A FOKKER-PLANCK CONTROL FRAMEWORK FOR STOCHASTIC SYSTEMS*

M. ANNUNZIATO† AND A. BORZÌ‡

Abstract. A new framework for the optimal control of probability density functions (PDF) of stochastic processes is reviewed. This framework is based on Fokker-Planck (FP) partial differential equations that govern the time evolution of the PDF of stochastic systems and on control objectives that may require to follow a given PDF trajectory or to minimize an expectation functional.

Corresponding to different stochastic processes, different FP equations are obtained. In particular, FP equations of parabolic, fractional parabolic, integro parabolic, and hyperbolic type are discussed. The corresponding optimization problems are deterministic and can be formulated in an open-loop framework and within a closed-loop model predictive control strategy. The connection between the Dynamic Programming scheme given by the Hamilton-Jacobi-Bellman equation and the FP control framework is discussed. Under appropriate assumptions, it is shown that the two strategies are equivalent. Some applications of the FP control framework to different models are discussed and its extension in a mean-field framework is elucidated.

This is a preprint of the paper
Mario Annunziato and Alfio Borzì
A Fokker–Planck control framework for stochastic systems
EMS Surveys In Mathematical Sciences, 5 (2018), 65 - 98. (DOI: 10.4171/EMSS/27)

Key words. Fokker-Planck equations, Hamilton-Jacobi-Bellman equation, stochastic process, hybrid systems, Markov renewal processes, probability density function, mean-field approach, optimal control theory, model predictive control, numerical analysis, scientific computing.

AMS subject classifications. 34A38, 35F21, 35K57, 35Q84, 49J20, 49K20, 49L20, 65C20, 65H10, 60H25, 65K15, 65M12, 65M55.

1. Introduction.

The modeling and control of stochastic processes is a very active research field because of many present and envisioned application in finance, sciences, and technology. We refer to different stochastic processes as Itō, subdiffusion, jump, and piecewise-deterministic models as stochastic systems. The research on stochastic systems is sustained by a well established mathematical theory [13, 48, 67, 114, 120] that provides tools for the investigation of the time evolution of random quantities in many practical cases. In particular, one of the main tools for analysing stochastic processes is the fact that the evolution of the probability density function (PDF) associated to the state of these processes is governed by a linear time-dependent partial differential equation (PDE), starting from a given initial PDF configuration; see, e.g., [52, 105, 107]. Indeed, the structure of this linear PDE depends on the features of the process as we illustrate in this paper. In particular, we remark that these so-called Fokker-Planck (FP) equations can be derived from the Chapman-Kolmogorov equation for the transition probability function of a Markov process. A possible extension of FP equations to model non-Markovian processes is also possible and results in PDEs with a special structure. Notice that FP equations have been investigated in many works and in the literature they are named after many famous scientists including Kolmogorov, Fokker, Planck, Einstein, and Smoluchowski. We use

*Supported in part by the EU Marie Curie International Training Network Multi-ITN STRIKE Projekt 'Novel Methods in Computational Finance' Grant Agreement Nr. 304617.

†Dipartimento di Matematica, Università degli Studi di Salerno, Via G. Paolo II 132, 84084 Fisciano, Italy (mannunzi@unisa.it)

‡Institut für Mathematik, Universität Würzburg, Emil-Fischer-Strasse 30, 97074 Würzburg, Germany (alfio.borzi@mathematik.uni-wuerzburg.de).

the term FP equation for convenience and refer to, e.g., [52] for an historical account of the subject.

However, while the FP equation has been considered for long time to model the time evolution of stochastic processes, it is only recently that a control framework for these processes based on the FP equation has been proposed; see [5] for an earlier publication. Following this publication, the Authors of this review have considerably developed this topic [5, 6, 7, 8, 9, 10, 12, 29, 65, 66, 95, 96, 108, 109, 116] and witnessed a surge of research work in this field focusing on FP models and related control problems; see, e.g., [25, 28, 56, 57, 58, 74, 77, 78, 125, 127].

For this reason, we believe that the review, presented in this paper, of these recent developments in an emerging field of applied mathematics is timely and appropriate and may boost further research on this subject.

In the following, we illustrate different stochastic systems and the corresponding FP equations. We start our discussion considering the Itō stochastic model. It is a continuous-time stochastic process described by the following multidimensional stochastic differential equation (SDE) with given initial condition

$$\begin{cases} dX_t = b(X_t, t)\, dt + \sigma(X_t, t)\, dW_t \\ X_{t_0} = X_0, \end{cases} \tag{1.1}$$

where the state variable $X_t \in \Omega \subseteq \mathbb{R}^d$ is subject to deterministic infinitesimal increments driven by the vector valued drift function b, and to random increments proportional to a multi-dimensional Wiener process $dW_t \in \mathbb{R}^m$, with stochastically independent components. We assume that the dispersion matrix $\sigma \in \mathbb{R}^{d\times m}$ is full rank. Concerning the existence and uniqueness of solutions X_t to (1.1), for a given realization of W_t; see, e.g., [67, 114]. As discussed in [60] Remark 2.1, pag. 161, we assume that the space of the stochastic processes is the one adapted to the filtration generated by the Wiener process.

The FP equation associated to the process (1.1) is given by

$$\partial_t f(x,t) - \sum_{i,j=1}^{d} \partial^2_{x_i x_j} (a_{ij}(x,t)\, f(x,t)) + \sum_{i=1}^{d} \partial_{x_i} \left(b_i(x,t)\, f(x,t)\right) = 0 \tag{1.2}$$

$$f(x,0) = f_0(x) \tag{1.3}$$

where f denotes the PDF of the process, f_0 represents the initial PDF distribution, and hence $f_0(x) \geq 0$ with $\int_\Omega f_0(x)\, dx = 1$. The diffusion coefficient is given by $a = \sigma\, \sigma^\top/2$, with elements

$$a_{ij} = \frac{1}{2} \sum_{k=1}^{m} \sigma_{ik}\, \sigma_{jk}.$$

Notice that in the FP equation (1.2), the ‘space’ dimension corresponds to the number of components of the stochastic process. We remark that by dealing with (1.2) - (1.3), we are restricting the statistical analysis to those processes that own an absolutely continuous probability measure.

While we focus our discussion on linear FP problems, at this point we mention that there exists a special class of problems with the structure (1.1) that leads to a nonlinear FP extension of (1.2) - (1.3). This class of problems is the focus of the mean-field approach that is discussed in Section 3 below.

The FP problem (1.2) - (1.3) and the following ones, can be defined in bounded or unbounded domains in $\mathbb{R}^d$. Existence and uniqueness of solutions to these problems often relay on the concept of uniform parabolicity. For the case $\Omega = \mathbb{R}^d$, we refer to, e.g., [14, 60, 88, 24] and the references therein. In the case of bounded domains, boundary conditions for the FP model must be chosen that ought to be meaningful for the underlying stochastic process, as for example in the case of absorbing and reflecting barriers [111]. Specifically, an absorbing barrier is one where the process leaves the domain Ω and the corresponding boundary condition for the FP equation corresponds to homogeneous Dirichlet boundary conditions. On the other hand, reflecting barriers let the process remain in Ω and thus the corresponding FP boundary conditions are modelled by the requirement that the flux of probability is zero. For this purpose, notice that the FP equation (1.2) can be written in flux form: $\partial_t f = \nabla \cdot F(x,t;f)$, where the ith component of the flux is given by

$$F_i(x,t;f) = \sum_{j=1}^{d} \partial_{x_j}(a_{ij}(x,t)\, f(x,t)) - b_i(x,t)\, f(x,t).$$

Therefore zero-flux (reflecting) boundary conditions are given by

$$F \cdot n = 0, \qquad \text{on } \partial\Omega \times (0,T), \tag{1.4}$$

where n is the unit outward normal to $\partial\Omega$.

The stochastic model (1.1) appears in, e.g., the simulation of Brownian motion with drift, as a Langevin equation, and it represents also a basic model in finance. However, in some applications in biology and physics, anomalous diffusion processes are observed that can be modelled by an extension of (1.1). The diffusion process is said to be normal when the variance of the process grows linearly in time, i.e. $\mathrm{Var}(X_t) \propto t$, which is the case of the Wiener process. If the variance grows in time as $\mathrm{Var}(X_t) \propto t^\alpha$, with exponent $\alpha \neq 1$, then the diffusion is said to be anomalous. In particular, a subdiffusion process is described by a state variable $Y(t) \in \mathbb{R}^d$ driven by the following model [91, 121]

$$\left\{ \begin{array}{rcl} Y_t & = & X_{S(t)} \\ dX_\tau & = & b(X_\tau,\tau)\, d\tau + \sigma(X_\tau,\tau)\, dW_\tau \\ X_{\tau_0} & = & X_0. \end{array} \right. \tag{1.5}$$

The inverse-time α-stable subordinator $S(t) \in \mathbb{R}$ is defined as a first-passage time process, $S(t) = \inf\{\tau, U(\tau) > t\}$, where U represents a strictly increasing α-stable Lévy motion, $\alpha \in (0,1)$. Moreover, the processes W_τ and $S(t)$ are assumed to be independent.

By denoting with $f(x,t)$ the PDF for the process $Y(t)$, driven by (1.5), the following fractional FP equation results [91, 93, 94]

$$\partial_t f(x,t) - {}_0D_t^{1-\alpha} \left[\sum_{i,j=1}^{d} \partial^2_{x_i x_j}(a_{ij}(x,t) f(x,t)) - \sum_{i=1}^{d} \partial_{x_i}(b_i(x,t) f(x,t)) \right] = 0$$
$$f(x,0) = f_0(x).$$

In this problem, the operator

$${}_0D_t^{1-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \partial_t \int_0^t (t-s)^{\alpha-1} g(s) ds,$$

represents the fractional Riemann-Liouville derivative. Notice that the non-Markovian process $Y(t)$ results in a nonlocal differential operator in the (fractional) FP equation.

We see that in the Itō model (1.1) and in the subdiffusion model (1.5), noise is added to a deterministic evolution equation to model random perturbations. On the other hand, random perturbations can also be modelled by events that change the deterministic motion at isolated instants of time as in, e.g., queueing and renewal processes [45].

A large effort has been put in the investigation of the dynamics of jump-diffusion processes; see, e.g., [13]. In this case, the time evolution of the state process X_t can be described by a stochastic differential equation that adds to the Itō model a compound Poisson process $P_t \in \mathbb{R}^d$ as follows

$$dX_t = b(X_t,t)dt + \sigma(X_t,t)dW_t + dP_t, \tag{1.6}$$

where P_t is exponentially distributed in time with $\lambda e^{-\lambda \Delta t}$, and λ represents the rate of jumps. In this process, the amplitude of the state jumps is distributed according to a PDF function $g = g(x)$.

The evolution of the PDF of (1.6) is modelled by a FP partial-integro differential equation [65], whose integral part is due to the compound Poisson process P_t, as follows

$$\begin{aligned}\partial_t f(x,t) - &\sum_{i,j=1}^{d} \partial^2_{x_i x_j}((a_{ij}(x,t)f(x,t)) + \sum_{i=1}^{d} \partial_{x_i}(b_i(x,t)f(x,t)) \\ &= \lambda \int_\Omega [f(x-y,t) - f(x,t)]g(y)dy.\end{aligned} \tag{1.7}$$

Next, we illustrate a less investigated point process where a dynamical system changes its deterministic structure at random points in time following a discrete Markov process. These processes were first discussed in [82, 100], whereas a first mathematical characterization of systems that switch randomly within a certain number of deterministic states at random times is given in [48]. In this reference, the name piecewise-deterministic processes (PDP) appears for the first time. PDP processes may also include stochastic hybrid systems and switching systems; see, e.g., [16, 37, 39, 44, 55].

