Consumption processes and positively homogeneous projection properties

Tom Fischer[*]

Heriot-Watt University, Edinburgh

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Abstract

We constructively prove the existence of time-discrete consumption processes for stochastic money accounts that fulfill a pre-specified positively homogeneous projection property (PHPP) and let the account always be positive and exactly zero at the end. One possible example is consumption rates forming a martingale under the above restrictions. For finite spaces, it is shown that any strictly positive consumption strategy with restrictions as above possesses at least one corresponding PHPP and could be constructed from it. We also consider numeric examples under time-discrete and -continuous account processes, cases with infinite time horizons and applications to income drawdown and bonus theory.

Key words: Consumption strategies, income drawdown, log-Lévy processes, martingale consumption, positive homogeneity, smooth bonus

JEL Classification: E21, G22, G23

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1 Introduction

The purpose of this paper is twofold. On the one hand, it introduces a general method to describe and construct consumption processes for stochastic money accounts where the consumption processes at the same time fulfill certain restrictions, e.g. non-negativity of the money account and total consumption in

the end. This method is general in the sense that at least in the case of finite spaces it can be shown that basically any meaningful consumption process can be described and constructed by it, even if the corresponding process is originally obtained by a completely different method, e.g. by expected utility maximization. On the other hand, the considered method itself allows to directly describe stochastic properties of consumption processes. These processes can then be constructed fulfilling at the same time the above restrictions. An example of such a property could be that the consumption rates form a martingale. This could be understood as a "smooth" form of consumption. In contrast to the economic approach of deriving consumption processes from maximization of expected utility, it will be explained below in what sense the direct postulation of stochastic properties for the consumption rates could be seen as an actuarial approach. Applications to actuarial problems like bonus theory and stochastic annuities will show the usefulness of the proposed methods. For instance, a "smooth", i.e. martingale bonus strategy for closed with-profits funds will be derived. Another example explains a benchmark procedure for determining upper consumption limits for pension policies under so-called income drawdown.

To be more specific, let us consider the following situation. A private or institutional agent invests an initial amount of currency units in a money account. We assume that without consumption the value of a positive amount of money in the account always stays strictly positive and develops in a non-deterministic way. The investor wants to take money regularly, e.g. yearly, out of the account. There are certain restrictions for the consumption procedure. Typically, the money account will not be allowed to become negative at any time, money will only be consumed and not added to the account, and after a certain period of time, e.g. 10 years, all funds will have to be consumed. Except for the consumption, the investor is not able to influence the dynamics of the account. For instance, if the money account represented a stock market index, we would assume that the investor is "small", and the consumption rates would therefore not influence the prices, i.e. the dynamics of the account between two successive consumption dates. There also is no alternative investment opportunity for our agent. Otherwise, he or she could influence the overall investment dynamics. This, however, we do not wish since such influence is not possible in the case of many real consumption situations. Under the assumptions made so far, one might be interested in the following questions:

(1) How could the investor specify desirable properties of the (in general non-deterministic) consumption process?

(2) Having described such properties and knowing the dynamics of the money account without consumption, does there exist a consumption process with these properties?

(3) If the second question was answered with "yes", how could the corresponding consumption process be constructed and pursued?

Regarding the first question, there exists a vast variety of ways in which such properties could be described. For instance, and first of all, we could

think of the already mentioned utility-optimizing approaches. Utility-based methods play for good reasons a predominant role in economic theory. One could mention applications in portfolio optimization (selection by expected utility optimization), utility-hedging and insurance premium calculation (utility indifference pricing). However, alternatives exist, and they are widely used. To mention some, we could think of portfolio performance or risk optimization with respect to measures like RORAC, Value-at-Risk or Expected Shortfall. For hedging we could think of quadratic and quantile hedging approaches. Talking of insurance premiums, calculation principles like the Expectation and the Variance Principle are commonly used. All mentioned alternatives have in common that a rather direct stochastic property or quantity - like (minimizing) a quantile, a variance, an expectation - is used to obtain a result like optimal risk or performance, a premium etc. Since premium calculation is supposedly the oldest financial application of such stochastic properties, (note: a risk measure can always be interpreted as a premium principle), it might be justified to call these alternatives "actuarial".

For the consumption problem, less work has been carried out on non-utility alternatives. In some way this is astonishing since especially in actuarial contexts consumers and/or companies seem to have great interest in "smoothness", e.g. in the sense of "smooth" consumption or bonus rates. However, bonus rates for instance are often declared in an *ad hoc* manner, and it is often not at all clearly defined what "smooth" is supposed to mean. Usually, it is understood to mean that rates from one year to the next are changing not too abruptly, i.e. in some way "smoothly". For the following, let us denote the absolute consumption rate at time k by X_k ($k = 0, 1, 2, \ldots, K$). As an example of a "smooth" consumption process, one might consider rates that form a martingale, i.e. $X_k = \mathbf{E}[X_{k+1}|\mathcal{F}_k]$ where $\mathcal{F}_k$ denotes the information up to time k. In other words, "consume every year what you can expect to consume in the following year". Since in the deterministic limit case a martingale is a constant function, this is perhaps as "smooth" as a discrete time process with certain restrictions in place can get. It is for these reasons that the martingale property is one of the prime examples in this paper.

The martingale property is local in the sense that a rate at time k, X_k, is obtained from a projection, or operator, applied to its successor at $k + 1$, X_{k+1}. In the course of the paper it will turn out that this feature is extremely useful for the definition, for the proof of the existence and for the construction of consumption processes under the earlier mentioned restrictions. In particular, we will consider consumption processes that possess what we call a *positively homogeneous projection property* (PHPP). A PHPP means that for any $k < K$ the rate X_k is determined by its successor X_{k+1} in the way that $X_k = \psi_k(X_{k+1})$, with ψ_k being an operator taking care of the right measurability of X_k with respect to the available information at k. We assume that ψ_k maps non-negative (non-positive) random variables to again non-negative (non-positive) ones. Furthermore, ψ_k is assumed to be positively homogeneous in the sense that $\psi_k(YX) = Y\psi_k(X)$ for X $\mathcal{F}_{k+1}$-measurable and $Y \geq 0$ and

$\mathcal{F}_k$-measurable. The final rate X_K is determined by a possibly random fraction $0 \leq \psi_K \leq 1$ of what is left in the account at time K. For instance, $\psi_K \equiv 1$ would mean that all funds are consumed at the end. Examples of ψ_k that are considered later are multiples of conditional expectations ("consume every year $x\%$ of what you can expect to consume in the following year", e.g. $x = 100$ would imply a martingale) and conditional quantiles ("consume this year X_k, and with probability 75% you are allowed to consume more next year"). A submartingale strategy (e.g. choosing $x = 95$ in the conditional expectation case) might be considered as to provide "smooth growth". As outlined before, institutional investors, in particular insurance companies, might be interested in such consumption solutions, especially in the "smooth" martingale consumption strategy. For insurance companies, the earlier mentioned applications to bonus theory and income drawdown should be of special interest. An additional benefit of the above examples is that they are comparatively easy to explain to customers or policyholders. While the communication of features of an insurance policy that come from a utility approach to a policyholder with a non-mathematical background might be difficult, most people seem to have a natural imagination of what an expectation (average) or a probability (percentage) is.

The main result of the paper shows that for any PHPP, $\psi = (\psi_0, \ldots, \psi_K)$, and any strictly positive stochastic process, S, for the money account without consumption, there exists a consumption process, $X = (X_k)_{k=0,\ldots,K}$, which possesses this property, never adds money to the account (i.e. is non-negative) and never lets the money account with consumption, A, become negative. We call an X with the latter two properties *regular*. We give a constructive proof which enables us to explicitly calculate strategies in many cases, e.g. when we have a tree model for S, or for certain PHPPs when S has i.i.d. growth factors (e.g. log-Lévy processes like geometric Brownian motions). Although the projection properties seem to be quite exclusive at first, it can be shown for finite probability spaces that *any* strictly positive regular consumption process (i.e. restrictions as above and no zero-consumption) possesses at least one such property and could therefore be constructed from it. In other words, the proposed method is not restricted to a certain type of consumption processes. It rather represents a general mathematical concept for the construction of arbitrary consumption processes fulfilling the usual restrictions.

