\rho(X)$ to be the extra minimum cash added to X such that the position becomes acceptable for the holder or a regulator. Hence, a *risk measure* on $L^p(\mathbb{Q})$, $1 \leq p \leq \infty$, is defined by a functional $\rho : L^p(\mathbb{Q}) \to \mathbb{R}$. ρ is called a *coherent* if the following properties hold.

(M) Monotonicity: If $X \geq 0$ then $\rho(X) \leq 0$.

(S) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

(PH) Positive homogeneity: For $\lambda \geq 0$ we have $\rho(\lambda X) = \lambda \rho(X)$.

(T) Translation: For constants a we have $\rho(a + X) = \rho(X) - a$.

For further motivation and information concerning coherence see Artzner et al. (1999) or Delbaen (2002). The discussion about suitable properties of risk measures continues and alternative approaches exist (e.g. Goovaerts, Kaas and Dhaene, 2003). However, in this paper we will consider coherent risk measures (CRM), only.

Now, consider the payoff $X(u) := \sum_{i=1}^n u_i X_i \in L^p(\mathbb{Q})$ of a portfolio $u = (u_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ persisting of assets with payoffs $X_i \in L^p(\mathbb{Q})$. A *portfolio base* in $L^p(\mathbb{Q})$ is a vector $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$. The components of B do not have to be linearly independent (cf. Fischer, 2003). For $B = (X_1, \ldots, X_n)$, a risk measure ρ on the payoffs $L^p(\mathbb{Q})$ directly implies a risk measure ρ_B on the portfolios $\mathbb{R}^n$. We define $\rho_B : \mathbb{R}^n \to \mathbb{R}$ by

$$\rho_B : u \mapsto \rho(X(u)). \tag{2}$$

When ρ_B is obtained from a CRM ρ on $L^p(\mathbb{Q})$ and X_n is the only constant component in B and not equal zero, ρ_B is also called coherent (cf. Denault, 2001).

The work Denault (2001) shows by application of results from game theory (Aubin, 1979) that for a given CRM ρ_B as above the gradient in u,

$(\frac{\partial \rho_B}{\partial u_i}(u))_{1\le i\le n}$, is the unique fair capital allocation principle (per-unit allocation). That means $\frac{\partial \rho_B}{\partial u_i}(u)$, the marginal risk of the assets of type i in u, is the unique fairly determined risk contribution of one such asset and one has

$$\rho_B(u) = \sum_{i=1}^{n} u_i \frac{\partial \rho_B}{\partial u_i}(u) \tag{3}$$

due to the Euler Theorem. See also Fischer (2003) for a brief explanation of the present theory (including Tasche's performance measurement approach, cf. Tasche (2000)). For this reason, we are interested in differentiability and explicit partial derivatives of the considered risk measures.

In Fischer (2003) it was shown that especially in the case of CRM differentiability in any portfolio in $\mathbb{R}^n$ might not be useful as the measures become linear (and minimal) then. Therefore, weaker differentiability properties which are still suitable for allocation purposes were introduced. In Fischer (2003) a coherent (or positively homogeneous) risk measure ρ on $L^p(\mathbb{Q})$ is called *suitable for risk capital allocation* by the gradient due to the portfolio base B if the function $\rho_B : \mathbb{R}^n \to \mathbb{R}$ with $\rho_B : u \mapsto \rho(X(u))$ is differentiable on the open set $\mathbb{R}^n \setminus U_e$, where $U_e = \bigcup_{i=1}^n \langle e_i \rangle$ and $\langle e_i \rangle \subset \mathbb{R}^n$ is the linear span of e_i. The rational of this principle is that for trivial portfolios (containing only one asset type) the allocation problem is trivial. Hence, a CRM can have nice allocation properties without being linear and minimal.

2 One-sided moments measures

We recapitulate some results of Fischer (2003) in this section.