For our discussion, we consider a class of PDP models described by a state function that is continuous in time and is driven by a discrete state Markov process as follows

$$\dot{X}(t) = A_{\mathcal{S}(t)}(X), \quad t \in [t_0, \infty), \tag{1.8}$$

where $\mathcal{S}(t) : [t_0,\infty) \to \mathbb{S}$ is the Markov process with a discrete set of states $\mathbb{S} = \{1,\dots,S\}$. This process is characterized by two random processes: 1) a Poisson process for the switching times having an exponential PDF of transition events as follows

$$\psi_s(t) = \mu_s e^{-\mu_s t}, \ \text{ with } \int_0^\infty \psi_s(t)\,dt = 1, \tag{1.9}$$

for each state $s \in \mathbb{S}$; and 2) at the jump times, the process $\mathcal{S}(t)$ changes its value based on a stochastic transition probability matrix, $\{q_{ij}\}$, with the following properties

$$0 \le q_{ij} \le 1, \qquad \sum_{i=1}^{M} q_{ij} = 1, \qquad i,j \in \mathbb{S}.$$

Given $s \in \mathbb{S}$, we say that the dynamics is in the (deterministic) state s, and it is driven by the function $A_s : \Omega \to \mathbb{R}^d$, which belongs to the set of functions $\{A_1, \ldots, A_S\}$. The state function satisfies the initial condition $X(t_0) = X_0 \in \Omega$, being in the initial state $s_0 = \mathcal{S}(t_0)$. These models include dichotomic noise, random telegraph processes, transport processes, and binary noise. Further, applications include reacting-diffusing systems [92], biological dispersal [2, 118], non-Maxwellian equilibrium [3, 11, 101], and filtered telegraph signal [104, 123].

Corresponding to the PDP model (1.8), we have the following FP system of first-order hyperbolic PDEs with coupling depending on the stochastic transition matrix as follows [8, 19, 43]

$$\partial_t f_s(x,t) + \partial_x(A_s(x)\, f_s(x,t)) = \sum_{j=1}^{S} Q_{sj}(x)\, f_j(x,t), \qquad s = 1, \ldots, S, \tag{1.10}$$

where Q_{sj}, depending on μ_j and q_{sj}, is given by

$$Q_{sj} = \begin{cases} \mu_j\, q_{sj} & \text{if } j \neq s, \\ \mu_s\, (q_{ss} - 1). \end{cases} \tag{1.11}$$

We see that the FP framework provides a unique bridge between SDEs and PDEs, and many of these PDEs constitute a focus of modern developments in applied mathematics. In fact, notice that FP equations of multi-dimensional stochastic processes give rise to high-dimensional PDEs, also of fractional type; moreover, notice that jump-diffusion processes give rise to integro-PDEs, etc. These are all emerging topics in applied mathematics.

It is the aim of this paper to illustrate a new control strategy for stochastic systems based on the corresponding FP models. As in any other control approach to stochastic processes, the first step in the formulation of a control mechanism is to include control functions in, e.g., the drift and/or dispersion coefficients of the stochastic differential model. Specifically, we focus on the case where the drift coefficient b is a function of a control u. However, the FP control framework accommodates equally well other control mechanisms that may enter in any of the coefficients characterizing the stochastic process and appearing in the corresponding FP models.

The next step in the formulation of any control scheme is to model the objective of the control. In particular, it may be required to drive the random process to follow a desired trajectory or attain a required terminal configuration. In the framework of stochastic optimal control, these tasks are formulated by introducing an objective functional that depends on the state and control variables. For non-deterministic processes the state evolution X_t is random, so that a direct insertion of a stochastic process into a deterministic objective functional results into a random variable. Therefore, to define a deterministic objective, the average on all possible trajectories X_t is required [61]. With this procedure, the following objective is usually considered

$$J(X,u) = \mathbb{E}[\int_0^T L(t, X_t, u(\cdot,t))\, dt + \Psi[X_T]], \tag{1.12}$$

where $\mathbb{E}[\cdot]$ represents the expectation on the measure of the stochastic process. This formulation is omnipresent in almost all stochastic optimal control problems considered in the scientific literature; see, e.g., [61, 102].

We notice that in this approach the control must be aware of all realization of the state at all times. On the other hand, the stochastic process can be characterized by its statistical features, described by the PDF distribution. This fact has motivated much work on different control strategies that consider the ensemble of all possible trajectories. In [62, 81, 84, 119] PDF-based control schemes were proposed, where the objective depends on the PDF of the stochastic state variable. In this way, a deterministic objective results and no average is needed, and along these lines, past scientific literature has dealt with alternative objectives as in [81, 84], where the objective is defined by the Kullback-Leibler distance between the state PDF and a desired one. On the other hand, in [62, 119] a square distance between the state PDF and a desired PDF is considered. Although these works consider deterministic objectives formulated with the PDF, they use stochastic models and the state PDF is obtained by interpolation techniques.

The last conceptually innovative step of using the FP equation to model the evolution of the PDF associated to a stochastic system appears for the first time in [5, 6, 7, 8], where a control framework that considers stochastic control problems from a statistical point of view, with the perspective to drive the collective behaviour of the process, is investigated. This alternative approach reformulates the control problem from stochastic to deterministic, based on the fact that the state of a stochastic process can be completely characterized by the PDF. Notice that solving the FP equation, a time-dependent PDF is obtained that can describe non-equilibrium statistics. We remark that independently of the references above, the possibility of formulating control problems with density function models as the FP equation was mentioned in [30, 31].

From the discussion above, it is clear that the formulation of control objectives in terms of the PDF and the use of the FP equation provide a consistent framework to formulate a robust optimal control strategy for stochastic processes. The working paradigm of the FP-based control of stochastic models is the following. First, one reasonably assumes that the initial PDF of the state variable is known at the initial time, and the state variable X_t evolves according to a stochastic differential model subject to the action of a multidimensional controller u. Corresponding to this controlled model, we have a FP equation that includes the same controls in its coefficients. This FP equation and a PDF-based objective define an open-loop FP optimal control problem whose solution provides the control sought. In this way, the problem of controlling a stochastic process is put in the realm of optimal control of PDE models where many theoretical results and powerful solution tools are available; see, e.g., [26, 89, 117] and references therein. In particular in [5, 6, 7, 8], a model predictive control (MPC) approach [70, 71] is pursued to construct fast closed-loop control schemes for the stochastic systems under consideration. These MPC schemes provide robust controllers that apply equally well to linear and nonlinear models and allow to accommodate different control- and state constraints [71, 79]. Recently, a more extensive theoretical analysis for space-time dependent controls has been presented in [56, 57, 108, 109].

We remark that the direct connection between stochastic models and the related FP equations clarifies also the meaning and choice of different functional dependencies of the control function with respect to the space and time variables. In fact, through the identification $X_t = x$ at time t, we can identify the control entering in the SDE model as $u = u(X_t, t)$ with the control function appearing in the FP equation as $u = u(x, t)$. Thus formally a space-time dependent control function may correspond to a time-dependent feedback law, and this fact immediately suggests a connection between the Hamilton-Jacobi-Bellman (HJB) and the FP control frameworks. In this paper, we discuss this connection and show that the FP-based strategy provides the same optimal control as the HJB method for an appropriate choice of the objectives.

In the following section, we introduce the FP control framework within the Lagrange formalism and discuss the optimality systems corresponding to specific choices of the objectives. We illustrate how the solution to the optimality system with forward- and adjoint FP equations and an optimality condition equation characterizes the optimal control solution. We also comment on appropriate discretization schemes for the FP equation. These schemes provide stable and accurate approximation, while guaranteeing positivity and conservation of total probability. In Section 3, we discuss the case of N coupled SDEs, with a special structure of the coupling, and discuss the limit $N \to \infty$. We show that with this limit a mean-field SDE is obtained whose PDF is governed by a nonlinear FP model, which we use to discuss the case of nonlinear FP control problems. In Section 4, the connection between the HJB control framework and the FP control strategy is discussed. In Section 5, we complete our discussion on the FP optimization strategy reviewing works on inverse problems (parameter identification, calibration) governed by the FP equation. Section 6 is devoted to applications. We consider the control of a quantum spin system described by a stochastic Lindblad master equation, the control of motion of a pedestrian in a crowd, and the optimal control of a PDP system arising in biology. A section of conclusions completes this work.

2. The Fokker-Planck control framework. In this section, we illustrate the Fokker-Planck control framework for different stochastic processes and discuss the derivation of the optimality systems characterizing the solutions to the FP optimal control problems. The formulation of the FP optimal control of a stochastic system requires the following terms: 1) The definition of a (or many) control function u that represents the driving mechanism of the stochastic system; 2) The FP equation corresponding to the stochastic system, that includes the control function, as parameter modelling the PDF of the controlled system, denoted by $f(u)$; 3) The objective that models the purpose of the control on the system.

We denote with u the control function belonging to a closed and convex set of admissible controls $U_{ad} \subset U$, where we assume that U is a real Hilbert space with inner product and norm denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively. The PDF of the system as a function of u is denoted by $f(u) \in F$, where F is a Hilbert space with inner product and norm denoted by $((\cdot, \cdot))$ and $||\cdot||$. The PDF f is given by the solution of the FP problem, which is formally expressed as $c(f, u) = 0$, including boundary- and initial conditions, where $c : F \times U \to F^*$, where F^* is the dual of F, and we assume that c is Frechét-differentiable. It is required that the solution of this equation with given u defines a continuous mapping $u \to f(u)$. Let us denote its first derivative at u in the direction δu by $f'(u, \delta u)$. It is characterized as the solution to the linearized equality constraint $c_f(f, u)\, f'(u, \delta u) + c_u(f, u)\, \delta u = 0$.

A cost functional is formally given by

$$J(\cdot, \cdot) : F \times U \to \mathbb{R}.$$

We assume that $J(f, u)$ is Frechét-differentiable, and using the mapping $u \to f(u)$, we can define the reduced cost functional $\hat{J}(u) = J(f(u), u)$. In particular, one can consider objectives of the following form

$$J(f, u) = h(f) + \nu\, g(u),$$

where $\nu \geq 0$ is the weight of the cost of the control, and h and g are required to be bounded from below and $g(u) \to \infty$ as $|u| \to \infty$.

A general formulation of the FP optimal control problem follows the same guidelines of any optimal control with PDE models; see, e.g., [26, 89, 117]. We have

$$\min_{u \in U_{ad}} J(f,u)$$
$$\text{s.t.} \quad c(f,u) = 0.$$

Equivalently, we have: Find $u \in U_{ad}$ such that $\hat{J}(u) = \inf_{v \in U_{ad}} \hat{J}(v)$.

A local solution $u \in U_{ad}$ to the optimal control problem can be characterized by the first order optimality condition as follows

$$\hat{J}'(u, v-u) \geq 0 \qquad \text{for all } v \in U_{ad}.$$

Now, to estimate this inequality, one introduces $p \in F$ as the unique solution to the following adjoint FP equation

$$c_f^*(f,u)\, p + h'(f) = 0,$$

where the adjoint operator $c_f^* : F \to F^*$, and p is the Lagrange multiplier, also called the adjoint variable. Using $c_f(f,u)\, f'(u, \delta u) + c_u(f,u)\, \delta u = 0$ and $\delta u = v - u$, we have

$$\hat{J}'(u, v-u) = (\nu\, g'(u) + c_u^*\, p\,, v-u) \geq 0 \text{ for all } v \in U_{ad}. \tag{2.1}$$

In the case $U_{ad} = U$, this condition becomes $\hat{J}'(u) = 0$.

Summarizing, the solution to the FP optimal control problem is characterized by the following optimality system

$$\begin{aligned} c(f,u) &= 0 \\ c_f(f,u)^*\, p + h'(f) &= 0 \\ (\nu\, g'(u) + c_u^*\, p\,, v-u) &\geq 0 \quad \text{ for all } v \in U_{ad}. \end{aligned} \tag{2.2}$$

We remark that the FP equation is a particular instance of the forward Kolmogorov equation and the adjoint FP equation resembles the backward Kolmogorov equation. In the FP optimality system (2.2), we refer to the third equation as the optimality condition, and $\nabla \hat{J}(u) = \nu\, g'(u) + c_u^*\, p(u)$ represents the reduced gradient.

Another way to derive the optimality system is by introducing the Lagrangian function

$$L(f,u,p) = J(f,u) + \langle c(f,u), p \rangle_{F^*,F}.$$

By formally equating to zero the Frechét derivatives of L with respect to the triple (f,u,p), we obtain the optimality system; see, e.g., [89, 97, 117]. Inequality constraints are treated by adding Lagrangian multipliers and corresponding complementarity conditions.

We remark that the FP control framework results in FP equations with control in the coefficients and, in this case, proving existence and uniqueness of the solution of this optimal control problem is a difficult task. The case of controlled drift of the form $b(x,u) = -\gamma(x) + u$ as been studied in [6]. By following the arguments in [1, 117] and subject to appropriate hypothesis on the structure of the FP control problem, existence of the optimal solutions is proved in [56, 57, 106, 108, 109, 112]. Further, because the control mechanism enters non-linearly in FP control problems, it is in general not possible to prove uniqueness of optimal control solutions; however, see [6] for a special case. Notice that solutions of optimality systems represent only extremal points and additional second-order conditions must be satisfied to guarantee that they are the minima sought; see, e.g., [36, 117] for additional details.