To compare the results of this paper with the existing literature seems to be somewhat difficult. Most papers regarding (optimal) consumption (see e.g. Merton (1971), Karatzas, Lehoczky and Shreve (1987), and Cox and Huang (1989)) work with utility frameworks, and the questions asked and/or solved are different from the ones addressed above. For example, the consumption problem is usually defined as the objective to maximize expected utility having at the same time the opportunity to invest in different kinds of assets. As we neither consider a utility framework, nor have an objective for optimization, our considerations are very different from this kind of investigation. Furthermore, we do not consider a portfolio problem as only one investment

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opportunity is given for the reasons indicated above.

The proof of the main result described above shows that for a given value process S a PHPP ψ uniquely defines a regular consumption process $X(S, \psi)$ which can be determined by a combined backward-forward recursion. Although this method is useful, for instance, when dealing with a tree model for S, its applications are limited since the number of paths might grow exponentially in time. Furthermore, the process S could be an almost arbitrary time-continuous process making such a calculation generally impossible. For these reasons, the possibility to simulate $X(S, \psi)$ would be helpful. However, assuming one can simulate paths of S, it is obvious that this is possible when relative consumption rates are deterministic (by a relative rate of consumption we mean the absolute consumption rate expressed as a fraction of the actual value of the money account at that time). The second part of the paper therefore takes a closer look at such strategies, derives more explicit results, and shows the implications for underlying PHPPs of the corresponding absolute consumption process. Examples illustrate that deterministic relative consumption rates are of foremost interest when S has i.i.d. growth factors (e.g. log-Lévy processes). In this part of the paper we also consider cases of consumption for infinite time horizons. Finally, we explain in two actuarial examples how the results of the paper can be applied to pensions with so-called income drawdown, and to what extent they can be applied to bonus strategies for closed with-profits funds.

The paper is organized as follows. In Section 2, we start with basic considerations of absolute consumption rates. The relationship between the consumption process, the money account process without consumption and the money account with consumption is derived. Section 3 explores relative rates of consumption. In Section 4, PHPPs are introduced, and the central theorem of the paper proves the existence of consumption strategies with such properties. Section 5 derives more explicit results for cases in which the relative consumption rates are deterministic. The section contains (partially numeric) examples of explicit consumption solutions when a binomial tree and a geometric Brownian motion model the dynamics of the money account without consumption. In Section 6, we consider consumption for infinite time horizons when relative consumption rates are again deterministic. Section 7 is on the mentioned example of income drawdown. Section 8 is on bonus for closed with-profits funds. In Section 9, we conclude.

2 Absolute rates of consumption

Given the positive real numbers $\mathbb{R}_0^+$ as time axis, let $S = (S_t)_{t \geq 0}$ be a strictly positive stochastic process adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ which is defined on an underlying probability space $(\Omega, \mathcal{F}_\infty, \mathbb{P})$. We consider S as the value process of a stochastic *money account without consumption*. For instance, S could represent the development of an initial investment of $S_0 = 100,000$

currency units into stocks, an index or a fund at time 0. S describes the dynamics of the value of this investment under the assumption that no money is taken out or paid in after time 0. As usual, the σ-algebra $\mathcal{F}_t$ represents all available information up to time t.

At integer times $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, a consumer intends to take money out of the account. The amount taken out at time k is denoted by X_k, the *absolute rate of consumption* at time k. The real-valued process $X = (X_k)_{k \in \mathbb{N}_0}$, which we call the *consumption process*, is adapted with respect to the filtration $(\mathcal{F}_k)_{k \in \mathbb{N}_0}$, so to speak a sub-filtration of $(\mathcal{F}_t)_{t \geq 0}$, which provides the information up to integer times, only. We now consider the value process of the money account *with* consumption. Our basic assumption here is that after the consumption of rate X_k, the process of the account is a rescaled version of the value process one would have without having consumed this particular rate. This is straightforward. In practice one would, for example, consume 10 shares out of a stock of 100 identical shares of a certain fund. Furthermore, we assume that the consumer is a "small investor", i.e. the volume of all consumption taking place is small in relation to the total trading volume of the market. We can therefore assume that consumption does not influence market prices.

Let $A = (A_t)_{t \geq 0}$ denote the value process of the money account with consumption. We have (in a mathematical sense, define)

$$A_t = \begin{cases} S_0 & \text{if } t = 0, \\ (A_k - X_k)S_t/S_k & \text{if } k < t \leq k+1 \ (k \in \mathbb{N}_0). \end{cases} \tag{1}$$

Figure 1 illustrates this relationship between S, X, and A. It is clear that $(A_t)_{t \geq 0}$ is an adapted process w.r.t. $(\mathcal{F}_t)_{t \geq 0}$. The discrete time process $(A_k)_{k \in \mathbb{N}_0}$ represents the value in the money account immediately *before* consumption at times $k \in \mathbb{N}_0$. From a mathematical point of view, it is not necessary to make any continuity assumptions for S in the following. However, if S is left-continuous, so is A. Provided left-continuity of S, (1) should be interpreted such that consumption takes place immediately *before* a possible jump in S at the same time. In the case of right-continuity of S, the interpretation should be that consumption takes place immediately *after* a possible jump in S at the same time.

LEMMA 1. *For $k \in \mathbb{N}_0$,*

$$A_t = \left(1 - \sum_{i=0}^{k} \frac{X_i}{S_i}\right) S_t, \quad k < t \leq k+1. \tag{2}$$

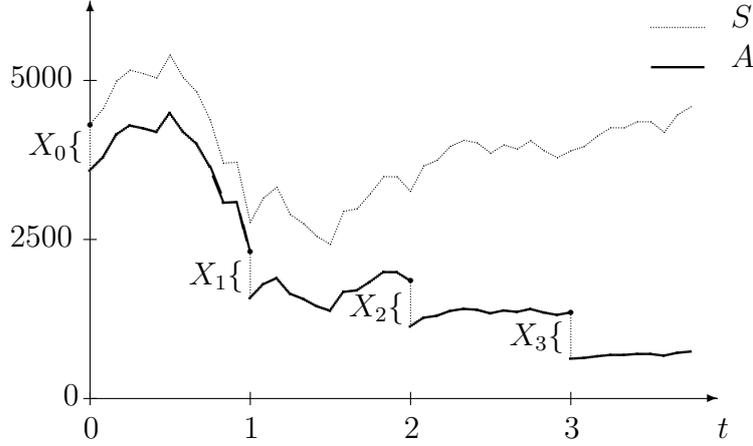*Proof.* By induction. (2) is true for $k = 0$. Now, assume (2) for some $k \in \mathbb{N}_0$.

Figure 1: Sample paths for $(S_t)_{t\geq 0}$ and $(A_t)_{t\geq 0}$

For $k+1 < t \leq k+2$,

$$A_t \overset{(1)}{=} (A_{k+1} - X_{k+1})S_t/S_{k+1} \tag{3}$$

$$\overset{(2)}{=} \left(\left(1 - \sum_{i=0}^{k} \frac{X_i}{S_i}\right) S_{k+1} - X_{k+1} \right) S_t/S_{k+1} \tag{4}$$

$$= \left(1 - \sum_{i=0}^{k+1} \frac{X_i}{S_i}\right) S_t.$$

$\square$

It is clear from (1) that

$$0 \leq X_k \leq (<) A_k, \quad k \in \mathbb{N}_0, \tag{5}$$

implies

$$A_t \geq (>) 0, \quad t > 0. \tag{6}$$

Note that (5) also implies that once all the money has been consumed, i.e. once one has had $X_{k_0} = A_{k_0}$ for some $k_0 \in \mathbb{N}_0$, the values of X_k for $k > k_0$ and the values of A_t for $t > k_0$ are zero.

DEFINITION 1. A consumption process $X = (X_k)_{k\in\mathbb{N}_0}$ fulfilling (5) is called *regular* with respect to S. We call X *strictly regular* w.r.t. S if

$$0 < X_k < A_k, \quad k \in \mathbb{N}_0. \tag{7}$$

The following result on scaling will be used in Section 8. It follows directly from (1).

LEMMA 2. *Let $(F_t)_{t\geq 0}$ be a strictly positive adapted process, and let X be a consumption process w.r.t. S (S as before). Define for $t \geq 0$*

$$\tilde{S}_t = F_t S_t \quad and \quad \tilde{X}_t = F_t X_t, \tag{8}$$

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and let $\tilde{A}_t$ be defined w.r.t. $\tilde{S}$ and $\tilde{X}$ in analogy to (1). Under these assumptions,

$$\tilde{A}_t = F_t A_t, \quad t \geq 0, \tag{9}$$

and $\tilde{X}$ is regular w.r.t. $\tilde{S}$ if and only if X is regular w.r.t. S.