PROPOSITION 2.1 (Fischer, 2003). *Let P be a random variable on a probability space $(\Omega', \mathbb{P})$ with range $P(\Omega') \subset [1,p]$ and assume that $1 \le p \le \infty$ and $0 \le a \le 1$. The risk measure*

$$\rho(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot \mathbf{E}_{\mathbb{P}}[\sigma_P^-(X)] \tag{4}$$

is coherent on $L^p(\mathbb{Q})$. We have $-\mathbf{E}_{\mathbb{Q}}[X] \le \rho(X) \le \text{ess.sup}\{-X\}$.

For a Dirac-measure $\mathbb{P}$ with atom p,

$$\rho_{p,a}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot \sigma_p^-(X) \tag{5}$$

is a special case of (4). It was shown in Fischer (2003) that for any X the risk measure ρ can be chosen such that $\rho(X)$ equals any value

$v \in [-\mathbf{E}_{\mathbb{Q}}[X], \text{ess.sup}\{-X\}]$, i.e. any value between the expected loss and the maximum loss. More precise, for $X \neq const$ a.s. and $v \in [-\mathbf{E}_{\mathbb{Q}}[X] + \sigma_1^-(X), \text{ess.sup}\{-X\}]$ there is a unique $p^* = p^*(v) \in [1, \infty]$ such that $\rho_{p^*,1}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + \sigma_{p^*}^-(X) = v$.

PROPOSITION 2.2 (Fischer, 2003). *Assume $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$, $1 < p < \infty$ and $0 \leq a \leq 1$. Let $1 < P \leq p$ be a random variable on a probability space with measure $\mathbb{P}$. The risk measures ρ_B implied by (4) are differentiable on $\mathbb{R}^n \setminus U_C(B)$, where $U_C(B)$ denotes the set of all $u \in \mathbb{R}^n$ for which $\sum_{i=1}^n u_i X_i = const$. The partial derivatives are*

$$\begin{aligned}\frac{\partial \rho_B}{\partial u_i}(u) &= -\mathbf{E}_{\mathbb{Q}}[X_i] + a \cdot \mathbf{E}_{\mathbb{P}}[\sigma_P^-(X(u))^{1-P} \cdot \\ &\qquad \mathbf{E}_{\mathbb{Q}}[(-X_i + \mathbf{E}_{\mathbb{Q}}[X_i]) \cdot ((X(u) - \mathbf{E}_{\mathbb{Q}}[X(u)])^-)^{P-1}]].\end{aligned} \tag{6}$$

COROLLARY 2.3 (Fischer, 2003). *Under the assumptions of 2.2, the risk measures ρ implied by (4) are suitable for risk capital allocation by the gradient due to the portfolio base B if the components $X_1, \ldots, X_n$ of B are linearly independent and $X_n \neq 0$ is constant. The per-unit allocations are given by (6).*

Proof. $U_C(B) = \langle (0, \ldots, 0, 1) \rangle \subset U_e$. □

3 Upper boundaries for shortfall probabilities

Notation: In the following Pr(.) denotes the probability of events (described in the brackets) due to the probability measure $\mathbb{Q}$ (which is defined on sets, only). $\mathbf{E}[.]$ is the expectation operator due to $\mathbb{Q}$.

A general problem of risk measures beside Value-at-Risk or Expected Shortfall is that they normally have no quantile-like meaning that corresponds to the shortfall probability $\Pr(X + \rho(X) < 0)$ of the portfolio $X + \rho(X)$, that is the portfolio X plus its particular risk capital $\rho(X)$. In the following we consider the so-called one-sided Chebyshev inequality, which is a tool that allows statements on upper boundaries of shortfall probabilities.

LEMMA 3.1 (One-sided Chebyshev inequality). *Let $X \in L^p(\mathbb{Q})$, $p \geq 2$, be a random variable with expectation 0 and variance σ^2. If $t \geq 0$ we have*

$$\Pr(X \geq t) \leq \frac{\sigma^2}{\sigma^2 + t^2}. \tag{7}$$

(7) is sharp, i.e. there are X such that (7) is an equality.

In the literature many proofs of this inequality can be found. The presented proof is rather analogue to the proof of the discrete case in Uspensky (1937).