Now, we illustrate the FP control framework for a Itō process. Consider the problem to determine a control $u = u(x,t)$ such that starting with an initial distribution f_0, the process evolves towards a desired target probability density $f_d(x,t)$ at time $t = T$. We have

$$\min J(f,u) := \frac{1}{2}\|f(\cdot,T) - f_d(\cdot,T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(Q)} \tag{2.3}$$

$$\partial_t f(x,t) - \frac{1}{2}\sum_{i,j=1}^{d} \partial^2_{x_i x_j}(a_{ij}(x,t)\, f(x,t)) + \sum_{i=1}^{d} \partial_{x_i}(b_i(x,t;u)\, f(x,t)) = 0 \tag{2.4}$$

$$f(x,0) = f_0(x). \tag{2.5}$$

The first-order necessary optimality conditions that characterize the optimal solution to (2.3) - (2.5) are given by the following optimality system

$$\begin{aligned}
\partial_t f - \tfrac{1}{2}\textstyle\sum_{i,j=1}^{d} \partial^2_{x_i x_j}(a_{ij}\, f) + \sum_{i=1}^{d} \partial_{x_i}(b_i(u)\, f) = 0 &\quad \text{in } Q,\\
f(x,0) = f_0(x) &\quad \text{in } \Omega,\\
-\partial_t p - \tfrac{1}{2}\textstyle\sum_{i,j=1}^{d} a_{ij}\, \partial^2_{x_i x_j} p - \sum_{i=1}^{d} b_i(u)\, \partial_{x_i} p = 0 &\quad \text{in } Q,\\
p(x,T) = f(x,T) - f_d(x,T) &\quad \text{in } \Omega,\\
\nu\, u_l + \textstyle\sum_{i=1}^{d} p\, \partial_{x_i}(\frac{\partial b_i}{\partial u_l}\, f) = 0 \quad \text{in } Q, \quad l = 1,\dots,\ell &
\end{aligned} \tag{2.6}$$

where $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$.

Notice that the case of piecewise constant controls discussed in [6], in the framework of a MPC procedure, corresponds to the following optimality condition

$$\nu\, u_l + \int_{t_k}^{t_{k+1}} \int_\Omega \sum_{i=1}^{d} p\, \partial_{x_i}\left(\frac{\partial b_i}{\partial u_l}\, f\right) dx\, dt = 0,$$

where (t_k, t_{k+1}) corresponds to a time interval where the control is constant. The case $u = u(t)$ would require to remove the time integration from this formula.

Notice that in (2.3) - (2.5) we have not specified the boundary conditions for the FP equation. In the case of absorbing boundary conditions, $f = 0$ on $\partial\Omega$, the same conditions result for the adjoint variable. On the other hand, flux zero boundary conditions result in homogeneous Neumann conditions for the adjoint variable.

The implementation of the FP control strategy requires discretization schemes that are appropriate for approximating the FP forward and adjoint problems. For this purpose, it appears essential that these approximation schemes guarantee positivity of the FP solution, together with stability and accuracy. In particular, in the case of Itō processes that have a corresponding convection-diffusion Fokker-Planck equation for the PDF, a second-order space discretization scheme that guarantees all these properties is based on an exponential fitting technique that was proposed independently by Scharfetter & Gummel [110] and Chang & Cooper (CC) [38], and analyzed in [6, 32, 33, 95, 65]. This scheme appears also appropriate for the discretization of generalized FP equations, i.e. for fractional FP equations and for FP equations with an integral operator that appears in the case of jump-diffusion processes [65] . The CC scheme has been further analysed in [108, 109] in combination with the alternate direction method. We remark that an additional advantage of the CC scheme is that, consistently to the discretize-before-optimize strategy [26], the transpose of the FP CC stencil provides an appropriate discretization of the adjoint FP equation.

In the case of PDP processes, the Fokker-Planck equation is a system of first order hyperbolic PDEs, and in this case a first-order time-explicit discretization scheme preserving the required structural properties of the PDF solution is discussed in [4, 8]. Further schemes for FP PDP problems are discussed in [42, 54]. Recently, the approximation of FP optimality systems on unbounded domains based on Hermite polynomials has been investigated in [96]. Clearly, the solution of FP optimality systems becomes very challenging when high-dimensional stochastic processes are considered. For this reason, special techniques for solving high-dimensional PDEs are under investigation; see, e.g., [50, 72, 126].

The FP optimal control strategy has been applied successfully to many different systems. Concerning Itō stochastic processes, we refer to [6] for application of the FP control framework to a stochastic Lotka-Volterra model, to [7] for the FP control of a stochastic quantum spin model, to [108, 109] for the control of crowd motion and to [77, 78] for that of the statistics of the spike emission of a neural membrane. For stochastic Itō systems that include random jumps, e.g., for finance modelling, and sparsity of the control we quote [66] and [9] for sub-diffusion models. Concerning other stochastic systems, the FP control approach has been applied successfully also to PDP models such as [8], to the optimization of antibiotic subtilin production [116], and to discrete random walks [29].

3. The mean-field approach. In this section, we discuss a special case of multi-dimensional Itō processes that allows to investigate the limit when the number of dimensions goes to infinity. For this purpose, we explicitly refer to a system of N identical interacting particles whose motion is subject to Wiener noises in a $\mathbb{R}^d$ space, such that the following system results

$$dX_t^i = \frac{1}{N}\sum_{j=1}^{N} b(X_t^i, X_t^j)\,dt + \frac{1}{N}\sum_{j=1}^{N}\sigma(X_t^i, X_t^j)\,dW_t^i \tag{3.1}$$

$$X_{t_0}^i = X_0^i, \qquad i = 1,\dots,N, \tag{3.2}$$

where $X_t^i \in \mathbb{R}^d$ denotes the position (state) of the ith particle. Notice that the structure of (3.1) assumes that in the coefficients an average of particle interactions appears and, for σ =const., each particle is subject to an independent Wiener noise.

A special case of $b(X_t^i, X_t^j)$ has been considered in, e.g., [49] as follows

$$b(x^i, x^j) = -(x^i)^3 + x^i - \theta\,(x^i - x^j),$$

where $\theta > 0$. This choice corresponds to a system of coupled nonlinear oscillators.

As already discussed in the Introduction, in correspondence to (3.1)-(3.2), we have the following dN-dimensional FP equation

$$\partial_t f_N - \frac{1}{2}\sum_{i=1}^{N}\Delta_i\left[\left(\frac{1}{N}\sum_{j=1}^{N}\sigma(x^i,x^j)\right)^2 f_N\right] + \sum_{i=1}^{N}\nabla_i\cdot\left[\left(\frac{1}{N}\sum_{j=1}^{N} b(x^i,x^j)\right) f_N\right] = 0 \tag{3.3}$$

where $f_N = f_N(x,t)$, $x = (x^1, \dots, x^N)$, $x^i \in \mathbb{R}^d$. We denote with Δ_i, resp. ∇_i, the $\mathbb{R}^d$ Laplacian, resp. the $\mathbb{R}^d$ gradient for the variable coordinates of the ith particle.

With (3.3), we formulate a Cauchy problem specifying an initial PDF $f_N(x,0) = f_{0N}(x)$. Since f_{0N} represents the initial PDF distribution, we have $f_{0N}(x) \geq 0$ with $\int_{\mathbb{R}^{dN}} f_{0N}(x)\, dx = 1$. However, the numerical solution of this problem is, in general, practically impossible to compute even for a moderate value of N. On the other hand, a powerful idea in order to reduce the dimensionality of this problem can be borrowed from physics, namely a mean-field strategy [49, 83]. This strategy considers the limit of (3.1)-(3.2) as $N \to \infty$ such that

$$\frac{1}{N}\sum_{j=1}^{N} b(x^i, x^j) \to \mathbb{E}(b(x^i, \cdot)), \tag{3.4}$$

and similarly for σ we have $\frac{1}{N}\sum_{j=1}^{N} \sigma(x^i, x^j) \to \mathbb{E}(\sigma(x^i, \cdot))$. If these limits hold, then the stochastic differential equations (3.1) appear as decoupled and equivalent to each other in the sense that any of the X_t^i represents the same process.

The validity of (3.4) has been rigorously discussed in, e.g., [27, 49, 115]. In particular, the following empirical measure process

$$X_N(A,t) := \frac{1}{N}\sum_{j=1}^{N} \mathbb{1}_A(X_t^j), \tag{3.5}$$

where A denotes any Borel set of $\mathbb{R}^d$ and $\mathbb{1}_A(\cdot)$ is the indicator function of A, is proved converge to a unique deterministic measure $\mu_t(A)$.

We remark that the above results are valid under the condition of indistinguishability, which means that the probability law (3.5) is invariant under exchange of particles. This is possible if the initial conditions X_0^i are independently and identically distributed and all the drift and dispersion functions are the same and symmetric under exchange of particles; see, e.g., [115].

Based on these consideration, in the limit $N \to \infty$, one considers the following Itō process, where X denotes any of the X^i. We have

$$dX_t = \mathbb{E}_{\mu_t}\left[b(X_t, \cdot)\right]\, dt + \mathbb{E}_{\mu_t}\left[\sigma(X_t, \cdot)\right]\, dW_t \tag{3.6}$$

$$X_{t_0} = X_0. \tag{3.7}$$

As in [27] and under suitable conditions on b and σ, the measure μ becomes absolutely continuous and we can write $\mu_t(dx) = f(\cdot,t)\, dx$, where f is the time dependent PDF of (3.6)-(3.7). Corresponding to this process, we have the following mean-field FP model

$$\partial_t f(x,t) - \frac{1}{2}\Delta\left[f(x,t)\left(\int_{\mathbb{R}^d}\sigma(x,y)f(y,t)dy\right)^2\right] + \nabla\cdot\left[f(x,t)\left(\int_{\mathbb{R}^d} b(x,y)f(y,t)dy\right)\right] = 0,$$

where Δ, resp. ∇, represent the Laplacian, resp. the gradient, in $\mathbb{R}^d$. Notice that the nonlinear FP equation above can be written in a more compact form as follows

$$\partial_t f(x,t) - \frac{1}{2}\Delta\left[f(x,t)\left(\mathbb{E}_{f_t}\left[\sigma(x,\cdot)\right]\right)^2\right] + \nabla\cdot\left[f(x,t)\left(\mathbb{E}_{f_t}\left[b(x,\cdot)\right]\right)\right] = 0. \tag{3.8}$$

However, the explicit form better shows the non-linearity of the mean-field FP equation with respect to its PDF solution f.

In [27], it is proved and demonstrated numerically by Monte Carlo simulation that the empirical PDF obtained with (3.1)-(3.2) converges to the PDF given by (3.8) with a rate of $1/\sqrt{N}$. Therefore we can state that

$$\int_{\mathbb{R}^{d(N-1)}} f_N(x, x^2, \ldots, x^N, t)\, dx^2 \ldots dx^N \approx f(x,t),$$

for N sufficiently large and any choice of the $(N-1)$ integration variables.

Now, following the focus of our work, we discuss the presence of a control function in (3.1)-(3.2). In fact, the condition of indistinguishability suggests that one should consider a unique control function entering in the drift and also only one entering in the dispersion coefficient. For simplicity, we discuss only the former case; the extension to the case of control in the dispersion is similar.

Let us augment the drift in (3.1) by a control function $u = u(x,t)$ as follows: $b = b(u(x^i,t); x^i, x^j)$. With this setting, and following the above discussion, we obtain a controlled mean-field FP equation as follows

$$\partial_t f(x,t) - \frac{1}{2}\Delta\left[f(x,t)\,\mathbb{E}_{f_t}\left[\sigma(x,\cdot)\right]^2\right] + \nabla\cdot\left[f(x,t)\,\mathbb{E}_{f_t}\left[b(u(x,t);x,\cdot)\right]\right] = 0. \tag{3.9}$$

Next, we define a class of cost functionals, for the N-particle setting, that appears appropriate in our mean-field framework. We have

$$J_N(f_N,u) = \int_0^T \int_{\mathbb{R}^{dN}} \left(\frac{1}{N}\sum_{j=1}^N \ell(x^j, u(x^j,t))\right) f_N(x^1,\ldots,x^N,t)\, dx^1 \ldots dx^N. \tag{3.10}$$

This functional models the purpose of the control and its cost.