3 Relative rates of consumption

So far we have described consumption by means of absolute consumption rates. This is reasonable as in the following we will be interested in properties of the consumption process X. However, some consumption properties under consideration can be described more easily when considering relative consumption rates. By a *relative rate of consumption* we mean the absolute consumption rate expressed as the percentage of the value of the money account at the time of consumption. For a regular consumption process X w.r.t. S, we therefore call

$$Z_k = \begin{cases} r \in \mathbb{R} & \text{if } A_k = 0 \\ \frac{X_k}{A_k} & \text{else} \end{cases} \tag{10}$$

a relative consumption rate belonging to the absolute consumption rate X_k. We easily obtain the following lemma, whose proof is left to the reader.

LEMMA 3. *Any adapted process $Z = (Z_k)_{k \in \mathbb{N}_0}$ with*

$$0 \leq (<) \, Z_k \leq (<) \, 1, \quad k \in \mathbb{N}_0, \tag{11}$$

determines by $X_k = Z_k A_k$ a (strictly) regular consumption process w.r.t. S. For $k < t \leq k+1$ ($k \in \mathbb{N}_0$),

$$A_t = \left(\prod_{i=0}^{k} (1 - Z_i) \right) S_t, \tag{12}$$

and

$$X_k = Z_k \left(\prod_{i=0}^{k-1} (1 - Z_i) \right) S_k. \tag{13}$$

Note that for regular consumption, (2) and (12) imply

$$\prod_{i=0}^{k} (1 - Z_i) = \left(1 - \sum_{i=0}^{k} \frac{X_i}{S_i} \right), \quad k \in \mathbb{N}_0. \tag{14}$$

4 PHPPs for finite time horizons

Consider a finite time axis $[0, K] \subset \mathbb{R}$ where $K \in \mathbb{N}_0$. One could consider processes $S = (S_t)_{0 \leq t \leq K}$, $X = (X_k)_{k=0,\dots,K}$, and $A = (A_t)_{0 \leq t \leq K}$ similar to their counterparts on the infinite time axis. It is easy to check that all statements

made and derived above are also true when restricted to such a finite time horizon. For instance, a consumption process regular w.r.t. $S = (S_t)_{0 \le t \le K}$ is then a process $X = (X_k)_{k=0,\ldots,K}$ adapted to the discrete time filtration $(\mathcal{F}_k)_{k=0,\ldots,K}$ fulfilling (5) for $k = 0, \ldots, K$.

We now describe operators that will help us to define what we will call a *positively homogeneous projection property* (PHPP) of a consumption process X.

Let $L^0(\mathcal{F}_k)$ ($k = 0, \ldots, K$) denote the set of real-valued $\mathcal{F}_k$-measurable random variables, and let $L^0_+(\mathcal{F}_k)$ ($L^0_-(\mathcal{F}_k)$) denote the cone of non-negative (non-positive) $\mathcal{F}_k$-measurable random variables. For $k = 0, \ldots, K-1$, let ψ_k be a mapping

$$\psi_k : L^0(\mathcal{F}_{k+1}) \longrightarrow L^0(\mathcal{F}_k), \tag{15}$$

such that

$$L^0_+(\mathcal{F}_{k+1}) \xrightarrow{\psi_k} L^0_+(\mathcal{F}_k), \tag{16}$$

$$L^0_-(\mathcal{F}_{k+1}) \xrightarrow{\psi_k} L^0_-(\mathcal{F}_k), \tag{17}$$

and

$$\psi_k(YX) = Y\psi_k(X) \tag{18}$$

for $X \in L^0(\mathcal{F}_{k+1})$ and $Y \in L^0_+(\mathcal{F}_k)$. Furthermore, let

$$\psi_K \in L^0_+(\mathcal{F}_K) \text{ such that } \psi_K \le 1. \tag{19}$$

Note that ψ_K, in contrast to $\psi_0, \ldots, \psi_{K-1}$, is defined as a random variable and not as an operator. Property (18) implies for any $F \in \mathcal{F}_k$ that $\psi_k(X)|_F$ only depends on $X|_F$ since we can choose $Y = \mathbf{1}_F$ in (18). The operators ψ_k in (15) are not necessarily projections in the usual geometric sense. Neither do they have to be linear, nor is necessarily $\psi_k \circ \psi_k = \psi_k$. For $|\mathcal{F}_K| = \infty$ it might not always be possible to define $\psi_0, \ldots, \psi_{K-1}$ on the whole $L^0(\mathcal{F}_1), \ldots, L^0(\mathcal{F}_K)$. If this is the case, one can often restrict the considered spaces and ranges the operators and properties are defined on, such that most of the results showed in the following are still applicable.

THEOREM 1. *An adapted process $(X_k)_{k=0,\ldots,K}$ is non-negative and fulfills the conditions*

$$X_k = \psi_k(X_{k+1}), \quad k = 0, \ldots, K-1, \tag{20}$$

and

$$X_K = \psi_K A_K \tag{21}$$

if and only if

$$X_k = Z_k A_k, \quad k = 0, \ldots, K, \tag{22}$$

where

$$Z_{k-1} = \frac{\psi_{k-1}(Z_k S_k)}{S_{k-1} + \psi_{k-1}(Z_k S_k)}, \quad k = 1, \ldots, K, \tag{23}$$

and

$$Z_K = \psi_K. \tag{24}$$

Furthermore, the uniquely determined process X is a regular consumption process w.r.t. S.

Note that (23) is a pointwise equation, i.e. we actually define $Z_{k-1}(\omega)$ for $\omega \in \Omega$, but leave out the argument for convenience.

Proof. First some observations on regularity and non-negativity of X. A non-negative consumption process X fulfilling (20) and (21) must be regular. If X was not regular, there would be a smallest k such that $X_k > A_k \geq 0$ on some $F \in \mathcal{F}_k$. For $k = K$, (21) would be a contradiction since $\psi_K \leq 1$. For $k < K$, this would imply $A_l < 0$ on F for $l = k+1, \ldots, K$ since X is non-negative. From (17) and (18) we would obtain

$$X_k \mathbf{1}_F = \psi_k(\psi_{k+1}(\ldots \psi_{K-1}(\psi_K A_K \mathbf{1}_F) \ldots)) \leq 0, \tag{25}$$

contradicting $X_k > 0$ on F. For non-negative, regular X, (20) and (21) imply $A > 0$, i.e. $A_t > 0$ for $0 \leq t \leq K$. To prove this observe that since $S > 0$, $A > 0$ is equivalent to $(A_k)_{k=1,\ldots,K} > 0$. If $(A_k)_{k=1,\ldots,K}$ was zero anywhere, there would exist some $k \in \{0, \ldots, K-1\}$ and $F \in \mathcal{F}_k$ such that $X_k = A_k > 0$ on F. Since A_K had to be zero on F as well, (25) would lead to the contradiction $X_k \mathbf{1}_F = 0$. Regarding the second part of the theorem, observe that $Z = (Z_k)_{k=0,\ldots,K}$ as given by (23) and (24) is a non-negative process adapted to $(\mathcal{F}_k)_{k=0,\ldots,K}$. From the definition of the ψ_k and from (23) and (24), it is clear that $0 \leq Z_K \leq 1$ and

$$0 \leq Z_k < 1, \quad k = 0, \ldots, K-1. \tag{26}$$

Therefore, any consumption process X with (22) is regular.