Proof. We have

$$\mathbf{E}[X-t] = -t \tag{8}$$

and

$$\mathbf{E}[(X-t)^2] = \sigma^2 + t^2. \tag{9}$$

From this we obtain by Hölder's inequality

$$\begin{aligned} t^2 &\overset{(8)}{\leq} \left(\int_{X<t} (X-t)d\mathbb{Q} \right)^2 \\ &\overset{\text{Hölder}}{\leq} \int_{X<t} 1 d\mathbb{Q} \cdot \int_{X<t} (X-t)^2 d\mathbb{Q} \\ &\overset{(9)}{\leq} \Pr(X<t)(\sigma^2+t^2). \end{aligned} \tag{10}$$

For equality in (7) just assume a discrete random variable X with $x_1 = t$, $p_1 = \frac{\sigma^2}{\sigma^2+t^2}$ and $x_2 = -\frac{\sigma^2}{t}$, $p_2 = \frac{t^2}{\sigma^2+t^2}$. □

Let us consider a risk measure of type

$$\rho(X) = -\mathbf{E}[X] + t, \quad t > 0, \tag{11}$$

where $X \in L^2(\mathbb{Q})$ and $\mathrm{Var}(X) = \sigma^2$. We can have $t = t(X)$, here. Obviously, the CRM defined in Proposition 2.1 fulfill this requirement.

COROLLARY 3.2. *For risk measures of type* (11) *it holds that*

$$Pr(X + \rho(X) \leq 0) \;\leq\; \frac{\sigma^2}{\sigma^2 + (\rho(X) + \mathbf{E}[X])^2}. \tag{12}$$

Proof. From Lemma 3.1 we obtain

$$\Pr(X - \mathbf{E}[X] + t \leq 0) \;\leq\; \frac{\sigma^2}{\sigma^2 + t^2} \tag{13}$$

since $\mathbf{E}[-X + \mathbf{E}[X]] = 0$ and $\mathrm{Var}(-X + \mathbf{E}[X]) = \mathrm{Var}(X) = \sigma^2$. □

The right side of (12) is a quick method to determine a boundary for $\Pr(X + \rho(X) \leq 0)$ when σ, $\mathbf{E}[X]$ and $\rho(X)$ are known (and the distribution function of X not or not well enough). Furthermore, one has a closed expression for this boundary.

The other way round, given a certain shortfall probability $\alpha \in (0,1)$, this α is undergone when chosing

$$\rho(X) + \mathbf{E}[X] = t \; \geq \; \sigma\sqrt{\frac{1-\alpha}{\alpha}}. \tag{14}$$

Having in mind regulatory purposes we are interested in small probabilities like 5 or even 1 percent. However, $\alpha = 0.05$ means $t \approx 4.36\sigma$, analogously $t \approx 9.95\sigma$ for the 1 percent shortfall probability. So, for small α the positive term t in addition to the negative expectation $-\mathbf{E}[X]$ must be big compared to the standard deviation of the payoff X.

Consider the risk measures of Proposition 2.1. From Equation (14) the upper boundary α for the shortfall probability is implied by the condition

$$a \cdot \mathbf{E}_{\mathbb{P}} ||(X - \mathbf{E}_{\mathbb{Q}}[X])^{-}||_P \; \geq \; \sigma\sqrt{\frac{1-\alpha}{\alpha}}. \tag{15}$$

(15) can be fulfilled as long as

$$\text{ess.sup}\{(X - \mathbf{E}_{\mathbb{Q}}[X])^{-}\} \; \geq \; \sigma\sqrt{\frac{1-\alpha}{\alpha}} \tag{16}$$

(cf. Fischer, 2003). Hence, in case of portfolios with almost constant payoff the one-sided Chebyshev inequality, respectively (15), might be rather useless. On the other side, having an X that fulfills (16) one can choose a risk measure fulfilling (15), so that the upper boundary α for the shortfall probability is given.