Now, we can exploit the symmetric structure of our evolution problem to obtain the following limit objective

$$J(f,u) = \int_0^T \int_{\mathbb{R}^d} \ell(x,u(x,t))\, f(x,t)\, dxdt. \tag{3.11}$$

Therefore, within the FP control framework, we can determine the optimal control u by solving an optimization problem that requires to minimize (3.11) subject to the differential constraint given by (3.9). In particular, considering the case of a constant σ, the adjoint mean-field FP equation for minimizing (3.11) subject to (3.9) is given by

$$\begin{aligned}\partial_t p(x,t) + \frac{\sigma^2}{2}\Delta p(x,t) + \nabla p(x,t)\cdot\int_{\mathbb{R}^d} b(u(x,t);x,y) f(y,t)\, dy \qquad (3.12)\\ + \int_{\mathbb{R}^d} \left(b(u(y,t);y,x)\cdot\nabla p(y,t)\right)\, f(y,t)\, dy + \ell(x,u(x,t)) = 0,\end{aligned}$$

and the terminal condition $p(x,T) = 0$.

With this adjoint variable and without bounds on the control, we obtain the following optimality condition

$$f(x,t)\left(\nabla p(x,t)\cdot\int_{\mathbb{R}^d} \partial_u b(u(x,t);x,y)\, f(y,t)\, dy + \partial_u \ell(x,u(x,t))\right) = 0. \tag{3.13}$$

While we elaborate further on this result in the next section, we can already point out one of the important outcomes of the mean-field approach for determining an optimal control to the N-particle problem (3.1)-(3.2) with objectives given by $\mathbb{E}_{f_N}[\ell(X,u)]$. In fact, while this problem is intractable due to its high dimensionality, We could solve the problem of minimising (3.11) subject to (3.9) and find a control u that is the optimal one when $N \to \infty$.

4. The connection between the HJB and FP control frameworks. In this section, we illustrate the connection between the FP control framework [5, 6, 8] and the Hamilton-Jacobi-Bellman (HJB) control strategy [22, 60, 90, 113]. The present discussion outlines some of the results in [10] with some additional remarks concerning a Merton portfolio problem, the issue of chosing boundary conditions in HJB problems, and the mean-field framework. Our purpose is to show that the HJB control approach emerges naturally from the FP control framework when considering cost functionals of expectation type.

Consider the following d-dimensional controlled Itō stochastic process

$$\left\{ \begin{array}{ll} dX_t & = b(X_t, u(X_t,t))dt + \sigma(X_t)\, dW_t, \qquad t \in (t_0, T] \\ X_{t_0} & = x_0 \end{array} \right. \tag{4.1}$$

We denote with $\mathcal{A}$ the set of Markovian controls that contains all jointly measurable functions u with $u(x,t) \in A \subset \mathbb{R}^l$. Controls of this kind are called Markov control policies [60].

In a closed-loop control setting, the function u uses the current value X_t to affect the dynamics of the stochastic process by adjusting the drift function. Corresponding to (4.1), we consider the following functional

$$C_{t_0,x_0}(u) = \mathbb{E}[\int_{t_0}^{T} \ell(X_s, u(X_s,s))ds + g(X_T) \ \mid \ X_{t_0} = x_0], \tag{4.2}$$

which is a conditional expectation to the process X_t taking the value x_0 at time t_0. We refer to the functions ℓ and g as the running cost and the terminal cost functions, respectively.

Now, the optimal control u^* that minimizes $C_{t_0,x_0}(u)$ for the process (4.1) is given by

$$u^* = \operatorname{argmin}_{u\in\mathcal{A}} C_{t_0,x_0}(u). \tag{4.3}$$

Further, we define the following value function

$$q(x,t) := \min_{u\in\mathcal{A}} C_{t,x}(u) = C_{t,x}(u^*). \tag{4.4}$$

The following theorem states that q is the solution to a HJB equation; see, e.g., [22, 60].

THEOREM 4.1. *Assume that X_t solves (4.1) with a control function u and that the function q defined by (4.4) is bounded and smooth. Then q satisfies the following HJB equation*

$$\left\{ \begin{array}{l} \partial_t q + H(x,t,Dq,D^2q) = 0, \\ q(x,T) = g(x), \end{array} \right. \tag{4.5}$$

with the Hamiltonian function

$$H(x,t,Dq,D^2q) := \min_{v\in A}[\sum_{i=1}^{d} b_i(x,v)\partial_{x_i} q(x,t) + \sum_{i,j=1}^{d} a_{ij}(x)\partial^2_{x_i x_j} q(x,t) + \ell(x,v)]. \tag{4.6}$$

Notice that, assuming differentiability with respect to the control function in (4.6), the optimal control u^* satisfies at each time t and for each x the following optimality condition

$$\sum_{i=1}^{d} \partial_u b_i(x, u^*(x,t)) \partial_{x_i} q(x,t) + \partial_u \ell(x, u^*(x,t)) = 0. \tag{4.7}$$

As in the FP case, existence and uniqueness of solutions to the HJB equation often involve the concept of uniform parabolicity; see [14, 46, 60]. If this non-degeneracy condition holds, results from the theory of PDEs of parabolic type imply existence and uniqueness of solutions to the HJB problem (4.5) with the properties required in the Verification Theorem [60].

Now, we discuss the Fokker-Planck optimal control strategy based on the same optimization setting. In fact, we start from the functional (4.2) and notice that the expectation is performed with respect to the probability measure induced by the process X_t of (4.1). Therefore, following our assumption that this process owns an absolutely continuous probability measure, we can explicitate the expectation in (4.2) in terms of the PDF governed by the FP problem with initial density distribution $f_0(x) = \delta(x - x_0)$ at $t = t_0$. Thus, the functional (4.2) becomes

$$J(f(u), u) := \int_{t_0}^{T} \int_{\mathbb{R}^d} \ell(x, u(x,s))\, f(x,s)\, ds\, dx + \int_{\mathbb{R}^d} g(x) f(x,T)\, dx. \tag{4.8}$$

Therefore the optimization problem (4.3) can be equivalently stated as a FP optimal control problem where an optimal control u in the admissible set $\mathcal{A}$ is sought that minimizes (4.8). In doing this, we are identifying the chosen admissible set of Markov control policies with the admissible set of controls in the FP optimal control formulation.

Next, to characterize the optimal FP solution to this problem, we introduce the following Lagrange function

$$\begin{aligned} \mathcal{L}(f, p, u) := & \int_{t_0}^{T} \int_{\mathbb{R}^d} \ell(x, u(x,s)) f(x,s) dx ds + \int_{\mathbb{R}^d} g(x) f(x,T) dx \\ & + \int_{t_0}^{T} \int_{\mathbb{R}^d} p(x,s) \big[-\partial_s f(x,s) - \sum_{i=1}^{d} \partial_{x_i} (b_i(x, u(x,s)) f(x,s)) \\ & + \sum_{ij=1}^{d} \partial_{x_i x_j} (a_{ij}(x) f(x,s)) \big] dx ds. \end{aligned} \tag{4.9}$$

Thus, we obtain that the optimal control solution is characterized as the solution to the following optimality system

$$\begin{aligned} & -\partial_t f(x,t) - \textstyle\sum_{i=1}^{d} \partial_{x_i} (b_i(x, u(x,t)) f(x,t)) + \sum_{ij=1}^{d} \partial_{x_i x_j} (a_{ij}(x) f(x,t)) = 0 \\ & f(x, t_0) = f_0(x), \end{aligned} \tag{4.10}$$

$$\begin{aligned} & \partial_t p(x,t) + \textstyle\sum_{i=1}^{d} b_i(x, u(x,t)) \partial_{x_i} p(x,t) + \sum_{ij=1}^{d} a_{ij}(x) \partial_{x_i x_j} p(x,t) + \ell(x, u(x,t)) = 0 \\ & p(x,T) = g(x), \end{aligned} \tag{4.11}$$

and

$$f(x,t)\left(\sum_{i=1}^{d}\partial_u b_i(x,u(x,t))\partial_{x_i}p(x,t)+\partial_u \ell(x,u(x,t))\right)=0. \tag{4.12}$$

Notice that a sufficient condition for (4.12) to hold is that the optimality condition (4.7) for the minimization of the Hamiltonian in the HJB formulation is satisfied. We also remark that, assuming uniform parabolicity of the FP operator, the resulting PDF is almost everywhere non-negative and therefore the HJB condition (4.7) appears to be also a necessary condition for optimality. Indeed, the HJB-FP connection can be shown in a broader sense working with the Pontryagin's maximum principle framework that, in fact, can be proven using dynamic programming and the related HJB equation; see, e.g., [18].

The result above demonstrates that we can identify the FP Lagrange multiplier p with the HJB value function q, since at optimality, the p and q differential problems coincide. Further, using (4.7) we could replace the optimal control u in terms of p in the backward FP adjoint equation (4.11) and obtain the HJB equation in a nonlinear form that is also common in the literature. Therefore the control u does not depend explicitly on the density f and this fact explains why the feedback control is based only on the value function.

The investigation of the HJB-FP connection may result very fruitful in order to extend the HJB approach to accommodate different costs (see Section 6) of the controls and different control constraints. Moreover, we remark that the HJB-FP connection can be instrumental for the development of efficient numerical schemes for solving HJB problems. On the other hand, it provides a framework that helps establishing appropriate boundary conditions for HJB models.

To also illustrate this latter fact, we exploit the HJB-FP connection to obtain an alternative formulation of the optimal Merton portfolio problem [60]. The evolution process corresponding to the Merton Portfolio problem is modelled by the following stochastic differential equation

$$\begin{aligned} &dX_t=\big[((1-u_1)r+u_1\mu)X_t-u_2\big]\,dt+\sigma u_1 X_t\,dW_t,\\ &X_0=x_0, \end{aligned} \tag{4.13}$$

together with the following maximization problem: Find u_1, u_2 such that

$$\max_{u_1,u_2} J(X_t,u_1,u_2):=\mathbb{E}\left[\int_0^T e^{-\beta t} l(u_2)dt\right],$$

where $X_t \geq 0$ represents the wealth of the portfolio. The objective can also be written as follows

$$J(f,u_1,u_2):=\int_0^\infty\int_0^T e^{-\beta t} l(u_2) f(x,t)\,dxdt. \tag{4.14}$$

Here, $r<\mu$ is the interest rate in the riskless market, μ is the expected return, $\sigma>0$ is the volatility of the stock market, $\beta>0$ is the discount rate. Further, $u_1=u_1(x,t)\in[0,1)$, is the fraction of wealth in the risky asset and $u_2=u_2(x,t)\geq 0$ is the consumption rate. The function $l(z)$ is the utility function that satisfies the following conditions: $l(0)=0$, $l'(0^+)=\infty$, $l'(z)>0, l''(z)<0$.

The stochastic problem (4.13) corresponds to the following FP equation

$$\partial_t f(x,t) - \frac{1}{2}\partial_x^2(\sigma^2 u_1^2 x^2 f(x,t)) + \partial_x\left[(((\mu-r)u_1 + r)x - u_2)f(x,t)\right] = 0, \tag{4.15}$$

$$f(x,0) = \delta(x - x_0). \tag{4.16}$$

This problem is defined for all $x > 0$ and the FP equation becomes degenerate at the boundary $x = 0$.

In the FP control framework, we define the following optimization problem

$$\max_{u_1,u_2} J(f, u_1, u_2), \qquad \text{subject to} \qquad (4.15). \tag{4.17}$$

To this FP optimal control problem corresponds an optimality system with (4.15) and the following adjoint FP problem

$$\begin{aligned} -\partial_t q(x,t) =& \frac{1}{2}\sigma^2 u_1^2 x^2 \partial_x^2 q(x,t) + (((\mu-r)u_1 + r)x - u_2(x,t))\partial_x q \\ &+ e^{-\beta t} l(u_2(x,t)) \\ q(x,T) =& 0. \end{aligned} \tag{4.18}$$

Further, the following optimality conditions corresponding to u_1 and u_2 are obtained

$$-[\sigma^2 x^2 u_1(x,t)\partial_x^2 q(x,t) + (\mu - r)x\partial_x q(x,t)]\, f(x,t) = 0,$$
$$[\partial_x q(x,t) - e^{-\beta t} l'(u_2(x,t))]\, f(x,t) = 0.$$

As we have seen, in the present setting with a cost functional of expectation type, at optimality the adjoint variable q, which solves (4.18), corresponds to the value function (4.4). However, the Merton model has a boundary in $x = 0$ where, because of the degeneracy in the FP model, the value of the PDF (or its derivative) cannot be assigned. However, this fact and the derivation of the adjoint problem lead to the boundary condition $q(0,t) = 0$. Furthermore, requiring $u_2(0,t) = 0$ appears possible and compatible with the above boundary conditions. No requirements on $u_1(x,t)$ result and also the case where u_1 models borrowing and shorting of stocks can be tackled. Notice that these conditions are similar to those of (4.17) in Chapter X of [60].