"If": Assume (22), (23) and (24). (21) follows from (22) and (24). From (23) and (26) we obtain $\psi_{k-1}(Z_k S_k) = \frac{S_{k-1} Z_{k-1}}{1 - Z_{k-1}}$ for $k = 1, \ldots, K$. By Lemma 3, for $k = 1, \ldots, K$,

$$\psi_{k-1}(X_k) \stackrel{(22)}{=} \psi_{k-1}(Z_k A_k) \stackrel{(12)\&(18)}{=} \psi_{k-1}(Z_k S_k) \prod_{i=0}^{k-1}(1 - Z_i) \tag{27}$$

$$= \frac{S_{k-1} Z_{k-1}}{1 - Z_{k-1}} \prod_{i=0}^{k-1}(1 - Z_i) = S_{k-1} Z_{k-1} \prod_{i=0}^{k-2}(1 - Z_i)$$

$$\stackrel{(12)}{=} Z_{k-1} A_{k-1} \stackrel{(22)}{=} X_{k-1}.$$

"Only if" is obtained by backward induction. For $k = K$, (22) with (24) follows from (21). From k to $k-1$ we get as follows.

$$\begin{aligned} X_{k-1} &= \psi_{k-1}(X_k) = \psi_{k-1}(Z_k A_k) \\ &\stackrel{(1)}{=} \psi_{k-1}(Z_k S_k)\frac{A_{k-1} - X_{k-1}}{S_{k-1}}. \end{aligned} \tag{28}$$

Therefore,

$$X_{k-1} = \frac{\psi_{k-1}(Z_k S_k)}{S_{k-1} + \psi_{k-1}(Z_k S_k)} A_{k-1}. \tag{29}$$

$\square$

DEFINITION 2. *Define $\psi = (\psi_0, \ldots, \psi_K)$ where $\psi_0, \ldots, \psi_K$ are as above. We say that a consumption process X for a value process S has the* positively homogeneous projection property (PHPP) *ψ if it fulfills (20) and (21). We denote such an X by $X(S, \psi)$, and the corresponding Z by $Z(S, \psi)$.*

COROLLARY 1. *For any non-negative $X(S, \psi)$ one has for $k = 0, \ldots, K-1$ that $Z_k < 1$, or equivalently $X_k < A_k$. Furthermore, $A_t > 0$ for $0 \le t \le K$.*

Proof. See proof of Theorem 1. $\square$

Theorem 1 shows that given some value process S and a PHPP ψ, there exists one and only one non-negative consumption process $X(S, \psi)$ with this property. Furthermore, this $X(S, \psi)$ is regular. The theorem provides us with a backward-forward recursion scheme for X. First, by backward recursion, the process $Z(S, \psi)$ of the relative consumption rates can be determined (Eq. (23) and (24)). Then, Z is used in the forward recursion formula (13), or, by (14), in

$$X_k = Z_k \left(1 - \sum_{i=0}^{k-1} \frac{X_i}{S_i} \right) S_k, \quad k = 0, \ldots, K. \tag{30}$$

The number of examples of positively homogeneous projection properties is vast. For instance, if $(\psi_0, \ldots, \psi_K)$ is a PHPP, any $(B_0 \psi_0, \ldots, B_K \psi_K)$ for $\mathcal{F}_k$-measurable B_k ($0 \le B_k \le 1$, $k = 0, \ldots, K$) is one, too. In the following, we briefly consider two particular classes. More detailed examples (a binomial tree model and a geometric Brownian motion for S) are considered in Section 5. Note that the following two examples could be numerically implemented for any reasonably sized finite stochastic tree modelling S.

Example 1 (Conditional expectations). Assume $|\mathcal{F}_K| < \infty$. For $k = 0, \ldots, K-1$, $X \in L^0(\mathcal{F}_{k+1})$ and $c > 0$ let

$$\psi_k(X) = c\mathbf{E}[X|\mathcal{F}_k] \tag{31}$$

and $\psi_K \equiv d$, $0 \le d \le 1$. It is easy to check that (15) to (19) are fulfilled. The consumption process with the corresponding PHPP fulfills $X_k = c\mathbf{E}[X_{k+1}|\mathcal{F}_k]$ for $k = 0, \ldots, K-1$ and $X_K = dA_K$. A consumer following this strategy consumes at each time k exactly $c \times 100\%$ of what he or she can expect to consume one step ahead at time $k + 1$. The final rate X_K would consume $d \times 100\%$ of the capital left in the account at time K. Note that $c = 1$ means that X is a martingale, $c < 1$ implies a submartingale, i.e. the consumer can expect growing consumption rates.

Example 2 (Conditional quantiles). Assume $|\mathcal{F}_K| < \infty$ and set $\psi_K \equiv d$, $0 \leq d \leq 1$, and

$$\psi_k(X) = q_\alpha(X|\mathcal{F}_k), \tag{32}$$

where $q_\alpha(X|\mathcal{F}_k)$ denotes a conditional quantile. For instance, for $X_k = q_{0.4}(X_{k+1}|\mathcal{F}_k)$, the consumption rate at $k+1$ would be higher than the one at k with probability 60%. The standard definition of an α-quantile ($\alpha \in [0,1]$) being

$$q_\alpha(X) = \inf\{y : \mathbb{P}(X \leq y) \geq \alpha\}, \tag{33}$$

one might choose to define

$$q_\alpha(X|\mathcal{F}_k) = \text{ess inf}\{Y : Y \in L^0(\mathcal{F}_k) \text{ and } \mathbf{E}[\mathbf{1}_{X \leq Y}|\mathcal{F}_k] \geq \alpha \text{ a.s.}\}, \tag{34}$$

where $\mathbf{E}[\mathbf{1}_{X \leq Y}|\mathcal{F}_k]$ is the probability of $\{X \leq Y\}$ conditional on $\mathcal{F}_k$. In the finite case, (34) is equivalent to

$$q_\alpha(X|\mathcal{F}_k) = \sum_{F \in \mathcal{A}_k} \mathbf{1}_F \inf\{y : \mathbb{P}(X \leq y|F) \geq \alpha\}, \tag{35}$$

where $\mathcal{A}_k$ denotes the partition of Ω consisting of the atoms of $\mathcal{F}_k$. Note that (32) is an example of a non-linear (non-additive) PHPP.

The examples above were restricted to $|\mathcal{F}_K| < \infty$ since a definition of the considered operators on the whole $L^0(\mathcal{F}_K)$ is generally not possible. However, consideration of consumption in more general spaces using these properties is often still possible when restricting the considered spaces in a reasonable way, e.g. working on integrable subspaces in the conditional expectations case, and when the dynamics of S are suitable. The elements that a restricted support of the operators ψ_{k-1} ($k = 1, \ldots, K$) should definitely contain in order to make Theorem 1 work are $Z_k A_k$ and $Z_k S_k$ (Z_k as in Theo. 1). The minimal functionality the operators should have (besides mapping non-negative/-positive functions to again non-negative/-positive ones) would be given by $\psi_{k-1}(Z_k A_k) = \psi_{k-1}(Z_k S_k) \prod_{i=0}^{k-1}(1 - Z_i)$. If these minimal requirements are satisfied, the proof of Theo. 1 shows that $X(S, \psi)$ exists.

It is remarkable that not only every positively homogeneous projection property leads to a regular consumption process, but also the inverse is true under certain restrictions.

PROPOSITION 1. *Assume $|\mathcal{F}_K| < \infty$. Let X be a regular consumption process such that $(X_k)_{k=0,\ldots,K-1}$ strictly regular and $X_K > 0$. Then there exists a PHPP ψ such that $X = X(S, \psi)$. This statement is in general not true for only regular consumption processes.*

Proof. Let

$$\psi_K = X_K/A_K, \tag{36}$$

and for $k = 0, \ldots, K - 1$ and $Y \in L^0(\mathcal{F}_{k+1})$

$$\psi_k(Y) = \frac{X_k \mathbf{E}[Y|\mathcal{F}_k]}{\mathbf{E}[X_{k+1}|\mathcal{F}_k]}. \tag{37}$$

Clearly, these ψ_k fulfill (15) to (20). To show that we can not find a positively homogeneous projection property ψ for every regular consumption process, it is sufficient to take a strictly regular one, called X here, e.g. obtained from the relative consumption rates $Z_0 = \ldots = Z_K = 0.5$, select a time k_0 ($0 < k_0 \leq K$) and define $Y_k = X_k$ for $k < k_0$ and $Y_k = 0$ for $k \geq k_0$. $Y = (Y_k)_{k=0,\ldots,K}$ is now a regular consumption process, but has no corresponding PHPP since this would lead to the contradiction $0 < Y_{k_0-1} = \psi_{k_0-1}(Y_{k_0}) = \psi_{k_0-1}(0) = 0$. $\qquad\square$

The relation between the strictly positive regular consumption processes (as in Proposition 1) and PHPPs is no bijection. For instance, (37) is linear, but we have seen that there are genuinely non-linear examples of PHPPs, e.g. conditional quantiles. The fact that we can find a PHPP for any strictly positive regular consumption process might be useful for the construction of such a process fulfilling a certain additional property (e.g. being optimal w.r.t. some objective like an expected utility). If from this latter property the corresponding positively homogeneous projection property could be derived, we could construct the process by backward-forward recursion as outlined before.