It was shown in Fischer (2003) and mentioned below Proposition 2.1 that the above risk measures actually can be adapted to any value in $[-\mathbf{E}_{\mathbb{Q}}[X], \text{ess.sup}\{-X\}]$ and therefore also to the Value-at-Risk or Expected Shortfall due to a given level α (see Fischer (2003) for examples). Hence, Corollary 3.2 is reasonably used when an upper shortfall probability boundary due to a risk measure (4) must be computed quickly.

The following lemma states a further inequality of Chebyshev-type.

LEMMA 3.3. *For a random variable $X \in L^p(\mathbb{Q})$, $p \geq 2$, we have*

$$\Pr(X < \mathbf{E}[X]) \; \geq \; \left(\frac{\sigma_1^-}{\sigma_2^-}\right)^2. \tag{17}$$

Proof. By Hölder's inequality we obtain

$$\begin{aligned} (\sigma_1^-)^2 &= \left(\int\limits_{X<\mathbf{E}[X]} (X - \mathbf{E}[X]) d\mathbb{Q} \right)^2 \qquad (18) \\ &\overset{\text{Hölder}}{\leq} \int\limits_{X<\mathbf{E}[X]} 1 d\mathbb{Q} \cdot \int\limits_{X<\mathbf{E}[X]} (X - \mathbf{E}[X])^2 d\mathbb{Q} \\ &= \Pr(X < \mathbf{E}[X]) \cdot (\sigma_2^-)^2. \end{aligned}$$

□

COROLLARY 3.4. *For a random variable $X \in L^p(\mathbb{Q})$, $p \geq 2$, we have*

i) $\Pr(X < \mathbf{E}[X]) \geq \left(\frac{\sigma_1^-}{\sigma_2^-}\right)^2$

ii) $\Pr(X \geq \mathbf{E}[X]) \leq 1 - \left(\frac{\sigma_1^-}{\sigma_2^-}\right)^2$

iii) $\Pr(X > \mathbf{E}[X]) \geq \left(\frac{\sigma_1^+}{\sigma_2^+}\right)^2$

iv) $\Pr(X \leq \mathbf{E}[X]) \leq 1 - \left(\frac{\sigma_1^+}{\sigma_2^+}\right)^2$.

Proof. The relations iii) and iv) follow from i) and ii) by the substitution $X' = -X$, where $\sigma_p^-(X') = \sigma_p^+(X)$. □

Note that $\sigma_1^- = \sigma_1^+$. An example in Section 4 gives an application of inequality ii) due to shortfall probabilities. The example concerns a class of coherent risk measures that is defined by recurrence.

4 Risk measures obtained by recurrence

In Proposition 2.1 (4) different attitudes towards risk can be expressed by certain choices of the variable a and the measure $\mathbb{P}$. However, the stress given to the certain moments can be chosen quite arbitrarily. Furthermore, it is hard to find an illustrative interpretation for these risk measures, although higher moments seem to be connected to stochastic dominance and degrees of risk aversion (cf. Levy, 1992). Clearly, risk measures are subjective. Nonetheless, the philosophy behind the choice of a certain measure should be able to be communicated. The underlying attitude towards risk should be objective in the sense of a reasonable stochastic interpretation. Also the step from a certain risk measure towards a more conservative one should be somehow objective.

The present section is driven by the intention to find a class of coherent risk measures that features discrete degrees of attitude towards risk. We intend to applicate one and the same stochastic philosophy when going from one degree to the next, more conservative one.

DEFINITION 4.1. *For $1 \le p \le \infty$ and an integer $n > 0$ we define the* **one-sided $\mathbf{L}^p$-norm risk measure of degree n** *on $L^p(\mathbb{Q})$ by*

$$\rho_{p,n}(X) = \rho_{p,n-1}(X) + ||(X + \rho_{p,n-1}(X))^-||_p \tag{19}$$

where $\rho_{p,0}(X) := -\mathbf{E}_{\mathbb{Q}}[X]$.

For $p = 1$ the different degrees of attitude towards risk follow a consistent stochastic philosophy in the sense that the definition of risk measures by recurrence in (19) assures the application of one stochastic principle or risk measurement method: The measure at level n is obtained as sum of the one at level $n-1$ plus the *expected shortfall* (the expectation not conditioned on the shortfall, here) due to the risk capital given at level $n-1$.