We argue that the FP-HJB connection has general validity as far as linear FP equations and expectation cost functionals are considered. We refer to [10] for an example involving a dichotomic PDP process. The FP-HJB connection has been already exploited in [108, 109] to develop a feedback control-constrained approach for crowd motion and in [116] to model and control the micro-biological process of antibiotic subtilin production.

On the other hand, in the case of nonlinear FP models, it seems difficult to formulate a dynamic principle and thus establish a general FP-HJB connection. However, we can discuss this issue further considering the mean-field framework discussed in the previous section and consider the mean-field FP control problem governed by (3.9) with the objective functional given by (4.8). Also in this case, assuming that σ is a constant function, the optimality condition is given by (3.12) and (3.13), and assuming that the PDF is almost everywhere positive, we obtain

$$\nabla p(x,t) \cdot \int_{\mathbb{R}^d} \partial_u b(u(x,t); x, y)\, f(y,t)\, dy + \partial_u \ell(x, u(x,t)) = 0.$$

Now, to simplify our discussion, consider the case

$$b(u(x,t);x,y) = u(x,t) + \tilde{b}(x,y) \qquad \text{and} \qquad \ell(x,u(x,t)) = \frac{\nu}{2}u(x,t)^2 + \tilde{\ell}(x),$$

then $\partial_u b = 1$ and $\partial_u \ell = \nu u$, and the following optimality condition results

$$\nabla p(x,t) + \nu\, u(x,t) = 0. \tag{4.19}$$

As discussed above, we can replace the optimal control given by this equation ($u = -\nabla p/\nu$) into the adjoint mean-field FP equation (3.12) and obtain the following

$$\begin{aligned}\partial_t p(x,t) + \frac{\sigma^2}{2}\Delta p(x,t) - \frac{1}{2\nu}|\nabla p(x,t)|^2 + \nabla p(x,t)\cdot \int_{\mathbb{R}^d} \tilde{b}(x,y) f(y,t)\,dy \\ - \frac{1}{\nu}\int_{\mathbb{R}^d} |\nabla p(y,t)|^2\, f(y,t)\,dy \\ + \int_{\mathbb{R}^d} \left(\tilde{b}(y,x)\cdot \nabla p(y,t)\right)\, f(y,t)\,dy + \tilde{\ell}(x) = 0,\end{aligned} \tag{4.20}$$

with the terminal condition $p(x,T) = g(x)$.

We see that, in the case when b does not model a two particle interaction but a drift of the form $b(u(x,t);x) = u(x,t) + \tilde{b}(x)$, then the last two integral terms in (4.20) become functions of time only and this adjoint does not reduce to the one in the standard case. This result actually shows that the entire mean-field control framework becomes meaningless if we remove particle interactions. Further notice that in the general mean-field setting, with $b = b(u;x,y)$, the equation (4.20) cannot be considered a true HJB equation because it depends on the forward PDF solution that enters in the integral term. This fact appears to be a common feature of all mean-field control works, including mean-field games [21, 87] where a simplified version of (4.20) is usually considered that still includes f among its coefficients.

Another important and better known connection between the value function $q(x,t)$ obtained solving the Hamilton-Jacobi equation, and the Pontryagin's maximum principle is discussed in [41]. In this reference, it is proved that the adjoint function is equal to the negative of the derivative of the value function with respect to the initial state x. In fact, this correspondence is not in contradiction with the HJB-FP connection established in the framework of control of stochastic models: we obtain the same correspondence if we formulate our FP control problem in terms of the distribution function $F(x,t) = \int_{-\infty}^{x} f(x,t)\,dx$ rather than for the PDF. Thus in the Lagrange function $\mathcal{L}(F,Q,u)$, we find that the multiplier $Q(x,t)$ equals the minus derivative of the value function, i.e. $-\partial_x q(x,t)$.

To illustrate this fact, consider the following one-dimensional Fokker-Planck problem for the distribution function

$$\partial_t F(x,t) = \partial_x(a(x)\partial_x F(x,t)) - b(x,u)\partial_x F(x,t) \tag{4.21}$$

$$F(-\infty,t) = 0, \quad F(+\infty,t) = 1, \tag{4.22}$$

$$F(x,0) = F_0(x) = \int_{-\infty}^{x} f_0(x)\,dx. \tag{4.23}$$

We introduce the Lagrange multiplier $Q(x,t)$ and the Lagrange function

$$\mathcal{L}(F,Q,u) = \int_{\mathbb{R}}\int_0^T \ell(x,u)F_x(x,t)dx\, ds + \int_{\mathbb{R}} g(x)F_x(x,T)\, dx + \int_{\mathbb{R}}\int_0^T Q(x,s)\left(-F_t - bF_x + \partial_x(aF_x)\right) dx\, ds. \tag{4.24}$$

The resulting adjoint equation is given by

$$\partial_t Q(x,t) + \partial_x(b(x,u)Q(x,t)) + \partial_x(a(x)\partial_x Q(x,t)) - \partial_x \ell(x,t) = 0 \tag{4.25}$$

$$Q(x,T) = -\,\partial_x g(x). \tag{4.26}$$

Further, we obtain the following optimality condition

$$\partial_x F(x,t)\left[\partial_u \ell(x,u) - Q(x,t)\partial_u b(x,u)\right] = 0. \tag{4.27}$$

A direct comparison with (4.12) reveals that $Q(x,t) \equiv -\partial_x p(x,t)$ and thus $Q(x,t) = -\partial_x q(x,t)$ where $q(x,t)$ is given in (4.4). Finally, the correspondence with the HJB equation (4.5) can be established as follows

$$-\partial_t Q + \min_{v\in A}[-\partial_x(b(x,v)Q(x,t)) - \partial_x(a(x)\partial_x Q(x,t)) + \partial_x \ell(x,v)] = 0 \tag{4.28}$$

$$Q(x,T) = -\,\partial_x g(x).$$

5. The FP framework and inverse problems. In this section, we discuss the use of the FP control framework for parameter- and functions identification in stochastic models. In fact, following the widely used PDE optimization formulation of PDE inverse problems, one can immediately recognize that in the FP framework, the cost functionals may include measures of discrepancy between simulated PDFs and measured ones, and between measured and simulated stochastic states and their statistical properties. Moreover, these objectives can have additional regularization terms of the functions to be identified.

The FP approach to parameter identification in stochastic models appears to be a much less investigated topic, with only a few contributions in the last decade. A pioneering work in this field can be found in [17]. In this work, the estimation of space-time function coefficients in the FP equation is considered with application to structured population models. The first attempts to use the FP equation and its adjoint to calibrate financial models are presented in [80, 53]. In [80], a FP parameter identification problem with parametrized drift and volatility and a least-square functional of exchange rates is considered. In [53], the identification of local volatility in the Black-Scholes/Dupire equation from market prices of European Vanilla options is considered. Further developments in this field in the context of financial mathematics are discussed in [63].

Another work on parameter identification of drift coefficients in stochastic models using the FP equation is presented in [51]. This work considers the identification of a state-dependent drift with the objective to maximize the likelihood of given observations.

The FP control framework discussed in this paper is also the main focus of the work [12] devoted to the problem of parameter calibration of Lévy processes. In this reference, the Lévy measure is approximated in the linear space of splines and the calibration parameters are the coefficients of the linear combination of compound Poisson processes. The optimal values of these parameters are obtained by solving the problem of minimizing a functional representing a Kullback-Leibler distance of sample measurements.

6. Applications. In this section, we report results of numerical experiments to illustrate the ability of the FP control framework to provide robust control functions that drive stochastic systems to achieve given objectives. Specifically, we discuss some challenging control problems related to Itō stochastic models and deal with the control of a biological PDP problem.

One of the fundamental problems in quantum mechanics is the modelling of the interaction of a quantum system with an external measurement device. For this purpose, the Liouville - von Neumann master equation, governing the evolution of the statistical ensemble of a quantum state, is augmented with a 'dissipator' term and results in a Lindblad master equation [8, 15]. Furthermore, in order to model the action of a measurement operation on the Lindblad dynamics, two types of stochastic Schrödinger equations have been investigated that correspond to measurements in continuous time (diffusion process) and to measurements at different instants of time (jump process) [47, 68].

Based on the results in [7], we illustrate the FP control of a two-level spin system in the diffusive case [122]. In this case, the stochastic master equation governing the orientation of the spin components in spherical coordinates is given by

$$\begin{cases} d\varphi(t) = & B_\varphi(\varphi,\theta,u,v)\,dt + \sigma_{11}(\varphi,\theta)dW_1 + \sigma_{12}(\varphi,\theta)dW_2 \\ d\theta(t) = & B_\theta(\varphi,\theta,u,v)\,dt + \sigma_{21}(\varphi,\theta)dW_1 + \sigma_{22}(\varphi,\theta)dW_2, \end{cases} \tag{6.1}$$

where

$$\begin{aligned}
B_\varphi(\varphi,\theta,u,v) &= \omega + a\cot(\theta)(u\sin(\varphi) + v\cos(\varphi)) \\
B_\theta(\varphi,\theta,u,v) &= -a(u\cos(\varphi) - v\sin(\varphi)) + g\frac{1+\cos(\theta)}{\sin(\theta)}(1-(1+\cos(\theta))\cos(\theta)/4) \\
\sigma_{11}(\varphi,\theta) &= -\sqrt{\tfrac{g}{2}}\frac{1+\cos(\theta)}{\sin(\theta)}\sin(\varphi), \qquad \sigma_{12}(\varphi,\theta) = \sqrt{\tfrac{g}{2}}\frac{1+\cos(\theta)}{\sin(\theta)}\cos(\varphi), \\
\sigma_{21}(\varphi,\theta) &= \sqrt{\tfrac{g}{2}}(1+\cos(\theta))\cos(\varphi), \qquad \sigma_{22}(\varphi,\theta) = \sqrt{\tfrac{g}{2}}(1+\cos(\theta))\sin(\varphi),
\end{aligned}$$

and u and v denote magnetic control fields.

Corresponding to the stochastic Bloch equation (6.1), the following FP equation on the Bloch sphere is obtained

$$\begin{aligned}
\partial_t f = &-\partial_\varphi(B_\varphi(\varphi,\theta,u,v)\,f) - \partial_\theta[(B_\theta(\varphi,u,v)\,f] \\
&+ \frac{g}{4}\partial_\varphi^2\left(\frac{1+\cos(\theta)}{1-\cos(\theta)}\,f\right) + \frac{g}{4}\partial_\theta^2((1+\cos(\theta))^2\,f),
\end{aligned} \tag{6.2}$$

where $\varphi \in [0, 2\pi]$, $\theta \in (0,\pi)$, and the solution $f(\varphi,\theta,t) \geq 0$ is required to be non-negative and its integral on the domain be conserved and normalized to one.

We consider the optimal control problem governed by the FP equation (6.2) with initial PDF given by a narrow normalized bi-dimensional Gaussian placed at the equator at $(\theta,\phi) = (\pi/2,\pi)$ with variances equal to $\sigma = \pi/20$. The aim is to reach a final desired Gaussian PDF target at the south pole with variances $\sigma = \pi/8$ in a time horizon of $T = 4$. In the MPC procedure $N = 10$ time windows are considered.