5 Deterministic relative consumption rates

The results of the previous section have shown that for a value process S a given positively homogeneous projection property uniquely defines a regular consumption process. This process can be determined by a combined backward-forward recursion. It has been mentioned in the introduction that this method is useful when dealing with tree models for S, but the applications are limited since the number of paths might grow too fast. Also, S could be an (almost) arbitrary time-continuous process such that calculations like these are generally impossible. Therefore, the possibility to simulate the consumption process $X(S, \psi)$ would be very useful.

From the results of the previous section, we see that if Z itself is deterministic, i.e. $Z_k \equiv z_k \in [0, 1]$, $k = 0, \ldots, K$, formula (13) provides a direct way to calculate X – provided one can simulate paths of S. In this section we will have a closer look at such strategies and will show that in many cases deterministic relative consumption strategies can be interpreted as absolute consumption derived from a particular positively homogeneous projection property. An example illustrates that these concepts are of foremost interest when S is a log-Lévy process, for instance a geometric Brownian motion.

PROPOSITION 2. *Let $a_0, \ldots, a_K$ be a family of non-negative real numbers. For any PHPP ψ with*

$$\psi_k(S_{k+1}/S_k) \equiv a_k, \quad k = 0, \ldots, K - 1, \tag{38}$$

and

$$\psi_K \equiv a_K \tag{39}$$

one has for $Z(S, \psi)$

$$Z_k \equiv \frac{\prod_{j=k}^K a_j}{1 + \sum_{j=k}^{K-1} \prod_{i=j}^K a_i}, \quad k = 0, \ldots, K. \tag{40}$$

Furthermore, for $X(S, \psi)$ one has that for $k = 0, \ldots, K$

$$X_k = \frac{\prod_{j=k}^K a_j}{1 + \sum_{j=k}^{K-1} \prod_{i=j}^K a_i} A_k \tag{41}$$

$$= \frac{\prod_{j=k}^K a_j}{1 + \sum_{j=0}^{K-1} \prod_{i=j}^K a_i} S_k. \tag{42}$$

Proof. We prove (40) by backward induction. Assume (38) and (39). For $k = K$, (40) is true since $Z_K = \psi_K = a_K$. Now suppose (40) is true for some $k > 0$. Then

$$\psi_{k-1}(Z_k S_k) \overset{(18)}{=} \left(\frac{\prod_{j=k}^K a_j}{1 + \sum_{j=k}^{K-1} \prod_{i=j}^K a_i} \right) S_{k-1} \cdot \psi_{k-1} \left(\frac{S_k}{S_{k-1}} \right) \tag{43}$$

$$= \left(\frac{\prod_{j=k-1}^K a_j}{1 + \sum_{j=k}^{K-1} \prod_{i=j}^K a_i} \right) S_{k-1}.$$

Therefore,

$$Z_{k-1} = \frac{\left(\frac{\prod_{j=k-1}^K a_j}{1 + \sum_{j=k}^{K-1} \prod_{i=j}^K a_i} \right) S_{k-1}}{S_{k-1} + \left(\frac{\prod_{j=k-1}^K a_j}{1 + \sum_{j=k}^{K-1} \prod_{i=j}^K a_i} \right) S_{k-1}} = \left(\frac{\prod_{j=k-1}^K a_j}{1 + \sum_{j=k-1}^{K-1} \prod_{i=j}^K a_i} \right). \tag{44}$$

Observe that (41) follows directly from (40) and (22). To prove (42) we again utilize induction. Obviously, (42) holds for $k = 0$. Now, assuming (42) to be valid up to k, we obtain from (41) and (2)

$$X_{k+1} = \left(1 - \sum_{i=0}^k \frac{X_i}{S_i} \right) \left(\frac{\prod_{j=k+1}^K a_j}{1 + \sum_{j=k+1}^{K-1} \prod_{i=j}^K a_i} \right) S_{k+1} \tag{45}$$

$$= \left(1 - \sum_{i=0}^k \left(\frac{\prod_{j=i}^K a_j}{1 + \sum_{j=0}^{K-1} \prod_{h=j}^K a_h} \right) \right) \left(\frac{\prod_{j=k+1}^K a_j}{1 + \sum_{j=k+1}^{K-1} \prod_{i=j}^K a_i} \right) S_{k+1}$$

$$= \left(\frac{1 + \sum_{j=k+1}^{K-1} \prod_{i=j}^K a_i}{1 + \sum_{j=0}^{K-1} \prod_{i=j}^K a_i} \right) \left(\frac{\prod_{j=k+1}^K a_j}{1 + \sum_{j=k+1}^{K-1} \prod_{i=j}^K a_i} \right) S_{k+1}$$

$$= \left(\frac{\prod_{j=k+1}^K a_j}{1 + \sum_{j=0}^{K-1} \prod_{i=j}^K a_i} \right) S_{k+1}.$$

$$\square$$

The next result inverts Proposition 2 in some way.

PROPOSITION 3. *Let a regular consumption process $X(S, \psi)$ with PHPP ψ have corresponding deterministic relative consumption rates $Z(S, \psi)$ with*

$$Z_k \equiv z_k, \quad (k = 0, \dots, K) \tag{46}$$

where $z_k \in [0, 1)$, $k = 0, \dots, K-1$, and $z_K \in [0, 1]$. Let k_0 be the maximal k such that $z_k = 0$. Then,

$$\psi_k(S_{k+1}/S_k) \equiv \frac{z_k}{(1 - z_k)z_{k+1}}, \quad k = k_0, \dots, K-1, \tag{47}$$

and

$$\psi_K \equiv z_K. \tag{48}$$

Proof. Z is given by (23) and (24). Note that (23) implies $Z_0 = \dots = Z_{k_0} \equiv z_{k_0} = 0$. We first prove (47) and (48) for k_0 equal to K or $K-1$. If $k_0 = K$, then $\psi_K \equiv z_K = 0$. If $k_0 = K-1$, then $\psi_K \equiv z_K$ and $z_{K-1} = 0$. From (23) and (18) we derive $\psi_{K-1}(S_K) = 0$, and by (18) $\psi_{K-1}(S_K/S_{K-1}) = \psi_{K-1}(S_K)/S_{K-1} = 0$. For $k_0 < K-1$, (23) implies inductively that $\psi_{k-1}(S_k/S_{k-1})$, $k = k_0+1, \dots, K$, is constant and strictly positive. From (23) we easily derive (47). $\square$

COROLLARY 2. *Assume $|\mathcal{F}_K| < \infty$.*
(i) There exists for any family $a_0, \dots, a_K$ of real numbers with $a_k \geq 0$ ($k = 0, \dots, K-1$), and $a_K \in [0, 1]$ a positively homogeneous projection property ψ such that (38) and (39) hold.
(ii) For any family $z_0, \dots, z_K$ of real numbers fulfilling $z_k \in [0, 1)$ for $k = 0, \dots, K-1$, $z_K \in [0, 1]$, and $z_k = 0$ for $k = 0, \dots, k_0$ (where k_0 is as in Prop. 3), there exists a positively homogeneous projection property ψ such that for $Z(S, \psi)$ one has $Z_k \equiv z_k$, $k = 0, \dots, K$.

Corollary 2 and Proposition 2 and 3 answer the question in what way deterministic relative consumption can be interpreted as absolute consumption under specific positively homogeneous projection properties. For strictly positive regular consumption processes, the mere existence of at least one corresponding PHPP follows already from Prop. 1. The following proof of the corollary makes use of Prop. 2.