PROPOSITION 4.2. *The risk measures defined by* (19) *are coherent.*

Proof. By induction: Axiom (PH) and (T) are trivial. Axiom (M): Assume $X \ge 0$. $\rho_{p,0}(X) \le 0$ follows. Now, assume $\rho_{p,n}(X) \le 0$. We get $X + \rho_{p,n}(X) \ge \rho_{p,n}(X)$ and therefore $0 \le (X + \rho_{p,n}(X))^- \le -\rho_{p,n}(X)$, which implies $||(X + \rho_{p,n}(X))^-||_p \le -\rho_{p,n}(X)$. This implies $\rho_{p,n+1}(X) \le 0$.
Axiom (S) by induction: Again, $n = 0$ is trivial. Assume

$$\rho_{p,n}(X + Y) + \varepsilon = \rho_{p,n}(X) + \rho_{p,n}(Y) \tag{20}$$

for some $\varepsilon \ge 0$. Now,

$$\begin{aligned} & \rho_{p,n+1}(X + Y) \\ = \;& \rho_{p,n}(X) + \rho_{p,n}(X) - \varepsilon + ||(X + Y + \rho_{p,n}(X) + \rho_{p,n}(Y) - \varepsilon)^-||_p \\ \le \;& \rho_{p,n}(X) + \rho_{p,n}(X) - \varepsilon + ||(X + \rho_{p,n}(X))^-||_p + ||Y + \rho_{p,n}(Y))^-||_p + \varepsilon \\ = \;& \rho_{p,n+1}(X) + \rho_{p,n+1}(Y) \; . \end{aligned} \tag{21}$$

□

For fixed $X \in L^p(\mathbb{Q})$ the sequence of risk measures (19) is monotone and converges from below to the maximum loss. So, the measures feature the desired discrete degrees of attitude towards risk and at the same time they can (pointwise in X) be chosen arbitrarily close to the maximum loss.

PROPOSITION 4.3. *Given a payoff $X \in L^p(\mathbb{Q})$, $1 \leq p \leq \infty$, one has*

$$\rho_{p,n}(X) \quad \stackrel{n\to\infty}{\longrightarrow} \quad \text{ess.sup}\{-X\}. \tag{22}$$

Hence, if ess.sup$\{-X\} < \infty$, *one has*

$$||(X + \rho_{p,n}(X))^-||_p \quad \stackrel{n\to\infty}{\longrightarrow} \quad 0. \tag{23}$$

The sequences are monotone (increasing/decreasing). If $X \neq$ const a.s., $1 \leq p < \infty$ and $n \in \mathbb{N}_0$, we have $\rho_{p,n}(X) <$ ess.sup$\{-X\}$.

Proof. Clearly, $\rho_{p,n}(X) \leq \rho_{p,n+1}(X)$ for $n \geq 0$ and $\rho_{p,0}(X) = -\mathbf{E}[X] \leq$ ess.sup$\{-X\}$. Assume $\rho_{p,n}(X) \leq$ ess.sup$\{-X\}$. Since

$$\begin{aligned} ||(X + \rho_{p,n}(X))^-||_p &\leq \text{ess.sup}\{-(X + \rho_{p,n}(X))\} \qquad (24)\\ &= \text{ess.sup}\{-X\} - \rho_{p,n}(X), \end{aligned}$$

(19) implies $\rho_{p,n+1}(X) \leq$ ess.sup$\{-X\}$. So, the considered sequence is increasing and dominated by ess.sup$\{-X\}$. If the limit was finite and not given by ess.sup$\{-X\}$, we would have

$$\varepsilon \leq ||(X + \lim_{n\to\infty} \rho_{p,n}(X))^-||_p \leq ||(X + \rho_{p,n}(X))^-||_p \tag{25}$$

for some $\varepsilon > 0$ and all integers $n \geq 0$. Now, choose n_0 big enough such that $\lim_{n\to\infty} \rho_{p,n}(X) - \rho_{p,n_0}(X) < \varepsilon$, i.e.