Now, to demonstrate the ability of the FP framework to control the stochastic model (6.1), we insert the resulting sequence of FP optimal control functions in the stochastic model and perform Monte Carlo simulations. The stochastic trajectories are computed using the Euler-Maruyama scheme [75] and in each realization the same controls are used; see Figure 6.1 for a plot of two controlled stochastic trajectories on

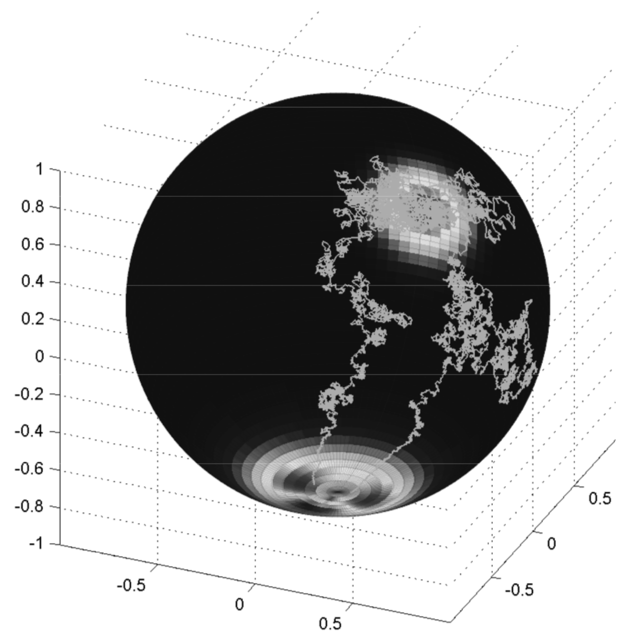

Fig. 6.1. *Two controlled stochastic trajectories starting at the equator and reaching the south pole of the Bloch sphere.*

the Bloch sphere. We obtain that all trajectories on the Bloch sphere, although very different, converge towards the south pole with the desired distribution.

Next, based on results given in [109], we discuss an application of the FP control framework to crowd motion. Efforts to investigate crowd movement, both empirically and theoretically, are motivated by many applications as, e.g., emergency evacuation procedures and efficient planning and designing of structures like bridges, stairways; see, e.g., [20, 34, 64].

Let us consider the motion of a pedestrian in a crowd [109], whose position at time t is denoted with $X(t)$, and its velocity field, depending on position, is given by $u(x,t)$. By assuming that the individual is subject to random collisions, the following stochastic model appears appropriate

$$\begin{aligned} &dX(t) = u(X(t),t)dt + \sigma dW(t), \\ &X(0) = X_0. \end{aligned} \tag{6.3}$$

In correspondence to this SDE model, we have the following FP problem

$$\begin{aligned}\partial_t f(x,t) - \frac{\sigma^2}{2}\sum_{i=1}^n \partial^2_{x_i x_i} f(x,t) + \sum_{i=1}^n \partial_{x_i}(u_i(x,t)f(x,t)) =&0,\\ f(x,0) =&f_0(x).\end{aligned} \tag{6.4}$$

Now, assume that the domain is bounded and convex with reflecting barriers for the process (a closed room). This setting results in flux zero boundary conditions, i.e. $(\frac{\sigma^2}{2}\nabla f - uf)\cdot \hat{n} = 0$, where $\hat{n}$ is the unit outward normal to $\partial\Omega$.

The control framework consists in determining the control velocity u such that the process follows as close as possible a desired trajectory $\bar{x}(t)$ in $(0,T)$ and reaches a desired terminal position x_T at final time. This objective can be formulated as the minimization of the following tracking functional

$$\begin{aligned}J(f,u) = \alpha \int_0^T \int_\Omega V(x-\bar{x}(t))\, f(x,t)\, dx\, dt + \beta \int_\Omega V(x-x_T)\, f(x,T)\, dx +\\ \frac{\nu}{2}\int_0^T \int_\Omega B(u(x,t))\, dx\, dt \quad \alpha,\beta,\nu>0,\end{aligned} \tag{6.5}$$

where V denotes a convex function (potential) of its arguments, and for $B(u)$ we consider the following two choices of the cost of the control

$$B(u(x,t)) = |u(x,t)|^2 + |\nabla u(x,t)|^2 \tag{6.6}$$

$$B(u(x,t)) = (|u(x,t)|^2 + |\nabla u(x,t)|^2) f(x,t). \tag{6.7}$$

These choices are considered in [109] in order to compare a standard setting of $H^1(Q)$ cost of the control with its expectation counterpart that corresponds to a setting where the HJB-FP connection holds. In fact, notice that in the second case (6.7), the adjoint equation reads as follows

$$-\partial_t p(x,t) - \frac{\sigma^2}{2}\sum_{i=1}^n \partial^2_{x_i x_i} p - \sum_{i=1}^n u_i \partial_{x_i} p = -\alpha V(x-x_t) - \frac{\nu}{2}(|u(x,t)|^2 + |\nabla u(x,t)|^2),$$

with $p(x,T) = -\beta V(x-x_T)$. Further, because the objective is linear in f, the functional (6.5) becomes an expectation cost functional. Now, we have that, in the unconstrained-control case, the optimality condition is given by

$$f\left(\nu u_k - \nu\Delta u_k - \frac{\partial p}{\partial x_k}\right) = 0, \quad k = 1\ldots n.$$

Equating to zero the term in parenthesis (an elliptic equation augmented with homogeneous Dirichlet boundary conditions [109]), we obtain a sufficient condition for optimality. In this case, the control u is determined by this optimality condition and the adjoint equation and thus the resulting control can be regarded as a closed-loop control for our stochastic model. Notice that due to the presence of the gradient of the control $|\nabla u(x,t)|^2$ in the cost function, there is no the exact correspondence to the HJB equation discussed in Section 4, rather it represents a suitable extension.

We solve the optimal control problem (6.4), (6.5), (6.6) with the values of $\alpha = 1$, $\beta = 1$, and $\nu = 0.01$ in (6.5). We take $\Omega = (-L,L)\times(-L,L)$ with $L = 6$. Let $x = (x_1,x_2)$. The diffusion parameter is $\sigma = 1$. The initial PDF $f_0(x)$ is given as follows $f_0(x) = \hat{C} e^{-2\{(x_1-A_1)^2+(x_2-A_2)^2\}}$, where $(A_1, A_2) = \bar{x}(0)$ is the starting point of the trajectory $\bar{x}$ and $\hat{C}$ is a normalization constant such that $\int_\Omega f_0(x) = 1$. We choose the control bounds $u_a = -5$ and $u_b = 5$. The total number of spatial grid points is $N_x = 60$ and the number of temporal grid points is $N_t = 60$. The desired trajectory is given by $\bar{x}(t) = (1.5t, 0)$ and the potential V is given by

$$V(x,t) = \begin{cases} 100, & (x_1-3)^2 + x_2^2 \le 0.2^2 \\ (x_1 - 1.5t)^2 + x_2^2, & \text{otherwise}, \end{cases} \tag{6.8}$$

where we also model the presence of an obstacle by a cylinder centered at (3,0) and radius 0.2 (a concave function). The time interval is chosen as $[0, T] = [0, 2]$. In correspondence to this setting, the solution of the optimization problem gives the evolution of the controlled PDF as depicted in Figure 6.2 (left), and the control u. The latter is used for Monte-Carlo simulations of the stochastic process for which a few trajectories are shown in Figure 6.2 (right).

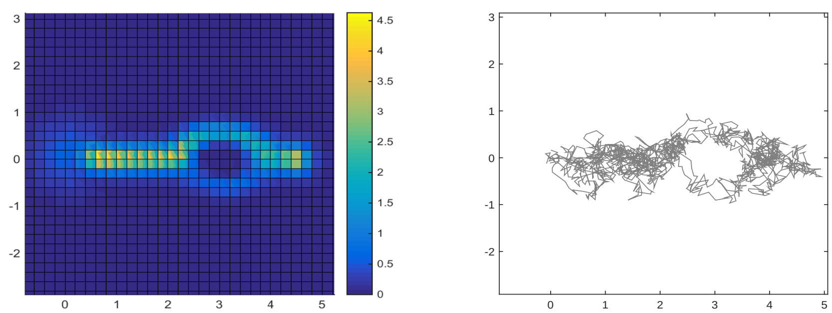

FIG. 6.2. *Evolution of the PDF related to the controlled random process along a trajectory with an obstacle represented by a high potential.*

We see that the control u drives the crowd along the desired path while avoiding the obstacle until the terminal point is reached. Notice that similar results are presented in [109] for the case of control costs given by (6.7).

We complete this section by illustrating the case of FP control of a PDP model of production of antibiotics (subtilin) that is synthetized by the Bacillus subtilis [76, 40]. This case refers to the results of [116]. A subtilin PDP model with the structure (1.8) is specified as follows

$$A_1(x, u_1) = \begin{pmatrix} -\tilde{k}_1\, x_1 + k_2\, \xi\, x_3 + u_1 \\ \chi_{(-\infty,\, \eta D_{max})}(x_1)\, k_3 - \lambda_1 x_2 \\ -\lambda_3 x_3 \end{pmatrix}, \tag{6.9}$$

$$A_2(x, u_2) = \begin{pmatrix} -\tilde{k}_1\, x_1 + k_2\, \xi\, x_3 + u_2 \\ \chi_{(-\infty,\, \eta D_{max})}(x_1)\, k_3 - \lambda_1 x_2 \\ k_5 - \lambda_3 x_3 \end{pmatrix}, \tag{6.10}$$

where x_1 denotes the amount of nutrients, x_2 denotes the concentration of SigH (a sigma factor that regulates gene expressions), and x_3 denotes the concentration of SpaS (antibiotics, subtilin structural peptide). The controls u_1, u_2 model an increase or decrease of concentration of the nutrients. The switching law for this 2-states process is given by (1.9) with $\mu_s = 5$.

The Fokker-Planck system of our subtilin PDP model is given by (1.10) with (6.9) and (6.10), and the stochastic matrix depends on x_2 (see [116] for details). The functions $f_1(x,t)$ and $f_2(x,t)$ are the two marginal PDFs related to the two dynamical states.

Now, assume that the purpose of the control is to maximize the production of subtilin. This objective can be formulated as the minimization of the following cost functional

$$J(f,u) = \frac{1}{2}\sum_{s=1}^{2}\int_0^T\int_\Omega |u_s(x,t)|^2\, f_s(x,t)\,dx\,dt - \sum_{s=1}^{2}\int_\Omega \frac{\alpha}{2\sigma\sqrt{2\pi}}\, e^{-\dfrac{(x_3-d_3)^2}{2\sigma^2}}\, f_s(x,T)\,dx \tag{6.11}$$

The first term in this functional can be interpreted as the mean nutrition effort represented by the control $u = (u_1, u_2)$ and the second term models an attractive potential to a desired value d_3 for the final value of SpaS.

The FP optimal control formulation requires to minimize (6.11) subject to the constraint given by the PDP FP system (1.10) with (6.9)-(6.10). We obtain the following adjoint FP system

$$\frac{1}{2}|u_s(x,t)|^2 + \partial_t p_s(x,t) + \sum_{i=1}^{3} A_s^i(x,u_s)\partial_{x_i} p_s(x,t) = -\sum_{l=1}^{2} Q_{sl}(x) p_l(x,t) \tag{6.12}$$

$$p_s(x,T) = g_s(x) \tag{6.13}$$

$$u_s(x,t) + \partial_{x_1} p_s(x,t) = 0, \qquad s = 1,2. \tag{6.14}$$

Notice that also in this case we have factored out the PDFs multiplying the optimality condition. Thus, we obtain that the adjoint FP problem does not depend on the PDFs and is defined backwards in time. By including the optimality condition in the adjoint FP equation, we obtain

$$\partial_t p_s(x,t) + \sum_{i=1}^{3} A_s^i(x)\partial_{x_i}\, p_s(x,t) + \frac{1}{2}\big(\partial_{x_1} p_s(x,t)\big)^2 = -\sum_{l=1}^{2} Q_{sl}(x)\, p_l(x,t)$$

$$p_s(x,T) = g_s(x), \qquad s = 1,2.$$

From the resulting adjoint variables, we compute the controls using (6.14).

In the numerical experiments, we consider a time horizon $T = 10$ and $\Omega = (1\,,7) \times (0\,,4) \times (-0.5\,,5.5)$, with settings $\mu_s = 5$, $s = 1,2$, $\eta D_{max} = 4.0$, $d_3 = 3$, $\alpha = 10$, $\sigma = 0.3$. The optimal controls u_1 and u_2 are determined solving the FP optimal control problem and thereafter are inserted in the PDP model to perform Monte Carlo validation. Figure 6.3 shows the first 20 runs of the Monte-Carlo simulation and the resulting relative frequencies at terminal time $T = 10$. We see that the control is able to steer the subtilin to increase antibiotic production towards the desired value.

7. Conclusions. An overview of recent developments in the field of control of stochastic systems based on the corresponding probability density functions (PDFs) and the related Fokker-Planck (FP) equations was presented. Many different classes of stochastic systems and the corresponding FP models were considered.