Proof. (i) Define $\psi_K = a_K$ and for $k = 0, \dots, K-1$

$$\psi_k(X) = \frac{a_k S_k \mathbf{E}[X|\mathcal{F}_k]}{\mathbf{E}[S_{k+1}|\mathcal{F}_k]} \tag{49}$$

These ψ_k fulfill (15) to (19), as well as (38) and (39).
(ii) Define a family of positive real numbers $a_0, \dots, a_K$ with

$$a_k = \frac{z_k}{(1 - z_k)z_{k+1}}, \quad k = k_0, \dots, K-1, \tag{50}$$

and $a_K = z_K$. Note that $a_0, \ldots, a_{k_0-1}$ can be arbitrarily chosen. Apply now part (i) and Prop. 2. From (40) one obtains for $k = k_0, \ldots, K-1$

$$
\begin{aligned}
Z_k &\equiv \frac{\prod_{j=k}^{K} a_j}{1 + \sum_{j=k}^{K-1} \prod_{i=j}^{K} a_i} = \frac{z_K \prod_{j=k}^{K-1} \frac{z_j}{(1-z_j)z_{j+1}}}{1 + \sum_{j=k}^{K-1} z_K \prod_{i=j}^{K-1} \frac{z_i}{(1-z_i)z_{i+1}}} \qquad (51) \\[2mm]
&= \frac{\frac{z_k}{\prod_{j=k}^{K-1}(1-z_j)}}{1 + \sum_{j=k}^{K-1} \frac{z_j}{\prod_{i=j}^{K-1}(1-z_i)}} \\[2mm]
&= z_k \left(\prod_{j=k}^{K-1}(1-z_j) + \sum_{j=k}^{K-1} z_j \prod_{i=k}^{j-1}(1-z_i) \right)^{-1} = z_k.
\end{aligned}
$$

Since $a_{k_0} = 0$, we obtain $Z_k \equiv 0$ $(k = 0, \ldots, k_0)$ from (40). $\qquad \square$

COROLLARY 3. *Assume in Prop. 2 that $a_0 = \ldots = a_{K-1} = a$. Then, for $k = 0, \ldots, K$,*

$$
X_k = \left(\frac{a_K a^{K-k}}{1 + a_K \sum_{j=1}^{K-k} a^j} \right) A_k = \left(\frac{a_K a^{K-k}}{1 + a_K \sum_{j=1}^{K} a^j} \right) S_k. \qquad (52)
$$

Remark 1. (42) and (52) enable us to directly calculate the consumption process $X(S, \psi)$, provided S can be simulated. Note that (42) and (52) also imply that the relative standard deviation of X_k and S_k are the same, i.e. $\mathbf{SD}[X_k]/\mathbf{E}[X_k] = \mathbf{SD}[S_k]/\mathbf{E}[S_k]$ (**SD** denoting the standard deviation). In other words, the fluctuations of X_k and S_k are identical in relative terms.

If the discrete time process $(\log S_k)_{k=0,\ldots,K}$ derived from S has i.i.d. increments, then all $\frac{S_k}{S_{k-1}}$, $k = 1, \ldots, K$, are i.i.d. and strictly positive. Geometric Brownian motions, or generally log-Lévy processes would be classical examples of such S. From previous results, especially Prop. 2 and the PHPPs in the Examples 1 and 2, it seems obvious that such processes for S are suitable candidates for applications.

Example 3 (Binomial tree). Let us work with a binomial tree for $(S_k)_{k=0,\ldots,K}$, i.e. all S_k/S_{k-1} are i.i.d. $(k = 1, \ldots, K)$, and $\mathbb{P}[S_k/S_{k-1} = u_1] = p$ and $\mathbb{P}[S_k/S_{k-1} = u_2] = 1 - p$, where $0 < u_1 < u_2$ and $p \in (0, 1)$. Consider now Example 1. Clearly, we are here in a situation where Corollary 3 can be applied with

$$
\psi_k(S_{k+1}/S_k) \equiv c(pu_1 + (1-p)u_2) = a \qquad (53)
$$

and $\psi_K = a_K = d$. Eq. (52) calculates paths of X for known paths of $(S_k)_{k=0,\ldots,K}$. Calculations can now be carried out by standard spreadsheet programs. Note that we do not need to separately calculate each path of the tree. The outcomes at times $k = 0, \ldots, K$ are sufficient, i.e. calculations do not grow exponentially, but in second power, only. Table 1 shows two paths of $(S_k)_{k=0,\ldots,K}$ $(S_0 = 10,000)$ and the corresponding consumption process X for $K = 10$, $c = 1/1.02$, $d = 1$, $u_1 = 1.02$, $u_2 = 1.06$, and $p = 0.5$. This

Table 1: Example 3 (binomial tree)

k	worst path			best path		
	S_k	X_k	A_k	S_k	X_k	A_k
0	10,000.00	999.90	10,000.00	10,000.00	999.90	10,000.00
1	10,200.00	1,000.28	9,180.11	10,600.00	1,039.51	9,540.11
2	10,404.00	1,000.66	8,343.42	11,236.00	1,080.69	9,010.64
...	...	...	...	...	...	...
8	11,716.59	1,002.98	2,951.44	15,938.48	1,364.38	4,014.94
9	11,950.93	1,003.36	1,987.43	16,894.79	1,418.43	2,809.59
10	12,189.94	1,003.75	1,003.75	17,908.48	1,474.63	1,474.63

particularly means that we have a submartingale with total consumption in the end. The value of c, that means an expected annual growth of X by 2%, is chosen such that even the worst case scenario, $(u_1, u_1, \ldots, u_1)$, provides a slightly increasing rate of consumption.

Example 4 (Geometric Brownian motion). We reconsider Example 2 with S being a geometric Brownian motion, i.e.

$$S_t = S_0 \exp(\mu t + \sigma W_t), \quad S_0, \mu, \sigma > 0, \tag{54}$$

where $W = (W_t)_{t \geq 0}$ denotes a standard Brownian motion. Let $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by W. We have

$$S_k/S_{k-1} = \exp(\mu + \sigma Y_k), \tag{55}$$

where $Y_k = W_k - W_{k-1} \sim N(0,1)$ independent from $\mathcal{F}_{k-1}$ $(k = 1, \ldots, K)$. From the definition of q_α in Example 2, it is not difficult to derive that $q_\alpha(S_k/S_{k-1}|\mathcal{F}_{k-1}) \equiv q_\alpha(S_k/S_{k-1})$. Assume now that $\psi_K \equiv 1$ and for $k = 0, \ldots, K-1$

$$\psi_k(X) = q_\alpha(X|\mathcal{F}_k), \tag{56}$$

where (56) is only considered for X where $q_\alpha(X|\mathcal{F}_k)$ is measurable. Although the ψ_k are now not defined on the whole $L^0(\mathcal{F}_{k+1})$, it is easy to check that Corollary 3 still applies. For instance, we could choose $d = 1$ (total consumption at the end), $S_0 = 10,000$, $\mu = 0.02$, $\sigma = 0.1$ and $\alpha = 0.1$ (alternatively, $\alpha = 1/3$). We obtain $a = q_{0.1}(S_k/S_{k-1}) = \exp(0.02 + 0.1\Phi^{-1}(0.1)) \approx 0.897488$ ($a \approx 0.977191$). Eq. (52) provides us with values of X when S is simulated by standard Monte-Carlo methods. Figure 2 shows for $K = 10$ a sample path for the corresponding money account without consumption, S. The corresponding consumption rates for $\alpha = 0.1$ ($\alpha = 1/3$) are plotted. Pursuing the outlined strategy, the probability of being allowed to consume more one year later is always 90% (66.67%). The chosen path shows an unpleasant scenario in which the money account is less worth after ten years than in the beginning. The graphs for X illustrate how these kind of consumption strategies can cope with such situations.
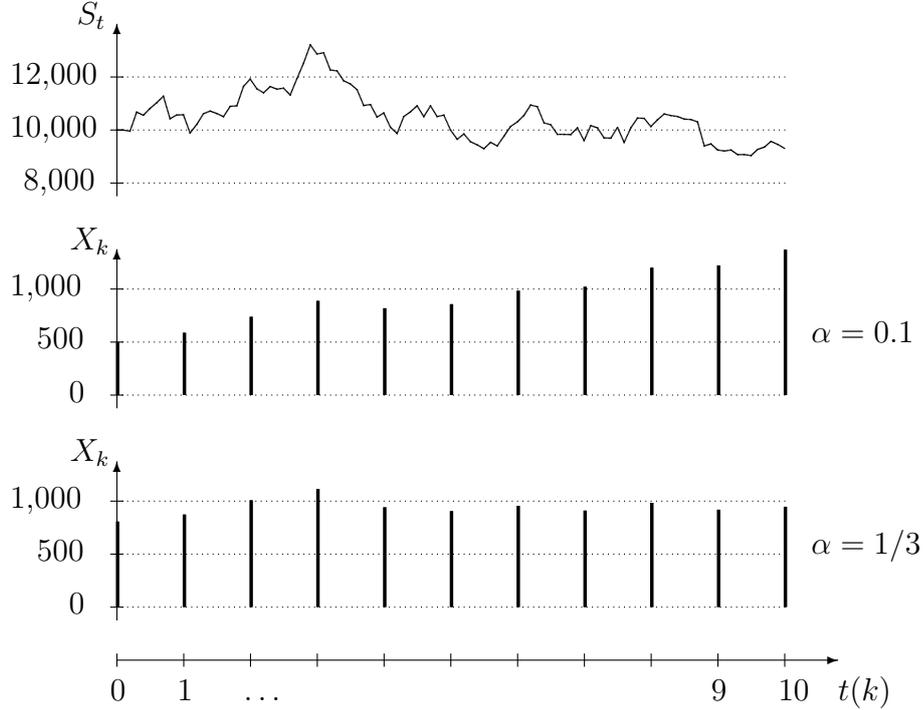
Figure 2: Example 4 (geometric Brownian motion). Sample paths for S and X ($K = 10$) under the conditional quantile property.