$$\lim_{n\to\infty} \rho_{p,n}(X) - \varepsilon < \rho_{p,n_0}(X). \tag{26}$$

Combining (26) with (25) we obtain

$$\lim_{n\to\infty} \rho_{p,n}(X) < \rho_{p,n_0}(X) + ||(X + \rho_{p,n_0}(X))^-||_p = \rho_{p,n_0+1}(X),$$

which is a contradiction. We can now turn to the last statement. In (24) equality, i.e. $\rho_{p,n+1}(X) =$ ess.sup$\{-X\}$, occurs if and only if $(X+\rho_{p,n}(X))^- =$ *const* a.s. ($p < \infty$). This in turn is equivalent to $X =$ *const* or $\rho_{p,n}(X) \geq$ ess.sup$\{-X\}$. However, $\rho_{p,0}(X) = -\mathbf{E}_{\mathbb{Q}}[X] <$ ess.sup$\{-X\}$ under the assumption $X \neq$ *const*. □

PROPOSITION 4.4. *Under the conditions of Proposition 2.2 the risk measures implied by* (19) *are differentiable on $\mathbb{R}^n \setminus U_C$. Under the assumptions of Corollary 2.3 they are suitable for risk capital allocation.*

Proof by induction. The case $n = 1$ was shown in Fischer (2003). Assume $\rho_{p,n}(X(u))$ to be proven as well. As we know, $\rho_{p,n}(X(u)) < \text{ess.sup}\{-X(u)\}$. Since $\rho_{p,n}(X(u))$ is differentiable on an open subset of $\mathbb{R}^n$, we can find for every $u \in \mathbb{R}^n \setminus U_C$ an open interval U with $u \in U \subset \mathbb{R}^n \setminus U_C$ such that the partial derivatives are bounded on U. Now, we apply Lemma A.3 in Fischer (2003) by setting $y(u) = \rho_{p,n}(X(u))$. The proposition follows from the fact that $\mathbb{R}^n \setminus U_C$ can be covered by such U. $\square$

Explicit partial derivatives can be obtained by the results of Fischer (2003). The choice $p = 1.01$ for example implies risk measures (19) which have the above allocation properties but which are still close to the case $p = 1$ where the expected shortfall philosophy is recurrently applied (as explained below Definition 4.1).

Boundaries for shortfall probabilities. We now turn to an application of Corollary 3.4. We are interested in shortfall probabilities due to risk capital given by $\rho_{1,n}$, i.e. in probabilities of form $\Pr(X + \rho_{1,n}(X) < 0)$. By inequality ii) of Corollary 3.4 and Equality (19) we obtain for $n > 1$ and $X \in L^2$

$$\begin{aligned} & \Pr(X + \rho_{1,n}(X) < 0) \\ = \; & \Pr(X + \rho_{1,n-1} < -\mathbf{E}[(X + \rho_{1,n-1}(X))^-]) \\ = \; & \Pr((X + \rho_{1,n-1})^- > \mathbf{E}[(X + \rho_{1,n-1}(X))^-]) \\ \leq \; & \Pr((X + \rho_{1,n-1})^- \geq \mathbf{E}[(X + \rho_{1,n-1}(X))^-]) \\ \leq \; & 1 - \frac{\sigma_1^-(Y)^2}{\sigma_2^-(Y)^2}\,, \end{aligned} \tag{27}$$

where $Y := (X + \rho_{1,n-1}(X))^-$.

The following example demonstrates the problem that especially for discrete distributions the shortfall probability and also the last expression in (27) above must not converge to zero.

Take $\Omega = \{\omega_1, \omega_2\}$, $X(\omega_1) = -1000$, $X(\omega_2) = 0$, $\Pr(\omega_1) = 0.5$ and $\Pr(\omega_2) = 0.5$. Easily, we find $\rho_0 = 500$, $\rho_1 = 750$, $\rho_2 = 875$ and so on, where $\rho_n < 1000$ for all n. So, the shortfall probability due to any ρ_n is constant 0.5.

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