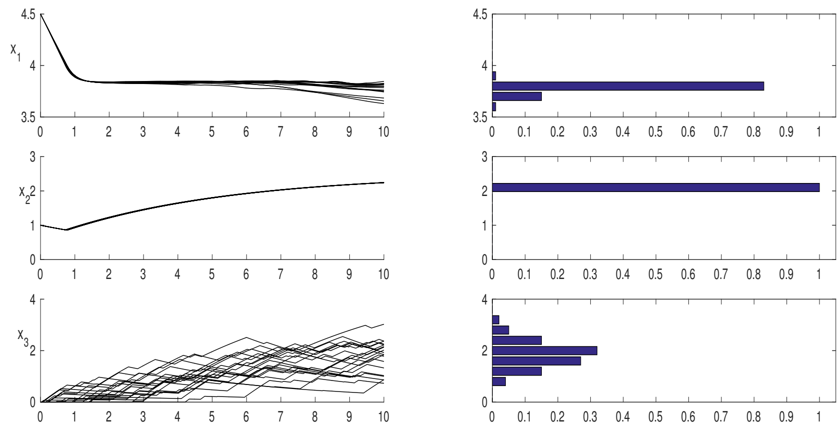

FIG. 6.3. *Left,* 20 *trajectories of Monte Carlo simulation of the controlled system states. Right, relative frequency of* 100 *runs. The control of the nutrients acts to increase the value of the production towards the desired value of production of SpaS.*

In this control framework, starting from the controlled stochastic model, a controlled FP equation is derived and objectives of the control are formulated that may require to follow a given PDF trajectory or to minimize an expectation functional. The resulting controls were validated with the stochastic models by Monte Carlo simulations.

While this work was devoted to stochastic models that result in linear FP equations, the case of N interacting systems and its mean-field limit $N \to \infty$ was discussed to show that in this case a nonlinear FP equation arises.

The fact was discussed that using expectation functionals, the FP controls are equivalent to the ones obtained within the dynamic programming Hamilton-Jacobi-Bellman scheme. Furthermore, a brief review of recent contributions on inverse problems governed by the FP equation was given. This work was completed showing results of the FP control framework applied to challenging control problems with stochastic models.

Acknowledgements. We would like to gratefully acknowledge the collaboration with Gabriele Ciaramella, Christian Clason, Matthias Ehrhardt, Arthur Fleig, Andrei Fursikov, Hanno Gottschalk, Lars Grüne, Abderrahmane Habbal, Marcin Magdziarz, Fabio Nobile, Kees Osterlee, Souvik Roy, Julien Salomon, Georg Stadler, Raul Tempone, Veronika Thalhofer, and Aleksander Weron.

REFERENCES

[1] A. Addou, A. Benbrik, Existence and uniqueness of optimal control for a distributed-parameter bilinear system, J. Dynamical Contr. Systems 8 (2002) 141–152.
[2] K. Aihara, H. Suzuki,*Introduction: Theory of hybrid dynamical systems and its applications to biological and medical systems*, Philos. Trans. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci., 368 (2010) 4893–4914.
[3] M. Annunziato, *Non-Gaussian equilibrium distributions arising from the Langevin equation*, Phys. Rev. E, 65 (2002), 21113–1–6.
[4] M. Annunziato, *Analysis of upwind method for piecewise deterministic Markov processes*, Comp. Meth. Appl. Math., **8** (2008), 3–20.

[5] M. Annunziato and A. Borzì, *Optimal control of probability density functions of stochastic processes*, Math. Mod. Analysis, 15 (2010), 393–407.
[6] M. Annunziato and A. Borzì, *A Fokker-Planck control framework for multidimensional stochastic processes*, Journal of Computational and Applied Mathematics, 237 (2013), 487–507.
[7] M. Annunziato and A. Borzì, *Fokker-Planck-Based Control of a Two-Level Open Quantum System*, Math. Models and Meth. in Appl. Sci. (M3AS), 23 (2013), 2039–2064.
[8] M. Annunziato and A. Borzì, *Optimal Control of a Class of Piecewise Deterministic Processes*, European Journal of Applied Mathematics, 25 (2014), 1–25.
[9] M. Annunziato, A. Borzì, M. Magdziarz, A. Weron, *A fractional Fokker-Planck control framework for subdiffusion processes*, Optimal Contr. Appl. and Meth. Vol. 37 (2016), 290–304.
[10] M. Annunziato, A. Borzì, F. Nobile, and R. Tempone, *On the connection between the Hamilton-Jacobi-Bellman and the Fokker-Planck control frameworks*, Applied Mathematics, 5 (2014), 2476–2484.
[11] M. Annunziato, P. Grigolini and B. J. West, *Canonical and noncanonical equilibrium distribution*, Phys. Rev. E, 66 (2001), 011107–1–13.
[12] M. Annunziato, H. Gottschalk, *Calibration of Lévy Processes using Optimal Control of Kolmogorov Equations with Periodic Boundary Conditions*, e-print:arXiv:1506.08439 (2015).
[13] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge University Press, Cambridge, 2004.
[14] D.G. Aronson, *Non-negative solutions of linear parabolic equations*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^a Serie, 22 (1968), 607–694.
[15] A. Barchielli and M. Gregoratti, *Quantum Trajectories and Measurements in Continuous Time*, Springer, Berlin, 2009.
[16] N. Bäurle and U. Rieder, *MDP Algorithms for portfolio optimization problems in pure jump markets*, Finance and Stochastics, 13 (2009), 591–611.
[17] H.T. Banks, H. T. Tran, and D. E. Woodward, *Estimation of Variable Cefficients in the Fokker–Planck Quations Using Moving Node Finite Elements*, SIAM J. Numer. Anal., 30 (1993), 1574–1602.
[18] E.N. Barron and R. Jensen, *The Pontryagin Maximum Principle From Dynamic Programming and Viscosity Solutions to First-Order Partial Differential Equations*, Transactions of the American Mathematical Society, 298 (1986): 635-41.
[19] J. Bect, *A unifying formulation of the Fokker-Planck-Kolmogorov equation for general stochastic hybrid systems*, Nonlinear Analysis: Hybrid Systems, 4 (2010), 357–370.
[20] N. Bellomo, D. Clarke, L. Gibelli, P. Townsend and B. J. Vreugdenhil, *Human behaviours in evacuation crowd dynamics: From modelling to big data toward crisis management*, Physics of Life Reviews, 18 (2016), 55–65.
[21] A. Bensoussan, J. Frehse, and P. Yam, *Mean Field Games and Mean Field Type Control Theory*, Springer, 2013.
[22] D. Bertsekas, *Dynamic Programming and Optimal Control, Vols. I and II*, Athena Scientific, 2007.
[23] F. J. Beutler and K. W. Ross, *Optimal policies for controlled Markov chains with a constraint*, JMAA, 112 (1985) 236–252.
[24] V.I. Bogachev, N.V. Krilov, M. Röckner, S.V. Shaposhnikov, *Fokker-Planck-Kolmogorov Equations*, Mathematical Surveys and Monographs, Vol 207, American Mathematical Society, 2015.
[25] V.I. Bogachev, M. Röckner, S.V. Shaposhnikov, *Distances between transition probabilities of diffusions and applications to nonlinear Fokker-Planck-Kolmogorov equations*, Journal of Functional Analysis 271 (2016) 1262–1300.
[26] A. Borzì and V. Schulz, *Computational optimization of systems governed by partial differential equations*, SIAM book series on Computational Science and Engineering 08, SIAM, Philadelphia, PA, 2012.
[27] M. Bossy and D. Talay, *A Stochastic Particle Method for the Mckean-Vlasov and the Burgers Equation*, Mathematics of Computation, 66 (1997) 157–192.
[28] T. Breiten, K. Kunisch and L. Pfeiffer, *Control strategies for the Fokker-Planck equation* ESAIM: Control, Optimisation and Calculus of Variations, 2017.
[29] T. H. Breitenbach, M. Annunziato, A. Borzì, *On the optimal control of random walks*, IFAC papers online 49 (2016) 248–253.
[30] R. W. Brockett, *New Issues in the Mathematics of Control*, Mathematics Unlimited–2001 and Beyond, Springer-Berlin (2001) 189–219.
[31] R. W. Brockett, *Optimal control of the Liouville equation*, AMS/IP Studies in Advanced Mathematics, 39 (2007) 23–35.
[32] C. Buet, S. Cordier, and V. Dos Santos, *A conservative and entropy scheme for a simplified model of granular media*, Transport Theory Statist. Phys., 33 (2004) 125–155.
[33] C. Buet and S. Dellacherie, *On the Chang and Cooper scheme applied to a linear Fokker-Planck equation*, Commun. Math. Sci., 8 (2010) 1079–1090.
[34] D. Canter, editor: Fires and Human Behaviour. *David Fulton Publishers, London, 2nd edition,* 1990.
[35] G. Carlier and J. Salomon, *A monotonic algorithm for the optimal control of the Fokker-Planck equation*, CDC, (2008).
[36] E. Casas and F. Tröltzsch, *Second Order Optimality Conditions and Their Role in PDE Control*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 117 (2015) 3–44.
[37] C.G. Cassandras and J. Lygeros, *Stochastic Hybrid Systems*, CRC Press Taylor & Francis

group, 2007.
[38] J. S. Chang and G. Cooper, *A practical Scheme for Fokker-Planck Equations*, Journal of computational Physics, 6 (1970) 1–16.
[39] J. Chiquet, N. Limnios and M. Eid, *Piecewise deterministic markov processes applied to fatigue crack growth modelling*, Journal of Statistical Planning and Inference, (Special Issue on Degradation, Damage, Fatigue and Accelerated Life Models in Reliability Testing), 139 (2009) 1657–1667.
[40] E. Cinquemani, R. Porreca, G. Ferrari-Trecate, J. Lygeros, *subtilin Production by Bacillus Subtilis: Stochastic Hybrid Models and Parameter Identification*, IEEE Trans. on Circuits and Systems-I 53 (2008) 38–50.
[41] F. H. Clarke and R. B. Vinter, *The relationship between the maximum principle and dynamic programming*, SIAM J. Contr. and Opt. 25 (1987) 1291–1311.
[42] C. Cocozza-Thivent, R. Eymard and S. Mercier, *A finite volume scheme for dynamic reliability models*, IMA J. Numer. Analysis, **26** (2006) 446–471.
[43] C. Cocozza-Thivent, R. Eymard, S. Mercier and M. Roussignol, *Characterization of the marginal distributions of Markov processes used in dynamic reliability*, Journal of Applied Mathematics and Stochastic Analysis, Article ID 92156 (2006) 1–18.
[44] O.L.V. Costa and F. Dufour, *Average continuous control of piecewise deterministic Markov processes*, SIAM J. Control Optim., 48 (2010) 4262–4291.
[45] D.R. Cox and H.D. Miller, *The Theory of Stochastic Processes* Chapman & Hall CRC, 2001.
[46] M. Crandall, H. Ishii and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[47] E. B. Davies, *Quantum theory of open systems*, Academic Press, London, 1976.
[48] M. H. A. Davis, *Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models*, Journal of the Royal Statistical Society. Series B (Methodological), **46** (1984) 353–388.
[49] D.A. Dawson, *Critical dynamics and fluctuations for a mean-field model of cooperative behavior*, Journal of Statistical Physics, 31 (1983) 29–85.
[50] S.V. Dolgov, B.N. Khoromskij, and I.V. Oseledets, *Fast solution of parabolic problems in the tensor train/quantized tensor train format with initial application to the Fokker-Planck equation*, SIAM J. Sci. Comput., 34 (2012) A3016–A3038.
[51] F. Dunker and T. Hohage, *On parameter identification in stochastic differential equations by penalized maximum likelihood*, Inverse Problems, 30 (2014) 095001.
[52] W. Ebeling, E. Gudowska-Nowak, and I.M. Sokolov, *On stochastic dynamics in physics - Remarks on history and terminology*, Acta Physica Polonica B, 39 (2008) 1003–1018.
[53] H. Egger and H.W. Engl, *Tikhonov Regularization Applied to the Inverse Problem of Option Pricing: Convergence Analysis and Rates*, I nverse Problems, 21 (2005) 1027–1045.
[54] R. Eymard, S. Mercier , A. Prignet, *An implicit finite volume scheme for a scalar hyperbolic problem with measure data related to piecewise deterministic Markov processes*, Journal of Computational and Applied Mathematics, 222 (2008) 293–323.
[55] A. Faggionato, D. Gabrielli and M. Ribezzi Crivellari, *Non-equilibrium thermodynamics of piecewise deterministic Markov processes*, Journal of Statistical Physics, 137 (2009) 259–304.
[56] A. Fleig, L. Grüne, R. Guglielmi, *Some results on Model Predictive Control for the Fokker-Planck equation*, Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems, (2014) 1203–1206.
[57] A. Fleig, R. Guglielmi, *Bilinear Optimal Control of the Fokker-Planck Equation*, 2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations, 49 (2016) 254–259.
[58] A. Fleig and R. Guglielmi, *Optimal Control of the Fokker–Planck Equation with Space-Dependent Controls*, Journal of Optimization Theory and Applications, 174 (2017) 408–427, http://dx.doi.org/10.1007/s10957-017-1120-5.
[59] A. Fleig, L. Grüne, *Estimates on the Minimal Stabilizing Horizon Length in Model Predictive Control for the Fokker-Planck Equation*, 2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations, 49 (2016) 260–265.
[60] W.H. Fleming and H.M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, Berlin, 2006.
[61] W.H. Fleming and R.W. Rishel, Deterministic and stochastic optimal control, Springer-Verlag, Berlin, 1975.
[62] M.G. Forbes, M. Guay, and J.F. Forbes, *Control design for first-order processes: shaping the probability density of the process state*, Journal of process control 14 (2004) 399–410.
[63] P. Foschi and A. Pascucci, *Kolmogorov equations arising in finance: direct and inverse problems*, Lecture Notes of Seminario Interdisciplinare di Matematica, Università degli Studi della Basilicata, VI, pp. 145–156, 2007.
[64] H. Frantzich, Study of movement on stairs during evacuation using video analysing techniques. *Technical report, Department of Fire Safety Engineering, Lund Institute of Technology, Lund University*, 1996.
[65] B. Gaviraghi, M. Annunziato, A. Borzì, *Analysis of splitting methods for solving a partial integro-differential Fokker-Planck equation*, Applied Mathematics and Computation, 294 (2017) 1–17.
[66] B. Gaviraghi, A. Schindele, M. Annunziato, A. Borzì, *On Optimal Sparse-Control Problems*