Remark 2. Using Corollary 3, it is easy to derive that for a fixed interest scenario with $S_t = S_0 \times (1 + i)^t$, $i > -1$, $S_0 \equiv const > 0$, the martingale property (Ex. 1) with $c = d = 1$ implies that

$$X_k = \frac{S_0}{\ddot{a}_{\overline{K+1|}}}, \quad k = 0, \ldots, K. \tag{57}$$

Here, $\ddot{a}_{\overline{K+1|}}$ denotes in international actuarial notation the present value of an annuity-certain of 1 currency unit annually in advance for $K + 1$ years. E.g., for $i = 4\%$ and $S_0 = 10,000$, one has $\ddot{a}_{\overline{K+1|}} = 9.1109$ and therefore $X_0 = \ldots = X_K = 1097.59$. This illustrates why X_0 calculated for a martingale strategy for a stochastic account could be interpreted as a stochastic generalisation of the benefit of an annuity-certain with present value S_0. One could also interpret S_0/X_0 as the generalisation of $\ddot{a}_{\overline{K+1|}}$, the capital at time 0 that is necessary to have an (expected) annual benefit of 1.

6 Cases of perpetual consumption

In this section, we will examine deterministic relative consumption rates for infinite time horizons. As an application, we could think of the Nobel Prize Committee that has to make sure that sufficiently stable amounts of money can be drawn from an existing fund every year, with no time limit being known.

For an infinite time horizon, we denote a consumption process with respect to S by $X(S, \psi)$ if $X_k = \psi_k(X_{k+1})$, $k \in \mathbb{N}_0$, where ψ_k, $k \in \mathbb{N}_0$, satisfies (15) to (18). We then say that X has the positively homogeneous projection property $\psi = (\psi_0, \psi_1, \ldots)$.

PROPOSITION 4. *Let $\{a_k : k \in \mathbb{N}_0\}$ be a family of strictly positive real numbers and assume that ψ is a positively homogeneous projection property with*

$$\psi_k(S_{k+1}/S_k) \equiv a_k, \quad k \in \mathbb{N}_0. \tag{58}$$

There exists a strictly regular consumption process $X(S, \psi)$ fulfilling

$$X_k = z_k A_k, \quad k \in \mathbb{N}_0, \tag{59}$$

for a family $\{z_k : k \in \mathbb{N}_0\}$ of real numbers if and only if

$$0 < z_0, \tag{60}$$

$$z_k = z_0 \left/ \left(\prod_{i=0}^{k-1} a_i - z_0 \sum_{i=0}^{k-1} \prod_{j=i}^{k-1} a_j \right) \right., \quad k \in \mathbb{N}_0, \tag{61}$$

and

$$z_0 < \frac{\prod_{i=0}^{k-1} a_i}{1 + \sum_{i=0}^{k-1} \prod_{j=i}^{k-1} a_j}, \quad k \in \mathbb{N}_0. \tag{62}$$

Observe that (62) implies $z_0 < 1$.

Proof. "Only if": We obtain

$$X_k \overset{(59)}{=} z_k A_k = \psi_k(X_{k+1}) \overset{(59)}{=} \psi_k(z_{k+1} A_{k+1}) \tag{63}$$

$$\overset{(1)}{=} z_{k+1} \psi_k([A_k - z_k A_K] S_{k+1}/S_k)$$

$$= z_{k+1} \psi_k([1 - z_k] A_k S_{k+1}/S_k)$$

$$\Rightarrow \quad \frac{z_k}{(1 - z_k) z_{k+1}} = \psi_k(S_{k+1}/S_k), \tag{64}$$

and hence

$$z_{k+1} \overset{(58)}{=} \frac{z_k}{(1 - z_k) a_k}, \quad k \in \mathbb{N}_0. \tag{65}$$

In (65), no division by zero can occur since strict regularity of X implies $0 < z_k < 1$, $k \in \mathbb{N}_0$. We now show (61) and (62) for $k \in \mathbb{N}_0$ by induction. Assume for any $k \in \mathbb{N}_0$ (61) and (62). Observe that (61) is valid for $k = 0$, and (62) implies that the denominator in (61) is greater than z_0, which is greater than 0. Now,

$$z_{k+1} = \frac{z_k}{(1 - z_k) a_k} = \frac{z_0}{\left(\prod_{i=0}^{k-1} a_i - z_0 \sum_{i=0}^{k-1} \prod_{j=i}^{k-1} a_j - z_0 \right) a_k} \tag{66}$$

$$= \frac{z_0}{\prod_{i=0}^{k} a_i - z_0 \sum_{i=0}^{k} \prod_{j=i}^{k} a_j}. \tag{67}$$

From $z_{k+1} < 1$ and (67) we obtain

$$z_0 < \frac{\prod_{i=0}^{k} a_i}{1 + \sum_{i=0}^{k} \prod_{j=i}^{k} a_j}. \tag{68}$$

"If": As mentioned before, (60) and (62) imply that the denominator in (61) is positive and greater z_0. Hence, $0 < z_k < 1$, $k \in \mathbb{N}_0$. Furthermore, (66) shows that (61) implies (65). Now,

$$\psi_k(z_{k+1} A_{k+1}) \overset{(1)}{=} z_{k+1}[1 - z_k]\psi_k(S_{k+1}/S_k)A_k \tag{69}$$
$$\overset{(58)\&(65)}{=} z_k A_k.$$

That means the strictly regular consumption process defined by (59) has the PHPP ψ. □

For infinite time horizons, the number of possible regular consumption processes $X(S, \psi)$ for a given value process S and a positively homogeneous projection property ψ may be infinite – in contrast to the uniqueness proved for the finite time case (Theo. 1). The reason for this is the missing "boundary condition" (21). For instance, any existing regular consumption solution X in the infinite case can simply be rescaled by a factor between zero and one and still be a regular solution for the same value process and the same property ψ, the only implication being that the money account A would have greater values than in the original case. In the case of Prop. 4, one could ask for the upper boundary of all possible z_0. By (62), the answer is

$$z_0^{\max} = \inf_{k \in \mathbb{N}_0} \left\{ \frac{\prod_{i=0}^{k-1} a_i}{1 + \sum_{i=0}^{k-1} \prod_{j=i}^{k-1} a_j} \right\}. \tag{70}$$

All strictly regular consumption solutions $X(S, \psi)$ for the given process S and PHPP ψ as stated in Prop. 4 can therefore be obtained by starting with any $z_0 \in (0, z_0^{\max})$, using (61) to obtain z_k ($k > 0$). Whether $z_0^{\max}$ itself can be used as a starting point depends on whether the infimum is actually reached in (70) (no), or not (yes).

Example 5. Let us consider the case of Prop. 4 when $a_k = a$ for all $k \in \mathbb{N}_0$. In this case, (70) boils down to

$$z_0^{\max} = \inf_{k \in \mathbb{N}_0} \left\{ \frac{a^k}{\sum_{i=0}^{k} a^i} \right\} = \lim_{k \to \infty} \left(\sum_{i=0}^{k} a^{-i} \right)^{-1} = 1 - \frac{1}{a}. \tag{71}$$

This makes sense for $a > 1$ only, since regularity is wanted. From (65) we get that

$$z_k = 1 - \frac{1}{a}, \quad k \in \mathbb{N}_0. \tag{72}$$

Therefore, the "maximal" solution (z_0 maximal) is a constant solution. Observe that $\psi_k(z_{k+1}A_{k+1}) = z_k A_k$ for identical z_k ($k \in \mathbb{N}_0$) implies

$$\psi_k(A_{k+1}) = A_k, \quad k \in \mathbb{N}_0. \tag{73}$$

In other words, the money account at integer times, i.e. the process $(A_k)_{k \in \mathbb{N}_0}$, has the positively homogeneous projection property ψ, too. In particular, if $\psi_k(\cdot) = \mathbf{E}[\cdot|\mathcal{F}_k]$ and X is a martingale, $(A_k)_{k \in \mathbb{N}_0}$ is a martingale, too.