Governed by Jump-Diffusion Processes, Applied Mathematics, 7 (2016) 1978–2004.
[67] Ĭ. I. Gihman and A. V. Skorohod, *Stochastic differential equations*, Springer-Verlag, New York, 1972.
[68] N. Gisin and I.C. Percival, *The quantum-state diffusion model applied to open systems*, J. Phys. A: Math. Gen., 25 (1992) 5677.
[69] L. Grüne, *Approximation Properties of Receding Horizon Optimal Control*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 118 (2016) 3–37.
[70] L. Grüne and J. Pannek, *Nonlinear Model Predictive Control, Theory and Algorithms*, Communications and Control Engineering, Springer Verlag, 2011.
[71] L. Grüne and A. Rantzer, *On the infinite horizon performance of receding horizon controllers*, IEEE Trans. Aut. Control, 53 (2008) 2100–2111.
[72] M. Gustafsson and S. Holmgren, *An Implementation Framework for Solving High-Dimensional PDEs on Massively Parallel Computers*, Numerical Mathematics and Advanced Applications, (2010), 417–424. In G. Kreiss et al. (eds.), Numerical Mathematics and Advanced Applications 2009, Springer-Verlag Berlin Heidelberg 2010.
[73] B. Hankin and R. Wright, Passenger flow in subways. *Operational Research Society* 9 (1958) 81–88.
[74] C. Hartmann, B. Schäfer-Bung and A. Thöns-Zueva, *Balanced Averaging of Bilinear Systems with Applications to Stochastic Control*, SIAM Journal on Control and Optimization, 51 (2013) 2356–2378.
[75] D.J. Higham, *An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations*, SIAM Review, 43 (2001) 525–546.
[76] J. Hu, W.C. Wu, S. Sastry, Modeling subtilin Production in Bacillus subtilis Using Stochastic Hybrid Systems., Hybrid Systems: Computation and Control, Lecture Notes in Computer Science, Ed.: R. Alur, G.J. Pappas, Springer (2004) 417–431.
[77] A. Iolov, S, Ditlevsen and A. Longtin, *Stochastic optimal control of single neuron spike trains*, Journal of Neural Engineering 11 (2014) 46004.
[78] A. Iolov, S. Ditlevsen, and A. Longtin, *Optimal Design for Estimation in Diffusion Processes from First Hitting Times*, SIAM/ASA J. Uncertainty Quantification, 5 (2017) 88–110.
[79] K. Ito and K. Kunisch, *Receding horizon optimal control for infinite dimensional systems*, ESAIM: Control, Optimisation and Calculus of Variations, 8 (2002) 741–760.
[80] S. Jäger and E.A. Kostina, *Parameter Estimation for Forward Kolmogorov Equation with Application to Nonlinear Exchange Rate Dynamic*, PAMM, 5 (2005) 745–746.
[81] G. Jumarie, *Tracking control of nonlinear stochastic systems by using path cross-entropy and Fokker-Planck equation*, Internat. J. Systems Sci., 23 (1992) 1101–1114.
[82] M. Kac, *A stochastic model related to the telegrapher's equation*, Rocky Mountain Journal of Mathematics, 4 (1974) 497–510.
[83] L. P. Kadanoff, *More is the Same; Mean Field Theory and Phase Transitions*, Journal of Statistical Physics, 137 (2009) 777–797.
[84] M. Kárný, *Towards Fully Probabilistic Control Design*, Automatica, 32 (1996) 1719–1722.
[85] H. Klüpfel and T. Meyer-König, Simulation of the evacuation of a football stadium. In *Pedestrian and Evacuation Dynamics, Berlin, Springer*, (2002) 423–428.
[86] A. Lachapelle, J. Salomon and G. Turinici. *Computation of mean field equilibria in economics*, Mathematical Models and Methods in Applied Sciences, 20 (2010) 567-588.
[87] J.M. Lasry and P.L. Lions, *Mean field games*, Japanese Journal of Mathematics, 2 (2007) 229–260.
[88] C. Le Bris and P.-L. Lions, *Existence and Uniqueness of Solutions to Fokker-Planck Type Equations with Irregular Coefficients*, Communications in Partial Differential Equations, 33 (2008) 1272–1317.
[89] J.L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, Berlin, 1971.
[90] J.L. Lions, *On the Hamilton-Jacobi-Bellman Equations*, Acta Appl. Mathem., 1 (1983) 17–41.
[91] M. Magdziarz, A. Weron, K. Weron, *Fractional Fokker-Planck dynamics: Stochastic representation and computer simulation*, Physical Review E, 75 (2007) 016708.
[92] V. Méndez, S. Fedotov and W. Horsthemke, *Reaction-Transport Systems: Mesoscopic Foundations, Fronts, and Spatial Instabilities*, Springer-Verlag, Berlin, 2010.
[93] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Physics Reports, 339 (2000) 1–77.
[94] R. Metzler and J. Klafter, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, Journal of Physics A: Mathematical and Theoretical, 37 (2004) R161–R208.
[95] M. Mohammadi and A. Borzì, *Analysis of the Chang-Cooper Discretization Scheme for a Class of Fokker-Planck Equations*, Journal of Numerical Mathematics, 23 (2015) 271–288.
[96] M. Mohammadi and A. Borzì, *A Hermite spectral method for a Fokker-Planck optimal control problem in an unbounded domain*, International Journal for Uncertainty Quantification, 5 (2015) 233–254.
[97] H. Maurer and J. Zowe, *First and second order necessary and sufficient optimality conditions for infinite-dimensional programming problems*, Mathematical Programming, 16 (1979) 98–110.
[98] D.Q. Mayne and H. Michalska, *Receding horizon control for nonlinear systems*, IEEE Trans. Aut. Control, 35 (1990) 814–824.
[99] L. Magni, D.M. Raimondo and F. Allgöwer, *Nonlinear Model Predictive Control*, (Springer,

Berlin, 2009).
[100] G. N. Mil'shtein and Yu. M. Repin, *Action of a Markov process on a system of differential equations*, Differential Equation (translated from russian), 5 (1972) 1010–1019.
[101] A. Morita, *Free Brownian motion of a particle driven by a dichotomous random force*, Phys. Rev A, **41** (1990) 754–760.
[102] B. Øksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, Springer Berlin Heidelberg, 2007.
[103] M. Orlowski, *1-D random walk in presence of a totally reflecting barrier*, Physica status solidi (b) 239 (2003) 158–167.
[104] R. F. Pawula and O. Rice, *On Filtered Binary Processes*. IEEE Trans. Inf. Th., IT-32 (1986) 63–72.
[105] S. Primak, V. Kontorovich, and V. Lyandres, *Stochastic Methods and Their Applications to Communications*, John Wiley & Sons, Chichester, 2004.
[106] D. W. Reid, K. L. Teo, Existence of optimal controls for systems governed by parabolic partial differential equations with Cauchy boundary conditions, Annali di Matematica Pura ed Applicata, 124 (1980) 13–38.
[107] R. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications*, Springer, Berlin, 1996.
[108] S. Roy, M. Annunziato, A. Borzì, *A Fokker-Planck Feedback Control-Constrained Approach for Modeling Crowd Motion*, J. Comp. and Theor. Transport, 45 (2016) 442–458.
[109] S. Roy, M. Annunziato, A. Borzì, C. Klingenberg, *A Fokker-Planck approach to control collective motion*, Comput. Optim. Appl. (2017), https://doi.org/10.1007/s10589-017-9944-3.
[110] D. L. Scharfetter and H. K. Gummel, Large signal analysis of a silicon Read diode, IEEE Trans. Electron. Dev., 16 (1969) 64–77.
[111] Z. Schuss, *Theory and applications of stochastic processes - an analytical approach*, New York, NY, Springer, 2010.
[112] T. I. Seidman, H–X. Zhou, Existence and uniqueness of optimal controls for a quasilinear parabolic equation, SIAM J. Control and Optimization, 20 (1982) 747–762.
[113] R.C. Seydel, *Existence and uniqueness of viscosity solutions for QVI associated with impulse control of jump-diffusions*, Stochastic Processes and their Applications, 119 (2009) 3719–3748.
[114] N. Shiryaev, Probability, Springer Science 1996.
[115] A.-S. Sznitman, *Topics in propagation of chaos*, In P.L. Hennequin, ed., Ecole d'Eté de Probabilités de Saint-Flour XIX - 1989, vol. 1464 of Lecture Notes in Math., pp. 165-251. Springer- Verlag, Berlin, Heidelberg, New York, 1991.
[116] V. Thalhofer, M. Annunziato, A. Borzì, *Stochastic modelling and control of antibiotic subtilin production*, J. Math. Biol., 73 (2016) 727–749.
[117] F. Tröltzsch, *Optimal Control of Partial Differential Equations*, AMS, USA, 2010.
[118] M. Ullah and O. Wolkenhauer, *Stochastic Approaches for Systems Biology*, Springer, 2011.
[119] H. Wang, *Robust control of the output probability density functions for multivariable stochastic systems with guaranteed stability*, IEEE Trans. on Automatic Control, 44 (1999) 2103–2107.
[120] A. Weron and M. Magdziarz, *Anomalous diffusion and semimartingales*, Europhysics Letters, 86 (2009), 60010.
[121] A. Weron,M. Magdziarz,K. Weron, Modeling of subdiffusion in space-time-dependent force fields beyond the fractional Fokker-Planck equation. *Physical Review E*, 77 (2008) 036704.
[122] H. M. Wiseman and G. J. Milburn. Interpretation of quantum jump and diffusion processes illustrated on the Bloch sphere, Phys. Rev. A, 47 (1993) 1652–1666.
[123] W. M. Wonham and A. T. Fuller, *Probability densities of the smoothed random telegraph signal*, J. Electron. Contr., 4 (1958) 567–576.
[124] X. Zhang and C.-W. Shu, *A genuinely high order total variation diminishing scheme for one-dimensional scalar conservation laws*, SIAM J. Numer. Anal. 48 (2010) 772–795.
[125] J. Zhou, G. Li, H. Wang, *Robust tracking controller design for non-Gaussian singular uncertainty stochastic distribution systems*, Automatica 50 (2014) 1296–1303.
[126] H. bin Zubair, C.C. Oosterlee, and R. Wienands, *Multigrid for high dimensional elliptic partial differential equations on non-equidistant grid*, SIAM J. Sci. Comput., 29 (2007) 1613–1636.
[127] E.I. Biao, E.A. Alamou, A. Afouda *Improving rainfall–runoff modelling through the control of uncertainties under increasing climate variability in the Ouémé River basin (Benin, West Africa)*, Hydrological Sciences Journal 61 (2016) 2902–2915.