7   Income drawdown

Income drawdown, sometimes also called *annuity drawdown* or *pension fund drawdown*, was introduced in the UK in 1995 to provide annuity holders with better income flexibility. The basic idea is the following. Instead of paying into a pension fund until retirement (e.g. at age 65) and then immediately buying an annuity, the policyholder has the opportunity to start taking regular benefits from his/her accumulated funds already from the age of 50, or older (an increase to 55 is planned). However, at the latest by the age of 75, an annuity has to be purchased from the remaining funds to provide a secure lifetime income. In the UK, the monthly, quarterly or yearly benefit during the drawdown period must be chosen between limits that are set by a government actuary each year. The policyholder can only vary the income within these limits. More information on income drawdown can be found through the consumer information web pages of the UK Financial Services Authority (FSA), or the web pages of the Government Actuary's Department (GAD). Also various private companies in the UK provide online information.

Let us assume that at time $k = 0$ a life of age x (e.g. $x = 55$) has the amount S_0 invested in a pension fund. Without consumption (income drawdown) the fund would develop like S, where S_0 and S are defined as before. Assume that the policyholder wants to draw income yearly and finally invests the remaining funds A_K into an annuity at age $x + K$ (e.g. $K = 10$, $x + K = 65$). If we further assume that the policyholder draws income using a martingale consumption strategy as in Example 1 ($c = 1$, "consume every year what you can expect to consume the following year"), then

$$X_k = d\mathbf{E}[A_K|\mathcal{F}_k], \quad k = 0, \dots, K. \tag{74}$$

The amount A_K (A_{10}) left at the end of the drawdown period is important since it is used to purchase the obligatory annuity and therefore determines the amount of the annuity paid later on. In international actuarial notation, $\ddot{a}_{x+K}$ ($\ddot{a}_{65}$) denotes the expected present value (and therefore actuarial price) of an annuity of one currency unit annually in advance for a life aged $x + K$ (65). Estimated values of $\ddot{a}_{x+K}$ can be found in professionally used life tables, e.g. $\ddot{a}_{65} = 13.666$ in the PMA92C20-table of the Actuarial Profession in the UK for a fixed effective annual interest rate of 4%. The annual amount of the

annuity purchased at age $x + K$ for a sum of A_K is given by $B = A_K / \ddot{a}_{x+K}$. By (74),

$$X_k = d \ddot{a}_{x+K} \mathbf{E}[B | \mathcal{F}_k], \quad k = 0, \dots, K. \tag{75}$$

Once we have picked a value $d \in [0, 1]$, the martingale strategy is fully determined. For instance, under the assumptions of Prop. 2 one could then derive X_0, and pursue the strategy. But what might be a reasonable choice of d? For example, $d = 1/\ddot{a}_{x+K}$ would mean that a rate X_k at any time k is exactly the expected annual benefit of the annuity that will finally be bought at time K (from the funds A_K).

Especially the mentioned GAD, which has to determine the upper limit of what a policyholder is allowed to consume each year, could make use of (75). Obviously, X_0 depends on d. Provided the GAD decided on an upper consumption limit L, it is, if L is not too large, possible to determine d_L such that $X_0(d_L) \approx L$. By (75), $L/(d_L \ddot{a}_{x+K})$ would then be the expected amount of the annuity, $\mathbf{E}[B]$, provided a "smooth" consumption strategy (martingale) was pursued from then on. The GAD could use $L/(d_L \ddot{a}_{x+K})$ as a benchmark value and lower L if $L/(d_L \ddot{a}_{x+K})$ was too small. The same procedure could be applied in the submartingale case ($c < 1$), allowing for consumption rates that grow on average.

8 Smooth bonus for a closed with-profits fund

An endowment policy with term K years is a life insurance product that pays the policyholder the sum assured (SA), e.g. $SA = 100,000$, at the end of the year of death if the policyholder dies before K, or it pays SA on survival to K. We assume now that at time 0 N_0 of such policies are gathered in a closed with-profits fund. The fund is closed in the sense that no new policies are issued and no premiums are paid after time 0. We denote the actuarial value (e.g. the expected present value) of such a contract at time 0 for a life aged x by $SA \times A_{x:\overline{K}|}$ ($0 < A_{x:\overline{K}|} < 1$). With this amount, we assume, the policy can be hedged sufficiently by buying corresponding securities, or even by buying reinsurance, i.e. selling the liabilities to another insurance company. Furthermore, assume that the funds belonging to one such policy are $SA \times A_{x:\overline{K}|} + B$ ($B > 0$) at time 0. Since B is not directly needed to secure the liabilities, this amount is invested in a stochastic asset R, represented by a strictly positive stochastic process, $(R_t)_{0 \le t \le K}$ ($R_0 \equiv 1$). The amount B belongs to the insured and must therefore be redistributed over time. One way to achieve this is to annually declare a bonus rate b_k such that the current sum assured for the survivors at this time is increased by $b_k \times SA$. Our goal is now to achieve that the process of bonus rates $b = (b_k)_{k=0,\dots,K}$ has a certain property, e.g. is "smooth" in the sense of being a martingale. For this purpose we will apply Lemma 2.

Let N_k denote the number of survivors to time k. We make the technical

assumption $N_k > 0$. Define now

$$F_k = SA \times N_k \times A_{x+k:\overline{K-k}|} \,, \quad k = 0, \ldots, K, \qquad (76)$$

and

$$S_k = \frac{N_0 \times B \times R_k}{F_k}, \qquad (77)$$

where we assume that $A_{x+k:\overline{K-k}|}$ is strictly positive ($k = 0, \ldots, K$). Note that $\tilde{S}_k := F_k S_k = N_0 \times B \times R_k$ is the value of the investment of the N_0 individual amounts B in R without consumption. $(\tilde{S}_t)_{0 \le t \le K}$ is therefore from what the insurer can consume to pay bonus. Choose now a PHPP for b consuming from S (note: S is artificial). Assume now that b_k is the bonus rate guaranteed at k. Then,

$$N_k \times (b_k \times SA) \times A_{x+k:\overline{K-k}|} = F_k b_k = \tilde{b}_k \qquad (78)$$

must be taken from the R-investment at time k (identical to $\tilde{S}$ with consumption, or $\tilde{A}$) to hedge or reinsure the additional insurance benefit $b_k SA$, that is guaranteed to all N_k surviving policyholders by declaring the bonus rate b_k. The problem is now to clarify whether this is possible at any k without ever getting into negative numbers. The answer is given by Lemma 2. Any b regular w.r.t. S guarantees regularity of $\tilde{b}$ w.r.t. $\tilde{S}$. Therefore, b defines a bonus strategy that is (a) smooth, e.g. when using a martingale with total consumption at K, and (b) fair for the insured and the insurer, since every policyholder gets in expectation the same amount paid as bonus, all money available for bonus (=R-investment) is redistributed until K, and regularity ensures that the insurance company can sufficiently hedge the liabilities evolving from this bonus strategy.

9 Conclusion

This paper has introduced a general method to describe properties of consumption processes for stochastic money accounts by means of certain positively homogeneous operators ("positively homogeneous projection properties"). The main result, Theorem 1, not only provided an existence theorem, but at the same time a backward-forward recursion method to explicitly calculate such strategies in many cases. Relative consumption rates played a predominant role, as opposed to absolute consumptions rates that were actually aimed for. The properties seemed to be rather restrictive at first sight. However, a result for finite spaces showed that almost any reasonable consumption strategy (i.e. also such possibly obtained from an optimality criterion) can be described and constructed by these properties, proving, in fact, how general the approach is. Furthermore, the conditional expectation and conditional quantile strategies that we described in examples are suitable for applications and are easy to communicate, even to non-mathematicians. The second part of the paper focused on the case when relative rates of consumption are deterministic.

In this case, we could derive much more explicit results that are particularly suitable for applications with i.i.d. growth rates for the original investment dynamics (e.g. log-Lévy processes). Two numeric examples, binomial tree and geometric Brownian motion, illustrated the results and the potential of the described methods. In particular, the potential to plan and provide "smooth" or "smoothly growing" income under adverse market conditions should be emphasized. We further showed that similar results and methods can be derived for perpetual consumption problems. Finally, the sections on income drawdown and on bonus for closed with-profits funds showed applicability to actuarial problems.

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